



Symmetric products of a semistable degeneration of surfaces

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Abstract We explicitly construct a V -normal crossing Gorenstein canonical model of the relative symmetric products of a local semistable degeneration of surfaces without a triple point by means of toric geometry. Using this model, we calculate the stringy E -polynomial of the relative symmetric product. We also construct a minimal model of degeneration of Hilbert schemes explicitly.

Introduction

Let S be a smooth (quasi-)projective algebraic surface. A theorem of Fogarty [8] says that the Hilbert scheme $\text{Hilb}^n(S)$ of 0-dimensional subschemes on S of length n is a smooth (quasi-)projective algebraic variety of dimension $2n$. This construction gives a very nice and interesting way to produce higher dimensional algebraic varieties. For example, if S is a K3 surface (resp. an abelian surface), then $\text{Hilb}^n(S)$ (resp. the Albanese fiber of $\text{Hilb}^{n+1}(S)$) gives an example of higher dimensional irreducible symplectic compact Kähler manifold [2]. Besides the holomorphic symplectic geometry, the Hilbert scheme of points on a surface is related to many branches of mathematics, such as differential geometry, singularity theory, and representation theory.

As Hilbert schemes behave nicely in family, it is quite natural to think of the relative Hilbert scheme $\text{Hilb}^n(\mathcal{S}/B)$ for a flat family of surfaces $\pi : \mathcal{S} \rightarrow B$. If the family π degenerates at some point $b \in B$, one naturally expects that the family of Hilbert schemes $\text{Hilb}^n(\mathcal{S}/B) \rightarrow B$ also degenerates at b . In this setting, one of the fundamental questions is to ask how much singular the induced degeneration of Hilbert schemes is. Of course, it will depend on the singularity of the degeneration of the original family. To get a modest degeneration of Hilbert schemes, it is natural to assume that the family of surfaces $\pi : \mathcal{S} \rightarrow B$ is *semistable*. In such a situation, another natural question is to find a *good* birational model of

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a degeneration of Hilbert schemes that is semistable (or very near to semistable) and minimal [9, 14], and to understand the behavior of the family.

However, even if the family $\pi : \mathcal{S} \rightarrow B$ is a semistable degeneration, and hence \mathcal{S} is smooth, it seems difficult, at least to the author, to study the relative Hilbert scheme $\text{Hilb}^n(\mathcal{S}/B)$ directly by the ring theoretic approach as in [8] or [4], in contrast to Ran’s work [18] on the case of semistable degeneration of curves in this direction. In fact, our relative Hilbert scheme can be seen as a closed subscheme $\text{Hilb}^n(\mathcal{S}/B) \subset \text{Hilb}^n(\mathcal{S})$, while $\text{Hilb}^n(\mathcal{S})$ can be very singular for $\dim \mathcal{S} \geq 3$ and n large.

In this article, we will focus on the *relative symmetric product* $\text{Sym}^n(\mathcal{S}/B)$ rather than the relative Hilbert scheme, and study its singularity and birational geometry. For that purpose, we start from a local model of semistable degeneration of surfaces $\mathcal{S} \rightarrow B$ and describe the symmetric product $\text{Sym}^n(\mathcal{S}/B)$ as a quotient of certain affine toric variety by an action of the symmetric group (§1). This description leads us to a Gorenstein canonical model with only quotient singularities for a degeneration without a triple point (§2, Theorem 2.10). The Gorenstein canonical model is obtained as a quotient by a natural action of the symmetric group of the total space of a rank two toric vector bundle over a toric variety associated with the Coxeter complex of a root system of type A (Proposition 2.5). It is noteworthy that such a toric variety was studied by several authors from an interest of combinatorics and representation of symmetric groups [6, 17, 21]. This toric-quotient description also enables us to calculate the stringy E -polynomial of the Gorenstein canonical model, which encodes cohomological information of the degeneration (§3, Theorem 3.14, Proposition 3.17). In the last section, we discuss an explicit construction of a \mathbb{Q} -factorial terminal minimal model of the relative symmetric product (Theorem 4.1), which turns out to be a V -normal crossing degeneration of Hilbert schemes of general fibers. This gives a good birational model of a degeneration of Hilbert schemes in the case where the singular fiber of the original semistable degeneration of surfaces has no triple point. We also relate our minimal model to the relative Hilbert scheme explicitly in the case where the length $n = 2$.

1 Local description of symmetric product

1.1 Let $S_2 = S_3 = \mathbb{C}^3$ and define $p_d : S_d \rightarrow B = \mathbb{C}$ by

$$p_d(x_1, x_2, x_3) = \begin{cases} x_1x_2 & \text{if } d = 2, \\ x_1x_2x_3 & \text{if } d = 3. \end{cases}$$

The origin of S_2 is the local model at the general point of the singular locus of the singular fiber of semistable degeneration of surfaces, while the origin of S_3 is the maximally degenerate point. Let us denote the n -fold self-products of S_d relative to p_d by

$$\tilde{X}_d^{(n)} = (S_d/B)^n = S_d \times_B S_d \times_B \cdots \times_B S_d.$$

and let $\tilde{\pi}_d^{(n)} : \tilde{X}_d^{(n)} \rightarrow B$ be the natural morphism. Then, the symmetric group \mathfrak{S}_n acts on $\tilde{X}_d^{(n)}$ and $\tilde{\pi}_d^{(n)}$ is \mathfrak{S}_n -invariant. The quotient variety

$$\pi_d^{(n)} : X_d^{(n)} = \tilde{X}_d^{(n)}/\mathfrak{S}_n \rightarrow B$$

is the n th *relative symmetric product* of $p_d : S_d \rightarrow B$.

The fiber $(\pi_d^{(n)})^{-1}(b)$ for $b \in B, b \neq 0$ is just $\text{Sym}^n(\mathbb{C}^* \times \mathbb{C})$ for $d = 2$ and $\text{Sym}^n((\mathbb{C}^*)^2)$ for $d = 3$. It is also easy to see the combinatorics of the fiber over the origin. It has $\binom{n+d-1}{d}$

components each of which corresponds to a partition of n ,

$$\mathbf{a} = \left(a_i \mid 1 \leq i \leq d, \sum a_i = n \right).$$

The component $X_{\mathbf{a}}^{(n)}$ is an image of a birational morphism

$$\text{Sym}^{\mathbf{a}}(\mathbb{C}^2) = \prod_{i=1}^d \text{Sym}^{a_i}(\mathbb{C}^2) \rightarrow X_{\mathbf{a}}^{(n)}.$$

The morphism has finite fibers over the double curves except for the case $a_i = n$ for some i . It implies that $X_{\mathbf{a}}^{(n)}$ is *non-normal* for general \mathbf{a} . The geometry of the intersection of these components seems difficult to describe directly.

In contrast, it is easy to describe the product variety $\tilde{X}_d^{(n)}$ by affine equations. Let $(z_{11}, z_{12}, z_{13}, \dots, z_{n1}, z_{n2}, z_{n3})$ be the coordinate of $(\mathbb{C}^3)^n = \mathbb{C}^{3n}$. Then, $\tilde{X}_d^{(n)}$ is nothing but the complete intersection

$$z_{11} \cdots z_{1d} = z_{21} \cdots z_{2d} = \cdots = z_{n1} \cdots z_{nd} \quad (d = 2, 3) \tag{1}$$

of dimension $2n + 1$ and the projection $\tilde{\pi}_d^{(n)} : \tilde{X}_d^{(n)} \rightarrow B = \mathbb{C}$ is the function defined by the value of these monomials. If $d = 2$, the variety split into a product of the closed subvariety $\tilde{X}_2^{(n)'} \subset \mathbb{C}^{2n}$ defined by the same Eq. (1) and \mathbb{C}^n .

Proposition 1.2 $X_d^{(n)}$ is \mathbb{Q} -Gorenstein, i.e., the canonical divisor of $X_d^{(n)}$ is \mathbb{Q} -Cartier.

Proof The locus F of points with non-trivial stabilizers with respect to the action of \mathfrak{S}_n is the union of linear subspaces defined by $z_{i1} = z_{j1}$, $z_{i2} = z_{j2}$, and $z_{i3} = z_{j3}$ for $i \neq j$. Therefore, the codimension of F inside $\tilde{X}_d^{(n)}$ is two. This implies that the quotient map $\tilde{X}_d^{(n)} \rightarrow X_d^{(n)}$ is étale in codimension 1 and the proposition follows from Proposition 5.20 of [12]. □

1.3. For later use, we give a description of $\tilde{X}_d^{(n)}$ as a toric variety. Let us first consider the case in which $d = 2$. Let $M = \mathbb{Z}^{n+1}$ and $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ its dual. We denote by $[a_1 a_2 \dots a_l]_{lr}$ the sequence a_1, a_2, \dots, a_l recurred r times. For example,

$$[0011]_{12} = (0, 0, 1, 1, 0, 0, 1, 1, 0, 0, 1, 1).$$

Using this notation, we define $C_2^{(n)}$ as the $(n + 1) \times 2^n$ matrix whose rows are given by

$$\begin{aligned} & [10]_{2^n} \\ & [01]_{2^n} \\ & [0011]_{2^n} \\ & \vdots \\ & [0^{2^{n-1}} 1^{2^{n-1}}]_{2^n}, \end{aligned}$$

where “ 0^r ” means 0 repeated r -times and the same for “ 1^r ”. For example,

$$C_2^{(3)} = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{pmatrix}.$$

Under the standard identification $N \cong \mathbb{Z}^{n+1}$, we define $\sigma_2^{(n)} \subset N \otimes \mathbb{R}$ to be a rational polyhedral convex cone generated by the column vectors in $C_2^{(n)}$. The cone $\sigma_2^{(n)}$ is a non-simplicial cone of maximal dimension for $n \geq 2$.

Proposition 1.4 *The affine variety $\tilde{X}_2^{(n)'} \subset \mathbb{C}^{2n}$ defined above is the affine toric variety $X(\sigma_2^{(n)}) = \text{Spec } \mathbb{C}[\sigma_2^{(n)\vee} \cap M]$.*

This proposition is implicitly given in [23], §4. Here we give a proof of slightly different flavor.

Lemma 1.5 *Let $\sigma_1, \sigma_2, \bar{\sigma}$ be strictly convex rational polyhedral cones on lattices N_1, N_2, \bar{N} , respectively. Assume that we have surjective homomorphisms $h_1 : N_1 \rightarrow \bar{N}$ and $h_2 : N_2 \rightarrow \bar{N}$ such that $\bar{\sigma} = h_{1,\mathbb{R}}(\sigma_1) = h_{2,\mathbb{R}}(\sigma_2)$. Let $\pi_i : X(\sigma_i) \rightarrow X(\bar{\sigma})$ ($i = 1, 2$) be the corresponding toric morphisms of affine toric varieties. Then, the fiber product $X(\sigma_1) \times_{X(\bar{\sigma})} X(\sigma_2)$ is an affine toric variety corresponding to the cone*

$$\sigma_1 \times_{\bar{N}} \sigma_2 = \{(v_1, v_2) \in (N_1 \oplus N_2)_{\mathbb{R}} \mid v_i \in \sigma_i \ (i = 1, 2), h_1(v_1) = h_2(v_2)\}$$

on the lattice $N_1 \times_{\bar{N}} N_2 = \{(v_1, v_2) \in N_1 \oplus N_2 \mid h_1(v_1) = h_2(v_2)\}$.

Proof Let M_1, M_2, \bar{M} be dual lattices of N_1, N_2, \bar{N} , respectively. Since h_i ($i = 1, 2$) is surjective, \bar{M} is a direct summand of M_i . The tensor product

$$\mathbb{C}[\sigma_1^{\vee} \cap M_1] \otimes_{\mathbb{C}[\bar{\sigma}^{\vee} \cap \bar{M}]} \mathbb{C}[\sigma_2^{\vee} \cap M_2]$$

has a basis consisting of monomials (m_1, m_2) ($m_i \in M_i$) subject to a relation

$$(m_1, m_2) = (m'_1, m'_2) \text{ if and only if } m_1 - m'_1 = m'_2 - m_2 \in \bar{M}.$$

This implies that the tensor ring is the monoid ring corresponding to a cone C that is the image of the product cone $\sigma_1^{\vee} \times \sigma_2^{\vee}$ under the surjective homomorphism to the fiber co-product

$$M_1 \oplus M_2 \rightarrow M_1 +_{\bar{M}} M_2 = (M_1 \oplus M_2) / \bar{M}.$$

In particular, C is spanned by the vectors $(m_1, m_2) \in M_1 +_{\bar{M}} M_2$ with m_i a generator of a ray of σ_i^{\vee} for each $i = 1, 2$. Passing to the dual, the dual cone C^{\vee} is cut out by positive half-planes defined by (m_1, m_2) as above on the fiber product of lattices $N_1 \times_{\bar{N}} N_2$. This immediately implies that C^{\vee} is nothing but the fiber product of cones $\sigma_1 \times_{\bar{N}} \sigma_2$. □

