



Generalized metrics and generalized twistor spaces

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Abstract The twistor construction for Riemannian manifolds is extended to the case of manifolds endowed with generalized metrics (in the sense of generalized geometry à la Hitchin). The generalized twistor space associated to such a manifold is defined as the bundle of generalized complex structures on the tangent spaces of the manifold compatible with the given generalized metric. This space admits natural generalized almost complex structures whose integrability conditions are found in the paper. An interesting feature of the generalized twistor spaces discussed in it is the existence of intrinsic isomorphisms.

Keywords Generalized complex structures · Twistor spaces

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1 Introduction

The concept of generalized complex geometry has been introduced by Nigel Hitchin [19] and further developed by his students M. Gualtieri [15], G. Cavalcanti [4], F. Witt [27] as well as by many other mathematicians and physicists (including Hitchin himself). A generalized almost complex structure in the sense of Hitchin [19] on a smooth manifold M is an endomorphism \mathcal{J} of the bundle $TM \oplus T^*M$ satisfying $\mathcal{J}^2 = -Id$ and compatible with the metric $\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X)$. Similar to the case of a usual almost complex structure, the integrability condition for a generalized almost complex structure \mathcal{J} is defined as the vanishing of its Nijenhuis tensor. However, for \mathcal{J} this tensor is defined by means of the

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bracket, introduced by Courant [8], instead of the Lie bracket. If \mathcal{J} is integrable, it is called a generalized complex structure. Every complex and every symplectic structure determines a generalized complex structure in a natural way. There are several examples of generalized complex structures which are not defined by means of a complex or a symplectic structure, to quote just a few of them [5–7, 15, 20]. In [3, 9–11, 14, 23] such examples have been given by means of the Penrose twistor construction [24, 25] as developed by Atiyah, Hitchin and Singer [1] in the framework of Riemannian geometry. While the base manifold of the twistor space considered in [9, 10] is not equipped with a metric, the base manifold in [11] is a four-dimensional Riemannian manifold M and the one in [3, 14] is a hyper-Kähler manifold. The fiber of the twistor space in [11] consists of (linear) generalized complex structures on the tangent spaces of the base manifold compatible with the metric on $TM \oplus T^*M$ induced by the metric of M . This construction can be placed and generalized in the framework of the concept of a generalized metric, introduced by Gualtieri [15] and Witt [27].

A generalized metric on a vector space T is a subspace E of $T \oplus T^*$ such that $\dim E = \dim T$ and the metric $\langle \cdot, \cdot \rangle$ is positive definite on E . Every generalized metric is uniquely determined by a positive definite metric g and a skew-symmetric 2-form Θ on T so that $E = \{X + \iota_X g + \iota_X \Theta : X \in T\}$. It is convenient to set $E' = E$ and $E'' = E^\perp$, the orthogonal complement of E with respect to $\langle \cdot, \cdot \rangle$. Then $T \oplus T^* = E' \oplus E''$ and the restrictions to E' and E'' of the projection $pr_T : T \oplus T^* \rightarrow T$ are bijective maps sending the metrics $\langle \cdot, \cdot \rangle|_{E'}$ and $\langle \cdot, \cdot \rangle|_{E''}$ to g and $-g$. A generalized complex structure \mathcal{J} on T is called compatible with E if $\mathcal{J}E = E$; in this case $\mathcal{J}E'' = E''$. Define a generalized complex structure \mathcal{J}^2 on T by $\mathcal{J}^2 = \mathcal{J}$ on E' , $\mathcal{J}^2 = -\mathcal{J}$ on E'' , and set $\mathcal{J}^1 = \mathcal{J}$. Then $(\mathcal{J}^1, \mathcal{J}^2)$ is a pair of commuting generalized complex structures for which the metric $\langle -\mathcal{J}^1 \circ \mathcal{J}^2(v), w \rangle = \langle \mathcal{J}^2(v), \mathcal{J}^1(w) \rangle$ on $T \oplus T^*$ is positive definite. Recall that such a pair is called linear generalized Kähler structure [15, 17]. Conversely, for every linear generalized Kähler structure $(\mathcal{J}^1, \mathcal{J}^2)$, the $+1$ -eigenspace of the involution $-\mathcal{J}^1 \mathcal{J}^2$ is a generalized metric compatible with \mathcal{J}^1 . Note also that if g is a positive definite metric on T , then a generalized complex structure on T is compatible with the generalized metric $E = \{X + \iota_X g : X \in T\}$ if and only if it is compatible with the metric on $T \oplus T^*$ induced by g .

A generalized metric on a manifold M is a subbundle E of $TM \oplus T^*M$ such that $\text{rank } E = \dim M$ and the metric $\langle \cdot, \cdot \rangle$ is positive definite on E . Given a generalized metric E , denote by $\mathcal{G}(E)$ the bundle over M whose fibre at every point $p \in M$ consists of all generalized complex structures on the tangent space $T_p M$ compatible with the generalized metric E_p , the fibre of E at p . Equivalently, the fibre of $\mathcal{G}(E)$ is the set of linear generalized Kähler structures on $T_p M$ yielding the given generalized metric E_p . We call $\mathcal{G}(E)$ the generalized twistor space of the generalized Riemannian manifold (M, E) . Let $\mathcal{Z}(E')$ be the bundle over M whose fibre at $p \in M$ consists of complex structures on the vector space E'_p compatible with the metric $g' = \langle \cdot, \cdot \rangle|_{E'}$. Similarly, let $\mathcal{Z}(E'')$ be the bundle of complex structures on the spaces E''_p compatible with the positive definite metric $g'' = -\langle \cdot, \cdot \rangle|_{E''}$. Then the bundle $\mathcal{G}(E)$ is isomorphic to the product bundle $\mathcal{Z}(E') \times \mathcal{Z}(E'')$. Given connections D' and D'' on the bundles E' and E'' one can define a generalized almost complex structure \mathcal{J}_1 on $\mathcal{G}(E)$ following the general scheme of the twistor construction. This structure is an analog of the Atiyah-Hitchin-Singer almost complex structure on the usual twistor space [1]. One can also define three generalized almost complex structures $\mathcal{J}_i, i = 2, 3, 4$, on $\mathcal{G}(E)$ which are analogs of the Eells-Salamon almost complex structure [13]. As one can expect, the structures \mathcal{J}_i are never integrable. As far as \mathcal{J}_1 is concerned, we discuss the integrability conditions for \mathcal{J}_1 in the case when the connections D' and D'' are determined by the generalized metric E as follows. Using the Courant bracket one can define a metric connection D' on the bundle $E' = E$ [22]. Transferring this connection by means of the isomorphism $pr_{TM}|_E : E \rightarrow TM$ we obtain

a connection ∇ on TM compatible with the metric g whose torsion 3-form is $d\Theta$, g and Θ being the metric and the 2-form determined by E [ibid.]. The connection on $TM \oplus T^*M$ induced by ∇ may not preserve the bundle E'' , so we transfer ∇ to a connection D'' on E'' by means of the isomorphism $(pr_{TM}|_{E''})^{-1} : TM \rightarrow E''$. The manifold $\mathcal{G}(E)$ has four connected components and we find the integrability conditions for the restriction of \mathcal{J}_1 to each of these components when $\dim M = 4k$. One of the integrability conditions is $d\Theta = 0$ and the others impose restrictions on the curvature of the Riemannian manifold (M, g) . In the case of an oriented four-dimensional manifold M these curvature restrictions coincide with those found in [11] when $\Theta = 0$. The reason is that if $d\Theta = 0$, ∇ is the Levi-Civita connection of (M, g) used therein to define the twistor space. Another explanation of this fact is that if Θ is closed, the generalized almost complex structures corresponding to the generalized metrics $E = \{X + \iota_X g + \iota_X \Theta : X \in TM\}$ and $\widehat{E} = \{X + \iota_X g : X \in TM\}$ are equivalent (see Sect. 7).

A specific property of generalized twistor spaces that the usual twistor spaces do not possess is that the generalized twistor spaces admit naturally defined (intrinsic) isomorphisms. One of these reflects the so-called B -transforms (the latter being an important feature of the generalized geometry), the others come from the decomposition $TM \oplus T^*M = E' \oplus E''$. In particular, if E and \widehat{E} are generalized metrics on a manifold determined by the same metric g and 2-forms $\Theta, \widehat{\Theta}$ such that the 2-form $\Theta - \widehat{\Theta}$ is closed, the natural generalized almost complex structures on the generalized twistor spaces $\mathcal{G}(E)$ and $\mathcal{G}(\widehat{E})$ are equivalent.

This paper is organized as follows. In Sect. 2, we collect several known facts for generalized geometry used in the paper. The generalized almost complex structures $\mathcal{J}_\varepsilon, \varepsilon = 1, \dots, 4$, on $\mathcal{G}(E)$ mentioned above are defined in the third section. The fourth one contains technical lemmas needed for computing the Nijenhuis tensors of the structures \mathcal{J}_ε . Coordinate-free formulas for the Nijensuis tensors are given in Sect. 5. These formulas are used in Sect. 6 to obtain integrability conditions for \mathcal{J}_ε . Section 7 is devoted to natural isomorphisms of generalized twistor spaces.

2 Preliminaries

2.1 Generalized complex structures on vector spaces

Let T be a n -dimensional real vector space. Suppose we are given a metric g and a complex structure J on T . Let $J^* : T^* \rightarrow T^*$ be the dual map of J . Then the complex structure J is compatible with g , i.e. g -orthogonal, if and only if $J = -J^*$ under the identification $T \cong T^*$ determined by the metric g . Replacing T by the vector space $T \oplus T^*$, note that we have a canonical isomorphism $T \oplus T^* \cong (T \oplus T^*)^*$.

Definition A generalized complex structure on T is a complex structure \mathcal{J} on the space $T \oplus T^*$ such that $\mathcal{J} = -\mathcal{J}^*$ under the identification $T \oplus T^* \cong (T \oplus T^*)^*$.

The latter isomorphism is determined by the metric $\langle X + \alpha, Y + \beta \rangle = \alpha(Y) + \beta(X)$, $X, Y \in T, \alpha, \beta \in T^*$, of signature (n, n) . Thus the condition $\mathcal{J} = -\mathcal{J}^*$ is equivalent to the requirement that \mathcal{J} is compatible with this metric. It turns out that it is convenient to consider one half of that metric, so we set

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}[\alpha(Y) + \beta(X)], \quad X, Y \in T, \quad \alpha, \beta \in T^*.$$

We note also that if a real vector space admits a generalized complex structure it is of even dimension [15].

Notation. The map $T \rightarrow T^*$ determined by a bilinear form φ on T will be denoted again by φ ; thus $\varphi(X)(Y) = \varphi(X, Y)$.

Here are some standard examples of generalized complex structures [15, 16].

Examples 1. Every complex structure J on T determines a generalized complex structure \mathcal{J} defined by

$$\mathcal{J}X = JX, \quad \mathcal{J}\alpha = -J^*\alpha \quad \text{for } X \in T, \alpha \in T^*.$$

2. If ω is a symplectic form on T (a non-degenerate skew-symmetric 2-form), the map $\omega : T \rightarrow T^*$ is an isomorphism and we set

$$\mathcal{J}X = -\omega(X), \quad \mathcal{J}\alpha = \omega^{-1}(\alpha).$$

Then \mathcal{J} is a generalized complex structure on T .

3. Let J be a complex structure on T . Let $T^{\mathbb{C}} = T^{1,0} \oplus T^{0,1}$ be the decomposition of the complexification of T into the direct sum of $(1, 0)$ and $(0, 1)$ -vectors with respect to J . Take a 2-vector $\pi \in \Lambda^2 T^{\mathbb{C}}$. Then, for $\xi \in (T^{1,0})^*$, there is a unique vector $\pi^{\sharp}(\xi) \in T^{\mathbb{C}}$ such that

$$\eta(\pi^{\sharp}(\xi)) = (\xi \wedge \eta)(\pi) \quad \text{for every } \eta \in (T^{1,0})^*.$$

In fact $\pi^{\sharp}(\xi) \in T^{1,0}$ and depends only on the $\Lambda^2 T^{1,0}$ -component of π . Then we can define a generalized complex structure \mathcal{J} on T setting

$$\mathcal{J}X = JX + 2(Im \pi^{\sharp})(\alpha), \quad \mathcal{J}\alpha = -J^*\alpha,$$

where $(Im \pi^{\sharp})(\alpha)$ is the vector in T determined by the identity $\beta((Im \pi^{\sharp})(\alpha)) = (\alpha \wedge \beta)(Im \pi)$ for every $\beta \in T^*$.

4. The direct sum of generalized complex structures is also a generalized complex structure.

5. Any 2-form $B \in \Lambda^2 T^*$ acts on $T \oplus T^*$ via the inclusion $\Lambda^2 T^* \subset \Lambda^2(T \oplus T^*) \cong so(T \oplus T^*)$; in fact this is the action $X + \alpha \rightarrow B(X)$, $X \in T, \alpha \in T^*$. Denote the latter map again by B . Then the invertible map e^B is given by $X + \alpha \rightarrow X + \alpha + B(X)$ and is an orthogonal transformation of $T \oplus T^*$ called a *B-transform*. Thus, given a generalized complex structure \mathcal{J} on T , the map $e^B \mathcal{J} e^{-B}$ is also a generalized complex structure on T , called the *B-transform of \mathcal{J}* .

We refer to [15, 16] for more linear algebra of generalized complex structures on vector spaces.

2.2 Generalized metrics on vector spaces

Let T be a n -dimensional real vector space. Every metric g on T is completely determined by its graph $E = \{X + g(X) : X \in T\} \subset T \oplus T^*$. The restriction to E of the metric $\langle \cdot, \cdot \rangle$ on $T \oplus T^*$ is

$$\langle X + g(X), Y + g(Y) \rangle = g(X, Y).$$

In particular, $\langle \cdot, \cdot \rangle|_E$ is positive definite if g is so. This motivates the following definition [15, 27].

Definition A generalized metric on T is a subspace E of $T \oplus T^*$ such that

1. $dim E = dim T$
2. The restriction of the metric $\langle \cdot, \cdot \rangle$ to E is positive definite.

Set

$$E' = E, \quad E'' = E^\perp = \{w \in T \oplus T^* : \langle w, v \rangle = 0 \text{ for every } v \in E\}.$$

Then $T \oplus T^* = E' \oplus E''$ since the bilinear form $\langle \cdot, \cdot \rangle$ is non-degenerate. Moreover the metric $\langle \cdot, \cdot \rangle$ is negative definite on E'' .

It is easy to see that to determine a generalized metric on T is equivalent to defining an orthogonal, self-adjoint with respect to the metric $\langle \cdot, \cdot \rangle$, linear operator $\mathcal{G} : T \oplus T^* \rightarrow T \oplus T^*$ such that $\langle \mathcal{G}w, w \rangle$ is positive for $w \in T \oplus T^*, w \neq 0$. Such an operator \mathcal{G} is an involution different from \pm the identity and the generalized metric corresponding to it is the $+1$ -eigenspace of \mathcal{G} .

If E is a generalized metric, we have $T^* \cap E = \{0\}$ since the restriction of the metric $\langle \cdot, \cdot \rangle$ to T^* vanishes, while its restriction to E is positive definite. Thus $T \oplus T^* = E \oplus T^*$ since $\dim E = \dim T^* = n$. Then E is the graph of a map $\alpha : T \rightarrow T^*, E = \{X + \alpha(X) : X \in T\}$. Let g and Θ be the bilinear forms on T determined by the symmetric and skew-symmetric parts of α . Under this notation

$$E' = E = \{X + g(X) + \Theta(X) : X \in T\}, \quad E'' = \{X - g(X) + \Theta(X) : X \in T\}. \tag{1}$$

The restriction of the metric $\langle \cdot, \cdot \rangle$ to E is

$$\langle X + g(X) + \Theta(X), Y + g(Y) + \Theta(Y) \rangle = g(X, Y), \quad X, Y \in T. \tag{2}$$

Hence the bilinear form g on T is positive definite. Thus every generalized metric E is uniquely determined by a positive definite metric g and a skew-symmetric 2-form Θ on T such that E has the representation (1). Let $pr_T : T \oplus T^* \rightarrow T$ be the natural projection. The restriction of this projection to E is an isomorphism since $E \cap T^* = \{0\}$. Identity (2) tells us that the isomorphism $pr_T|_E : E \rightarrow T$ is an isometry when E is equipped with the metric $\langle \cdot, \cdot \rangle|_E$ and T with the metric g . Similarly, the map $pr_T|_{E''}$ is an isometry of the metrics $\langle \cdot, \cdot \rangle|_{E''}$ and $-g$.

2.3 Generalized Hermitian structures on vector spaces

Let $E = \{X + g(X) : X \in T\}$ be the generalized metric determined by a positive definite metric g on T and let \mathcal{J} be the generalized complex structure determined by a complex structure J on $T, \mathcal{J}X = JX, \mathcal{J}\alpha = -J^*\alpha, X \in T, \alpha \in T^*$. Then J is compatible with g , i.e. g -orthogonal, if and only if $\mathcal{J}E \subset E$ (and so $\mathcal{J}E = E$). This leads to the following definition, see [15].

Definition A generalized complex structure \mathcal{J} on T is said to be compatible with a generalized metric E if the operator \mathcal{J} preserves the space E .

As usual, if \mathcal{J} is compatible with E , we shall also say that the generalized metric E is compatible with \mathcal{J} . A pair (E, \mathcal{J}) of a generalized metric and a compatible generalized complex structure is said to be a *generalized Hermitian structure*.

Suppose that a generalized metric E is determined by an orthogonal, self-adjoint linear operator $\mathcal{G} : T \oplus T^* \rightarrow T \oplus T^*$ with the property that $\langle \mathcal{G}w, w \rangle$ is positive for $w \neq 0$. Then a generalized complex structure \mathcal{J} is compatible with E if and only if the linear operators \mathcal{J} and \mathcal{G} commute. In this case $\mathcal{J}^2 = \mathcal{G} \circ \mathcal{J}$ is a compatible generalized complex structure on T commuting with the generalized complex structure $\mathcal{J}^1 = \mathcal{J}$. Moreover, the metric

$$\langle -\mathcal{J}^1 \circ \mathcal{J}^2(v), w \rangle = \langle \mathcal{J}^2(v), \mathcal{J}^1(w) \rangle$$

on $T \oplus T^*$ is positive definite. Recall that a pair of $(\mathcal{J}^1, \mathcal{J}^2)$ of commuting generalized complex structures such that the metric above is positive definite is called a *linear generalized Kähler structure* [15, 17]. Given such a structure, the operator $\mathcal{G} = -\mathcal{J}^1 \circ \mathcal{J}^2$ determines a generalized metric compatible with \mathcal{J}^1 and \mathcal{J}^2 . Thus the notion of a generalized Hermitian structure on a vector space is equivalent to the concept of a linear generalized Kähler structure. To fix a generalized metric E means to consider a linear generalized Kähler structure $(\mathcal{J}^1, \mathcal{J}^2)$ such that E is the $+1$ -eigenspace of the involution $\mathcal{G} = -\mathcal{J}^1 \circ \mathcal{J}^2$.

Example 6 Let J be a complex structure on T compatible with a metric g and let $\omega(X, Y) = g(X, JY)$. If \mathcal{J}^1 and \mathcal{J}^2 are the generalized complex structures determined by J and ω , respectively, then $(\mathcal{J}^1, \mathcal{J}^2)$ is a linear generalized Kähler structure. The generalized Hermitian structure defined by $(\mathcal{J}^1, \mathcal{J}^2)$ is (E, \mathcal{J}^1) , where

$$E = \{X + \alpha \in T \oplus T^* : \mathcal{J}^1(X + \alpha) = \mathcal{J}^2(X + \alpha)\} = \{X + g(X) : X \in T\}.$$

This generalized metric is determined by the operator $\mathcal{G} = -\mathcal{J}^1 \circ \mathcal{J}^2$; it is given by $\mathcal{G}(X + g(Y)) = Y + g(X)$, $X, Y \in T$.

Let (E, \mathcal{J}) be a generalized Hermitian structure with $E = \{X + g(X) + \Theta(X) : X \in T\}$. Then $\mathcal{J}E' = E'$, $\mathcal{J}E'' = E''$, where, as above, $E' = E$, $E'' = E^\perp$, and we can define two complex structures on T setting

$$J_1 = (pr_T|E') \circ \mathcal{J} \circ (pr_T|E')^{-1}, \quad J_2 = (pr_T|E'') \circ \mathcal{J} \circ (pr_T|E'')^{-1}. \tag{3}$$

These structures are compatible with the metric g . Thus we can assign a positive definite metric g , a skew-symmetric form Θ and two g -compatible complex structures J_1, J_2 on T to any generalized Hermitian structure (E, \mathcal{J}) . The generalized complex structure \mathcal{J} can be reconstructed from the the data g, Θ, J_1, J_2 by means of an explicit formula [15].

Proposition 1 *Let g be a positive definite metric, Θ -a skew-symmetric 2-form on T , and J_1, J_2 - two complex structures compatible with the metric g . Let $\omega_k(X, Y) = g(X, J_k Y)$ be the fundamental 2-forms of the Hermitian structure (g, J_k) , $k = 1, 2$. Then the block-matrix representation of the generalized complex structure \mathcal{J} determined by the data (g, Θ, J_1, J_2) is of the form*

$$\mathcal{J} = \frac{1}{2} \begin{pmatrix} I & 0 \\ \Theta & I \end{pmatrix} \begin{pmatrix} J_1 + J_2 & \omega_1^{-1} - \omega_2^{-1} \\ -(\omega_1 - \omega_2) & -(J_1^* + J_2^*) \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Theta & I \end{pmatrix},$$

where I is the identity matrix and $\Theta, \omega_1, \omega_2$ stand for the maps $T \rightarrow T^*$ determined by the corresponding 2-forms.

This follows from the identities $\omega_k^{-1} \circ g = J_k, \omega_k = -g \circ J_k, J_k^* \circ g = -g \circ J_k, k = 1, 2$, and the following facts, which will be used further on:

(a) the E' and E'' -components of a vector $X \in T$ are

$$\begin{aligned} X_{E'} &= \frac{1}{2} \{X - (g^{-1} \circ \Theta)(X) + g(X) - (\Theta \circ g^{-1} \circ \Theta)(X)\}, \\ X_{E''} &= \frac{1}{2} \{X + (g^{-1} \circ \Theta)(X) - g(X) + (\Theta \circ g^{-1} \circ \Theta)(X)\}; \end{aligned} \tag{4}$$

the components of a 1-form $\alpha \in T^*$ are given by

$$\begin{aligned} \alpha_{E'} &= \frac{1}{2} \{g^{-1}(\alpha) + \alpha + (\Theta \circ g^{-1})(\alpha)\}, \\ \alpha_{E''} &= \frac{1}{2} \{-g^{-1}(\alpha) + \alpha - (\Theta \circ g^{-1})(\alpha)\}. \end{aligned} \tag{5}$$

$$(b) \begin{aligned} \mathcal{J}(X + g(X) + \Theta(X)) &= J_1 X + g(J_1 X) + \Theta(J_1 X), \\ \mathcal{J}(X - g(X) + \Theta(X)) &= J_2 X - g(J_2 X) + \Theta(J_2 X). \end{aligned}$$

Example 7 Let J be a complex structure on T compatible with a metric g . Then, under the notation in the proposition above, \mathcal{J} is the generalized complex structure defined by J exactly when $J_1 = J_2 = J$ and $\Theta = 0$. The generalized complex structure defined by the 2-form $\omega(X, Y) = g(X, JY)$ is determined by the data $(g, \Theta = 0, J = J_1 = -J_2)$.

Remarks 1. The forms ω_k used here differ by a sign from those used in [15, Proposition 6.12].

2. Suppose that the generalized complex structure \mathcal{J} is determined by the data (g, Θ, J_1, J_2) . Let \mathcal{G} be the endomorphism of $T \oplus T^*$ corresponding to the generalized metric defined by means of (g, Θ) . Then the generalized complex structure $\mathcal{J}^2 = \mathcal{G} \circ \mathcal{J}$ is determined by the data $(g, \Theta, J_1, -J_2)$.

3. It follows from (4) and (5) that the block-matrix representation of the endomorphism \mathcal{G} is [15]

$$\mathcal{G} = \begin{pmatrix} I & 0 \\ \Theta & I \end{pmatrix} \begin{pmatrix} 0 & g^{-1} \\ g & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ -\Theta & I \end{pmatrix}.$$

4. According to Proposition 1, $\mathcal{J} = e^\Theta \mathcal{I} e^{-\Theta}$ where \mathcal{I} is the generalized complex structure on T with block-matrix

$$\mathcal{I} = \frac{1}{2} \begin{pmatrix} J_1 + J_2 & \omega_1^{-1} - \omega_2^{-1} \\ -(\omega_1 - \omega_2) & -(J_1^* + J_2^*) \end{pmatrix}.$$

The restriction to T^* of every B -transform of $T \oplus T^*$ is the identity map. It follows that \mathcal{J} preserves T^* exactly when $J_1 = J_2$ and \mathcal{J} sends T^* into T if and only if $J_1 = -J_2$. Thus, if $J_1 \neq J_2$, the generalized complex structure \mathcal{J} is not a B -transform of the generalized complex structure determined by a complex structure (Example 1), or by a complex structure and a 2-vector (Example 3). Also, if $J_1 \neq -J_2$, \mathcal{J} is not a B -transform of the generalized complex structure determined by a symplectic form (Example 2).

Proposition 1 in [11] and the fact that to define a generalized Hermitian structure is equivalent to defining a linear generalized Kähler structure imply the following

Proposition 2 *Let g be a positive definite metric on T and g^* the metric on T^* determined by g . A generalized complex structure \mathcal{J} on T is compatible with the generalized metric $E = \{X + g(X) : X \in T\}$ if and only if it is compatible with the metric $g \oplus g^*$ on $T \oplus T^*$.*

This can also be proved by means of (4) and (5).

2.4 Generalized almost complex structures on manifolds: the Courant bracket

A *generalized almost complex structure* on an even-dimensional smooth manifold M is, by definition, an endomorphism \mathcal{J} of the bundle $TM \oplus T^*M$ with $\mathcal{J}^2 = -Id$ which preserves the natural metric

$$\langle X + \alpha, Y + \beta \rangle = \frac{1}{2}[\alpha(Y) + \beta(X)], \quad X, Y \in TM, \quad \alpha, \beta \in T^*M. \tag{6}$$

Such a structure is said to be *integrable* or a *generalized complex structure* if its $+i$ -eigensubbundle of $(TM \oplus T^*M) \otimes \mathbb{C}$ is closed under the Courant bracket [19]. Recall that if

X, Y are vector fields on M and α, β are 1-forms, the Courant bracket [8] is defined by the formula

$$[X + \alpha, Y + \beta] = [X, Y] + \mathcal{L}_X\beta - \mathcal{L}_Y\alpha - \frac{1}{2}d(\iota_X\beta - \iota_Y\alpha),$$

where $[X, Y]$ on the right hand-side is the Lie bracket, \mathcal{L} means the Lie derivative, and ι stands for the interior product. Note that the Courant bracket is skew-symmetric like the Lie bracket but it does not satisfy the Jacobi identity.

Examples [15, 16]. 8. The generalized complex structure defined by an almost complex structure J on M is integrable if and only if J is integrable.

9. The generalized complex structure determined by a pre-symplectic form ω is integrable if and only if ω is symplectic, i.e. $d\omega = 0$.

10. Let J be an almost complex manifold on M and π a (smooth) section of $\Lambda^2 T^{1,0}M$. The generalized almost complex structure \mathcal{J} on M defined by means of J and π is integrable if and only if the almost complex structure J is integrable and the field π is holomorphic and Poisson.

As in the case of almost complex structures, the integrability condition for a generalized almost complex structure \mathcal{J} is equivalent to the vanishing of its Nijenhuis tensor N , the latter being defined by means of the Courant bracket:

$$N(A, B) = -[A, B] + [\mathcal{J}A, \mathcal{J}B] - \mathcal{J}[\mathcal{J}A, B] - \mathcal{J}[A, \mathcal{J}B],$$

where A and B are sections of the bundle $TM \oplus T^*M$.

Clearly $N(A, B)$ is skew-symmetric. That N is a tensor, i.e. $N(A, fB) = fN(A, B)$ for every smooth function f on M , follows from the following property of the Courant bracket [15, Proposition 3.17].

Proposition 3 *If f is a smooth function on M , then for every sections A and B of $TM \oplus T^*M$*

$$[A, fB] = f[A, B] + (Xf)B - \langle A, B \rangle df,$$

where X is the TM -component of A .

Let \mathcal{J} be a generalized almost complex structure on a manifold M and let Θ be a (skew-symmetric) smooth 2-form on M . Then, according to Example 5, $e^\Theta \mathcal{J} e^{-\Theta}$ is a generalized almost complex structure on M . The exponential map e^Θ is an automorphism of the Courant bracket (i.e. $[e^\Theta A, e^\Theta B] = e^\Theta[A, B]$) if and only if the form Θ is closed. This key property of the Courant bracket follows from the following formula given in the proof of [15, Proposition 3.23].

Proposition 4 *If Θ is a 2-form on M , then for every sections $A = X + \alpha$ and $B = Y + \beta$ of $TM \oplus T^*M$*

$$[e^\Theta A, e^\Theta B] = e^\Theta[A, B] - \iota_X \iota_Y d\Theta.$$

Thus, if the form Θ is closed, the structure $e^\Theta \mathcal{J} e^{-\Theta}$ is integrable exactly when the structure \mathcal{J} is so.

The diffeomorphisms also give symmetries of the Courant bracket [15].

Proposition 5 *If $f : M \rightarrow N$ is a diffeomorphism, then the Courant bracket is invariant under the bundle isomorphism $F = f_* \oplus (f^{-1})^* : TM \oplus T^*M \rightarrow TN \oplus T^*N$:*

$$[F(A), F(B)] = F([A, B]), \quad A, B \in TM \oplus T^*M.$$

Thus, if \mathcal{J} is a generalized almost complex structure on M and $f : M \rightarrow N$ is a diffeomorphism, then $F \circ \mathcal{J} \circ F^{-1}$ is a generalized almost complex structure, which is integrable if and only if \mathcal{J} is so.

Another important property of the Courant bracket is the following formula proved in [15, Proposition 3.18].

Proposition 6 *Let A, B, C be sections of the bundle $TM \oplus T^*M$ and X the TM -component of A . Then*

$$X \langle B, C \rangle = \langle [A, B] + d \langle A, B \rangle, C \rangle + \langle B, [A, C] + d \langle A, C \rangle \rangle.$$

2.5 Connections induced by a generalized metric

By definition, a *generalized metric* on a manifold M is a subbundle E of $TM \oplus T^*M$ such that $\text{rank } E = \dim M$ and the restriction of the metric $\langle \cdot, \cdot \rangle$ to E is positive definite. Every such a bundle E is uniquely determined by a Riemannian metric g and a 2-form Θ on M . The pair (M, E) will be called a generalized Riemannian manifold.

Let $E' = E$ be a generalized metric and, as above, denote E^\perp by E'' . For $X \in TM$, set

$$X'' = (pr_{TM}|_{E''})^{-1}(X) \in E'',$$

where $pr_{TM} : TM \oplus T^*M \rightarrow TM$ is the natural projection. It follows from Proposition 6 that if B and C are sections of the bundle E

$$X \langle B, C \rangle = \langle [X'', B]_E, C \rangle + \langle B, [X'', C]_E \rangle,$$

where the subscript E means “the E -component with respect to the decomposition $TM \oplus T^*M = E \oplus E''$ ”. The latter identity is reminiscent of the condition for a connection on the bundle E to be compatible with the metric $\langle \cdot, \cdot \rangle$. In fact, we have the following statement [21, 22].