Proof of Proposition 1.4 Let $C \rightarrow B = \mathbb{C}$ be a family of curves defined by $(x_1, x_2) \mapsto x_1 x_2$. This is a toric morphism corresponding to a surjective homomorphism of lattices

$$(1 \ 1) : N_C = \mathbb{Z}^2 \rightarrow N_B = \mathbb{Z}$$

that is compatible with the cones $\sigma_C = \mathbb{R}_{\geq 0} d_1 + \mathbb{R}_{\geq 0} d_2$ and $\sigma_B = \mathbb{R}_{\geq 0}$, where $d_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$

and $d_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Noting that $\tilde{X}_2^{(n)'}$ is an n -fold fiber product $(C/B)^n$, the cone $\sigma_2^{(n)}$ is nothing but the fiber product of cones

$$(\sigma_C/N_B)^n = \sigma_C \times_{N_B} \cdots \times_{N_B} \sigma_C$$

on the lattice $(N_C/N_B)^n$ by the lemma above. If we take a basis $(N_C/N_B)^n \cong \mathbb{Z}^{n+1}$ as

$$\begin{aligned} e_0 &= (d_1, d_1, \dots, d_1), & e_1 &= (d_2, d_1, \dots, d_1) \\ e_j &= (0, \dots, \underset{\hat{j+1}}{d_2 - d_1}, 0, \dots, 0) \end{aligned}$$

one immediately sees that the cone $(\sigma_C/N_B)^n$ is generated by the column vectors of the matrix $C_2^{(n)}$ defined above. \square

1.6. Let $\tilde{M} = \mathbb{Z}^2 \otimes \mathbb{Z}^n$. The symmetric group \mathfrak{S}_n acts on \tilde{M} by the permutation representation on the second factor. Define a $(n + 1) \times 2n$ integral matrix $P_2^{(n)}$ by

$$P_2^{(n)} = (e_0 \ e_1 \ e_0 + e_1 - e_2 \ e_2 \ \cdots \ e_0 + e_1 - e_n \ e_n), \tag{2}$$

where e_0, \dots, e_n is the standard basis of $M = \mathbb{Z}^{n+1}$. The cone $\sigma_2^{(n)\vee}$ is nothing but the image under the surjective linear map $P_2^{(n)} : \tilde{M} \otimes \mathbb{R} \rightarrow M \otimes \mathbb{R}$ of the cone spanned by the standard simplex in \tilde{M} . Therefore, we have an induced action of \mathfrak{S}_n on M and its dual N . More precisely, \mathfrak{S}_n acts on M by permuting n pairs of vectors

$$(e_0 \ e_1 \ | \ e_0 + e_1 - e_2 \ e_2 \ | \ \cdots \ | \ e_0 + e_1 - e_n \ e_n),$$

so that the action of \mathfrak{S}_n on N is represented by matrices

$$(1 \ 2) = \left(\begin{array}{ccc|c} 1 & 1 & -1 & \\ 0 & 0 & 1 & \\ 0 & 1 & 0 & \\ \hline & & & I_{n-2} \end{array} \right) \text{ and } (k \ k+1) = \left(\begin{array}{cc|c} I_k & & \\ \hline 0 & 1 & \\ 1 & 0 & \\ \hline & & I_{n-k-1} \end{array} \right) \tag{3}$$

for $k > 1$. The cone $\sigma_2^{(n)}$ and its dual $\sigma_2^{(n)\vee}$ are invariant under the action of \mathfrak{S}_n . Let σ be a cone in $(N \oplus \mathbb{Z}^n) \otimes \mathbb{R}$ spanned by $\sigma_2^{(n)}$ and the standard basis of \mathbb{Z}^n . Then, the associated affine toric variety $X(\sigma)$ is nothing but $\tilde{X}_2^{(n)}$. The action of \mathfrak{S}_n on $\tilde{X}_2^{(n)}$ coincides with the action induced by the diagonal \mathfrak{S}_n -action on N and \mathbb{Z}^n .

1.7. We also have a similar description of $\tilde{X}_3^{(n)}$ as a toric variety. Let $M = \mathbb{Z}^{2n+1}$ and $N = \text{Hom}_{\mathbb{Z}}(M, \mathbb{Z}) \cong \mathbb{Z}^{2n+1}$. Let $C_3^{(n)}$ be the $(2n + 1) \times 3^n$ matrix whose rows are given by

$$\begin{aligned} & [1 \ 0 \ 0]_{3^n} \\ & [0 \ 1 \ 0]_{3^n} \\ & [0 \ 0 \ 1]_{3^n} \\ & [0 \ 0 \ 0 \ 1 \ 1 \ 0 \ 0 \ 0]_{3^n} \\ & [0 \ 0 \ 0 \ 0 \ 0 \ 1 \ 1 \ 1]_{3^n} \\ & \vdots \\ & [0^{3^n-1} \ 1^{3^n-1} \ 0^{3^n-1}]_{3^n} \\ & [0^{3^n-1} \ 0^{3^n-1} \ 1^{3^n-1}]_{3^n}. \end{aligned}$$

and $\sigma_3^{(n)} \subset N \otimes \mathbb{R}$ the cone generated by the column vectors of $C_3^{(n)}$. Then, the associated toric variety $X(\sigma_3^{(n)}) = \text{Spec } \mathbb{C}[\sigma_3^{(n)\vee} \cap M]$ is nothing but $\tilde{X}_3^{(n)}$. The proof of this claim is completely parallel to the case of $\tilde{X}_2^{(n)}$; it is a direct consequence of Lemma 1.5.

2 Gorenstein canonical orbifold model

In this section, we construct a Gorenstein canonical model of $X_2^{(n)}$ with only quotient singularities. From now on, we concentrate on the case $d = 2$. We suppress the subscript and write $\tilde{X}^{(n)}$, $\tilde{X}^{(n) \prime}$ and $X^{(n)}$ instead of $\tilde{X}_2^{(n)}$, $\tilde{X}_2^{(n) \prime}$ and $X_2^{(n)}$, respectively, for better readability.

Proposition 2.1 *There is an \mathfrak{S}_n -equivariant small projective toric resolution*

$$\tilde{\mu}^{(n)'} : \tilde{Z}^{(n)'} \rightarrow \tilde{X}^{(n)'}$$

Proof $\tilde{X}^{(n)'}$ is the affine closed subvariety in \mathbb{C}^{2n} defined by

$$z_{11}z_{12} = z_{21}z_{22} = \cdots = z_{n1}z_{n2}. \tag{4}$$

The n -plane $\Sigma^{(n)}$ defined by $z_{11} = z_{21} = \cdots = z_{n1} = 0$ is an \mathfrak{S}_n -invariant non-Cartier Weil divisor on $X^{(n)'}$. Let

$$\tilde{f}^{(n)} : \tilde{W}^{(n)'} \rightarrow \tilde{X}^{(n)'}$$

be the blowing-up of $\tilde{X}^{(n)'}$ along $\Sigma^{(n)}$. $\tilde{W}^{(n)'}$ is the closed subvariety of $\mathbb{C}^{2n} \times \mathbb{P}^{n-1}$ defined by

$$z_{i1}y_j - z_{j1}y_i = 0 \quad (i \neq j)$$

along with (4), where $[y_1 : \cdots : y_n]$ is the homogeneous coordinate of \mathbb{P}^{n-1} . Let $P_i = [0 : \cdots : 0 : \overset{i}{1} : 0 : \cdots : 0] \in \mathbb{P}^{n-1}$ and $U_i = (y_i \neq 0) \subset \mathbb{P}^{n-1}$. Then, it is easily checked using coordinate that there is a natural isomorphism of affine varieties

$$\tilde{W}^{(n)'} \cap (\mathbb{C}^{2n} \times U_i) \cong \tilde{X}^{(n-1)'} \times \mathbb{C}.$$

Moreover,

$$D_i = \tilde{W}^{(n)'} \cap (\mathbb{C}^{2n} \times \{P_i\}) \quad (i = 1, 2, \dots, n)$$

is a non- \mathbb{Q} -Cartier Weil divisor of $\tilde{W}^{(n)'}$ that is the strict transform of the divisor on $\tilde{X}^{(n)'}$ defined by

$$z_{11} = \cdots = z_{i-1,1} = z_{i2} = z_{i+1,1} = \cdots = z_{n1} = 0.$$

D_i is identified with $\Sigma^{(n-1)} \times \mathbb{C} \subset \tilde{X}^{(n-1)'} \times \mathbb{C}$ under the isomorphism above. D_i 's are disjoint to each other and the union $D = \sqcup D_i$ is \mathfrak{S}_n -invariant. Therefore, the blowing-up of $\tilde{W}^{(n)'}$ along D is \mathfrak{S}_n -equivariant. As it is locally isomorphic to $\tilde{f}^{(n-1)}$, we get an \mathfrak{S}_n -equivariant small resolution of $\tilde{X}^{(n)'}$ by induction on n . The centers of the blowing-ups are strict transform of torus invariant (non- \mathbb{Q} -Cartier) divisors on $\tilde{X}^{(n)'}$, so the resolution is a toric morphism. □

Remark 2.1.1 The toric variety $\tilde{Z}^{(n)'}$ also appeared in [23] as the local model of ‘augmented relative Hilbert scheme’.

2.2. From this proposition, one immediately sees that the relative self-product $\tilde{X}^{(n)}$ admits an \mathfrak{S}_n -equivariant small projective toric resolution

$$\tilde{\mu}^{(n)} = (\tilde{\mu}^{(n)'} \times \text{id}) : \tilde{Z}^{(n)} = \tilde{Z}^{(n)'} \times \mathbb{C}^n \rightarrow \tilde{X}^{(n)} = \tilde{X}^{(n)'} \times \mathbb{C}^n.$$

Now we let $Z^{(n)} = \tilde{Z}^{(n)}/\mathfrak{S}_n = (\tilde{Z}^{(n)'} \times \mathbb{C}^n)/\mathfrak{S}_n$. Then, we get a small projective birational morphism

$$\mu^{(n)} : Z^{(n)} \rightarrow X^{(n)}$$

and an induced family

$$\rho^{(n)} = \pi^{(n)} \circ \mu^{(n)} : Z^{(n)} \rightarrow B.$$

We want to study the singular locus of $Z^{(n)}$. For that purpose, we need a description of the toric birational morphism $\tilde{\mu}^{(n)'} : \tilde{Z}^{(n)'} \rightarrow \tilde{X}^{(n)'}$ in terms of fan.

2.3. The blowing-up $\tilde{f}^{(n)}$ appeared above corresponds to the star subdivision (see [5], §11.1 for the definition) $\Theta^{(n)}$ of the cone $\sigma^{(n)}$ with respect to the ray spanned by ${}^t(1, 0, \dots, 0)$. The fact that the ray is a one dimensional face of $\sigma^{(n)}$ corresponds to the smallness of $\tilde{f}^{(n)}$. The cone $\sigma^{(n)}$ is spanned by the column vectors of the $(n + 1) \times 2^n$ matrix $C^{(n)}$. One can check that the resolution $\tilde{\mu}^{(n)'} : \tilde{Z}^{(n)'} \rightarrow \tilde{X}^{(n)'}$ is given by the consecutive star subdivisions of $\sigma^{(n)}$ with respect to first $(2^n - 1)$ column vectors (in this order).

Let $\Delta^{(n)}$ be the resulted fan in $N \otimes \mathbb{R}$ and $\tilde{Z}^{(n)'}$ is the toric variety $X(\Delta^{(n)})$. By the proof of Proposition 2.1, one sees that \mathfrak{S}_n acts transitively on the set of open subsets $\{\tilde{W}^{(n)'} \cap (\mathbb{C}^{2^n} \times U_i)\}_{i=1}^n$. This means that \mathfrak{S}_n acts transitively on the set of maximal cones $\{\theta_i\}_{i=1}^n$ of the fan $\Theta^{(n)}$ corresponding to $\tilde{W}^{(n)'}$. Actually, \mathfrak{S}_n acts on it via the permutation of the index set $\{1, 2, \dots, n\}$. Each cone θ_i can be identified with $\sigma^{(n-1)}$ and its stabilizer subgroup $\text{Stab}(\theta_i) \subset \mathfrak{S}_n$ is nothing but the subgroup of permutations that leave i invariant, which is naturally isomorphic to \mathfrak{S}_{n-1} . An inductive argument infers that the set of maximal cones of the fan $\Delta^{(n)}$ consists of $n!$ cones and \mathfrak{S}_n acts on the set transitively. By the construction, $\Delta^{(n)}$ contains a cone $\delta^{(n)}$ that is generated by the column vectors of

$$\begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 1 & \cdots & 1 \\ 0 & 0 & 1 & \cdots & 1 \\ & & & \ddots & \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}. \tag{5}$$

2.4. Let $\bar{\Delta}^{(n)}$ be the Coxeter fan of A_{n-1} -root system, namely the fan whose maximal cones are Weyl chambers of A_{n-1} -root system on the weight lattice $\bar{N} = \mathbb{Z}^{n-1}$. Here, we adopt somewhat non-standard realization of A_{n-1} -root system. Regardless of inner product, we set the non-zero vectors in $\bar{N} = \mathbb{Z}^{n-1}$ whose entries are 0 or 1 the positive primitive weight vectors, and the negative primitive weight vectors are the negation of the positive primitive weight vectors. Note that this determines an \mathfrak{S}_n -action on the lattice $\bar{N} = \mathbb{Z}^{n-1}$, namely for $k < n - 1$

$$(k \ k + 1) = \left(\begin{array}{c|cc|c} I_{k-1} & & & \\ \hline & 0 & 1 & \\ & 1 & 0 & \\ \hline & & & I_{n-k-2} \end{array} \right) \text{ and}$$

$$(n - 1 \ n) = \begin{pmatrix} 1 & 0 & \cdots & 0 & -1 \\ 0 & 1 & \cdots & 0 & -1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -1 \\ 0 & 0 & \cdots & 0 & -1 \end{pmatrix}.$$

Now we consider the projective toric variety $X(\bar{\Delta}^{(n)})$. It has been studied by several authors [7, 17, 22] in connection with combinatorics theory. Speaking in a geometric language, $X(\bar{\Delta}^{(n)})$ is the canonical elimination of indeterminacy of the standard Cremona transformation of degree $n - 1$ in \mathbb{P}^{n-1} ([6], Example 7.2.5). More precisely, we have a sequence of projective birational morphisms

$$g : X(\bar{\Delta}^{(n)}) = X_1 \xrightarrow{g^1} X \longrightarrow \cdots \longrightarrow X_{n-2} \xrightarrow{g^{n-2}} X_{n-1} = \mathbb{P}^{n-1},$$

where g_i is the blowing-up of the strict transform of the union of the linear subspaces defined by vanishing of $i + 1$ projective coordinates ([7], Lemma 5.1). We take a fan Φ in $\bar{N} = \mathbb{Z}^{n-1}$ spanned by

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{pmatrix}, \dots, v_{n-1} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{pmatrix}, v_n = \begin{pmatrix} -1 \\ -1 \\ \vdots \\ -1 \end{pmatrix}.$$

The fan $\bar{\Delta}^{(n)}$ is a subdivision of Φ and the associated toric morphism $X(\bar{\Delta}^{(n)}) \rightarrow X(\Phi) = \mathbb{P}^{n-1}$ is nothing but the above-mentioned birational morphism g .