Proposition 7 *If S is a section of E , then*

$$\nabla_X^E S = [X'', S]_E$$

defines a connection preserving the metric $\langle \cdot, \cdot \rangle$.

Suppose that E is determined by the Riemannian metric g and the 2-form Θ , so that $E = \{X + g(X) + \Theta(X) : X \in TM\}$. Transferring the connection ∇^E from the bundle E to the bundle TM via the isomorphism $pr_{TM}|_E : E \rightarrow TM$ we get a connection on TM preserving the metric g . Denote this connection by ∇ and let T be its torsion. Then we have [21, 22]:

Proposition 8 *The torsion T of the connection ∇ is skew-symmetric and is given by*

$$g(T(X, Y), Z) = d\Theta(X, Y, Z), \quad X, Y, Z \in TM.$$

Interchanging the roles of E and $E'' = \{X - g(X) + \Theta(X) : X \in TM\}$ we can get a connection ∇'' on TM preserving the Riemannian metric g and having torsion T'' with $g(T''(X, Y), Z) = -d\Theta(X, Y, Z)$.

If we set $\nabla' = \nabla$, then $\frac{1}{2}(\nabla' + \nabla'')$ is a metric connection with vanishing torsion, so it is the Levi-Civita connection of the Riemannian manifold (M, g) .

2.6 The space of compatible generalized complex structures

Let E be a generalized metric on the vector space T . As above, set $E' = E$ and $E'' = E^\perp$, the orthogonal complement being taken with respect to the metric $\langle \cdot, \cdot \rangle$ on $T \oplus T^*$.

Suppose that T is of even dimension $n = 2m$. Denote by $\mathcal{G}(E)$ the set of generalized complex structures compatible with E . Equivalently, $\mathcal{G}(E)$ is the set of linear generalized Kähler structures, which determine the generalized metric E . This (non-empty) set has the structure of an imbedded submanifold of the vector space $so(n, n)$ of the endomorphisms of $T \oplus T^*$, which are skew-symmetric with respect to the metric $\langle \cdot, \cdot \rangle$. The tangent space of $\mathcal{G}(E)$ at a point \mathcal{J} consists of the endomorphisms V of $T \oplus T^*$ anti-commuting with \mathcal{J} , skew-symmetric w.r.t. $\langle \cdot, \cdot \rangle$ and such that $VE \subset E$. Such an endomorphism V sends also E'' into itself. Note also that the smooth manifold $\mathcal{G}(E)$ admits a natural complex structure \mathcal{J} given by $V \rightarrow \mathcal{J} \circ V$.

For every $\mathcal{J} \in \mathcal{G}(E)$, the restrictions $J' = \mathcal{J}|E'$ and $J'' = \mathcal{J}|E''$ are complex structures on the vector spaces E' and E'' compatible with the positive definite metrics $g' = \langle \cdot, \cdot \rangle|E'$ and $g'' = -\langle \cdot, \cdot \rangle|E''$, respectively. Denote by $Z(E')$ and $Z(E'')$ the sets of complex structures on E' and E'' compatible with the metrics g' and g'' . Consider these sets with their natural structures of imbedded submanifolds of the vector spaces $so(E', g')$ and $so(E'', g'')$, where $so(E', g')$ is, as usual, the space of g' -skew-symmetric endomorphisms of E' , and similarly for $so(E'', g'')$. The tangent space of $Z(E')$ at J' is $T_{J'}Z(E') = \{V' \in so(E', g') : V'J' + J'V' = 0\}$; similarly for the tangent space $T_{J''}Z(E'')$. Recall that the manifold $Z(E')$ admits a complex structure \mathcal{J}' defined by $V' \rightarrow J' \circ V'$; similarly $V'' \rightarrow J'' \circ V''$ defines a complex structure \mathcal{J}'' on $Z(E'')$. The map $\mathcal{J} \rightarrow (J', J'')$ is a diffeomorphism sending a tangent vector V at \mathcal{J} to the tangent vector (V', V'') where $V' = V|E'$ and $V'' = V|E''$. Thus $\mathcal{G}(E) \cong Z(E') \times Z(E'')$ admits four complex structure defined by

$$K_1(V', V'') = (J' \circ V', J'' \circ V''), \quad K_2(V', V'') = (J' \circ V', -J'' \circ V''),$$

$$K_3 = -K_2, \quad K_4 = -K_1.$$

Clearly, the map $\mathcal{J} \rightarrow (J', J'')$ is biholomorphic with respect to the complex structures \mathcal{J} on $\mathcal{G}(E)$ and K_1 on $Z(E') \times Z(E'')$.

Let $G'(S'_1, S'_2) = -\frac{1}{2}Trace_{g'}(S'_1 \circ S'_2)$ be the standard metric on $so(E', g')$ induced by g' ; similarly denote by G'' the metric on $so(E'', g'')$ induced by g'' . Then, as is well-known, (G', \mathcal{J}') and (G'', \mathcal{J}'') are Kähler structures on $Z(E')$ and $Z(E'')$, so $(G = G' + G'', K_\varepsilon)$, $\varepsilon = 1, \dots, 4$, is a Kähler structure on $\mathcal{G}(E)$.

Let g and Θ be the positive definite metric and the skew-symmetric 2-form on T determined by E , so that $E = \{X + g(X) + \Theta(X) : X \in T\}$. Denote by $Z(T, g)$ the manifold of all complex structures on T compatible with the metric g considered as an imbedded submanifold of the space $so(g)$ of g -skew-symmetric endomorphisms of T . Endow $Z(T, g)$ with its natural complex structure and compatible metric. For $\mathcal{J} \in \mathcal{G}(E)$, let J_1 and J_2 be the g -compatible complex structures on T defined by means of \mathcal{J} :

$$J_1 = (pr_T|E') \circ \mathcal{J} \circ (pr_T|E')^{-1}, \quad J_2 = (pr_T|E'') \circ \mathcal{J} \circ (pr_T|E'')^{-1}.$$

Then the map $\mathcal{J} \rightarrow (J_1, J_2)$ is an isometry of $\mathcal{G}(E)$ onto $Z(T, g) \times Z(T, g)$. Moreover it sends a tangent vector V at $\mathcal{J} \in \mathcal{G}(E)$ to the tangent vector (V_1, V_2) , where

$$V_1 = (pr_T|E') \circ V \circ (pr_T|E')^{-1}, \quad V_2 = (pr_T|E'') \circ V \circ (pr_T|E'')^{-1}.$$

Hence $\mathcal{J} \rightarrow (J_1, J_2)$ is a biholomorphic map. The manifold $Z(T, g)$ has the homogeneous representation $O(2m)/U(m)$ where $2m = dim T$ and the group $O(2m) \cong O(g)$ acts by

conjugation. In particular, it has two connected components, each of them having the homogeneous representation $SO(2m)/U(m)$. Fix an orientation on the vector space T and denote by Z_{\pm} the space of complex structures on T compatible with the metric g and yielding \pm the orientation of T . Then Z_+ and Z_- are the connected components of $Z(T, g)$. Thus $\mathcal{G}(E)$ has four connected components biholomorphically isometric to $Z_+ \times Z_+$, $Z_+ \times Z_-$, $Z_- \times Z_+$, $Z_- \times Z_-$. If $\dim T = 4k$, the open subsets \mathcal{G}_+ and \mathcal{G}_- of $\mathcal{G}(E)$ biholomorphic to $(Z_+ \times Z_+) \cup (Z_- \times Z_-)$ and $(Z_+ \times Z_-) \cup (Z_- \times Z_+)$ can be described in terms of the generalized complex structures as follows. Recall first that the vector space $T \oplus T^*$ has a canonical orientation; if $\{a_i\}$ is an arbitrary basis of T and $\{\alpha_i\}$ is its dual basis, $i = 1, \dots, n$, the orientation of the space $T \oplus T^*$ defined by the basis $\{a_i, \alpha_i\}$ does not depend on the choice of the basis $\{a_i\}$. Let \mathcal{G} be the endomorphism of $T \oplus T^*$ determined by the generalized metric E . Then, by [15, Remark 6.14 and Proposition 4.7], $\mathcal{J} \in \mathcal{G}_{\pm}(E)$ if and only if the complex structures $\mathcal{J}_1 = \mathcal{J}$ and $\mathcal{J}_2 = \mathcal{G} \circ \mathcal{J}_1$ both induce \pm the canonical orientation of $T \oplus T^*$. We also note that if $\dim T = 4k + 2$, then, by [15, Propostion 6.8], one of the complex structures $\mathcal{J}_1 = \mathcal{J} \in \mathcal{G}(E)$ and $\mathcal{J}_2 = \mathcal{G} \circ \mathcal{J}_1$ induces the canonical orientation of $T \oplus T^*$, while the other one the opposite orientation.

3 Generalized twistor spaces

Let M be a smooth manifold of dimension $n = 2m$ equipped with a generalized metric E determined by a Riemannian metric g and a 2-form Θ on M . Denote by $\mathcal{G} = \mathcal{G}(E) \rightarrow M$ the bundle over M whose fibre at a point $p \in M$ consists of all generalized complex structures on $T_p M$ compatible with the generalized metric E_p , the fibre of E at p . We call \mathcal{G} generalized twistor space of the generalized Riemannian manifold (M, E) .

Set $E' = E$ and $E'' = E^{\perp}$, the orthogonal complement of E in $TM \oplus T^*M$ with respect to the metric $\langle \cdot, \cdot \rangle$. Denote by $\mathcal{Z}(E')$ the bundle over M whose fibre at a point $p \in M$ is constituted of all complex structures on the vector space E'_p compatible with the positive definite metric $g' = \langle \cdot, \cdot \rangle|_{E'}$. Define a bundle $\mathcal{Z}(E'')$ in a similar way, E'' being endowed with the metric $g'' = -\langle \cdot, \cdot \rangle|_{E''}$. Then \mathcal{G} is identified with the product bundle $\mathcal{Z}(E') \times \mathcal{Z}(E'')$ by the map $\mathcal{G}_p \ni J \rightarrow (J|_{E'_p}, J|_{E''_p})$.

Suppose we are given metric connections D' and D'' on E' and E'' , respectively, and let $D = D' \oplus D''$ be the connection on $E' \oplus E'' = TM \oplus T^*M$ determined by D' and D'' .

The bundle $\mathcal{Z}(E')$ is a subbundle of the vector bundle $A(E')$ of g' -skew-symmetric endomorphisms of E' , and similarly for $\mathcal{Z}(E'')$. Henceforth we shall consider the bundle $\mathcal{G} \cong \mathcal{Z}(E') \times \mathcal{Z}(E'')$ as a subbundle of the vector bundle $\pi : A(E') \oplus A(E'') \rightarrow M$. The connection on $A(E') \oplus A(E'')$ induced by the connection $D = D' \oplus D''$ on $E' \oplus E''$ will again be denoted by D . It is easy to see that the horizontal space of $A(E') \oplus A(E'')$ with respect to D at every point of \mathcal{G} is tangent to \mathcal{G} (cf. the next section). Thus the connection D gives rise to a splitting $\mathcal{V} \oplus \mathcal{H}$ of the tangent bundle of the bundle \mathcal{G} into vertical and horizontal parts. Then, following the standard twistor construction, we can define four generalized almost complex structures $\mathcal{J}_{\varepsilon}$ on the manifold \mathcal{G} ; when we need to indicate explicitly the bundle E we hall write $\mathcal{J}_{\varepsilon}^E$.

The vertical space \mathcal{V}_J of \mathcal{G} at a point $J \in \mathcal{G}$ is the tangent space at J of the fibre through this point. This fibre is the manifold $\mathcal{G}(E_{\pi(J)})$, which admits four complex structures K_{ε} defined in the preceding section. We define $\mathcal{J}_{\varepsilon}|(\mathcal{V}_J \oplus \mathcal{V}_J^*)$ to be the generalized complex structure determined by the complex structure K_{ε} . Thus

$$\mathcal{J}_{\varepsilon} = K_{\varepsilon} \text{ on } \mathcal{V}_J, \quad \mathcal{J}_{\varepsilon} = -K_{\varepsilon}^* \text{ on } \mathcal{V}_J^*, \quad \varepsilon = 1, 2, 3, 4.$$

The horizontal space \mathcal{H}_J is isomorphic to the tangent space $T_{\pi(J)}M$ via the differential π_{*J} . If $\pi_{\mathcal{H}}$ is the restriction of π_* to \mathcal{H} , the image of every $A \in T_pM \oplus T_p^*M$ under the map $\pi_{\mathcal{H}}^{-1} \oplus \pi_{\mathcal{H}}^*$ will be denoted by A^h . Thus, for $J \in \mathcal{G}$, $Z \in T_{\pi(J)}M$ and $\omega \in T_{\pi(J)}^*M$, we have $\omega_J^h(Z_J^h) = \omega_{\pi(J)}(Z)$. The elements of \mathcal{H}_J^* , resp. \mathcal{V}_J^* , will be considered as 1-forms on $T_J\mathcal{G}$ vanishing on \mathcal{V}_J , resp. \mathcal{H}_J .

Now we define a generalized complex structure \mathcal{J} on the vector space $\mathcal{H}_J \oplus \mathcal{H}_J^*$ as the lift of the endomorphism J of $T_{\pi(J)}M \oplus T_{\pi(J)}^*M$ by the isomorphism $\pi_{\mathcal{H}} \oplus (\pi_{\mathcal{H}}^{-1})^* : \mathcal{H}_J \oplus \mathcal{H}_J^* \rightarrow T_{\pi(J)}M \oplus T_{\pi(J)}^*M$:

$$\mathcal{J}A_J^h = (JA)_J^h, \quad A \in T_{\pi(J)}M \oplus T_{\pi(J)}^*M.$$

Finally, we set $\mathcal{J}_\varepsilon = \mathcal{J}$ on $\mathcal{H} \oplus \mathcal{H}^*$.

Remark 5 According to Remark 4, if $n \geq 2$, the generalized almost complex structures \mathcal{J}_ε are not B -transforms of generalized complex structures induced by complex or pre-symplectic structures.

4 Technical lemmas

To compute the Nijenhuis tensor of the generalized almost complex structures \mathcal{J}_ε , $\varepsilon = 1, 2, 3, 4$, on the twistor space \mathcal{G} we need some preliminary lemmas.

Let $(\mathcal{U}, x_1, \dots, x_{2m})$ be a local coordinate system of M and $\{Q'_1, \dots, Q'_{2m}\}, \{Q''_1, \dots, Q''_{2m}\}$ orthonormal frames of E' and E'' on \mathcal{U} , respectively. Define sections $S'_{ij}, S''_{ij}, 1 \leq i, j \leq 2m$, of $A(E')$ and $A(E'')$ by the formulas

$$S'_{ij}Q'_k = \delta_{ik}Q'_j - \delta_{kj}Q'_i, \quad S''_{ij}Q''_k = \delta_{ik}Q''_j - \delta_{kj}Q''_i. \tag{7}$$

Then S'_{ij} and S''_{ij} with $i < j$ form orthonormal frames of $A(E')$ and $A(E'')$ with respect to the metrics G' and G'' defined by

$$G'(a', b') = -\frac{1}{2} \text{Trace}_{g'}(a' \circ b')$$

for $a', b' \in A(E')$, and similarly for G'' .