Let D_{pos} (resp. D_{neg}) be a torus invariant (resp. negative) primitive weight vectors. We take a homogeneous coordinate $[x_1 : \dots : x_n]$ on \mathbb{P}^{n-1} such that the prime divisor corresponding to v_j is the hyperplane $(x_j = 0)$ for $1 \leq j \leq n$. Then, the \mathfrak{S}_n -action on the toric variety $X(\Phi)$ coincides with the natural permutation of coordinates on \mathbb{P}^{n-1} :

$$s \cdot [x_1 : \dots : x_n] = [x_{s(1)} : \dots : x_{s(n)}] \quad (s \in \mathfrak{S}_n).$$

Under this choice of coordinate, we have $D_{neg} = g^* \text{div}(x_n)$. Let Φ' be another \mathbb{P}^{n-1} -fan on \bar{N} spanned by $-v_1, -v_2, \dots, -v_n$. $\bar{\Delta}^{(n)}$ is again a subdivision of Φ' and let $h : X(\bar{\Delta}^{(n)}) \rightarrow X(\Phi') = \mathbb{P}^{n-1}$ be the associated birational morphism. Then, the composition $h \circ g^{-1} : \mathbb{P}^{n-1} \dashrightarrow \mathbb{P}^{n-1}$ is nothing but the standard Cremona transformation

$$[x_1 : \dots : x_n] \mapsto \left[\frac{1}{x_1} : \dots : \frac{1}{x_n} \right],$$

and we have $D_{pos} = h^* \text{div}(x_n)$. In the below, we sometimes denote the toric variety $X(\bar{\Delta}^{(n)})$ by $X(A_{n-1})$ for easy recognition.

Proposition 2.5 *The toric variety $\tilde{Z}^{(n)'} = X(\Delta^{(n)})$ is isomorphic to the total space of a rank 2 vector bundle*

$$\mathcal{O}(-D_{pos}) \oplus \mathcal{O}(-D_{neg})$$

over $X(\bar{\Delta}^{(n)}) = X(A_{n-1})$. Moreover, the projection

$$\eta^{(n)} : X(\Delta^{(n)}) \rightarrow X(\bar{\Delta}^{(n)})$$

is \mathfrak{S}_n -equivariant.

Proof Let $Q : N \rightarrow N$ be an automorphism defined by left multiplication of the matrix

$$Q = \begin{pmatrix} 1 & 1 & 0 & \dots & 0 & -1 \\ 0 & 1 & 0 & \dots & 0 & -1 \\ 0 & 0 & 1 & \dots & 0 & -1 \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \dots & 1 & -1 \\ 0 & 0 & 0 & \dots & 0 & 1 \end{pmatrix}.$$

Then, we see that

$$QC_2^{(n)} = \left(\begin{array}{c|ccc|ccc|c} 1 & 1 & \cdots & \cdots & 1 & 0 & \cdots & \cdots & 0 & 0 \\ \hline 0 & \text{positive primitive} & & & \text{negative primitive} & & & & & 0 \\ \vdots & \text{weight vectors} & & & \text{weight vectors} & & & & & \vdots \\ \hline 0 & & & & & & & & & 0 \\ \hline 0 & 0 & \cdots & \cdots & 0 & 1 & \cdots & \cdots & 1 & 1 \end{array} \right). \tag{6}$$

In particular, $Q\delta^{(n)}$ is generated by column vectors of

$$\left(\begin{array}{c|ccc|c} 1 & 1 & \cdots & 1 & 0 \\ \hline 0 & 1 & \cdots & 1 & 0 \\ \vdots & & \ddots & & \vdots \\ \hline 0 & 0 & \cdots & 1 & 0 \\ \hline 0 & 0 & \cdots & 0 & 1 \end{array} \right).$$

Let $p : N = \mathbb{Z}^{n+1} \rightarrow \bar{N} = \mathbb{Z}^{n-1}$ be the projection to middle $(n - 1)$ -factors. From the matrix representation (3), it is easy to see that the composition

$$\Pi = p \circ Q : N \rightarrow \bar{N}$$

is \mathfrak{S}_n -equivariant. As the maximal cones in $\Delta^{(n)}$ are \mathfrak{S}_n -translates of $\delta^{(n)}$ and Π is \mathfrak{S}_n -equivariant, we know that Π is compatible with fans $\Delta^{(n)}$ and $\bar{\Delta}^{(n)}$, and induces a toric morphism $\eta^{(n)} : X(\Delta^{(n)}) \rightarrow X(\bar{\Delta}^{(n)})$. Moreover, one sees from (6) and [5] Proposition 7.3.1 that $X(\Delta^{(n)})$ is the total space of the direct sum of line bundles $\mathcal{O}(-D_{pos}) \oplus \mathcal{O}(-D_{neg})$. □

2.6. The coordinate $[x_1 : \cdots : x_n]$ on \mathbb{P}^{n-1} gives a convenient system of local coordinate on $X(A_{n-1})$ as in the following way. Let $\bar{\delta}^{(n)}$ be the positive Weyl chamber generated by the column vectors of

$$\left(\begin{array}{c|ccc|c} 1 & 1 & \cdots & 1 \\ \hline 0 & 1 & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ \hline 0 & 0 & \cdots & 1 \end{array} \right).$$

This is a smooth cone and the corresponding affine open subset U , which is isomorphic to \mathbb{C}^{n-1} , has a toric coordinate

$$\left(\frac{x_1}{x_2}, \frac{x_2}{x_3}, \dots, \frac{x_{n-1}}{x_n} \right).$$

A maximal cone in the A_{n-1} -Coxeter fan $\bar{\Delta}^{(n)}$ is written as $s \cdot \bar{\delta}^{(n)}$ for some $s \in \mathfrak{S}_n$. It is immediate to see that the corresponding affine open subset U_s has a toric coordinate

$$\left(\frac{x_{s(1)}}{x_{s(2)}}, \frac{x_{s(2)}}{x_{s(3)}}, \dots, \frac{x_{s(n-1)}}{x_{s(n)}} \right).$$

2.7. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a partition of n , i.e., let λ satisfy $\lambda_1 \geq \dots \geq \lambda_r > 0$ and $\lambda_1 + \dots + \lambda_r = n$. We set $l_j = \sum_{i=1}^j \lambda_i$ for $1 \leq j \leq r$ (and $l_0 = 0$ for convenience). Let c_j ($1 \leq j \leq r$) be a cyclic permutation

$$c_j = (l_{j-1} + 1 \dots l_j)$$

of length λ_j and we take a standard element

$$s = s_\lambda = (1 \dots l_1)(l_1 + 1 \dots l_2) \cdots (l_{r-1} + 1 \dots l_r) = c_1 c_2 \dots c_r \tag{7}$$

of \mathfrak{S}_n in the conjugacy class determined by λ .

Let $\overline{N}^{(s)}$ be the sublattice of s -fixed vectors. If we denote the standard basis of $\overline{N} = \mathbb{Z}^{n-1}$ by e_1, \dots, e_{n-1} , $\overline{N}^{(s)}$ is generated by

$$e_1 + \dots + e_{l_1}, e_{l_1+1} + \dots + e_{l_2}, \dots, e_{l_{r-2}+1} + \dots + e_{l_{r-1}},$$

so that $\overline{N}^{(s)} \cong \mathbb{Z}^{r-1}$. Every s -fixed point on $X(A_{n-1})$ is contained in an open subset

$$X_{\overline{N}}(\overline{\Delta}^{(n)} \cap (N^{(s)} \otimes \mathbb{R})) \cong (\mathbb{C}^*)^{n-r} \times X(A_{r-1}),$$

which \mathfrak{S}_n -equivariantly birationally dominates $(\mathbb{C}^*)^{n-r} \times \mathbb{P}^{r-1}$. Let

$$t_{ji} = \frac{x_{l_{j-1}+i+1}}{x_{l_{j-1}+i}}, \mathbf{t}_j = (t_{j1}, \dots, t_{j,\lambda_j-1}), \text{ and } y_k = x_{l_{k-1}+1}.$$

Then,

$$((\mathbf{t}_1, \dots, \mathbf{t}_r), [y_1 : \dots : y_r]) = ((t_{11}, \dots, t_{1,\lambda_1-1}; \dots; t_{r1}, \dots, t_{r,\lambda_r-1}), [y_1 : \dots : y_r])$$

is a coordinate on $(\mathbb{C}^*)^{n-r} \times \mathbb{P}^{r-1}$ and the action of c_j is given by

$$\begin{aligned} & ((\mathbf{t}_1; \dots; t_{j1}, \dots, t_{j,\lambda_j-1}; \dots; \mathbf{t}_r), [y_1 : \dots : y_j : \dots : y_r]) \\ & \mapsto \left(\left(\mathbf{t}_1; \dots; t_{j2}, \dots, t_{j,\lambda_j-1}, \frac{1}{t_{j1} \cdots t_{j,\lambda_j-1}}; \dots; \mathbf{t}_r \right), [y_1 : \dots : t_{j1} y_j : \dots : y_r] \right). \end{aligned}$$

In particular, if a point on $(\mathbb{C}^*)^{n-r} \times X(A_{r-1})$ is fixed by s , we necessarily have

$$t_{j1} = \dots = t_{j,\lambda_j-1} = \alpha_j$$

for all $1 \leq j \leq r$, where α_j is a λ_j th root of unity. Let us fix such a point

$$\boldsymbol{\alpha} = (\alpha_1, \dots, \alpha_1; \dots; \alpha_r, \dots, \alpha_r) \in (\mathbb{C}^*)^{n-r}$$

and $p \in \mathfrak{S}_r$. Let $U_p \cong \mathbb{C}^{r-1}$ be the affine open subset of $X(A_{r-1})$ corresponding to p . At a point $(\boldsymbol{\alpha}, \xi) \in (\mathbb{C}^*)^{n-r} \times U_p$, the action of c_j is given by

$$\begin{aligned} & \left(\boldsymbol{\alpha}, \left(\frac{y_{p(1)}}{y_{p(2)}}, \dots, \frac{y_{p(r-1)}}{y_{p(r)}} \right) \right) \\ & \mapsto \left(\boldsymbol{\alpha}, \left(\frac{y_{p(1)}}{y_{p(2)}}, \dots, \frac{1}{\alpha_j} \cdot \frac{y_{p(p^{-1}(j)-1)}}{y_j}, \alpha_j \cdot \frac{y_j}{y_{p(p^{-1}(j)+1)}}, \dots, \frac{y_{p(r-1)}}{y_{p(r)}} \right) \right), \tag{8} \end{aligned}$$

thus the action of $s = c_1 \dots c_r$ is of the form

$$\left(\boldsymbol{\alpha}, \left(\frac{y_{p(1)}}{y_{p(2)}}, \dots, \frac{y_{p(r-1)}}{y_{p(r)}} \right) \right) \mapsto \left(\boldsymbol{\alpha}, \left(\frac{\alpha_{p(1)}}{\alpha_{p(2)}} \cdot \frac{y_{p(1)}}{y_{p(2)}}, \dots, \frac{\alpha_{p(r-1)}}{\alpha_{p(r)}} \cdot \frac{y_{p(r-1)}}{y_{p(r)}} \right) \right). \tag{9}$$

In particular, for the torus invariant point ξ_p , the origin of $U_p \cong \mathbb{C}^{r-1}$, $(\boldsymbol{\alpha}, \xi_p)$ is always an s -fixed point.

2.8. Now we consider the action of s on the fiber \mathbb{C}^2 of $\mathcal{O}(-D_{pos}) \oplus \mathcal{O}(-D_{neg})$ over a fixed point. In §2.4, we saw that $D_{neg} = g^* \operatorname{div}(x_n)$. Noting that $n = l_r$ and $s(n) = l_{r-1} + 1$, we have

$$(s^{-1})^* D_{neg} = g^* \operatorname{div}(x_{l_{r-1}+1}) = D_{neg} + \operatorname{div}\left(\frac{x_{l_{r-1}+1}}{x_n}\right),$$

hence an isomorphism of invertible sheaves

$$(s^{-1})^* \mathcal{O}(-D_{neg}) \xrightarrow{\frac{x_{l_{r-1}+1}}{x_n}} \mathcal{O}(-D_{neg}).$$

Let us take an s -fixed point $(\alpha, \xi) \in (\mathbb{C}^*)^{n-r} \times U_p$ ($p \in \mathfrak{S}_r$) and set $v = p^{-1}(r)$. Then, it is easy to see that

$$\mathcal{O}_{(\mathbb{C}^*)^{n-r} \times U_p}(-D_{neg}) = \frac{y_{p(v+1)}}{y_{p(v)}} \cdots \frac{y_{p(r)}}{y_{p(r-1)}} \mathcal{O}_{(\mathbb{C}^*)^{n-r} \times U_p}.$$

Therefore, for any $\xi \in U_p$, the action of s on the fiber $\mathcal{O}(-D_{neg}) \otimes \kappa(\alpha, \xi)$ is given by the composition

$$\begin{aligned} \mathcal{O}(-D_{neg}) \otimes \kappa(\alpha, \xi) &\xrightarrow{\frac{\alpha_{p(r)}}{\alpha_r}} (s^{-1})^* \mathcal{O}(-D_{neg}) \otimes \kappa(\alpha, \xi) \\ &\xrightarrow{\alpha_r} \mathcal{O}(-D_{neg}) \otimes \kappa(\alpha, \xi), \end{aligned}$$

namely by a multiplication of $\alpha_{p(r)}$. One also sees by the same argument that the action of s on the fiber $\mathcal{O}(-D_{pos}) \otimes \kappa(\alpha, \xi)$ is given by the multiplication of $\alpha_{p(1)}^{-1}$.

Lemma 2.9 *Let $s \in \mathfrak{S}_n$ as in (7) and take a fixed point $q = (q_1, q_2) \in \tilde{Z}^{(n)} = X(\Delta^{(n)}) \times \mathbb{C}^n$. Assume $\eta^{(n)}(q_1) = (\alpha, \xi) \in (\mathbb{C}^*)^{n-r} \times U_p \subset X(\Delta^{(n)})$. Then, the eigenvalues of the Jacobian matrix $J_q(s)$ of s at q is*

$$\begin{aligned} &\underbrace{(\zeta_1, \dots, \zeta_1^{\lambda_1-1}; \dots; \zeta_r, \dots, \zeta_r^{\lambda_r-1})}_{(i)}; \quad \underbrace{\frac{\alpha_{p(1)}}{\alpha_{p(2)}}, \dots, \frac{\alpha_{p(r-1)}}{\alpha_{p(r)}}}_{(ii)}; \\ &\underbrace{\alpha_{p(1)}^{-1}, \alpha_{p(r)}}_{(iii)}; \quad \underbrace{1, \zeta_1, \dots, \zeta_1^{\lambda_1-1}; \dots; 1, \zeta_r, \dots, \zeta_r^{\lambda_r-1}}_{(iv)}. \end{aligned}$$

In particular, we always have $\det(J_q(s)) = 1$.

Proof The part (i) comes from the action of s on $(\mathbb{C}^*)^{n-r}$. More precisely, as we saw in §2.7, s acts on $(\mathbb{C}^*)^{n-r}$ by

$$(\dots; t_{j1}, \dots, t_{j, \lambda_j-1}; \dots) \mapsto \left(\dots; t_{j2}, \dots, t_{j, \lambda_j-1}, \frac{1}{t_{j1} \cdots t_{j, \lambda_j-1}}; \dots \right),$$

therefore the corresponding Jacobian matrix at $t_{ji} = \alpha_j$ ($1 \leq j \leq r, 1 \leq i \leq \lambda_j - 1$) is a block matrix with components of the form

$$\begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \\ -1 & -1 & -1 & \cdots & -1 \end{pmatrix},$$

whose characteristic polynomial is $1 + t + \cdots + t^{\lambda_j - 1}$. This gives the part (i). The part (ii) comes from the action (9) of s on U_p , and the part (iii) is the action on the fiber of $\mathcal{O}(-D_{pos}) \oplus \mathcal{O}(-D_{neg})$ (§2.8). The part (iv) is the contribution from the permutation representation of \mathfrak{S}_n on the second factor \mathbb{C}^n in $\tilde{Z}^{(n)}$. \square

Theorem 2.10 *The quotient $Z^{(n)} = \tilde{Z}^{(n)}/\mathfrak{S}_n$ has only Gorenstein canonical quotient singularities and the singular fiber $(\rho^{(n)})^{-1}(0) \subset Z^{(n)}$ of the induced family $\rho^{(n)} : Z^{(n)} \rightarrow B$ is a divisor with V -normal crossings ([21], Definition (1.16)).*

Proof V -normal crossingness automatically follows from the fact that the morphism $\tilde{Z}^{(n)} = X(\Delta^{(n)}) \times \mathbb{C}^n \rightarrow B$ is toric and each $s \in \mathfrak{S}_n$ acts on $\tilde{Z}^{(n)}$ by a toric morphism. Therefore, it is enough to show that for every point $q \in \tilde{Z}^{(n)}$, the stabilizer subgroup $\text{Stab}_{\mathfrak{S}_n}(q)$ is contained in $SL(T_q \tilde{Z}^{(n)})$. This is equivalent to say that for any $s \in \mathfrak{S}_n$ and s -fixed point $q \in \tilde{Z}^{(n)}$, the determinant of the Jacobian matrix $J_q(s)$ for the action of s at q is 1, which is nothing but the last assertion of the lemma above. \square

Remark 2.10.1 It follows from the theorem that $X^{(n)}$ also has only Gorenstein canonical singularities (but not \mathbb{Q} -factorial, unlike $Z^{(n)}$). Since $\mu^{(n)} : Z^{(n)} \rightarrow X^{(n)}$ is small, and therefore $K_{Z^{(n)}} = \mu^{(n)*} K_{X^{(n)}}$, it is clear that $X^{(n)}$ has only canonical singularities. The following argument to show that $X^{(n)}$ is Gorenstein is suggested by the referee; Let D be an effective divisor such that $-D$ is $\mu^{(n)}$ -ample ([12], Lemma 6.28). Then, for a sufficiently small positive rational number ε , the pair $(Z^{(n)}, \varepsilon D)$ is klt and $\mu^{(n)}$ is a contraction of $(K_{Z^{(n)}} + \varepsilon D)$ -negative extremal curves. Cone theorem ([12], Theorem 3.25) implies that there exists a Cartier divisor B on $X^{(n)}$ such that $K_{Z^{(n)}} = \mu^{(n)*} B$ since $(K_{Z^{(n)}} \cdot C) = 0$ for every curve C that is contracted by $\mu^{(n)}$. This shows that $K_{X^{(n)}} \sim_{\mathbb{Q}} B$ and therefore $K_{X^{(n)}}$ is Cartier.

3 Stringy E -polynomial

3.1. The *stringy E -function* is a cohomological invariant defined for varieties with only log terminal singularities. We review the formula, which can be seen as a definition in our purpose, of the stringy E -function for (global) quotient variety. For the foundation of the theory of stringy E -functions, we refer [1, 24].

Let X be a variety. By general theory of mixed Hodge structures, the compact support cohomology $H_c^k(X, \mathbb{Q})$ carries a canonical mixed Hodge structure, and hence the *Hodge number* $h^{p,q}(H_c^k(X))$ is defined. We define the *E -polynomial* $E(X) \in \mathbb{Z}[u, v]$ of X by

$$E(X) = \sum_{p,q,k} (-1)^k h^{p,q}(H_c^k(X)) u^p v^q.$$

As customary, we denote the E -polynomial of the affine line \mathbb{A}^1 by \mathbb{L} :

$$\mathbb{L} = E(\mathbb{A}^1) = uv.$$

If X is a toric variety, X is stratified into tori of various dimensions in Zariski topology, therefore, $E(X)$ can be written as a polynomial in \mathbb{L} .

Let M be a non-singular algebraic variety of dimension n and G a finite group acting on M . We denote a set of complete representatives of the conjugacy classes of G by $\text{Conj}(G)$. Let F_g be the locus of g -fixed points on M for $g \in \text{Conj}(G)$. For each point $q \in F_g$, the Jacobian matrix $J_q(g)$ is diagonalizable. We list its eigenvalues as

$$\left(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_n} \right) \quad (0 \leq \theta_j < 1),$$

where θ_j is a rational number whose denominator is a divisor of the order of g . We define the *age* (or *shift number*) of g at q by

$$\text{age}(g; q) = \sum_{j=1}^n \theta_j.$$

The age gives a locally constant function

$$\text{age} : F_g \rightarrow \mathbb{Q}.$$

For $v \in \mathbb{Q}$, we define $F_{g,v} = \text{age}^{-1}(v)$. The age of g is an integer if and only if $J_q(g) \in SL(T_q M)$.

From now on, let us assume $J_q(g) \in SL(T_q M)$ for all $g \in G$ and $q \in M$, namely we assume M/G is Gorenstein canonical. We also assume that the quotient map $M \rightarrow M/G$ has no ramification divisor, i.e., $\text{codim } F_g > 1$ for any $g \in G$. Then, the *stringy E-polynomial of the quotient variety* M/G is given by

$$E_{st}(M/G) = E(M/G) + \sum_{\substack{\text{id} \neq g \in \text{Conj}(G) \\ v \in \mathbb{Z}}} E(F_{g,v}/Z(g)) \cdot \mathbb{L}^v, \quad (10)$$

where $Z(g)$ is the centralizer of $g \in G$. The right hand side is called the *orbifold E-function* of M/G (see [1], Definition 6.3), which is known to be the same as the stringy E -function that, in turn, is defined in a quite different way ([1] Theorem 7.5). The summands other than $E(M/G)$ are called *twisted sectors*, while we call $E(M/G)$ the *untwisted sector*.

3.2. Now, let us move on to the case $M = \tilde{Z}^{(n)}$ and $G = \mathfrak{S}_n$. The untwisted sector $E(\tilde{Z}^{(n)}/\mathfrak{S}_n) = E(Z^{(n)})$ can be calculated by the character formula for the cohomology group $H^*(X(A_{n-1}), \mathbb{Q})$ due to Procesi, Dolgachev–Lunts, and Stembridge.

Lemma 3.3 $E(X(A_{n-1})/\mathfrak{S}_n) = (1 + \mathbb{L})^{n-1}$.

Proof This is essentially Theorem 3.1 of [22], which states that the \mathfrak{S}_n -invariant part of the cohomology ring $H^*(X(A_{n-1}))^{\mathfrak{S}_n}$ has a basis $\{y_J \mid J \subset \{\text{simple roots}\}\}$ indexed by all the subset of the set of simple roots. Moreover the degree of y_J is $2 \cdot |J|$. Therefore, $H^{2k}(X(A_{n-1}))^{\mathfrak{S}_n}$ is of dimension $\binom{n-1}{k}$. Since $X(A_{n-1})$ is a smooth toric variety, the whole H^{2k} has Hodge type (k, k) ([5] Theorem 12.5.3), and we obtain

$$E(X(A_{n-1})/\mathfrak{S}_n) = \sum_{k=0}^{n-1} \binom{n-1}{k} (uv)^k = (1 + \mathbb{L})^{n-1}.$$

□

Proposition 3.4 $E(Z^{(n)}) = \mathbb{L}^{n+2}(1 + \mathbb{L})^{n-1}$.

Proof We recall that $\tilde{Z}^{(n)} = X(\Delta^{(n)}) \times \mathbb{C}^n$ and $X(\Delta^{(n)})$ is the total space of a rank 2 vector bundle over $X(A_{n-1})$. Poincaré duality implies

$$H_c^{2k+4}(X(\Delta^{(n)})) \cong H^{2k}(X(A_{n-1})) \otimes H_c^4(\mathbb{C}^2)$$

and therefore

$$H_c^{2k+2n+4}(\tilde{Z}^{(n)}) \cong H^{2k}(X(A_{n-1})) \otimes H_c^4(\mathbb{C}^2) \otimes H_c^{2n}(\mathbb{C}^n).$$

The one-dimensional space $H_c^{2m}(\mathbb{C}^m)$ is spanned by the fundamental class so that a finite group action always leaves it invariant. Thus we get

$$E(\tilde{Z}^{(n)})/\mathfrak{S}_n = E(X(A_{n-1})) \cdot \mathbb{L}^{n+2} = \mathbb{L}^{n+2}(1 + \mathbb{L})^{n-1}$$

by the previous lemma. □

3.5. For a subset $M \subset \{1, \dots, r\}$, we define $\mathfrak{S}M$ to be the subgroup consisting of elements in \mathfrak{S}_r that leave each element of $\{1, \dots, r\} \setminus M$ invariant. Let

$$\varphi : \{1, \dots, r\} \rightarrow T$$

be a map to a set T and $\{\beta_1, \dots, \beta_k\}$ the set of its values. Taking the level sets $M(\varphi)_j = \varphi^{-1}(\beta_j)$ ($1 \leq j \leq k$), we get a partition $M(\varphi) = \{M(\varphi)_j\}$,

$$\{1, \dots, r\} = \bigsqcup_{j=1}^k M(\varphi)_j,$$

and

$$m(\varphi) = (|M(\varphi)_1|, \dots, |M(\varphi)_k|)$$

is a (not necessarily non-increasing) partition of r . We will call $M(\varphi)$ (or $m(\varphi)$) the *multiplicity partition* of φ . We define

$$\mathfrak{S}_{M(\varphi)} = \mathfrak{S}M(\varphi)_1 \times \dots \times \mathfrak{S}M(\varphi)_k \subset \mathfrak{S}_r.$$

This is a Young subgroup of \mathfrak{S}_r .