For $a = (a', a'') \in A(E') \oplus A(E'')$, set

$$\tilde{x}_i(a) = x_i \circ \pi(a), \quad y'_{kl}(a) = G'(a', S'_{kl} \circ \pi(a)), \quad y''_{kl}(a) = G''(a'', S''_{kl} \circ \pi(a)). \tag{8}$$

Then $(\tilde{x}_i, y'_{jk}, y''_{jk}), 1 \leq i \leq 2m, 1 \leq j < k \leq 2m$, is a local coordinate system on the total space of the bundle $A(E') \oplus A(E'')$.

Let

$$V = \sum_{j < k} \left[v'_{jk} \frac{\partial}{\partial y'_{jk}}(J) + v''_{jk} \frac{\partial}{\partial y''_{jk}}(J) \right] \tag{9}$$

be a vertical vector of \mathcal{G} at a point J . It is convenient to set $v'_{ij} = -v'_{ji}, v''_{ij} = -v''_{ji}$ and $y'_{ij} = -y'_{ji}, y''_{ij} = -y''_{ji}$ for $i \geq j, 1 \leq i, j \leq 2m$. Then the endomorphism V of $T_pM \oplus T_p^*M, p = \pi(J)$, is determined by

$$V Q'_i = \sum_{j=1}^{2m} v'_{ij} Q'_j, \quad V Q''_i = \sum_{j=1}^{2m} v''_{ij} Q''_j.$$

Moreover

$$\mathcal{J}_\varepsilon V = (-1)^{\varepsilon+1} \sum_{j < k} \sum_{s=1}^{2m} \left[\pm v'_{js} y'_{sk} \frac{\partial}{\partial y'_{jk}} + v''_{js} y''_{sk} \frac{\partial}{\partial y''_{jk}} \right], \tag{10}$$

where the plus sign corresponds to $\varepsilon = 1, 4$ and the minus sign to $\varepsilon = 2, 3$.

Note also that, for every $A \in T_p M \oplus T_p^* M$, we have

$$\begin{aligned} A^h &= \sum_{i=1}^{2m} \left[\langle (A, Q'_i) \circ \pi \rangle Q_i'^h - \langle (A, Q''_i) \circ \pi \rangle Q_i''^h \right], \\ \mathcal{J}A^h &= \sum_{i,j=1}^{2m} \left[\langle (A, Q'_i) \circ \pi \rangle y'_{ij} Q_j'^h - \langle (A, Q''_i) \circ \pi \rangle y''_{ij} Q_j''^h \right]. \end{aligned} \tag{11}$$

For each vector field

$$X = \sum_{i=1}^{2m} X^i \frac{\partial}{\partial x_i}$$

on \mathcal{U} , the horizontal lift X^h on $\pi^{-1}(\mathcal{U})$ is given by

$$\begin{aligned} X^h &= \sum_{l=1}^{2m} (X^l \circ \pi) \frac{\partial}{\partial \bar{x}_l} \\ &\quad - \sum_{i < j} \sum_{k < l} \left[y'_{kl} (G'(D_X S'_{kl}, S'_{ij}) \circ \pi) \frac{\partial}{\partial y'_{ij}} + y''_{kl} (G''(D_X S''_{kl}, S''_{ij}) \circ \pi) \frac{\partial}{\partial y''_{ij}} \right]. \end{aligned} \tag{12}$$

Let $a = (a', a'') \in A(E') \oplus A(E'')$. Denote by $A(E'_{\pi(a)})$ the fiber of $A(E')$ at the point $\pi(a)$ and similarly for $A(E''_{\pi(a)})$. Then (12) implies that, under the standard identification of $T_a(A(E'_{\pi(a)}) \oplus A(E''_{\pi(a)}))$ with the vector space $A(E'_{\pi(a)}) \oplus A(E''_{\pi(a)})$, we have

$$[X^h, Y^h]_a = [X, Y]_a^h + R(X, Y)a, \tag{13}$$

where $R(X, Y)a = R(X, Y)a' + R(X, Y)a''$ is the curvature of the connection D (for the curvature tensor we adopt the following definition: $R(X, Y) = D_{[X, Y]} - [D_X, D_Y]$). Note also that (9) and (12) imply the well-known fact that

$$[V, X^h] \text{ is a vertical vector field.} \tag{14}$$

Notation. Let $J \in \mathcal{G}$ and $p = \pi(J)$. Take orthonormal bases $\{a'_1, \dots, a'_{2m}\}, \{a''_1, \dots, a''_{2m}\}$ of E'_p, E''_p such that $a'_{2l} = J a'_{2l-1}, a''_{2l} = J a''_{2l-1}$ for $l = 1, \dots, m$. Let $\{Q'_i\}, \{Q''_i\}, i = 1, \dots, 2m$, be orthonormal frames of E', E'' in the vicinity of the point p such that

$$Q'_i(p) = a'_i, Q''_i(p) = a''_i \quad \text{and} \quad D Q'_i|_p = 0, D Q''_i|_p = 0, \quad i = 1, \dots, 2m.$$

Define a section $S = (S', S'')$ of $A(E') \oplus A(E'')$ setting

$$S' Q'_{2l-1} = Q'_{2l}, \quad S'' Q''_{2l-1} = Q''_{2l}, \quad S' Q'_{2l} = -Q'_{2l-1}, \quad S'' Q''_{2l} = -Q''_{2l-1},$$

$l = 1, \dots, m$. Then,

$$S(p) = J, \quad DS|_p = 0.$$

In particular $X_J^h = S_*X$ for every $X \in T_pM$.

Clearly, the section S takes its values in \mathcal{G} , hence the horizontal space of $A(E') \oplus A(E'')$ with respect to the connection D at any $J \in \mathcal{G}$ is tangent to \mathcal{G} .

Further on, given a smooth manifold M , the natural projections of $TM \oplus T^*M$ onto TM and T^*M will be denoted by π_1 and π_2 , respectively. The natural projections of $\mathcal{H} \oplus \mathcal{H}^*$ onto \mathcal{H} and \mathcal{H}^* will also be denoted by π_1 and π_2 when this will not cause confusion. Thus if $\pi_1(A) = X$ for $A \in TM \oplus T^*M$, then $\pi_1(A^h) = X^h$ and similarly for $\pi_2(A)$ and $\pi_2(A^h)$.

We shall use the above notations throughout the next sections.

Note that, although $DS|_p = 0$, $D\pi_1(S)$ and $D\pi_2(S)$ may not vanish at the point p since the connection D may not preserve TM or T^*M .

Lemma 1 *If A and B are sections of the bundle $TM \oplus T^*M$ near p , then*

- (i) $[\pi_1(A^h), \pi_1(\mathcal{J}B^h)]_J = [\pi_1(A), \pi_1(SB)]_J^h + R(\pi_1(A), \pi_1(JB))J$.
- (ii) $[\pi_1(\mathcal{J}A^h), \pi_1(\mathcal{J}B^h)]_J = [\pi_1(SA), \pi_1(SB)]_J^h + R(\pi_1(JA), \pi_1(JB))J$.

Proof Set $X = \pi_1(A)$. By (12), we have $X_J^h = \sum_{l=1}^{2m} X^l(p) \frac{\partial}{\partial x_l}(J)$ since $DS'_{kl}|_p = DS''_{kl}|_p = 0$, $k, l = 1, \dots, 2m$. Then, using (11), we get

$$\begin{aligned}
 [X^h, \pi_1(\mathcal{J}B^h)]_J &= \sum_{i,j=1}^{2m} \left[\langle B, Q'_i \rangle_p y'_{ij}(J) [X^h, \pi_1(Q'_j)^h]_J + X_p(\langle B, Q'_i \rangle) y'_{ij}(J) (\pi_1(Q'_j))^h_J \right] \\
 &\quad - \sum_{i,j=1}^{2m} \left[\langle B, Q''_i \rangle_p y''_{ij}(J) [X^h, \pi_1(Q''_j)^h]_J + X_p(\langle B, Q''_i \rangle) y''_{ij}(J) (\pi_1(Q''_j))^h_J \right].
 \end{aligned}
 \tag{15}$$

We also have

$$SB = \sum_{i,j=1}^{2m} \left[\langle B, Q'_i \rangle (y'_{ij} \circ S) Q'_j - \langle B, Q''_i \rangle (y''_{ij} \circ S) Q''_j \right].
 \tag{16}$$

Therefore

$$\begin{aligned}
 [X, \pi_1(SB)]_p &= \sum_{i,j=1}^{2m} \left[\langle B, Q'_i \rangle_p y'_{ij}(J) [X, \pi_1(Q'_j)]_p + X_p(\langle B, Q'_i \rangle) y'_{ij}(J) (\pi_1(Q'_j))_p \right] \\
 &\quad - \sum_{i,j=1}^{2m} \left[\langle B, Q''_i \rangle_p y''_{ij}(J) [X, \pi_1(Q''_j)]_p + X_p(\langle B, Q''_i \rangle) y''_{ij}(J) (\pi_1(Q''_j))_p \right].
 \end{aligned}
 \tag{17}$$

Now formula (i) follows from (15), (13) and (17). A similar computation gives (ii). \square

For any (local) section $a = (a', a'')$ of $A(E') \oplus A(E'')$, denote by \tilde{a} the vertical vector field on \mathcal{G} defined by

$$\tilde{a}_J = \left(a'_{\pi(J)} + (J|E') \circ a'_{\pi(J)} \circ (J|E'), a''_{\pi(J)} + (J|E'') \circ a''_{\pi(J)} \circ (J|E'') \right).
 \tag{18}$$

Let us note that for every $J \in \mathcal{G}$ we can find sections a_1, \dots, a_s , $s = 2(m^2 - m)$, of $A(E') \oplus A(E'')$ near the point $p = \pi(J)$ such that $\tilde{a}_1, \dots, \tilde{a}_s$ form a basis of the vertical vector space at each point in a neighbourhood of J .

Lemma 2 *Let $J \in \mathcal{G}$ and let a be a section of $A(E') \oplus A(E'')$ near the point $p = \pi(J)$. Then, for any section A of the bundle $TM \oplus T^*M$ near p , we have (for the Lie brackets)*

- (i) $[\pi_1(A^h), \widetilde{a}]_J = (D_{\pi_1(A)}\widetilde{a})_J$.
- (ii) $[\pi_1(A^h), \mathcal{J}_\varepsilon\widetilde{a}]_J = K_\varepsilon(D_{\pi_1(A)}a)_J$.
- (iii) $[\pi_1(\mathcal{J}A^h), \widetilde{a}]_J = (D_{\pi_1(\mathcal{J}A)}a)_J - (\pi_1(\widetilde{a}(A)))_J^h$.
- (iv) $[\pi_1(\mathcal{J}A^h), \mathcal{J}_\varepsilon\widetilde{a}]_J = K_\varepsilon(D_{\pi_1(\mathcal{J}A)}a)_J - (\pi_1((K_\varepsilon\widetilde{a})(A)))_J^h$.

Proof Let $a'(Q'_i) = \sum_{j=1}^{2m} a'_{ij} Q'_j$, $a''(Q''_i) = \sum_{j=1}^{2m} a''_{ij} Q''_j$, $i = 1, \dots, 2m$. Then, in the local coordinates of $A(E') \oplus A(E'')$ introduced above,

$$\widetilde{a} = \sum_{i < j} \left[\widetilde{a}'_{ij} \frac{\partial}{\partial y'_{ij}} + \widetilde{a}''_{ij} \frac{\partial}{\partial y''_{ij}} \right],$$

where

$$\widetilde{a}'_{ij} = a'_{ij} \circ \pi + \sum_{k,l=1}^{2m} y'_{ik} (a'_{kl} \circ \pi) y'_{lj}, \quad a''_{ij} = a''_{ij} \circ \pi + \sum_{k,l=1}^{2m} y''_{ik} (a''_{kl} \circ \pi) y''_{lj}. \quad (19)$$

Let us also note that for every vector field X on M near the point p , we have in view of (12)

$$X^h_J = \sum_{i=1}^{2m} X^i(p) \frac{\partial}{\partial \widetilde{x}_i}(J),$$

$$\left[X^h, \frac{\partial}{\partial y'_{ij}} \right]_J = \left[X^h, \frac{\partial}{\partial y''_{ij}} \right]_J = 0$$

since $DS'_{ij}|_p = DS''_{ij}|_p = 0$. Moreover,

$$(D_{X_p}a')(Q'_i) = \sum_{j=1}^{2m} X_p(a'_{ij}) Q'_j, \quad (D_{X_p}a'')(Q''_i) = \sum_{j=1}^{2m} X_p(a''_{ij}) Q''_j \quad (20)$$

since $DQ'_i|_p = DQ''_i|_p = 0$. Now the lemma follows by simple computations making use of (10) and (11). □

Lemma 3 *Let A and B be sections of the bundle $TM \oplus T^*M$ near p , and let $Z \in T_pM$, $W \in \mathcal{V}_J$. Then*

- (i) $(\mathcal{L}_{\pi_1(A^h)}\pi_2(B^h))_J = (\mathcal{L}_{\pi_1(A)}\pi_2(B))_J^h$.
- (ii) $(\mathcal{L}_{\pi_1(A^h)}\pi_2(\mathcal{J}B^h))_J = (\mathcal{L}_{\pi_1(A)}\pi_2(SB))_J^h$.
- (iii) $(\mathcal{L}_{\pi_1(\mathcal{J}A^h)}\pi_2(B^h))_J(Z^h + W) = (\mathcal{L}_{\pi_1(SA)}\pi_2(B))_J^h(Z^h) + (\pi_2(B))_p(\pi_1(WA))$.
- (iv) $(\mathcal{L}_{\pi_1(\mathcal{J}A^h)}\pi_2(\mathcal{J}B^h))_J(Z^h + W) = (\mathcal{L}_{\pi_1(SA)}\pi_2(SB))_J^h(Z^h) + (\pi_2(JB))_p(\pi_1(WA))$.

Proof Formula (i) follows from (13) and (14); (ii) is a consequence of (i), (11) and (16). A simple computations involving (11), (13), (14) and (16) gives formula (iii); (iv) follows from (iii), (11) and (16). □

The proofs of the next lemmas are also easy and will be omitted.

Lemma 4 *Let A and B be sections of the bundle $TM \oplus T^*M$ near p . Let $Z \in T_pM$ and $W \in \mathcal{V}_J$. Then*

- (i) $(d \iota_{\pi_1(A^h)} \pi_2(B^h))_J = (d \iota_{\pi_1(A)} \pi_2(B))_J^h$.
- (ii) $(d \iota_{\pi_1(A^h)} \pi_2(\mathcal{J}B^h))_J(Z^h + W) = (d \iota_{\pi_1(A)} \pi_2(SB))_J^h(Z^h) + (\pi_2(WB))_p(\pi_1(A))$.
- (iii) $(d \iota_{\pi_1(\mathcal{J}A^h)} \pi_2(B^h))_J(Z^h + W) = (d \iota_{\pi_1(SA)} \pi_2(B))_J^h(Z^h) + (\pi_2(B))_p(\pi_1(WA))$.
- (iv) $(d \iota_{\pi_1(\mathcal{J}A^h)} \pi_2(\mathcal{J}B^h))_J(Z^h + W) = (d \iota_{\pi_1(SA)} \pi_2(SB))_J^h(Z^h) + (\pi_2(WB))_p(\pi_1(JA)) + (\pi_2(JB))_p(\pi_1(WA))$.

Lemma 5 *Let A be a section of the bundle $TM \oplus T^*M$ and V a vertical vector field on \mathcal{G} . Then*

- (i) $\mathcal{L}_V \pi_2(A^h) = 0; \quad \iota_V \pi_2(A^h) = 0$.
- (ii) $\mathcal{L}_V \pi_2(\mathcal{J}A^h) = \pi_2((VA)^h); \quad \iota_V \pi_2(\mathcal{J}A^h) = 0$.