3.6. Let $\lambda = (\lambda_1, \dots, \lambda_r)$ be a length r partition of n . It defines a non-increasing map $\lambda : \{1, \dots, r\} \rightarrow \mathbb{Z}$ by $j \mapsto \lambda_j$, and therefore we have the associated multiplicity partition $M(\lambda)$ and the Young subgroup $\mathfrak{S}_{M(\lambda)}$. If $m(\lambda) = (m_1, \dots, m_k)$, we have

$$M(\lambda)_j = \left\{ \sum_{i=1}^j m_{i-1} + 1, \dots, \sum_{i=1}^j m_i \right\},$$

where $m_0 = 0$ by convention. For each partition $\lambda = (\lambda_1, \dots, \lambda_r)$, we define

$$\Theta_\lambda = \left\{ \theta = (\theta_1, \dots, \theta_r) \mid \theta_j = \frac{a_j}{\lambda_j}, a_j \in \mathbb{Z}, 0 \leq a_j < \lambda_j \right\}$$

and call an element $\theta \in \Theta_\lambda$ an *angle type* associated with λ . We say that an angle type $\theta = (\theta_1, \dots, \theta_r) \in \Theta_\lambda$ is *standard* if

$$\theta_{\sum_{i=1}^j m_{i-1} + 1} \geq \dots \geq \theta_{\sum_{i=1}^j m_i}$$

for all $1 \leq j \leq k$. An angle type θ also determines a map

$$\theta : \{1, \dots, r\} \rightarrow \mathbb{Q} \cap [0, 1)$$

and we have the associated multiplicity partition $M(\theta)$ and the Young subgroup $\mathfrak{S}_{M(\theta)}$.

3.7. Each angle type $\theta \in \Theta_\lambda$ determines a point $\alpha(\theta) \in (\mathbb{C}^*)^{n-r}$ by

$$\alpha(\theta) = \underbrace{(e^{2\pi i\theta_1}, \dots, e^{2\pi i\theta_1})}_{(\lambda_1-1)\text{-times}}; \dots; \underbrace{(e^{2\pi i\theta_r}, \dots, e^{2\pi i\theta_r})}_{(\lambda_r-1)\text{-times}}.$$

Now let $s = s_\lambda \in \mathfrak{S}_n$ be the standard element in the conjugacy class determined by λ , as in (7). Let \overline{F}_s be the set of s -fixed points on $X(A_{n-1})$. As we saw in §2.7, a point in \overline{F}_s is of the form $(\alpha(\theta), \xi) \in (\mathbb{C}^*)^{n-r} \times X(A_{r-1}) \subset X(A_{n-1})$ for some angle type $\theta \in \Theta_\lambda$. Therefore, if we define

$$\overline{F}_s^\theta = \overline{F}_s \cap (\{\alpha(\theta)\} \times X(A_{r-1})),$$

then we have $\overline{F}_s = \coprod_{\theta \in \Theta_\lambda} \overline{F}_s^\theta$. We naturally identify \overline{F}_s^θ with the corresponding closed subset of $X(A_{r-1})$.

The centralizer $Z(s)$ for $s \in \mathfrak{S}_n$ is generated by the cyclic permutations c_1, \dots, c_r in the notation of (7) and the permutations of the cycles of the same length among $\{c_1, \dots, c_r\}$ (see [20], Proposition 1.1.1). The subgroup $H(s)$ generated by c_1, \dots, c_r in \mathfrak{S}_n is a normal subgroup of $Z(s)$ and we have $Z(s)/H(s) \cong \mathfrak{S}_{M(\lambda)}$.

Lemma 3.8 *Let $V \subset \overline{F}_s$ be a union of connected components that is invariant under the action of the centralizer subgroup $Z(s)$. Then, $Z(s)$ acts on the cohomology group $H^*(V)$ via a natural action of $\mathfrak{S}_{M(\lambda)}$, namely $H^*(V/Z(s)) \cong H^*(V/\mathfrak{S}_{M(\lambda)})$.*

Proof The permutations of the cycles of the same length in $\{c_1, \dots, c_r\}$ act on \mathbb{P}^{r-1} through corresponding permutations of homogeneous coordinates $[y_1 : \dots : y_r]$, and accordingly they naturally act on $X(A_{r-1})$. The action is identified with the natural action of $\mathfrak{S}_{M(\lambda)} \subset \mathfrak{S}_r$. Therefore, it is sufficient to show that the subgroup $H(s)$ acts on $H^*(V)$ trivially. $H(s)$ leaves $\alpha(\theta)$ invariant and it acts on the torus invariant closed subset \overline{F}_s^θ as a finite subgroup of the open dense torus associated to $N^{(s)}$ of $X(A_{r-1})$. $H(s)$ also leaves $H^*(\overline{F}_s^\theta \cap V)$ invariant, since the cohomology group $H^*(\overline{F}_s^\theta \cap V)$ is generated by the fundamental cycles of torus invariant closed subvarieties of a smooth projective toric variety $\overline{F}_s^\theta \cap V$ (see [5], Lemma 12.5.1 and Theorem 12.5.3). □

3.9 Let $\varphi : \{1, \dots, r\} \rightarrow T$ be a map. We define the adjacency function

$$\text{adj}(\varphi) : \{1, \dots, r\} \rightarrow \mathbb{Z}$$

of φ inductively by

$$\text{adj}(\varphi)(1) = 1, \quad \text{adj}(\varphi)(j) = \begin{cases} \text{adj}(\varphi)(j-1) & \text{if } \varphi(j) = \varphi(j-1) \\ \text{adj}(\varphi)(j-1) + 1 & \text{if } \varphi(j) \neq \varphi(j-1) \end{cases}.$$

Let λ be a length r partition of n , $\theta \in \Theta_\lambda$ an angle type, and $p \in \mathfrak{S}_r$. We define a function

$$\theta \star p : \{1, \dots, r\} \rightarrow \mathbb{Z}$$

by $\theta \star p = \text{adj}(\theta \circ p) \circ p^{-1}$ and let $M(\theta \star p)$ be the corresponding level set partition of $\{1, \dots, r\}$. The partition defines a Young subgroup $\mathfrak{S}_{M(\theta \star p)} \subset \mathfrak{S}_r$. As $M(\theta \star p)$ is a

refinement of the partition $M(\theta)$, $\mathfrak{S}_{M(\theta \star p)}$ is a subgroup of $\mathfrak{S}_{M(\theta)}$. Let $\bar{p} = \mathfrak{S}_{M(\theta \star p)} p$ be the right coset in \mathfrak{S}_r and $P(\theta) = \{\bar{p} \mid p \in \mathfrak{S}_r\}$. Then, it is easy to see that $P(\theta)$ is a partition of \mathfrak{S}_r . We define

$$\tau_{\bar{p}} = \bigcap_{q \in \bar{p}} q \bar{\delta}^{(r)},$$

where $\bar{\delta}^{(r)}$ is the positive Weyl chamber of the A_{r-1} -root system as in §2.6.

Proposition 3.10 *The set of connected components of \bar{F}_s^θ agrees with the set of orbit closures $\{V(\tau_{\bar{p}}) \mid \bar{p} \in P(\theta)\}$. Moreover, if $m(\theta \star p) = (r_1, \dots, r_k)$, then $V(\tau_{\bar{p}}) \cong X(A_{r_1-1}) \times \dots \times X(A_{r_k-1})$.*

Proof A codimension one face of $p\bar{\delta}^{(r)}$ is cut out by a hyperplane of invariant vectors under a transposition $(p(j) \ p(j+1))$ for some $1 \leq j < r$. The face corresponds to an affine line with coordinate $y_{p(j)}/y_{p(j+1)}$. Taking the action (9) into account, this affine line consists of s -fixed point if and only if $\theta_{p(j)} = \theta_{p(j+1)}$. On the other hand, it is easy to verify that

$$\mathfrak{S}_{M(\theta \star p)} = \langle (p(j) \ p(j+1)) \mid \theta_{p(j)} = \theta_{p(j+1)} \rangle \subset \mathfrak{S}_r$$

and if $q \in \bar{p}$, then $\mathfrak{S}_{M(\theta \star q)} = \mathfrak{S}_{M(\theta \star p)}$ as subgroups of \mathfrak{S}_r . By construction, the orbit $O(\tau_{\bar{p}})$ is a torus with the coordinates

$$\left\{ \frac{y_{p(j)}}{y_{p(j+1)}} \mid \theta_{p(j)} = \theta_{p(j+1)} \right\}.$$

In particular, every point in $O(\tau_{\bar{p}})$, and therefore of $V(\tau_{\bar{p}})$, is fixed by s , namely $V(\tau_{\bar{p}})$ is a connected component of \bar{F}_s^θ .

On the other hand, since s acts on $X(A_{r-1})$ via a cyclic subgroup of the torus, every connected component V of \bar{F}_s^θ is a torus invariant closed subset of $X(A_{r-1})$. In particular, V contains a torus invariant point ξ_p corresponding to a maximal cone $p\bar{\delta}^{(r)}$ for some $p \in \mathfrak{S}_r$. The above argument shows that $V(\tau_{\bar{p}})$ is the connected component of \bar{F}_s^θ containing ξ_p . Hence we know that $V = V(\tau_{\bar{p}})$.

The image fan on $\bar{N}^{(s)}/(\langle (\tau_{\bar{p}}) \rangle_{\mathbb{R}} \cap \bar{N}^{(s)})$ corresponding to $V(\tau_{\bar{p}})$ is the Coxeter complex of the root system corresponding to the Young subgroup $\mathfrak{S}_{M(\theta \star p)} \cong \mathfrak{S}_{r_1} \times \dots \times \mathfrak{S}_{r_k}$, where $m(\theta \star p) = (r_1, \dots, r_k)$. It immediately follows that $V(\tau_{\bar{p}}) \cong X(A_{r_1-1}) \times \dots \times X(A_{r_k-1})$. □

3.11. We denote the connected component $V(\tau_{\bar{p}}) \subset \bar{F}_s^\theta$ by $\bar{F}_s^{\theta, \bar{p}}$. Let F_s be the closed subset of s -fixed points on $X(\Delta^{(n)})$ and $F_s^{\theta, \bar{p}} \subset X(\Delta^{(n)})$ the union of connected components of F_s that map to $\bar{F}_s^{\theta, \bar{p}}$. If we define $\phi(\theta, \bar{p})$ to be the number of 0 in the set $\{\theta_{p(1)}, \theta_{p(r)}\}$, the calculation in §2.8 implies that $F_s^{\theta, \bar{p}} \rightarrow \bar{F}_s^{\theta, \bar{p}}$ is a vector bundle of rank $\phi(\theta, \bar{p})$. In particular $F_s^{\theta, \bar{p}}$ is connected. We know that the s -fixed point locus of $\tilde{Z}^{(n)}$ is a disjoint union of $F_s^{\theta, \bar{p}} \times (\mathbb{C}^n)^{(s)} \subset X(\Delta^{(n)}) \times \mathbb{C}^n = \tilde{Z}^{(n)}$.

Lemma 3.12 *The value of the age function at a point in $F_s^{\theta, \bar{p}} \times (\mathbb{C}^n)^{(s)}$ is given by*

$$a(s; \theta, \bar{p}) = n - r + \sum_{j=1}^{r-1} \{\theta_{p(j+1)} - \theta_{p(j)}\} + \{1 - \theta_{p(1)}\} + \theta_{p(r)}, \tag{11}$$

where $\{t\} = t - \lfloor t \rfloor$ is the fractional part of t .

Proof This is just a consequence of Lemma 2.9, noting that the contribution from the parts (i) and (iv) sum up to

$$\sum_{j=1}^r \sum_{i=1}^{\lambda_j-1} 2 \cdot \frac{i}{\lambda_j} = \sum_{j=1}^r (\lambda_j - 1) = n - r.$$

□

3.13 Let Θ_λ^{st} be the set of standard angle types. $\mathfrak{S}_{M(\lambda)} \subset \mathfrak{S}_r$ acts on the set of angle types Θ_λ by permutation of the factors and each orbit contains a unique standard element θ . Therefore, there is an identification $\Theta_\lambda / \mathfrak{S}_{M(\lambda)} \cong \Theta_\lambda^{st}$. The stabilizer subgroup of θ is $\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)}$. Therefore, we get

$$\overline{F}_s / \mathfrak{S}_{M(\lambda)} = \left(\prod_{\theta \in \Theta_\lambda} \overline{F}_s^\theta \right) / \mathfrak{S}_{M(\lambda)} \cong \prod_{\theta \in \Theta_\lambda^{st}} \overline{F}_s^\theta / (\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)}).$$

Let $\theta \in \Theta_\lambda^{st}$. Then, Proposition 3.10 says that we have a decomposition into connected components

$$\overline{F}_s^\theta = \coprod_{\bar{p} \in P(\theta)} \overline{F}_s^{\theta, \bar{p}},$$

where $P(\theta) = \{\bar{p} = \mathfrak{S}_{M(\theta \star p)} p \mid p \in \mathfrak{S}_r\}$, and the isomorphism class of $\overline{F}_s^{\theta, \bar{p}} = V(\tau_{\bar{p}})$ is determined only by the multiplicity partition $m(\theta \star p)$. $\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)}$ naturally acts on $P(\theta)$. Let $\overline{P}(\theta)$ be a complete system of representative for the quotient set $P(\theta) / (\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)})$. Then, we get

$$\overline{F}_s^\theta / (\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)}) \cong \coprod_{\bar{p} \in \overline{P}(\theta)} \overline{F}_s^{\theta, \bar{p}} / (\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta \star p)}).$$

Combining what we have obtained above, finally we get the following

Theorem 3.14 *We keep the notation above. The stringy E-polynomial of $Z^{(n)}$ is given by the following formula:*

$$\begin{aligned} E_{st}(Z^{(n)}) &= \mathbb{L}^{n+2} (1 + \mathbb{L})^{n-1} \\ &+ \sum_{\substack{\lambda \vdash n \\ \lambda \neq (1^n)}} \sum_{\theta \in \Theta_\lambda^{st}} \sum_{\bar{p} \in \overline{P}(\theta)} E(\overline{F}_{s_{\lambda}}^{\theta, \bar{p}} / (\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta \star p)})) \cdot \mathbb{L}^{\phi(\theta, \bar{p}) + r(\lambda) + a(s; \theta, \bar{p})}, \end{aligned} \tag{12}$$

where s_λ is the standard permutation associated with λ as in (7) and $r(\lambda)$ stands for the length of the partition λ .