Notation. Let $J \in \mathcal{G}$. For any fixed $\varepsilon = 1, \dots, 4$, take a basis $\{U_{2t-1}^\varepsilon, U_{2t}^\varepsilon = \mathcal{J}_\varepsilon U_{2t-1}^\varepsilon\}$, $t = 1, \dots, m^2 - m$, of the vertical space \mathcal{V}_J . Let a_{2t-1}^ε be sections of $A(E') \oplus A(E'')$ near the point $p = \pi(J)$ such that $a_{2t-1}^\varepsilon(p) = U_{2t-1}^\varepsilon$ and $Da_{2t-1}^\varepsilon|_p = 0$. Define vertical vector fields $\widetilde{a}_{2t-1}^\varepsilon$ by (18). Then $\{\widetilde{a}_{2t-1}^\varepsilon, \mathcal{J}_\varepsilon \widetilde{a}_{2t-1}^\varepsilon\}$, $t = 1, \dots, m^2 - m$, is a frame of the vertical bundle on \mathcal{G} near the point J . Denote by $\{\beta_{2t-1}^\varepsilon, \beta_{2t}^\varepsilon\}$ the dual frame of the bundle \mathcal{V}^* . Then $\beta_{2t}^\varepsilon = \mathcal{J}_\varepsilon \beta_{2t-1}^\varepsilon$.

Under these notations, we have the following.

Lemma 6 *Let A be a section of the bundle $TM \oplus T^*M$ near the point $p = \pi(J)$. Then for every $Z \in T_pM$, $s, r = 1, \dots, 2(m^2 - m)$ and $\varepsilon = 1, \dots, 4$, we have*

- (i) $(\mathcal{L}_{\pi_1(A^h)} \beta_s^\varepsilon)_J(Z^h + U_r^\varepsilon) = -\beta_s^\varepsilon(R(\pi_1(A), Z)J)$.
- (ii) $(\mathcal{L}_{\pi_1(\mathcal{J}A^h)} \beta_s^\varepsilon)_J(Z^h + U_r^\varepsilon) = -\beta_s^\varepsilon(R(\pi_1(JA), Z)J)$.
- (iii) $(\mathcal{L}_{\pi_1(A^h)} \mathcal{J}_\varepsilon \beta_s^\varepsilon)_J(Z^h + U_r^\varepsilon) = -(\mathcal{J}_\varepsilon \beta_s^\varepsilon)(R(\pi_1(A), Z)J)$.
- (iv) $(\mathcal{L}_{\pi_1(\mathcal{J}A^h)} \mathcal{J}_\varepsilon \beta_s^\varepsilon)_J(Z^h + U_r^\varepsilon) = -(\mathcal{J}_\varepsilon \beta_s^\varepsilon)(R(\pi_1(JA), Z)J)$.

Proof By (13), if $X = \pi_1(A)$,

$$(\mathcal{L}_{\pi_1(A^h)} \beta_s^\varepsilon)_J(Z^h + U_r^\varepsilon) = -\beta_s^\varepsilon(R(X, Z)J) - \frac{1}{2} \beta_s^\varepsilon \left(\left[X^h, \widetilde{a}_r^\varepsilon \right]_J \right).$$

By Lemma 2,

$$\begin{aligned} \left[X^h, a_{2t-1}^\varepsilon \right]_J &= \left(D_X \widetilde{a}_{2t-1}^\varepsilon \right)_J = 0, \\ \left[X^h, a_{2t}^\varepsilon \right]_J &= \left[X^h, \mathcal{J}_\varepsilon \widetilde{a}_{2t-1}^\varepsilon \right]_J = K_\varepsilon \left(D_X \widetilde{a}_{2t-1}^\varepsilon \right)_J = 0 \end{aligned}$$

since $Da_{2t-1}^\varepsilon|_p = 0$. This proves the first identity of the lemma. To prove the second one, we note that if f is a smooth function on \mathcal{G} and Y is a vector field on M , $(\mathcal{L}_{fY^h} \beta_s^\varepsilon)_J(Z^h + U_r^\varepsilon) = f(\mathcal{L}_{Y^h} \beta_s^\varepsilon)_J(Z^h + U_r^\varepsilon)$ since $\beta_s^\varepsilon(Y^h) = 0$. Now (ii) follows from (11) and the first identity of the lemma. Identities (iii) and (iv) are straightforward consequences from (i) and (ii), respectively, since $\mathcal{J}_\varepsilon \beta_{2t-1}^\varepsilon = \beta_{2t}^\varepsilon, \mathcal{J}_\varepsilon \beta_{2t}^\varepsilon = -\beta_{2t-1}^\varepsilon, t = 1, \dots, m^2 - m$. \square

5 The Nijenhuis tensor

Notation. We denote the Nijenhuis tensor of \mathcal{J}_ε by $N_\varepsilon, \varepsilon = 1, 2, 3, 4$.

Moreover, given $J \in \mathcal{G}$ and $A, B \in T_pM \oplus T_p^*M, p = \pi(J)$, we define 1-forms on \mathcal{V}_J setting

$$\omega_{A,B}^\varepsilon(W) = \langle (K_1W - K_\varepsilon W)(A), B \rangle - \langle (K_1W - K_\varepsilon W)(B), A \rangle, \quad W \in \mathcal{V}_J.$$

Also, let S be a section of \mathcal{G} in a neighbourhood of the point $p = \pi(J)$ such that $S(p) = J$ and $DS|_p = 0$ (S being considered as a section of $A(E') \oplus A(E'')$).

Proposition 9 *Let $J \in \mathcal{G}$, $A, B \in T_{\pi(J)}M \oplus T_{\pi(J)}^*M$, $V, W \in \mathcal{V}_J$, $\varphi, \psi \in \mathcal{V}_J^*$. Then, denoting the projection operators onto the horizontal and vertical components by $\mathcal{H} \oplus \mathcal{H}^*$ and $\mathcal{V} \oplus \mathcal{V}^*$, we have:*

- (i) $(\mathcal{H} \oplus \mathcal{H}^*)N_\varepsilon(A^h, B^h)_J = (-[A, B] + [SA, SB] - S[A, SB] - S[SA, B])_J^h.$
- (ii) $(\mathcal{V} \oplus \mathcal{V}^*)N_\varepsilon(A^h, B^h)_J = -R(\pi_1(A), \pi_1(B))J + R(\pi_1(JA), \pi_1(JB))J - K_\varepsilon R(\pi_1(JA), \pi_1(B))J - K_\varepsilon R(\pi_1(A), \pi_1(JB))J - \omega_{A,B}^\varepsilon.$
- (iii) $N_\varepsilon(A^h, V)_J = -(K_\varepsilon V)A + (K_1 V)A_J^h.$
- (iv) $N_\varepsilon(A^h, \varphi)_J \in \mathcal{H}_J \oplus \mathcal{H}_J^*$ and $\langle \pi_* N_\varepsilon(A^h, \varphi)_J, B \rangle = -\frac{1}{2}\varphi(\mathcal{V}N_\varepsilon(A^h, B^h)_J).$
- (v) $N_\varepsilon(V + \varphi, W + \psi)_J = 0.$

Proof Formula (i) follows from identity (13) and Lemmas 1, 3, 4. Also, the vertical part of $N_\varepsilon(A^h, B^h)_J$ is equal to

$$\mathcal{V}N_\varepsilon(A^h, B^h)_J = -R(\pi_1(A), \pi_1(B))J + R(\pi_1(JA), \pi_1(JB))J - \mathcal{J}_\varepsilon R(\pi_1(A), \pi_1(JB))J - \mathcal{J}_\varepsilon R(\pi_1(JA), \pi_1(B))J.$$

The part of $N_\varepsilon(A^h, B^h)_J$ lying in \mathcal{V}_J^* is the 1-form whose value at every vertical vector W is

$$\begin{aligned} (\mathcal{V}^*N_\varepsilon(A^h, B^h)_J)(W) &= -\frac{1}{2} [\pi_2(JA)(\pi_1(WB)) + \pi_2(WB)(\pi_1(JA)) \\ &\quad - \pi_2(B)(\pi_1((K_\varepsilon W)A)) - \pi_2((K_\varepsilon W)A)(\pi_1(B))] \\ &\quad + \frac{1}{2} [\pi_2(JB)(\pi_1(WA)) + \pi_2(WA)(\pi_1(JB)) \\ &\quad - \pi_2(A)(\pi_1((K_\varepsilon W)B)) - \pi_2((K_\varepsilon W)B)(\pi_1(A))] \\ &= -[\langle JA, WB \rangle - \langle (K_\varepsilon W)A, B \rangle] + [\langle JA, WB \rangle - \langle (K_\varepsilon W)B, A \rangle]. \end{aligned}$$

Note also that

$$\langle JA, WB \rangle = \langle JW(A), B \rangle = \langle K_1 W(A), B \rangle.$$

It follows that

$$\mathcal{V}^*N_\varepsilon(A^h, B^h)_J = -\omega_{A,B}^\varepsilon.$$

This proves (ii).

To prove (iii) take a section a of $A(M)$ near the point p such that $a(p) = V$ and $\nabla a|_p = 0$. Let \tilde{a} be the vertical vector field defined by (18). Then it follows from Lemmas 2 and 5 that

$$N_\varepsilon(A^h, V)_J = \frac{1}{2}N_\varepsilon(A^h, \tilde{a})_J = -(K_\varepsilon V)(A) + (J \circ V)A|_J^h.$$

To prove (iv) let us take the vertical co-frame $\{\beta_{2t-1}^\varepsilon, \beta_{2t}^\varepsilon\}, t = 1, \dots, m^2 - m$, defined before the statement of Lemma 6. Set $\varphi = \sum_{s=1}^{2(m^2-m)} \varphi_s^\varepsilon \beta_s^\varepsilon, \varphi_s \in \mathbb{R}$. Let E_1, \dots, E_{2m} be a basis of T_pM and ξ_1, \dots, ξ_{2m} its dual basis. Then, by Lemma 6, we have

$$\begin{aligned} N_\varepsilon(A^h, \varphi)_J &= \sum_{s=1}^{2(m^2-m)} \varphi_s^\varepsilon N_\varepsilon(A^h, \beta_s^\varepsilon)_J \\ &= \sum_{s=1}^{2(m^2-m)} \sum_{k=1}^{2m} \varphi_s^\varepsilon \left\{ [\beta_s^\varepsilon(R(\pi_1(A), E_k)J) + \beta_s^\varepsilon(K_\varepsilon R(\pi_1(JA), E_k)J)] (\xi_k)_J^h \right. \\ &\quad \left. + [\beta_s^\varepsilon(R(\pi_1(JA), E_k)J) - \beta_s^\varepsilon(K_\varepsilon R(\pi_1(A), E_k)J)] (J\xi_k)_J^h \right\}. \end{aligned} \tag{21}$$

Moreover, note that

$$\langle \xi_k, B \rangle = \frac{1}{2} \xi_k(\pi_1(B)) \quad \text{and} \quad \langle J\xi_k, B \rangle = -\langle \xi_k, JB \rangle = -\frac{1}{2} \xi_k(\pi_1(JB)).$$

Therefore

$$\sum_{k=1}^{2m} \langle \xi_k, B \rangle E_k = \frac{1}{2} \pi_1(B) \quad \text{and} \quad \sum_{k=1}^{2m} \langle J\xi_k, B \rangle E_k = -\frac{1}{2} \pi_1(JB).$$

Now (iv) is an obvious consequence of (21) and formula (ii).

Finally, identity (v) follows from the fact that the generalized almost complex structure \mathcal{J}_ε on every fibre of \mathcal{G} is induced by a complex structure. □

6 Integrability conditions for generalized almost complex structures on generalized twistor spaces

Proposition 10 *The generalized almost complex structures $\mathcal{J}_2, \mathcal{J}_3, \mathcal{J}_4$ are never integrable.*

Proof Let $p \in M$ and take orthonormal bases $\{Q'_1, \dots, Q'_{2m}\}, \{Q''_1, \dots, Q''_{2m}\}$ of E'_p and E''_p , respectively. Let J' and J'' be the complex structures on E'_p and E''_p for which $J'Q'_{2k-1} = Q'_{2k}$ and $J''Q''_{2k-1} = Q''_{2k}, k = 1, \dots, m$. Then $J = J' + J''$ is a generalized complex structure on the vector space T_pM compatible with the generalized metric E_p . Define endomorphisms S'_{ij} and S''_{ij} by (7). Then $V' = S'_{13} + S'_{42}$ and $V'' = S''_{13} + S''_{42}$ are vertical tangent vectors of \mathcal{G} at the point J . By Proposition 9 (iii), $N_2(Q_1^{h'}, V') = N_4(Q_1^{h''}, V'') = 2Q_4^{h'}, N_3(Q_1^{h'}, V') = 2Q_4^{h'}$. □

6.1 The case of the connection determined by a generalized metric

Let $D' = \nabla^{E'}$ be the connection on $E' = E$ determined by the generalized metric E (Proposition 7). The image of this connection under the isomorphism $pr_{TM}|_E : E \rightarrow TM$ will be denoted by ∇ . The connection ∇ has a skew-symmetric torsion $g(T(X, Y), Z) = d\Theta(X, Y, Z), X, Y, Z \in TM$.

We define a connection D'' on E'' transferring the connection ∇ on TM to E'' by means of the isomorphism $pr_{TM}|E'' : E'' \rightarrow TM$. Since this isomorphism is an isometry with respect to the metrics $g'' = -\langle \cdot, \cdot \rangle|E''$ and g , we get a metric connection on E'' . As in the preceding section, define a connection D on $TM \oplus T^*M$ setting $D = D'$ on $E' = E$ and $D = D''$ on E'' .

The connections induced by ∇ on the bundles obtained from TM by algebraic operations like $T^*M, TM \oplus T^*M$, etc. will also be denoted by ∇ .

Every section of E' is of the form $S' = X + g(X) + \Theta(X)$ for a unique vector field X and we have

$$D_Z S' = \nabla_Z X + g(\nabla_Z X) + \Theta(\nabla_Z X), \quad Z \in TM, \tag{22}$$

while

$$\nabla_Z S' = \nabla_Z X + g(\nabla_Z X) + \Theta(\nabla_Z X) + (\nabla_Z \Theta)(X).$$

Similarly for a section $S'' = X - g(X) + \Theta(X)$ of E''

$$D_Z S'' = \nabla_Z X - g(\nabla_Z X) + \Theta(\nabla_Z X), \quad Z \in TM, \tag{23}$$

and

$$\nabla_Z S'' = \nabla_Z X - g(\nabla_Z X) + \Theta(\nabla_Z X) + (\nabla_Z \Theta)(X).$$

Thus ∇ preserves E' or E'' if and only if $\nabla\Theta = 0$. Of course, this condition is not satisfied in general. For example, if Θ is a closed 2-form, which is not parallel with respect to the Levi-Civita connection ∇^{LC} , we have $\nabla\Theta = \nabla^{LC}\Theta \neq 0$.

It follows from (5), (22) and (23) that if α is a one form on M and $Z \in TM$,

$$D_Z \alpha = D_Z \alpha_{E'} + D_Z \alpha_{E''} = g(\nabla_Z g^{-1}(\alpha)). \tag{24}$$

Hence, $D_Z \alpha$ coincides with the covariant derivative of α with respect to the connection ∇ on T^*M :

$$D\alpha = \nabla\alpha. \tag{25}$$

On the other hand, if X is a vector field on M and $Z \in TM$, we have by (4), (22) and (23)

$$D_Z X = D_Z X_{E'} + D_Z X_{E''} = \nabla_Z X - g(\nabla_Z \{(g^{-1} \circ \Theta)(X)\}) + \Theta(\nabla_Z X).$$

Moreover, for every vector field Y ,

$$\begin{aligned} g(\nabla_Z \{(g^{-1} \circ \Theta)(X)\})(Y) &= Z(g((g^{-1} \circ \Theta)(X), Y)) - g((g^{-1} \circ \Theta)(X), \nabla_Z Y) \\ &= Z(\Theta(X, Y)) - \Theta(X, \nabla_Z Y) = (\nabla_Z \Theta)(X, Y) + \Theta(\nabla_Z X, Y). \end{aligned}$$

Thus

$$D_Z X = \nabla_Z X - (\nabla_Z \Theta)(X). \tag{26}$$

Therefore the connection D does not preserve TM in general. In particular, the connection D on $TM \oplus T^*M$ is different from the connection on this bundle induced by ∇ ; the two connections coincide if and only if $\nabla\Theta = 0$.