3.15. We remark that one can actually calculate the term $E(\overline{F}_{s_\lambda}^{\theta, \bar{p}} / (\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta \star p)}))$ in the formula (12) as follows.

We saw in Proposition 3.10 that

$$\overline{F}_s^{\theta, \bar{p}} \cong X(A_{r_1-1}) \times \cdots \times X(A_{r_k-1})$$

if $m(\theta \star p) = (r_1, \dots, r_k)$. Since $\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta \star p)}$ is a Young subgroup of

$$\mathfrak{S}_{M(\theta \star p)} \cong \mathfrak{S}_{r_1} \times \cdots \times \mathfrak{S}_{r_k},$$

there is a partition $\mu_j \vdash r_j$ for each $1 \leq j \leq k$ such that

$$\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta \star p)} \cong \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_k},$$

where $\mathfrak{S}_{\mu_j} \subset \mathfrak{S}_{r_j}$ is the Young subgroup associated with the partition $\mu_j \vdash r_j$. Therefore, we have

$$E(\overline{F}_{S_\lambda}^{\theta, \overline{p}} / (\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta \star p)})) = E(X(A_{r_1-1}) / \mathfrak{S}_{\mu_1}) \times \cdots \times E(X(A_{r_k-1}) / \mathfrak{S}_{\mu_k}).$$

On the other hand, $E(X(A_{r-1}) / \mathfrak{S}_\mu)$ for $\mu \vdash r$ can be calculated by the character formula of Procesi, Dolgachev–Lunts, and Stembridge: if we define $\chi_l(A_{n-1})$ to be the character of the \mathfrak{S}_n -representation $H^{2l}(X(A_{n-1}), \mathbb{Q})$ and $\chi[A_{n-1}, q] = \sum_{l=0}^{n-1} \chi_l(A_{n-1})q^l$ the generating function, then we know that

$$\begin{aligned} 1 + \sum_{n \geq 1} \chi[A_{n-1}, q]t^n &= \frac{1 + \sum_{m \geq 1} h_m t^m}{1 - \sum_{m \geq 2} (q + \cdots + q^{m-1})h_m t^m} \\ &= 1 + h_1 t + h_2(1 + q)t^2 + (h_3 + (h_1 h_2 + h_3)q + h_3 q^2)t^3 \\ &\quad + (h_4 + (h_2^2 + h_1 h_3 + h_4)(q + q^2) + h_4 q^3)t^4 + \cdots, \end{aligned}$$

where h_m is the character of the trivial representation of \mathfrak{S}_m , or rather, the complete symmetric function of degree m (see [22], Theorem 6.2, see also [7, 17]). Therefore, if $\mu = (m_1, \dots, m_k)$, the character inner product gives the formula

$$E(X(A_{r-1} / \mathfrak{S}_\mu)) = (h_\mu, \chi[A_{r-1}, \mathbb{L}]),$$

where $h_\mu = h_{m_1} \cdots h_{m_k}$.

3.16. Let X be a variety with only log terminal singularities. The change of variable formula for motivic integration ([1], Theorem 3.5, or [24], Theorem 66) implies that if $f : Y \rightarrow X$ is a proper birational morphism that is crepant, *i.e.*, $K_Y = f^* K_X$, the stringy E -function is invariant: $E_{st}(Y) = E_{st}(X)$ ([1], Theorem 3.8). In particular, if $f : Y \rightarrow X$ is a crepant resolution, we have $E_{st}(X) = E_{st}(Y) = E(Y)$.

In our situation, as the birational morphism $\mu^{(n)} : Z^{(n)} \rightarrow X^{(n)}$ is small, it is in particular crepant. Thus we have $E_{st}(X^{(n)}) = E_{st}(Z^{(n)})$. If the symmetric product $X^{(n)}$ admits a crepant resolution $Y \rightarrow X^{(n)}$, we have $E(Y) = E_{st}(Z^{(n)})$. Moreover if the induced family $Y \rightarrow B$ is semistable, and if we denote the singular fiber by Y_0 , Poincaré duality and homotopy invariance implies

$$H_c^p(Y)^* \cong H^{2n-p+2}(Y) \cong H^{2n-p+2}(Y_0).$$

Therefore, the polynomial $E(Y) = E_{st}(Z^{(n)})$ encodes the cohomological information of the singular fiber of a semistable model. In general, $X^{(n)}$ may not admit a crepant resolution; nevertheless the Gorenstein canonical orbifold model $Z^{(n)}$ is already a good substitute for a minimal semistable model of $X^{(n)}$ on the level of cohomology.

Proposition 3.17 *The stringy E-polynomial $E_{st}(Z^{(n)})$ for $n = 2, 3, 4, 5$ is as follows:*

$$\begin{aligned} E_{st}(Z^{(2)}) &= \mathbb{L}^5 + 2\mathbb{L}^4 + \mathbb{L}^3, \\ E_{st}(Z^{(3)}) &= \mathbb{L}^7 + 3\mathbb{L}^6 + 5\mathbb{L}^5 + 2\mathbb{L}^4, \\ E_{st}(Z^{(4)}) &= \mathbb{L}^9 + 4\mathbb{L}^8 + 11\mathbb{L}^7 + 14\mathbb{L}^6 + 4\mathbb{L}^5, \\ E_{st}(Z^{(5)}) &= \mathbb{L}^{11} + 5\mathbb{L}^{10} + 17\mathbb{L}^9 + 35\mathbb{L}^8 + 30\mathbb{L}^7 + 6\mathbb{L}^6. \end{aligned}$$

Proof We demonstrate the case $n = 4$. The other cases are similar (but much more complicated in the case $n = 5$). Nontrivial partition λ of 4 is one of $\lambda = (2, 1^2), (2^2), (3, 1), (4)$. We calculate the contributions to twisted sectors case by case. We represent $p \in \mathfrak{S}_r$ by a sequence $p = [p(1), p(2), \dots, p(r)]$ to make it short.

Case $\lambda = (2, 1^2)$. The length of the partition is $r = 3$. The set of standard angle type in this case is $\Theta_{(2,1^2)}^{st} = \{(0^3), (1/2, 0^2)\}$. We also note that $\mathfrak{S}_{M(\lambda)} = \mathfrak{S}\{2, 3\}$.

Case $\theta = (0^3)$: In this case, we have $\mathfrak{S}_{M(\theta)} = \mathfrak{S}_{M(\theta \star p)} = \mathfrak{S}_3$ for all $p \in \mathfrak{S}_3$. It immediately follows that $P(\theta) = \{\bar{1}\}$, $\overline{F}_{(1\ 2)}^{(0^3)} = X(A_2)$, and $\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta \star 1)} = \mathfrak{S}\{2, 3\}$. We need to know $E(X(A_2)/\mathfrak{S}\{2, 3\})$, which is calculated by the argument in §3.15 as follows:

$$\begin{aligned} E(X(A_2)/\mathfrak{S}\{2, 3\}) &= (h_1 h_2, \chi[A_2, \mathbb{L}]) \\ &= (h_1 h_2, h_3 + (h_1 h_2 + h_3)\mathbb{L} + h_3 \mathbb{L}^2) \\ &= 1 + 3\mathbb{L} + \mathbb{L}^2. \end{aligned}$$

Noting $\phi = 2, a = 1$, the associated twisted sector is $\mathbb{L}^8 + 3\mathbb{L}^7 + \mathbb{L}^6$.

Case $\theta = (1/2, 0^2)$: $\mathfrak{S}_{M(\theta)} = \mathfrak{S}\{2, 3\}$. $P(\theta)$ consists of

$$\overline{[1, 2, 3]} = \overline{[1, 3, 2]}, \overline{[2, 1, 3]}, \overline{[3, 1, 2]}, \text{ and } \overline{[2, 3, 1]} = \overline{[3, 2, 1]}.$$

However, $\overline{[2, 1, 3]}$ and $\overline{[3, 1, 2]}$ is in the same orbit under the action of $\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)} = \mathfrak{S}\{2, 3\}$, therefore we have $\overline{P(\theta)} = \{\bar{1}, \overline{(1\ 2)}, \overline{(1\ 2\ 3)}\}$. It is straightforward to get the following table:

p	$\mathfrak{S}_{M(\theta \star p)}$	$\theta \circ p$	ϕ	a	Twisted sector
1	$\mathfrak{S}\{2, 3\}$	$(1/2, 0, 0)$	1	2	$(1 + \mathbb{L})\mathbb{L}^6$
(1 2)	{1}	$(0, 1/2, 0)$	2	2	\mathbb{L}^7
(1 2 3)	$\mathfrak{S}\{2, 3\}$	$(0, 0, 1/2)$	1	2	$(1 + \mathbb{L})\mathbb{L}^6$

In total, the contribution is $3\mathbb{L}^7 + 2\mathbb{L}^6$.

Case $\lambda = (2^2)$. We have $r = 2, \mathfrak{S}_{M(\lambda)} = \mathfrak{S}_2$, and

$$\Theta_{(2^2)}^{st} = \{(0, 0), (1/2, 0), (1/2, 1/2)\}.$$

Case $\theta = (0^2)$: $\mathfrak{S}_{M(\theta)} = \mathfrak{S}_{M(\theta \circ p)} = \mathfrak{S}_2$ for all $p \in \mathfrak{S}_2$, and $P(\theta) = \{\bar{1}\}$ as before. $\overline{F}_{(1\ 2)(3\ 4)}^{(0^2)} = X(A_1)$ and $\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)} = \mathfrak{S}_2, \phi = 2$, and $a = 2$, so that the associated twisted sector is $E(X(A_1)/\mathfrak{S}_2)\mathbb{L}^6 = \mathbb{L}^7 + \mathbb{L}^6$.

Case $\theta = (1/2, 0)$: Since $\mathfrak{S}_{M(\theta)} = \{1\}$, $P(\theta) = \{\bar{1}, \overline{(1\ 2)}\}$ and $\overline{F}_{(1\ 2)(3\ 4)}^{(0^2)}$ consists of two points. As we have $\phi = 1$, $a = 3$ in both cases, the associated twisted sector is $2\mathbb{L}^6$.

Case $\theta = (1/2, 1/2)$: $\mathfrak{S}_{M(\theta)} = \mathfrak{S}_{M(\theta \star p)} = \mathfrak{S}_2$, and $P(\theta) = \{\bar{1}\}$. As before, we know $\overline{F}_{(1\ 2)(3\ 4)}^{(1/2, 1/2)} = X(A_1)$ and $\mathfrak{S}_{M(\lambda)} \cap \mathfrak{S}_{M(\theta)} = \mathfrak{S}_2$. Since $\phi = 0$, $a = 3$ the associated twisted sector is $E(X(A_1)/\mathfrak{S}_2)\mathbb{L}^5 = \mathbb{L}^6 + \mathbb{L}^5$.

Case $\lambda = (3, 1)$. In this case, $r = 2$, $\mathfrak{S}_{M(\lambda)} = \{1\}$, $\Theta_{(3,1)}^{st} = \{(0^2), (1/3, 0), (2/3, 0)\}$.

Case $\theta = (0^2)$: As before, $\mathfrak{S}_{M(\theta)} = \mathfrak{S}_{M(\theta \star p)} = \mathfrak{S}_2$ and $P(\theta) = \{1\}$. Therefore, $\overline{F}_{(1\ 2\ 3)}^{(0^2)} = X(A_1)$. As $\phi = 2$, $a = 2$, the twisted sector is $\mathbb{L}^7 + \mathbb{L}^6$.

Case $\theta = (1/3, 0)$: $\mathfrak{S}_{M(\theta)} = 1$, $P(\theta) = \{\bar{1}, \overline{(1\ 2)}\}$ and $\overline{F}_{(1\ 2\ 3)}^{(1/3, 0)}$ consists of two points. As $\phi = 1$, $a = 3$ in this case, the twisted sector is $2\mathbb{L}^6$.

Case $\theta = (2/3, 0)$: This case is completely the same as in the case $\theta = (1/3, 0)$. The twisted sector is $2\mathbb{L}^6$.

Case $\lambda = (4)$. In this case $r = 1$ and we always have $\mathfrak{S}_{M(\lambda)} = \mathfrak{S}_{M(\theta)} = \{1\}$. Therefore $P(\theta) = \{\bar{1}\}$ and $\overline{F}_{(1\ 2\ 3\ 4)}^\theta$ is just one point set. As we have the twisted sector in total is

θ	ϕ	a	Twisted sector
(0)	2	3	\mathbb{L}^6
(1/4), (2/4), (3/4)	0	4	\mathbb{L}^5

$\mathbb{L}^6 + 3\mathbb{L}^5$.