Denote by $\mathcal{Z} = \mathcal{Z}(TM, g)$ the bundle over M whose fibre at a point $p \in M$ consists of complex structures on $T_p M$ compatible with the metric g (the usual twistor space of (M, g)). Consider \mathcal{Z} as a submanifold of the bundle $A(TM)$ of g -skew-symmetric endomorphisms of TM . The projections $pr_{TM}|E' : E' \rightarrow TM$ and $pr_{TM}|E'' : E'' \rightarrow TM$ yield an isometric bundle-isomorphism $A(E') \oplus A(E'') \rightarrow A(TM) \oplus A(TM)$ sending the connection $D' \oplus D''$ to the connection $\nabla \oplus \nabla$. The restriction of this map to $\mathcal{Z}(E') \times \mathcal{Z}(E'') \cong \mathcal{G}$ yields an

isomorphism of \mathcal{G} onto $\mathcal{Z} \times \mathcal{Z}$ given by $\mathcal{G} \ni J \rightarrow (J_1, J_2)$, where $J_1 = (pr_T|E') \circ J \circ (pr_T|E')^{-1}$, $J_2 = (pr_T|E'') \circ J \circ (pr_T|E'')^{-1}$. In the case when M is oriented it identifies the connected components of \mathcal{G} with the four product bundles $\mathcal{Z}_\pm \times \mathcal{Z}_\pm$, \mathcal{Z}_\pm being the bundle over M whose sections are the almost complex structures on M compatible with the metric and \pm the orientation.

Proposition 11 $(\mathcal{H} \oplus \mathcal{H}^*)N_\varepsilon(A^h, B^h)_J = 0$ for every $J \in \mathcal{G}$ and every $A, B \in TM \oplus T^*M$ if and only if $d\Theta = 0$.

Proof Let $J \in \mathcal{G}$ and let $S = (S', S'')$ be a section of \mathcal{G} in a neighbourhood of the point $p = \pi(J)$ with the properties that $S(p) = J$ and $DS|_p = 0$.

According to Proposition 9 (i), $(\mathcal{H} \oplus \mathcal{H}^*)N_\varepsilon(A^h, B^h)_J = 0$ if and only if the Nijenhuis tensor N_S of the generalized almost complex structure S on M vanishes at the point p . Let S_1 and S_2 be the almost complex structures on M determined by S ,

$$S_1 = (\pi_1|E') \circ S' \circ (\pi_1|E')^{-1}, \quad S_2 = (\pi_1|E'') \circ S'' \circ (\pi_1|E'')^{-1}.$$

These structures are compatible with the metric g and we denote their fundamental 2-forms by Ω_1 and Ω_2 , respectively:

$$\Omega_1(X, Y) = g(X, S_1 Y), \quad \Omega_2(X, Y) = g(X, S_2 Y), \quad X, Y \in TM.$$

Denote by K the generalized complex structure on M with the block-matrix

$$K = \frac{1}{2} \begin{pmatrix} S_1 + S_2 & \Omega_1^{-1} - \Omega_2^{-1} \\ -(\Omega_1 - \Omega_2) & -(S_1^* + S_2^*) \end{pmatrix}.$$

By Proposition 1, the generalized complex structure S is the B -transform of K by means of the form Θ :

$$S = e^\Theta K e^{-\Theta}.$$

Let N_K be the Nijensuis tensor of the generalized almost complex structure K . Set

$$A = X + \alpha, \quad B = Y + \beta, \quad KA = \widehat{X} + \widehat{\alpha}, \quad KB = \widehat{Y} + \widehat{\beta}, \tag{27}$$

where $X, Y, \widehat{X}, \widehat{Y} \in TM$ and $\alpha, \beta, \widehat{\alpha}, \widehat{\beta} \in T^*M$. Then, by Proposition 4 and the fact that $e^{-\Theta}|T^*M = Id$,

$$\begin{aligned} N_S(e^\Theta A, e^\Theta B) &= e^\Theta N_K(A, B) - \iota_Y \iota_X d\Theta + \iota_{\widehat{Y}} \iota_{\widehat{X}} d\Theta \\ &\quad - e^\Theta K(\iota_Y \iota_{\widehat{X}} d\Theta + \iota_{\widehat{Y}} \iota_X d\Theta). \end{aligned}$$

It follows that $(\mathcal{H} \oplus \mathcal{H}^*)N_\varepsilon(A^h, B^h)_J = 0$ for every $A, B \in TM \oplus T^*M$ if and only if at the point $p = \pi(J)$

$$N_K(A, B) = \iota_Y \iota_X d\Theta - \iota_{\widehat{Y}} \iota_{\widehat{X}} d\Theta + K(\iota_Y \iota_{\widehat{X}} d\Theta + \iota_{\widehat{Y}} \iota_X d\Theta). \tag{28}$$

We have

$$\nabla S_1 = (\pi_1|E') \circ (DS') \circ (\pi_1|E')^{-1}$$

since the connection ∇ on TM is obtained from the connection $D|E' = \nabla^{E'}$ by means of the isomorphism $\pi_1|E' : E' \rightarrow TM$. In particular $\nabla S_1|_p = 0$. Similarly, $\nabla S_2|_p = 0$. Then $\nabla S_k^*|_p = 0$, $k = 1, 2$, and $\nabla \Omega_k|_p = -\nabla(g \circ S_k)|_p = 0$, $\nabla \Omega_k^{-1} = \nabla(S_k \circ g^{-1})|_p = 0$ since $\nabla g = \nabla g^{-1} = 0$. It follows that $\nabla K|_p = 0$. Extend X and α to a vector field X and a 1-form α on M such that $\nabla X|_p = 0$ and $\nabla \alpha|_p = 0$; similarly for Y and β . In this way we

obtain sections $A = X + \alpha$ and $B = Y + \beta$ of $TM \oplus T^*M$ such that $\nabla A|_p = \nabla B|_p = 0$ and $\nabla KA|_p = \nabla KB|_p = 0$.

In order to compute $N_K(A, B)$ we need the following simple observation: Let Z be a vector field and ω a 1-form on M such that $\nabla Z|_p = 0$ and $\nabla\omega|_p = 0$. Then, for every $Z' \in T_pM$,

$$\begin{aligned} (\mathcal{L}_Z\omega)(Z')_p &= (\nabla_Z\omega)(Z')_p + \omega(T(Z, Z')) = \omega(T(Z, Z')), \\ (d\iota_Z\omega)(Z')_p &= Z'_p(\omega(Z)) = (\nabla_{Z'}\omega)(Z)_p = 0, \end{aligned}$$

where $T(Z, Z')$ is the torsion tensor of the connection ∇ . For $Z \in TM$, let $\iota_Z T : TM \rightarrow TM$ be the map $Z' \rightarrow T(Z, Z')$. Then, under the notation in (27), we have

$$\begin{aligned} N_K(A, B) &= T(X, Y) - T(\widehat{X}, \widehat{Y}) + \alpha(\iota_Y T) - \beta(\iota_X T) - \widehat{\alpha}(\iota_{\widehat{Y}} T) + \widehat{\beta}(\iota_{\widehat{X}} T) \\ &\quad + K [T(\widehat{X}, Y) + T(X, \widehat{Y}) + \widehat{\alpha}(\iota_Y T) - \beta(\iota_{\widehat{X}} T) + \alpha(\iota_{\widehat{Y}} T) - \widehat{\beta}(\iota_X T)]. \end{aligned} \tag{29}$$

If $\alpha = g(X')$ for some (unique) $X' \in TM$, we have

$$\alpha(\iota_Y T) = g(T(X', Y)) = \iota_Y \iota_{X'} d\Theta, \quad Y \in TM,$$

and

$$K(\alpha \circ \iota_Y T) = \frac{1}{2} \left[(\Omega_1^{-1} - \Omega_2^{-1}) \iota_Y \iota_{X'} d\Theta - (S_1^* + S_2^*) \iota_Y \iota_{X'} d\Theta \right]$$

Moreover, $g(T(X, Y)) = \iota_Y \iota_X d\Theta$ for every $X, Y \in TM$, hence

$$K(T(X, Y)) = \frac{1}{2} \left[(\Omega_1^{-1} + \Omega_2^{-1}) \iota_Y \iota_X d\Theta - (S_1^* - S_2^*) \iota_Y \iota_X d\Theta \right].$$

Note also that

$$\begin{aligned} \widehat{X} &= \frac{1}{2} [S_1(X + X') + S_2(X - X')], \\ \widehat{\alpha} &= \frac{1}{2} [g(S_1(X + X') - S_2(X - X'))]. \end{aligned}$$

Now suppose that

$$(\mathcal{H} \oplus \mathcal{H}^*) N_\varepsilon(A^h, B^h)_J = 0, \quad A, B \in TM \oplus T^*M.$$

Then, by (28),

$$N_K(g(X), g(Y)) = \frac{1}{4} \iota_{(S_1 X - S_2 X)} \iota_{(S_1 Y - S_2 Y)} d\Theta. \tag{30}$$

Therefore the tangential component of $N_K(g(X), g(Y))$ vanishes. Hence by (29)

$$\begin{aligned} & -T(S_1 X - S_2 X, S_1 Y - S_2 Y) \\ & - \left(\Omega_1^{-1} - \Omega_2^{-1} \right) \iota_{(S_1 X - S_2 X)} \iota_Y d\Theta + \left(\Omega_1^{-1} - \Omega_2^{-1} \right) \iota_{(S_1 Y - S_2 Y)} \iota_X d\Theta = 0. \end{aligned}$$

Applying the map g to both sides of the latter identity we obtain by means of the identities $g \circ \Omega_k^{-1} = -S_k^*$ that for every $X, Y, Z \in T_pM$

$$\begin{aligned} & d\Theta(S_1 X - S_2 X, Y, S_1 Z - S_2 Z) + d\Theta(X, S_1 Y - S_2 Y, S_1 Z - S_2 Z) \\ & = -d\Theta(S_1 X - S_2 X, S_1 Y - S_2 Y, Z). \end{aligned} \tag{31}$$

Applying (31) for the generalized almost complex structure determined by the complex structures $(-S_1, S_2)$ on T_pM and comparing the obtained identity with (31) we see that

$$d\Theta(S_1X, Y, S_1Z) + d\Theta(S_2X, Y, S_2Z) + d\Theta(X, S_1Y, S_1Z) + d\Theta(X, S_2Y, S_2Z) = -d\Theta(S_1X, S_1Y, Z) - d\Theta(S_2X, S_2Y, Z). \tag{32}$$

Computing the co-tangential component of $N_K(g(X), g(Y))$ by means of (29), then applying identity (30) for the generalized almost complex structures determined by (S_1, S_2) and $(-S_1, S_2)$, we obtain

$$-d\Theta(S_1X, Y, S_1Z) + d\Theta(S_2X, Y, S_2Z) - d\Theta(X, S_1Y, S_1Z) + d\Theta(X, S_2Y, S_2Z) = d\Theta(S_1X, S_1Y, Z) - 3d\Theta(S_2X, S_2Y, Z). \tag{33}$$

It follows from (32) and (33) that

$$d\Theta(S_2X, Y, S_2Z) + d\Theta(X, S_2Y, S_2Z) = -2d\Theta(S_2X, S_2Y, Z).$$

Hence

$$2d\Theta(X, Y, Z) = d\Theta(X, S_2Y, S_2Z) + d\Theta(S_2X, Y, S_2Z).$$

The latter identity holds if and only if it holds for every $X, Y, Z \in T_pM$ with $|X| = |Y| = 1, X \perp Y$. Given three tangent vectors with these properties, there exists a complex structure S_2 on T_pM such that $Y = S_2X$. It follows that

$$d\Theta(X, Y, Z) = 0, \quad X, Y, Z \in T_pM.$$

Conversely, if $d\Theta = 0$, then $T = 0$ and we have $N_K = 0$ by (29). Thus the condition (28) is trivially satisfied. Therefore

$$(\mathcal{H} \oplus \mathcal{H}^*)N_\varepsilon(A^h, B^h)_J = 0, \quad A, B \in TM \oplus T^*M.$$

□

Suppose that M is oriented and $\dim M = 4k$. Then the above proof still holds true if we, instead of \mathcal{G} , consider a connected component of it. Indeed, the almost complex structures S_1 and $-S_1$ induce the same orientation and, moreover, the complex structure S_2 with the property $Y = S_2X$ used at the end of the proof can be chosen to induce the given or the opposite orientation of M . Thus we have the following.

Proposition 12 *If M is oriented and $\dim M = 4k$, then*

$$(\mathcal{H} \oplus \mathcal{H}^*)N_\varepsilon(A^h, B^h)_J = 0$$

*for every J in a connected component of \mathcal{G} and every $A, B \in TM \oplus T^*M$ if and only if $d\Theta = 0$.*

Considering the double orientable covering of M , if necessary, we may assume that M itself is orientable. Fix an orientation on M . Denote by \mathcal{G}_{++} the subbundle of \mathcal{G} whose fibre at a point $p \in M$ consists of generalized complex structures J on T_pM compatible with the generalized metric E_p and such that the complex structures J_1 and J_2 on T_pM determined by J via (3) induce the orientation of T_pM . We define subbundles $\mathcal{G}_{--}, \mathcal{G}_{+-}, \mathcal{G}_{-+}$ in a similar way. These are the connected components of the space \mathcal{G} .

Convention. Henceforth we assume that M is oriented and of dimension $4k$.

Recall that if R is the curvature tensor of the Levi-Civita connection of (M, g) , the curvature operator \mathcal{R} is the self-adjoint endomorphism of $\Lambda^2 TM$ defined by

$$g(\mathcal{R}(X \wedge Y), Z \wedge T) = g(R(X, Y)Z, T), \quad X, Y, Z, T \in TM.$$

The metric on $\Lambda^2 TM$ used in the left-hand side of the latter identity is defined by

$$g(X_1 \wedge X_2, X_3 \wedge X_4) = g(X_1, X_3)g(X_2, X_4) - g(X_1, X_4)g(X_3, X_2).$$

As is well-known, the curvature operator decomposes as (see, for example, [2, Section 1 G, H])

$$\mathcal{R} = \frac{s}{n(n-1)} Id + \mathcal{B} + \mathcal{W}, \tag{34}$$

where s is the scalar curvature of the manifold (M, g) and \mathcal{B}, \mathcal{W} correspond to its traceless Ricci tensor and Weyl conformal tensor, respectively. If $\rho : TM \rightarrow TM$ is the Ricci operator, $g(\rho(X), Y) = Ricci(X, Y)$, the operator \mathcal{B} is given by

$$\mathcal{B}(X \wedge Y) = \frac{1}{n-2} \left[\rho(X) \wedge Y + X \wedge \rho(Y) - \frac{2s}{n} X \wedge Y \right], \quad X, Y \in TM. \tag{35}$$

Thus, a Riemannian manifold is Einstein exactly when $\mathcal{B} = 0$; it is conformally flat when $\mathcal{W} = 0$.

If the dimension of M is four, the Hodge star operator defines an involution $*$ of $\Lambda^2 TM$ and we have the orthogonal decomposition

$$\Lambda^2 TM = \Lambda^2_- TM \oplus \Lambda^2_+ TM,$$

where $\Lambda^2_{\pm} TM$ are the subbundles of $\Lambda^2 TM$ corresponding to the (± 1) -eigenvalues of the operator $*$. Accordingly, the operator \mathcal{W} has an extra decomposition $\mathcal{W} = \mathcal{W}_+ + \mathcal{W}_-$ where $\mathcal{W}_{\pm} = \mathcal{W}$ on $\Lambda^2_{\pm} TM$ and $\mathcal{W}_{\pm} = 0$ on $\Lambda^2_{\mp} TM$. The operator \mathcal{B} does not have a decomposition of this type since it maps $\Lambda^2_{\pm} TM$ into $\Lambda^2_{\mp} TM$.