Summing up everything, finally we get

$$\begin{aligned}
 E_{st}(X^{(4)}) &= \mathbb{L}^6(1 + \mathbb{L})^3 \\
 &\quad + (\mathbb{L}^8 + 3\mathbb{L}^7 + \mathbb{L}^6) + (3\mathbb{L}^7 + 2\mathbb{L}^6) + (\mathbb{L}^7 + \mathbb{L}^6) + 2\mathbb{L}^6 \\
 &\quad + (\mathbb{L}^6 + \mathbb{L}^5) + (\mathbb{L}^7 + \mathbb{L}^6) + 2\mathbb{L}^6 + 2\mathbb{L}^6 + (\mathbb{L}^6 + 3\mathbb{L}^5) \\
 &= \mathbb{L}^9 + 4\mathbb{L}^8 + 11\mathbb{L}^7 + 14\mathbb{L}^6 + 4\mathbb{L}^5.
 \end{aligned}$$

□

Remark 3.17.1 In comparison with the case of $\text{Hilb}^n(S)$ for smooth algebraic surface S (see e.g. [16]), it is interesting to look for a formula of the generating function

$$\sum_{n \geq 0} E_{st}(Z^{(n)}) \cdot t^n.$$

Unfortunately, due to combinatorial complication in Theorem 3.14, the author does not yet have a good answer to this question at the time of writing.

4 Minimal model

In this section, we will discuss more birational modifications of $Z^{(n)}$, in particular minimal models of $Z^{(n)}$.

Theorem 4.1 *There exists a projective birational morphism $v^{(n)} : Y^{(n)} \rightarrow Z^{(n)}$ satisfying the following conditions:*

- (i) $Y^{(n)}$ has only Gorenstein terminal quotient singularities.
- (ii) $v^{(n)}$ is a crepant divisorial contraction, namely $K_{Y^{(n)}} = v^{(n)*} K_{Z^{(n)}}$ and the exceptional set of $v^{(n)}$ is a divisor.
- (iii) Let $\psi^{(n)} : Y^{(n)} \rightarrow B$ be the composition $\rho^{(n)} \circ v^{(n)}$. Then, its general fiber is $\text{Hilb}^n(\mathbb{C}^* \times \mathbb{C})$ and the singular fiber is a divisor with V -normal crossings.

Although the existence of a minimal model of $Z^{(n)}$ is a consequence of the general theory of minimal model program (MMP) of higher dimensional algebraic varieties [3], here we stick to an explicit construction of a minimal model $Y^{(n)}$ so that we have a good control on the singularities of the total space and the singular fiber of the resulted minimal model. The claims (i) and (iii) will not follow from a straightforward application of MMP.

4.2. From the description of §1.6, the natural morphism $\tilde{\rho}^{(n)'} : \tilde{Z}^{(n)'} = X(\Delta^{(n)}) \rightarrow B = \mathbb{C}$ is a toric morphism associated with the lattice homomorphism

$$g = (1 \ 1 \ 0 \ \dots \ 0) : N = \mathbb{Z}^{n+1} \rightarrow \mathbb{Z}.$$

If we take a basis of N consisting of the column vectors of Q in the proof of Proposition 2.5, g is represented by a matrix $(1 \ 0 \ \dots \ 0 \ 1)$. It immediately follows that the primitive generator for each ray in the fan $\Delta^{(n)}$, namely the column vectors of (6), has multiplicity one with respect to g . This means that the fiber of $\tilde{\rho}^{(n)'}$ over the origin is the union of all the torus invariant divisors of $X(\Delta^{(n)})$. It is easy to see from the description (3) the \mathfrak{S}_n -action on N that its restriction to $\text{Ker}(g)$ is the permutation representation. Therefore, the restriction of $\tilde{\rho}^{(n)'}$ to the torus $N \otimes \mathbb{C}^* \rightarrow \mathbb{C}^*$ is a trivial family of the permutation action on $(\mathbb{C}^*)^n$, and the \mathfrak{S}_n -quotient of $N \otimes \mathbb{C}^* \times \mathbb{C}^n \rightarrow \mathbb{C}^*$ is a trivial family of $\text{Sym}^n(\mathbb{C}^* \times \mathbb{C})$. It follows that $Z_n^{(n)\circ} = \rho^{(n)-1}(\mathbb{C}^*)$ has a crepant divisorial resolution

$$\psi^{(n)\circ} : Y^{(n)\circ} \rightarrow Z^{(n)\circ}$$

that is a family of Hilbert–Chow morphism $\text{Hilb}^n(\mathbb{C}^* \times \mathbb{C}) \rightarrow \text{Sym}^n(\mathbb{C}^* \times \mathbb{C})$. We prove that $\psi^{(n)\circ}$ extends to a crepant birational morphism $\psi^{(n)} : Y^{(n)} \rightarrow Z^{(n)}$.

Lemma 4.3 *Let $s \in \mathfrak{S}_n$. The connected component $F_s^{\theta, \bar{p}}$ (see §3.11) of the s -fixed point locus in $X(\Delta^{(n)})$ has intersection with the open dense torus $N \otimes \mathbb{C}^*$ if and only if θ is a zero-sequence (0^r) where r is the length of the partition λ associated with s .*

Proof Let us assume that $F_s^{\theta, \bar{p}}$ has a point in common with $N \otimes \mathbb{C}^*$. Then, we necessarily have $\phi(\theta, \bar{p}) = 2$, that is, $\theta_p(1) = \theta_p(r) = 0$. A point $\left(\alpha, \left(\frac{y_{p(1)}}{y_{p(2)}}, \dots, \frac{y_{p(r-1)}}{y_{p(r)}}\right)\right) \in X(A_{n-1})$ is in the open dense torus if and only if $\frac{y_{p(j)}}{y_{p(j+1)}} \neq 0$ for all j . Taking the action map (9) into account, it follows that $\theta_j - \theta_{j+1} \in \mathbb{Z}$. Therefore, we must have $\theta = (0^r)$. The converse also follows from the action map (9) and the definition of $F_s^{\theta, \bar{p}}$. □

4.4. Let F^0 be the union of the fixed point loci of trivial angle type $F_s^{(0^r)}$ for all $s \in \mathfrak{S}_n$ (here r is the length of the associated partition λ to $s \in \mathfrak{S}_n$). Since $F_s^{(0^r)}$ is the total space of

a rank 2 vector bundle over $\overline{F}_s^{(0^{r-1})} \cong X(A_{r-1})$, we know $\dim F_s^{(0^r)} = r + 1$. In particular F^0 is a divisor in $X(\Delta^{(n)})$.

Lemma 4.5 *Let $q = (q_1, q_2) \in \tilde{Z}^{(n)}$. We define*

$$\text{Stab}^0(q) = \{s \in \text{Stab}(q) \mid q_1 \in F_s^{(0^r)}\}.$$

Then,

- (i) $\text{Stab}^0(q)$ is a Young subgroup of \mathfrak{S}_n .
- (ii) $\text{Stab}^0(q)$ is a normal subgroup of $\text{Stab}(q)$.

Proof (i) Let \bar{q}_1 be an image of q_1 under the composition

$$X(\Delta^{(n)}) \rightarrow X(A_{n-1}) \rightarrow \mathbb{P}^{n-1}.$$

From the description in §2.7, one sees that q_1 is an s -fixed point of trivial angle type if and only if its coordinate satisfies the relation $\frac{x_s(i)}{x_i} = 1$ for all i such that $s(i) \neq i$. If q_1 is also t -fixed point of trivial angle type, we have $\frac{x_{t \circ s(i)}}{x_i} = \frac{x_{t(s(i))} x_s(i)}{x_{s(i)} x_i} = 1$, and hence $\text{Stab}^0(q)$ is a subgroup of $\text{Stab}(q)$. Also by the characterization above, one sees that if $s \in \text{Stab}^0(q)$ has a cycle decomposition

$$s = (i_1 \cdots i_{l_1})(i_{l_1+1} \cdots i_{l_2}) \cdots (i_{l_{r-1}+1} \cdots i_r),$$

all the elements in the subgroup

$$\mathfrak{S}\{i_1 \cdots i_{l_1}\} \times \mathfrak{S}\{i_{l_1+1} \cdots i_{l_2}\} \times \cdots \times \mathfrak{S}\{i_{l_{r-1}+1} \cdots i_r\}$$

is also contained in $\text{Stab}^0(q)$. It implies that $\text{Stab}^0(q) \subset \mathfrak{S}_n$ is a Young subgroup.

(ii) Assume that $s \in \text{Stab}(q) \subset \mathfrak{S}_n$ has the partition type λ of length r . Then, by Lemma 2.9, $s \in \text{Stab}^0(q)$ if and only if the multiplicity of 1 in the eigenvalues of the action of s on $T_q \tilde{Z}^{(n)}$ is $2r + 1$. As the eigenvalues are constant in a conjugacy class, we know that $\text{Stab}^0(q)$ is a normal subgroup of $\text{Stab}(q)$. □

4.6. Let $q = (q_1, q_1) \in \tilde{Z}^{(n)} = X(\Delta^{(n)}) \times \mathbb{C}^n$ and assume that $q_1 \in F^0$. Then, by Lemma 2.9, the action of an element $s \in \text{Stab}^0(q)$ on the tangent space $T_q \tilde{Z}^{(n)}$ has eigenvalues

$$(\zeta_1, \dots, \zeta_1^{\lambda_1-1}; \dots; \zeta_r, \dots, \zeta_r^{\lambda_r-1}; \underbrace{1, \dots, 1}_{r+1}; 1, \zeta_1, \dots, \zeta_1^{\lambda_1-1}; \dots; 1, \zeta_r, \dots, \zeta_r^{\lambda_r-1})$$

if s has partition type $\lambda = (\lambda_1, \dots, \lambda_r)$. In particular, the character of the representation $\text{Stab}^0(q) \rightarrow GL(T_q \tilde{Z}^{(n)})$ agrees with the character of the permutation representation of the Young subgroup $\text{Stab}^0(q) \subset \mathfrak{S}_n$ on $\mathbb{C}^n \oplus \mathbb{C}^n \oplus \mathbb{C}$, the direct sum of two copies of a permutation representation and a trivial representation. Let U_q be a sufficiently small $\text{Stab}(q)$ -invariant open neighborhood of $q \in \tilde{Z}^{(n)}$. If $\text{Stab}^0(q)$ is of the form \mathfrak{S}_μ for a partition $\mu = (\mu_1, \dots, \mu_k)$, $V_q = U_q / \text{Stab}^0(q)$ is locally isomorphic to a product of a neighborhood of a general cycle in $\text{Sym}^n(\mathbb{C}^2)$ of the form

$$\sum_{i=1}^k \mu_i a_i \quad (a_i \in \mathbb{C}^2),$$

and a complex line \mathbb{C} . Therefore, it admits a crepant resolution $\tilde{V}_q \subset \text{Hilb}^n(\mathbb{C}^2) \times \mathbb{C}$. By a theorem of Haiman ([10], Theorem 5.1), \tilde{V}_q can be regarded as an open subset of $\mathfrak{S}_n\text{-Hilb}(\mathbb{C}^{2n+1})$, so the quotient group $G_q = \text{Stab}(q)/\text{Stab}^0(q)$ naturally acts on \tilde{V}_q and the quotient \tilde{V}_q/G_q gives a partial resolution

$$\psi_{2,q}^{(n)} : \tilde{V}_q/G_q \rightarrow \bar{U}_q$$

of the image \bar{U}_q of U_q in $Z^{(n)} = \tilde{Z}^{(n)}$. Since $\bar{U}_q \subset Z^{(n)}$ has only canonical singularities, $\psi_{2,q}^{(n)}$ is again a crepant birational morphism. As the partial resolution $\psi_{2,q}^{(n)}$ naturally glue with the Hilbert-Chow morphism $\psi^{(n)\circ} : Y^{(n)\circ} \rightarrow Z^{(n)\circ}$ at the image of q in $Z^{(n)}$ for every $q = (q_1, q_2)$ with $q_1 \in F^0$, we get an extended crepant partial resolution $\psi^{(n)} : Y^{(n)} \rightarrow Z^{(n)}$.

4.7. To finish the proof of Theorem 4.1, we check that $\psi^{(n)}$ constructed above satisfies the conditions (i) and (iii).

Let $D_1, \dots, D_k \subset X(\Delta^{(n)})$ be torus invariant prime divisors. At a point $q \in F^0$, they are defined by $y_j = 0$ (not all but for some j 's) in the notation of §2.7. Lemma 2.9 implies that the intersection $D_1 \cap \dots \cap D_k$ is (if not empty) transversal to the action of \mathfrak{S}_n . Therefore, the strict transform D'_i of the image of D_i in \tilde{V}_q is a smooth divisor and intersecting transversally along the exceptional divisor. As the quotient group $G_q = \text{Stab}(q)/\text{Stab}^0(q)$ acts on the coordinate (y_1, \dots, y_r) via a torus $(\mathbb{C}^*)^r$, the normal crossing divisor $\sum D'_i$ is preserved by the action of G_q . Therefore, the singular fiber of $Y^{(n)} \rightarrow B$ is a divisor with V -normal crossings. This proves the condition (iii).

The condition (i) is a consequences of the characterization of terminal quotient singularity (see, [13] Theorem 2.3). Let $\tilde{\Gamma}$ be a unique smooth irreducible divisor on $\tilde{Z}^{(n)}$ dominating $F_{(1\ 2)}^{(0^{n-1})} \subset X(\Delta^{(n)})$ and Γ the image of $\tilde{\Gamma}$ in $Z^{(n)}$. Lemma 3.12 implies that the age function always satisfies $a(s; \theta, \bar{p}) \geq n - r$ if s has the partition type of length r (regardless of a choice of primitive root of unity). In particular, if $a(s; \theta, \bar{p}) = 1$, we necessarily have $r = n - 1$, therefore the associated partition should be $\lambda = (2, 1^{n-1})$. Moreover, in that case, one need to have $\theta_1 = \dots = \theta_r$ and $\theta_r = 0$, namely $\theta = (0^{n-1})$. This implies that an exceptional divisor of discrepancy 0 over $Z^{(n)}$ necessarily dominates Γ (see [19] Remark (3.2)). On the other hand, as $\psi^{(n)}$ is a crepant resolution of the singularity of $Z^{(n)}$ at the generic point of Γ , $Y^{(n)}$ has no crepant exceptional divisor over it. This implies (i) and completes the proof of Theorem 4.1.

4.8. The recent construction of degeneration of Hilbert schemes by Gulbrandsen et al. [9] seems to be strongly related to the problem. In this paragraph, we use the notation of [9] freely. We can show that, for the expanded degeneration $X[n] \rightarrow \mathbb{A}^{n+1}$, there is a natural isomorphism between the GIT quotient of the stable locus of the relative symmetric product $\text{Sym}^n(X[n]/\mathbb{A}^{n+1})^s$ by $G[n] = (\mathbb{C}^*)^n$ and our $Z^{(n)}$:

$$\varepsilon^{(n)} : \text{Sym}^n(X[n]/\mathbb{A}^{n+1})^s // G[n] \xrightarrow{\sim} Z^{(n)}.$$

Therefore, the relative Hilbert–Chow morphism

$$\text{Hilb}^n(X[n]/\mathbb{A}^{n+1})^s \rightarrow \text{Sym}^n(X[n]/\mathbb{A}^{n+1})^s$$

gives a birational morphism

$$\psi^{(n),GHH} : I^n(X/\mathbb{A}^1) = \text{Hilb}^n(X[n]/\mathbb{A}^{n+1})^s // G[n] \rightarrow Z^{(n)}$$

over the base $B = \mathbb{A}^1$. As the authors claim in [9] that $I^n(X/\mathbb{A}^1)$ has only (abelian) quotient singularities and has trivial canonical bundle, it will immediately follow that $\psi^{(n),GHH}$ is

an extension of $\psi^{(n)\circ}$ satisfying the conditions (i)–(iii) of Theorem 4.1. We will discuss the construction of $\varepsilon^{(n)}$ and the comparison of $\psi^{(n),GHH}$ and our $\psi^{(n)}$ in a forthcoming article [15].

4.9. The theorem asserts that $Y^{(n)}$ is a *relatively minimal model* of the symmetric product $X^{(n)}$ over B , as $K_{Y^{(n)}}$ is numerically trivial over $X^{(n)}$ and $K_{X^{(n)}} \equiv 0$. The general theory of MMP also suggests that there may be other minimal models of the symmetric product. Actually, if $n = 2$ or 3 , we can prove that the relative Hilbert scheme $\text{Hilb}^n(S_2/B)$ is irreducible and admits a small resolution

$$H^{(n)} \rightarrow \text{Hilb}^n(S_2/B) \rightarrow X^{(n)}$$

such that the natural map $H^{(n)} \rightarrow B$ is *semistable* (see [14], Theorem 4.3, for the case $n = 2$). Moreover, for $n = 2$, we can explicitly write down the flop $H^{(2)} \dashrightarrow Y^{(2)}$ as follows.

The singular fiber of $p_2 : S_2 \rightarrow B$ consists of two components $S_{2,1} = (x_1 = 0)$ and $S_{2,2} = (x_2 = 0)$. Let $C = S_{2,1} \cap S_{2,2} \cong \mathbb{A}^1$ be the double line. The fiber of the relative Hilbert scheme $\text{Hilb}^2(S_2/B) \rightarrow B$ consists of three components:

$$\text{Hilb}^2(S_{2,1}), \text{Hilb}^2(S_{2,2}), \text{ and a component } \overline{H}_{12} \text{ birational to } S_0 \times S_1.$$

A non-trivial fiber occurs over a cycle $\gamma = 2p \in X^{(2)}$. If the support of γ lies in the smooth locus of p_2 , the fiber of $\text{Hilb}^2(S_2/B) \rightarrow X^{(2)}$ is \mathbb{P}^1 . Let us denote the associated cycle by l_1 . If the support is in the double curve C , the fiber is \mathbb{P}^2 and we denote the class of a line in this \mathbb{P}^2 by l_2 .

The small resolution $h : H^{(2)} \rightarrow \text{Hilb}^2(S_2/B)$ is given by a blowing-up along the (non-Carrier) divisor $\text{Hilb}^2(S_{2,1})$.

Claim The singular fiber of $H^{(2)} \rightarrow B$ consists of

$$H_{ii} = Bl_{\text{Hilb}^2(C)}(\text{Hilb}^2(S_{2,i})) \ (i = 1, 2), \quad \text{and} \quad H_{12} = Bl_{\Delta_C}(S_{2,1} \times S_{2,2}),$$

where Δ_C is the diagonal of C in $C \times C \subset S_{2,1} \times S_{2,2}$.

Proof Let D be the diagonal of $S_2 \times S_2$ and W the strict transform of $(p_2 \times p_2)^{-1}\Delta_B$ in $Bl_D(S_2 \times S_2)$, where $\Delta_B \cong B$ is the diagonal of $B \times B$. Then, $\text{Hilb}^2(S_2/B)$ is nothing but the quotient W/\mathfrak{S}_2 . The fiber of $W \rightarrow \Delta_B \cong B$ over the origin $0 \in B$ consists of four components

$$W_{ii} = Bl_{\Delta_{S_{2,i}}}(S_{2,i} \times S_{2,i}), \quad W_{ij} = Bl_{\Delta_C}(S_{2,i} \times S_{2,j})$$

with $i, j \in \{1, 2\}$, $i \neq j$. If we take $\tilde{W} = Bl_{W_{11}}W = Bl_{W_{22}}W$, we have an isomorphism $H^{(2)} \cong \tilde{W}/\mathfrak{S}_2$. Let us denote by $\tilde{W}_{ii}, \tilde{W}_{ij}$ the strict transforms of W_{ii}, W_{ij} , respectively. It immediately follows that the fiber of $H^{(2)} \rightarrow B$ over $0 \in B$ consists of

$$H_{ii} \cong \tilde{W}_{ii}/\mathfrak{S}_2 = Bl_{\text{Hilb}^2(C)}(\text{Hilb}^2(S_{2,i})) \quad \text{and} \\ H_{12} \cong \tilde{W}_{12} \cong W_{12} = Bl_{\Delta_C}(S_{2,1} \times S_{2,2}),$$

noting that W_{12} is smooth and $W_{12} \cap W_{11}$ is a Cartier divisor on W_{12} . □

The exceptional divisor E of $Bl_{\Delta_C}(S_{2,1} \times S_{2,2})$ is isomorphic to $\mathbb{P}^2 \times \Delta_C$ and the class of a line on the fiber \mathbb{P}^2 is l_2 . Here, we remark that the morphism h restricted to the strict transform $C \times C \subset H_{12}$ of $C \times C \subset S_{2,1} \times S_{2,2}$ is the canonical morphism

$$C \times C \rightarrow \text{Sym}^2(C),$$

while $h|_{H_{12}}$ is birational. It in particular implies that the component \overline{H}_{12} is *non-normal*.¹ One also sees from the description given in the proof of the claim above that $H_{11} \cap H_{22}$ is \mathbb{P}^1 -bundle over $\text{Hilb}^2(C) = \text{Sym}^2(C)$:

$$H_{11} \cap H_{22} = \mathbb{P}(N_{\text{Hilb}^2(C)/\text{Hilb}^2(S_{2,1})}),$$

and $H_{11} \cap H_{22} \cap H_{12}$ is isomorphic to $C \times C$ that is embedded in $H_{11} \cap H_{22}$ as a double section over $\text{Sym}^2(C)$. All the exceptional fibers of h are isomorphic to \mathbb{P}^1 , whose numerical class we denote by l_3 . The fiber \mathbb{P}^1 of $H_{11} \cap H_{22} \rightarrow \text{Sym}^2(C)$ is numerically equivalent to l_3 .

Now, the relative cone of curves $NE(H^{(2)}/X^{(2)})$ is spanned by l_1 , l_2 , and l_3 . An easy calculation shows that

$$H_{12} \cdot l_1 = 0, \quad H_{12} \cdot l_2 = -2, \quad \text{and} \quad H_{12} \cdot l_3 = 2.$$

As the canonical bundle of $H^{(2)}$ is trivial by [14], Theorem 4.3, $(H^{(n)}, \varepsilon H_{12})$ is klt for a sufficiently small positive rational number ε , and Cone Theorem guarantees that there is an extremal contraction of l_2 , which is a small contraction that contracts E . One sees that $Y^{(2)}$ is nothing but its flop. Actually, the flop produces family of \mathbb{P}^1 over Δ_C passing through a $\frac{1}{2}(1, 1, 1, 1)$ -singularity coming from the fixed point locus with the angle type $\theta = (1/2)$. This is a locally trivial family of toric flop that is called ‘‘Francia flop’’ in [11] (Example 5.1 and Definition 4.1).

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References

1. Batyrev, V.V.: Non-archimedean integrals and stringy euler numbers of log-terminal pairs. J. Eur. Math. Soc. (JEMS) **1**(1), 5–33 (1999). <https://doi.org/10.1007/PL00011158>
2. Beauville, A.: Variétés kähleriennes dont la première classe de chern est nulle. J. Differ. Geom. **18**(4), 755–782 (1984). (French)
3. Birkar, C., Cascini, P., Hacon, C.D., McKernan, J.: Existence of minimal models for varieties of log general type. J. Am. Math. Soc. **23**(2), 405–468 (2010). <https://doi.org/10.1090/S0894-0347-09-00649-3>
4. Brianc¸on, J.: Description de $\text{Hilb}^n \mathbb{C}\{x, y\}$. Invent. Math. **41**(1), 45–89 (1977)
5. Cox, D.A., Little, J.B., Schenck, H.K.: Toric Varieties, Graduate Studies in Mathematics, vol. 124. American Mathematical Society, Providence (2011)
6. Dolgachev, I.V.: Classical Algebraic Geometry. A Modern View. Cambridge University Press, Cambridge (2012)
7. Dolgachev, I., Lunts, V.: A character formula for the representation of a weyl group in the cohomology of the associated toric variety. J. Algebra **168**(3), 741–772 (1994). <https://doi.org/10.1006/jabr.1994.1251>
8. Fogarty, J.: Algebraic families on an algebraic surface. Am. J. Math **90**, 511–521 (1968)
9. Gulbrandsen, M.G., Halle, L. H., Hulek, K.: A GIT construction of degenerations of Hilbert schemes of points. Preprint [arXiv:1604.00215](https://arxiv.org/abs/1604.00215) (2016)
10. Haiman, M.: Hilbert schemes, polygraphs and the macdonald positivity conjecture. J. Am. Math. Soc. **14**(4), 941–1006 (2001). <https://doi.org/10.1090/S0894-0347-01-00373-3>
11. Kawamata, Y.: Francia’s Flip and Derived Categories. Algebraic Geometry, pp. 197–215. de Gruyter, Berlin (2002)
12. Kollár, J., Mori, S.: *Birational geometry of algebraic varieties*, Cambridge Tracts in Mathematics, vol. 134, Cambridge University Press, Cambridge, 1998. With the collaboration of C. H. Clemens and A. Corti; Translated from the 1998 Japanese original

¹ We remark that the description of \overline{H}_{12} in [14], p.419 (appeared as ‘ $Y_{i,i+1}$ ’) is erroneous.

13. Morrison, D.R., Stevens, G.: Terminal quotient singularities in dimensions three and four. *Proc. Am. Math. Soc.* **90**(1), 15–20 (1984). <https://doi.org/10.2307/2044659>
14. Nagai, Y.: On monodromies of a degeneration of irreducible symplectic kähler manifolds. *Math. Z.* **258**(2), 407–426 (2008). <https://doi.org/10.1007/s00209-007-0179-3>
15. Nagai, Y.: Gulbrandsen–Halle–Hulek degeneration and Hilbert–Chow morphism. Preprint (2017)
16. Nakajima, H.: *Lectures on Hilbert Schemes of Points on Surfaces*, University Lecture Series, vol. 18. American Mathematical Society, Providence (1999)
17. Procesi, C.: The toric variety associated to Weyl chambers. *Mots, Lang. Raison. Calc.*, pp. 153–161. Hermès, Paris (1990)
18. Ran, Z.: *Cycle Map on Hilbert Schemes of Nodal Curves*. Projective Varieties with Unexpected Properties, pp. 361–378. Walter de Gruyter GmbH & Co. KG, Berlin (2005)
19. Reid, M.: Canonical 3-folds, *Journées de Géométrie Algébrique d'Angers, Juillet 1979/Algebraic Geometry, Angers, 1979*, pp. 273–310. Sijthoff & Noordhoff. Alphen aan den Rijn-Germantown, Md (1980)
20. Sagan, B.E.: *The symmetric group*, 2nd edn, Graduate Texts in Mathematics. Representations, Combinatorial Algorithms, and Symmetric Functions, vol. 203, Springer-Verlag, New York (2001)
21. Steenbrink, J.H.M.: Mixed Hodge Structure on the Vanishing Cohomology, *Real and Complex Singularities (Proc. Ninth Nordic Summer School/NAVF Sympos. Math., Oslo, 1976)*, pp. 525–563. Sijthoff and Noordhoff, Alphen aan den Rijn (1977)
22. Stembridge, J.R.: Some permutation representations of Weyl groups associated with the cohomology of toric varieties. *Adv. Math.* **106**(2), 244–301 (1994). <https://doi.org/10.1006/aima.1994.1058>
23. Wang, J.: Degenerations of symmetric products of curves. Preprint http://math.uga.edu/~jiewang/Jies_Website/Research_files/7degenerationsofsym.pdf. Accessed 10 Aug 2017
24. Yasuda, T.: Motivic integration over deligne-mumford stacks. *Adv. Math.* **207**(2), 707–761 (2006). <https://doi.org/10.1016/j.aim.2006.01.004>