Recall also that a Riemannian manifold (M, g) is called self-dual (anti-self-dual), if $\mathcal{W}_- = 0$ (resp. $\mathcal{W}_+ = 0$).

According to Propositions 9 and 12, the restriction of the generalized almost complex structure \mathcal{J}_1 to a connected component $\tilde{\mathcal{G}}$ of \mathcal{G} is integrable if and only if $d\Theta = 0$ and for every $p \in M, A, B \in T_p M$, and for every generalized complex structure $J \in \tilde{\mathcal{G}}$ on $T_p M$

$$\begin{aligned} & -R(\pi_1(A), \pi_1(B))J + R(\pi_1(JA), \pi_1(JB))J \\ & -K_1 R(\pi_1(JA), \pi_1(B))J - K_1 R(\pi_1(A), \pi_1(JB))J = 0, \end{aligned}$$

where R is the curvature tensor of the connection D on the bundle $A(E') \oplus A(E'')$. If (J_1, J_2) are the complex structures on $T_p M$ determined by J , the latter identity is equivalent to the identities

$$\begin{aligned} & -R(\pi_1(A), \pi_1(B))J_r + R(\pi_1(JA), \pi_1(JB))J_r \\ & -J_r \circ R(\pi_1(JA), \pi_1(B))J_r - J_r \circ R(\pi_1(A), \pi_1(JB))J_r = 0, \quad r = 1, 2, \end{aligned} \tag{36}$$

where R is the curvature tensor on the bundle $A(TM)$ of skew-symmetric endomorphism of TM induced by the connection ∇ .

Assume that $d\Theta = 0$. Then ∇ is the Levi-Civita connection of the Riemannian manifold (M, g) . Every $A \in E'_p$ is of the form $A = X + g(X) + \Theta(X)$ for some (unique) $X \in T_p M$ and $JA = J_1 X + g(J_1 X) + \Theta(J_1 X)$. Similarly, if $B \in E''_p$, then $B = Y - g(Y) + \Theta(Y)$, $Y \in T_p M$ and $JB = J_2 Y - g(J_2 Y) + \Theta(J_2 Y)$. It follows that the identity (36) is equivalent

to the condition that for every $X, Y, Z, U \in T_pM$ and every complex structures (J_1, J_2) on T_pM corresponding to a generalized complex structure J in $\tilde{\mathcal{G}}$

$$\begin{aligned} &g(\mathcal{R}(X \wedge Y - J_j X \wedge J_l Y), Z \wedge U - J_r Z \wedge J_r U) \\ &= g(\mathcal{R}(J_j X \wedge Y + X \wedge J_l Y), J_r Z \wedge U + Z \wedge J_r U), \\ &j, l, r = 1, 2. \end{aligned} \tag{37}$$

The complex structures (J_1, J_2) in the latter identities are compatible with the metric g and, moreover, they induce the orientation of T_pM if we consider the connected component $\tilde{\mathcal{G}} = \mathcal{G}_{++}$, while (J_1, J_2) induce the opposite orientation in the case $\tilde{\mathcal{G}} = \mathcal{G}_{--}$. If $\tilde{\mathcal{G}} = \mathcal{G}_{+-}$, the complex structure J_1 induces the given orientation of T_pM and J_2 yields the opposite one, and vice versa if $\tilde{\mathcal{G}} = \mathcal{G}_{-+}$.

For $j = l = r$ identity (37) coincides with the integrability condition for the Atiyah-Hitchin-Singer almost complex structure [1] on the positive or negative twistor space of (M, g) , the fibre bundles over M whose fibre at every point $p \in M$ consists of the complex structures on T_pM compatible with the metric and \pm the orientation of T_pM (see, for example, [26, Section 5.19]). It is also well known that this integrability condition is equivalent to (M, g) being conformally flat if $\dim M \geq 6$. If $\dim M = 4$ the integrability condition is equivalent to anti-self-duality of (M, g) in the case of positive twistor spaces and to its self-duality when considering the negative twistor space.

Theorem 1 (I) *Suppose that $\dim M = 4$.*

(a) *The restriction of the generalized complex structure \mathcal{J}_1 to \mathcal{G}_{++} is integrable if and only if (M, g) is anti-self-dual and Ricci flat.*

(b) *The restriction $\mathcal{J}_1|_{\mathcal{G}_{--}}$ is integrable if and only if (M, g) is self-dual and Ricci flat.*

(II) *If $\dim M = 4k \geq 6$, each of the restrictions of \mathcal{J}_1 to \mathcal{G}_{++} and \mathcal{G}_{--} is integrable if and only if the manifold (M, g) is flat.*

Proof Let E_1, \dots, E_n be an oriented orthonormal basis of a tangent space T_pM . It is convenient to set $E_{ab} = E_a \wedge E_b$ and $\rho_{ab} = \text{Ricci}(E_a, E_b)$, $a, b = 1, \dots, n$.

Suppose that the structure $\mathcal{J}_1|_{\mathcal{G}_{++}}$ is integrable. Let J_1 and J_2 be complex structures on T_pM for which $J_1 E_1 = E_3, J_1 E_2 = -E_4$ and $J_2 E_1 = E_4, J_2 E_2 = E_3$. Identity (37) with $j = l = 1, r = 2$, and $(X, Y, Z, U) = (E_1, E_2, E_3, E_4)$ gives

$$g(\mathcal{R}(E_{12} + E_{34}), E_{12} + E_{34}) + g(\mathcal{R}(E_{14} + E_{23}), E_{13} + E_{42}) = 0. \tag{38}$$

If $\dim M = 4$, then $E_{12} + E_{34}, E_{14} + E_{23} \in \Lambda^2_+ T_pM$ and $\mathcal{W}_+ = 0$, hence

$$\mathcal{W}(E_{12} + E_{34}) = \mathcal{W}(E_{14} + E_{23}) = 0.$$

If $\dim M \geq 6$, we have $\mathcal{W} = 0$. Thus, in both cases by (34),

$$\begin{aligned} &g(\mathcal{R}(E_{12} + E_{34}), E_{12} + E_{34}) + g(\mathcal{R}(E_{14} + E_{23}), E_{13} + E_{42}) \\ &= \frac{2s}{n(n-1)} + g(\mathcal{B}(E_{12} + E_{34}), E_{12} + E_{34}) + g(\mathcal{B}(E_{14} + E_{23}), E_{13} + E_{42}). \end{aligned}$$

By (35)

$$\begin{aligned} &g(\mathcal{B}(E_{12} + E_{34}), E_{12} + E_{34}) = \frac{1}{n-2} \left[\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} - \frac{4s}{n} \right], \\ &g(\mathcal{B}(E_{14} + E_{23}), E_{13} + E_{42}) = 0. \end{aligned}$$

Then by (38)

$$\frac{2s}{n(n-1)} + \frac{1}{n-2} \left[\rho_{11} + \rho_{22} + \rho_{33} + \rho_{44} - \frac{4s}{n} \right] = 0.$$

In a similar way we see that

$$\frac{2s}{n(n-1)} + \frac{1}{n-2} \left[\rho_{4i-3,4i-3} + \rho_{4i-2,4i-2} + \rho_{4i-1,4i-1} + \rho_{4i,4i} - \frac{4s}{n} \right] = 0$$

for $i = 1, 2, \dots, k$. Summing up these identities we get $s = 0$.

In order to show that $\mathcal{B} = 0$ we apply identity (37) with $j = 1, l = 2$ and take J_1, J_2 to be the complex structures introduced above. Subtracting the identities corresponding to $X = Y = E_2$ and $X = Y = E_3$, we get

$$g(\mathcal{R}(E_{13} - E_{42}), Z \wedge U - J_r Z \wedge J_r U) = -g(\mathcal{R}(E_{12} - E_{34}), J_r Z \wedge U + Z \wedge J_r U).$$

Subtraction of the identities corresponding to $X = Y = E_1$ and $X = Y = E_4$ gives

$$g(\mathcal{R}(E_{13} - E_{42}), Z \wedge U - J_r Z \wedge J_r U) = g(\mathcal{R}(E_{12} - E_{34}), J_r Z \wedge U + Z \wedge J_r U).$$

Thus

$$g(\mathcal{R}(E_{13} - E_{42}), Z \wedge U - J_r Z \wedge J_r U) = 0 = g(\mathcal{R}(E_{12} - E_{34}), J_r Z \wedge U + Z \wedge J_r U). \tag{39}$$

If $\dim M = 4$, every 2-vector of the form $Z \wedge U - J_r Z \wedge J_r U$ lies in $\Lambda^2_+ T_p M$ since J_r is compatible with the metric and orientation of $T_p M$. Therefore $\mathcal{W}(Z \wedge U - J_r Z \wedge J_r U) = \mathcal{W}(J_r Z \wedge U + Z \wedge J_r U) = 0$. If $\dim M \geq 6$, this is obvious. Then the first identity in (39) with $r = 1$ and $(Z, U) = (E_1, E_3)$ gives

$$g(\mathcal{B}(E_{13} - E_{42}), E_{13} + E_{42}) = 0.$$

It follows by (35) that

$$\rho_{11} - \rho_{22} + \rho_{33} - \rho_{44} = 0.$$

Applying the latter identity for the basis $E_1, E_3, E_4, E_2, E_5, \dots, E_n$, we get

$$\rho_{11} - \rho_{22} - \rho_{33} + \rho_{44} = 0.$$

Therefore $\rho_{11} = \rho_{22}$. It follows that $\rho_{11} = \rho_{aa}$ for $a = 1, \dots, n$. This implies $\rho_{aa} = 0$ for every $a = 1, \dots, n$ since the scalar curvature vanishes. Moreover, the first identity in (39) for $r = 2$ and $(Z, U) = (E_1, E_2)$ reads as

$$g(\mathcal{B}(E_{13} - E_{42}), E_{12} + E_{34}) = 0.$$

This gives $-\rho_{14} + \rho_{23} = 0$. Similarly, it follows from the second identity in (39) with $r = 1$ and $(Z, U) = (E_2, E_2)$ that $\rho_{14} + \rho_{23} = 0$. Hence $\rho_{14} = \rho_{23} = 0$. It follows that $\rho_{ab} = 0, a \neq b$. Therefore $Ricci = 0$.

Conversely, it is obvious that identity (37) is satisfied if (M, g) is flat. In the case when $s = 0$ and $\mathcal{B} = \mathcal{W}_+ = 0$, identity (37) is also trivially satisfied since for every $X, Y \in T_p M$ and every complex structure J on $T_p M$ compatible with the metric and orientation, the 2-vector $X \wedge Y - JX \wedge JY$ lies in $\Lambda^2_+ T_p M$, so $\mathcal{R}(X \wedge Y - JX \wedge JY) = 0$.

This proves statements I (a) and II. Statement I (b) is an obvious corollary of I (a) by reversing the orientation of M . □

Remark 6 By a result of Hitchin [18] if M is a compact anti-self-dual, Ricci flat four-dimensional manifold, then either M is flat or is a $K3$ -surface, an Enriques surface or the quotient of an Enriques surface by a free anti-holomorphic involution.

Theorem 2 *Each of the restrictions $\mathcal{J}_1|_{\mathcal{G}_{+-}}$ and $\mathcal{J}_1|_{\mathcal{G}_{-+}}$ is an integrable generalized almost complex structure if and only (M, g) is of constant sectional curvature.*

Proof Suppose that $\mathcal{J}_1|_{\mathcal{G}_{+-}}$ is integrable. Then, by the preceding remarks, if $\dim M = 4$, (M, g) is both anti-self-dual and self-dual, hence $\mathcal{W} = 0$; if $\dim M \geq 6$, we also have $\mathcal{W} = 0$. Take an orthonormal oriented basis E_1, \dots, E_n of a tangent space T_pM and consider (37) with $j = 1, l = r = 2$. Take for J_1 and J_2 the complex structures on T_pM for which $J_1E_1 = E_3, J_1E_4 = E_2$ and $J_2E_1 = E_4, J_2E_2 = -E_3$. Adding the identities corresponding to $(X, Y) = (E_1, E_2)$ and $(X, Y) = (E_3, E_4)$, we get

$$g(\mathcal{R}(E_{12} + E_{34}), Z \wedge U - J_2Z \wedge J_2U) + g(\mathcal{R}(E_{14} + E_{23}), J_2Z \wedge U + Z \wedge J_2U) = 0.$$

For $(Z, U) = (E_1, E_2)$, this gives

$$g(\mathcal{R}(E_{12} + E_{34}), E_{12} - E_{34}) - g(\mathcal{R}(E_{14} + E_{23}), E_{13} - E_{42}) = 0.$$

Then, since $\mathcal{W} = 0$, we obtain by means of (35)

$$\rho_{11} + \rho_{22} - \rho_{33} - \rho_{44} - 2(\rho_{12} + \rho_{34}) = 0.$$

Applying this identity for the basis $(-E_1, E_2, -E_3, E_4, E_5, \dots, E_n)$ we have

$$\rho_{11} + \rho_{22} - \rho_{33} + \rho_{44} + 2(\rho_{12} + \rho_{34}) = 0.$$

Hence

$$\rho_{11} + \rho_{22} = \rho_{33} + \rho_{44}, \quad \rho_{12} = -\rho_{34}.$$

The first of these identities imply

$$\rho_{11} + \rho_{22} = \rho_{aa} + \rho_{bb} \text{ for } a \neq b, \quad a, b = 1, \dots, n.$$

It follows that

$$\rho_{aa} + \rho_{bb} = \frac{s}{2k}, \quad a \neq b.$$

Applying the above obtained identity $\rho_{12} = -\rho_{34}$ for the basis $(E_2, E_1, -E_3, E_4, E_5, \dots, E_n)$ we get $\rho_{12} = \rho_{34}$, thus $\rho_{12} = \rho_{34} = 0$. It follows that

$$\rho_{ab} = 0, \quad a \neq b.$$

Now we note that the condition $\mathcal{B} = 0$ is equivalent to

$$\rho_{aa} + \rho_{bb} - \frac{2s}{n} = 0, \quad \rho_{ab} = 0, \quad a \neq b, \quad a, b = 1, \dots, n.$$

Thus we can conclude that $\mathcal{B} = 0$. Therefore $\mathcal{R} = \frac{s}{n(n-1)}Id$, i.e. (M, g) is of constant sectional curvature.

Conversely, if $\mathcal{R} = \frac{s}{n(n-1)}Id$, a straightforward computation shows that identity (37) is satisfied. □

7 Natural isomorphisms of generalized twistor spaces

(I) Let $f : A(E') \oplus A(E'') \rightarrow A(E') \oplus A(E'')$ be the bundle isomorphism $a = (a', a'') \rightarrow (a', -a'')$. The differential of this isomorphism preserves the horizontal lifts, $f_*X_a^h = X_{f(a)}^h$, and if $V = (V', V'')$ is a vertical vector, $f_*V = (V', -V'')$. The restriction of f to the generalized twistor space \mathcal{G} is an automorphism of \mathcal{G} . The automorphism $F = f_* \oplus (f^{-1})^*$ of $T\mathcal{G} \oplus T^*\mathcal{G}$ preserves the horizontal and vertical subbundles and sends the generalized almost complex structure \mathcal{J}_ε to the structure $\tilde{\mathcal{J}}_\varepsilon$ given by $\tilde{\mathcal{J}}_\varepsilon A_J^h = (F^{-1}(J)A)_J^h$ for $A \in T_{\pi(J)}M \oplus T_{\pi(J)}^*M$ and $\tilde{\mathcal{J}}_\varepsilon = \mathcal{J}_\varepsilon$ on $\mathcal{V} \oplus \mathcal{V}^*$. By Proposition 5, $\tilde{\mathcal{J}}_\varepsilon$ is integrable if and only if \mathcal{J}_ε is so.

(II) Now for $a = (a', a'') \in A(E') \oplus A(E'')$ set

$$a_1 = pr_{TM}|E' \circ a' \circ (pr_{TM}|E')^{-1}, \quad a_2 = pr_{TM}|E'' \circ a'' \circ (pr_{TM}|E'')^{-1}.$$

Let φ be the automorphism $a \rightarrow b = (b', b'')$ of $A(E') \oplus A(E'')$ defined by

$$b' = (pr_{TM}|E')^{-1} \circ a_2 \circ pr_{TM}|E', \quad b'' = (pr_{TM}|E'')^{-1} \circ a_1 \circ pr_{TM}|E''.$$

The differential φ_* preserves the horizontal lifts. Clearly, if $J \in \mathcal{G}$ gives rise to the complex structures (J_1, J_2) on $T_{\pi(J)}M$, then $\varphi(J) \in \mathcal{G}$ is the generalized complex structure on $T_{\pi(J)}M$ determined by the pair (J_2, J_1) . Moreover, if $V \in \mathcal{V}_J$ gives rise to the tangent vector (V_1, V_2) of $Z(T_{\pi(J)}M, g) \times Z(T_{\pi(J)}M, g)$ at (J_1, J_2) , then φ_*V is the vertical vector of \mathcal{G} at $\varphi(J)$ determined by (V_2, V_1) .

(III) Let (M, \mathcal{J}) and (N, \mathcal{K}) be two generalized complex manifolds. Every diffeomorphism $f : M \rightarrow N$ induces a bundle isomorphism $F = f_* \oplus f^{*-1} : TM \oplus T^*M \rightarrow TN \oplus T^*N$ and the identity $F \circ \mathcal{J} = \mathcal{K} \circ F$ is a natural generalization of the condition for a map between complex manifolds to be holomorphic. The diffeomorphisms are not the only symmetries of the generalized complex structures, the B -transforms are also symmetries. Thus we say that (M, \mathcal{J}) and (N, \mathcal{K}) are equivalent if there is a diffeomorphism $f : M \rightarrow N$ and a closed 2-form B on M such that $F \circ e^B \mathcal{J} e^{-B} = \mathcal{K} \circ F$ (this is really an equivalence relation). Since the form B is closed, each of two equivalent generalized almost complex structures \mathcal{J} and \mathcal{K} is integrable if and only if the other one is so.

Let \hat{E} be the B -transform of E by a 2-form Ψ on M . Then we have a natural diffeomorphism β of the generalized twistor spaces $\mathcal{G} = \mathcal{G}(E)$ and $\hat{\mathcal{G}} = \mathcal{G}(\hat{E})$ sending a generalized complex structure $J \in \mathcal{G}$ to its B -transform $\hat{J} = e^\Psi J e^{-\Psi}$.

Denote by D and \hat{D} the connections on $TM \oplus T^*M$ determined by the generalized metrics E and \hat{E} , respectively, as in Sect. 6.1. Let $\mathcal{J} = \mathcal{J}_1^E$ and $\hat{\mathcal{J}} = \mathcal{J}_1^{\hat{E}}$ be the generalized almost complex structures on \mathcal{G} and $\hat{\mathcal{G}}$ defined by means of the connections D and \hat{D} . If the form Ψ is closed, these generalized almost complex structures are equivalent in a natural way. Indeed, set $E' = E$, $\hat{E}' = \hat{E}$. The B -transform by Ψ is an orthogonal transformation of $TM \oplus T^*M$, thus it sends $E'' = E^\perp$ onto $\hat{E}'' = \hat{E}^\perp$, the orthogonal complements being taken with respect to the metric $\langle \cdot, \cdot \rangle$. Let ∇ and $\hat{\nabla}$ be the connections on TM obtained by transferring $D' = D|E$ and $\hat{D}' = \hat{D}| \hat{E}$. Recall that, on a Riemannian manifold (M, g) , there is a unique metric connection with a given torsion T (for an explicit formula see, for example, [12, Sec. 3.5, formula (14)]). If the torsion 3-form $T(X, Y, Z) = g(T(X, Y), Z)$ is skew-symmetric this connection can be written as $\nabla^{LC} + \frac{1}{2}T$ where ∇^{LC} is the Levi-Civita

connection of (M, g) . Thus

$$\begin{aligned} \nabla_X Y &= \nabla_X^{LC} Y - \frac{1}{2} g^{-1} (i_X i_Y d\Theta), \\ \widehat{\nabla}_X Y &= \nabla_X^{LC} Y - \frac{1}{2} g^{-1} (i_X i_Y d\Theta) - \frac{1}{2} g^{-1} (i_X i_Y d\Psi). \end{aligned}$$

Hence

$$\widehat{\nabla}_X Y = \nabla_X Y - \frac{1}{2} g^{-1} (i_X i_Y d\Psi).$$

Suppose that the form Ψ is closed, so that $\widehat{\nabla}_X Y = \nabla_X Y$. Then the B -transform e^Ψ sends the connection D to the connection \widehat{D} since $pr_{TM}|E' = pr_{TM}|\widehat{E}' \circ e^\Psi$ and $pr_{TM}|E'' = pr_{TM}|\widehat{E}'' \circ e^\Psi$.

It follows that $\beta : L \rightarrow \widehat{L} = e^\Psi L e^{-\Psi}$ is an isometry of $A(E') \oplus A(E'')$ onto $A(\widehat{E}') \oplus A(\widehat{E}'')$ sending the connection D on $A(E') \oplus A(E'')$ induced by the connection $D|E' \oplus D|E''$ to the connection \widehat{D}' on $A(\widehat{E}') \oplus A(\widehat{E}'')$ induced by $\widehat{D}|\widehat{E}' \oplus \widehat{D}|\widehat{E}''$. In particular, β_* preserves the horizontal spaces,

$$\beta_* X_L^h = X_{\widehat{L}}^h, \quad X \in TM, \tag{40}$$

where $X^{\widehat{h}}$ is the horizontal lift of X to $T(A(\widehat{E}') \oplus A(\widehat{E}''))$.

The restriction of β to \mathcal{G} is a diffeomorphism of \mathcal{G} onto $\widehat{\mathcal{G}}$ whose differential preserves the horizontal spaces. Clearly, β_* preserves also the vertical spaces sending a vertical vector V at $J \in \mathcal{G}$ to the vertical vector $\widehat{V} = e^\Psi V e^{-\Psi}$ at \widehat{J} . Then, if $\alpha \in T_p^* M, Z \in T_p M$

$$\left((\beta^{-1})^* \alpha_J^h \right) \left(Z_{\widehat{J}}^h \right) = \alpha^h \left(\beta_*^{-1} Z_{\widehat{J}}^h \right) = \alpha_J^h (Z_J^h) = \alpha(Z) = \alpha_{\widehat{J}}^h \left(Z_{\widehat{J}}^h \right),$$

where $\alpha^{\widehat{h}}$ is the horizontal lift of α to $T(A(\widehat{E}') \oplus A(\widehat{E}''))$. Also

$$\left((\beta^{-1})^* \alpha_J^h \right) (\widehat{V}) = \alpha^h (\beta_*^{-1} \widehat{V}) = 0 = \alpha_{\widehat{J}}^h (\widehat{V})$$

for every vertical vector \widehat{V} at \widehat{J} . Thus

$$(\beta^{-1})^* \alpha_J^h = \alpha_{\widehat{J}}^h, \quad \alpha \in T^* M. \tag{41}$$

Note also that if $\Upsilon \in \mathcal{V}_{\widehat{J}}^*$,

$$((\beta^{-1})^* \Upsilon) (\widehat{V}) = \Upsilon (e^{-\Psi} \widehat{V} e^\Psi).$$

Set

$$\mathcal{B} = \beta_* \oplus (\beta^{-1})^*, \quad \widetilde{\Psi} = \pi^* \Psi,$$

where, as before, π is the projection to M of the bundle $A(E') \oplus A(E'')$ restricted to \mathcal{G} . Taking into account the fact that B -transforms act as the identity on 1-forms, we have

$$\mathcal{B} \left(e^{\widetilde{\Psi}} \mathcal{J} e^{-\widetilde{\Psi}} (\Upsilon) \right) (\widehat{V}) = \mathcal{B} (K_1^* \Upsilon) (\widehat{V}) = (K_1^* \Upsilon) (e^{-\Psi} \widehat{V} e^\Psi) = \Upsilon (J e^{-\Psi} \widehat{V} e^\Psi)$$

and

$$(\widehat{\mathcal{J}} \mathcal{B} (\Upsilon)) (\widehat{V}) = \mathcal{B} (\Upsilon) (\widehat{\mathcal{J}} \widehat{V}) = \Upsilon (e^{-\Psi} \widehat{\mathcal{J}} \widehat{V} e^\Psi) = \Upsilon (J e^{-\Psi} \widehat{V} e^\Psi).$$

Thus

$$\mathcal{B} (e^{\widetilde{\Psi}} \mathcal{J} e^{-\widetilde{\Psi}} (\Upsilon)) = \widehat{\mathcal{J}} (\mathcal{B} (\Upsilon)).$$

Also

$$\mathcal{B}\left(e^{\tilde{\Psi}}\mathcal{J}e^{-\tilde{\Psi}}(V)\right)=\mathcal{B}(JV)=e^{\Psi}JVe^{-\Psi}=e^{\Psi}Je^{-\Psi}e^{\Psi}Ve^{-\Psi}=\widehat{\mathcal{J}}\mathcal{B}(V)=\widehat{\mathcal{J}}\mathcal{B}(V)$$

since $\tilde{\Psi}(V)=0$. For $J\in\mathcal{G}$, let (J_1, J_2) be the complex structures on $T_{\pi(J)}M$ determined by J . Let $A=X+g(X)+\Theta(X)\in E'_{\pi(J)}$. Noting that $\tilde{\Psi}(X^h)=(\Psi(X))^h$, we have

$$\begin{aligned} e^{\tilde{\Psi}}\mathcal{J}e^{-\tilde{\Psi}}(A^h_j) &= e^{\tilde{\Psi}}\mathcal{J}(X+g(X)+\Theta(X)-\Psi(X))^h_j \\ &= e^{\tilde{\Psi}}(J_1X+g(J_1X)+\Theta(J_1X)-J\Psi(X))^h_j \\ &= [J_1X+g(J_1X)+\Theta(J_1X)-J\Psi(X)-\Psi(\pi_1(J\Psi(X)))]^h_j \\ &= [e^{\Psi}Je^{-\Psi}(A)]^h_j = (\widehat{J}(A))^h_j. \end{aligned}$$

Then, by (40) and (41),

$$\mathcal{B}\left(e^{\tilde{\Psi}}\mathcal{J}e^{-\tilde{\Psi}}(A^h_j)\right)=\left(\widehat{J}(A)\right)^h_{\widehat{J}}=\widehat{\mathcal{J}}\left(A^h_{\widehat{J}}\right)=\widehat{\mathcal{J}}\left(\mathcal{B}(A^h_j)\right).$$

Similarly, for $A=X-g(X)+\Theta(X)\in E''_{\pi(J)}$ in which case $JA=J_2X-g(J_2X)+\Theta(J_2X)$.

This shows that $\mathcal{B}\circ(e^{\tilde{\Psi}}\mathcal{J}e^{-\tilde{\Psi}})=\widehat{\mathcal{J}}\circ\mathcal{B}$ where the 2-form $\tilde{\Psi}$ is closed.

A similar identity holds for another closed 2-form $\bar{\Psi}$ under certain restrictions on the curvature of M . This form is defined by

$$\begin{aligned} \bar{\Psi}(X^h, Y^h)_J &= \Psi(X, Y)_{\pi(J)}, \quad \bar{\Psi}(X^h, V) = \bar{\Psi}(V, X^h) = 0, \\ \bar{\Psi}(V, W)_J &= G(V, K_1W), \end{aligned}$$

where $X, Y\in T_{\pi(J)}$, $V, W\in\mathcal{V}_J$. To prove the identity $\mathcal{B}\circ(e^{\bar{\Psi}}\mathcal{J}e^{-\bar{\Psi}})=\widehat{\mathcal{J}}\circ\mathcal{B}$ we have only to show that $\mathcal{B}(e^{\bar{\Psi}}\mathcal{J}e^{-\bar{\Psi}}(V))=\widehat{\mathcal{J}}\mathcal{B}(V)$. But this follows from the identity

$$e^{\bar{\Psi}}\mathcal{J}e^{-\bar{\Psi}}(V)=K_1V-K_1^*(\bar{\Psi}(V))+\bar{\Psi}(K_1V)=J\circ V-G(V)+G(V)=J\circ V.$$

The standard formula for the differential in terms of the Lie bracket and identity (13) imply $d\bar{\Psi}(X^h, Y^h, Z^h)_J=0$. Let \tilde{a}, \tilde{b} be the vertical vector fields obtained from sections a, b of $A(E')\oplus A(E'')$ such that $a(p)=V, b(p)=W, Da|_p=D b|_p=0$ for $p=\pi(J)$. Then, by Lemma 2, $d\bar{\Psi}(X^h, V, W)_J=X^h_jG(\tilde{a}, \tilde{b})$ and it is easy to see that $X^h_jG(\tilde{a}, \tilde{b})=0$ using formulas given in the proof of Lemma 2. Next, $d\bar{\Psi}(X^h, Y^h, V)_J=0$ if and only if $G(R(X, Y)J, JV)=0$. Therefore $d\bar{\Psi}=0$ if and only if $R(X, Y)J=0$ for every $J\in\mathcal{G}$ and $X, Y\in T_{\pi(J)}M$. The latter condition is equivalent to

$$g(\mathcal{R}(X\wedge Y), J_kZ\wedge U+Z\wedge J_kU)=0, \quad k=1, 2, \quad X, Y, Z, U\in T_{\pi(J)}M, \quad (42)$$

where (J_1, J_2) are the complex structures on $T_{\pi(J)}M$ determined by J .

Let $\dim M=4$. In this case, identity (42) for J running over \mathcal{G}_{++} (\mathcal{G}_{--}) is equivalent to (M, g) being Ricci flat and anti-self-dual (self-dual, respectively). This identity holds on \mathcal{G}_{+-} or \mathcal{G}_{-+} if and only if (M, g) is flat.

If $\dim M\geq 6$, identity (42) is equivalent to the flatness of (M, g) .

Finally, note that the complex structures on a tangent space of M determined by $J\in\mathcal{G}$ and $\widehat{J}=e^{\Psi}Je^{-\Psi}$ via (3) are the same. Therefore the diffeomorphism β sends the connected components $\mathcal{G}_{++}, \dots, \mathcal{G}_{-+}$ of \mathcal{G} onto the corresponding connected components $\widehat{\mathcal{G}}_{++}, \dots, \widehat{\mathcal{G}}_{-+}$ of $\widehat{\mathcal{G}}$. In the case $\Psi=-\Theta$ we have $\widehat{E}=\{X+g(X):X\in TM\}$. Thus if Θ is closed the integrability conditions for the generalized almost complex structure \mathcal{J} are the same as those for $\widehat{\mathcal{J}}$.

We summarize the considerations above as follows.

Theorem 3 *Let E and \widehat{E} be generalized metrics on a manifold M determined by the same metric g and 2-forms Θ and $\widehat{\Theta}$. If the 2-form $\Theta - \widehat{\Theta}$ is closed, the generalized almost complex structures \mathcal{J}_1^E and $\mathcal{J}_1^{\widehat{E}}$ on the generalized twistor spaces $\mathcal{G}(E)$ and $\mathcal{G}(\widehat{E})$ are equivalent.*

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