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Boundedness of spectral multipliers of generalized Laplacians on compact manifolds with boundary

Mayukh Mukherjee[1,2]

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Abstract Consider a second order, strongly elliptic negative semidefinite differential operator L (may be a system) on a compact Riemannian manifold $\overline{M}$ with smooth boundary, where the domain of L is defined by a coercive boundary condition. Classically known results, and also recent work in Duong et al. (J Funct Anal 196:443–485, 2002) and Duong and McIntosh (Rev Math Iberoam 15:233–265, 1999) establish sufficient conditions for $L^\infty - \mathrm{BMO}_L$ continuity of $\varphi(\sqrt{-L})$, where $\varphi \in S^0_1(\mathbb{R})$, and L is a suitable elliptic operator. Using a variant of the Cheeger–Gromov–Taylor functional calculus due to Mauceri et al. (Math Res Lett 16:861–879, 2009), and short time upper bounds on the integral kernel of e^{tL} due to Greiner (Arch Ration Mech Anal 41:168–218, 1971), we prove that a variant of such sufficient conditions holds for our operator L.

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1 Introduction

Consider a compact Riemannian manifold $\overline{M}$ with smooth metric and smooth boundary, and a second order strongly elliptic differential operator $L\colon L^2(M, E) \to L^2(M, E)$ with smooth coefficients, where E is a complex vector bundle with Hermitian metric. Assume a regular elliptic boundary condition $B(x, \partial_x)u = 0$ on ∂M (see Proposition 11.9, Chapter 5 of [24]; see also [13] for general background information) which makes L into a negative semidefinite

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✉ Mayukh Mukherjee
mathmukherjee@gmail.com

1 Max Planck Institute for Mathematics, Bonn, Germany

2 Present Address: The Technion, Haifa, Israel

self-adjoint operator with domain $\mathcal{D}(L) \subset H^2(M, E)$. For notational convenience, we will drop the letter E when denoting the space of sections; for example, $L^2(M)$, or just L^2, will stand for $L^2(M, E)$ henceforth. Also, $T: B_1 \to B_2$ will mean that T is a bounded linear operator from B_1 to B_2, where B_i are Banach spaces.

Given a bounded continuous function $\psi: \mathbb{R} \longrightarrow \mathbb{R}$, the spectral theorem defines

$$\psi(\sqrt{-L}): L^2(M) \longrightarrow L^2(M) \tag{1.1}$$

as a bounded self-adjoint operator. Following [20,23], here we consider functions φ in the pseudodifferential function class $S^0_1(\mathbb{R})$, which means that

$$\varphi \in S^0_1(\mathbb{R}) \Longrightarrow |\varphi^{(k)}(\lambda)| \lesssim (1+|\lambda|)^{-k}, \quad k = 0, 1, 2, \ldots \tag{1.2}$$

We wish to prove that

Theorem 1.1

$$\varphi(\sqrt{-L}): L^\infty \longrightarrow BMO_L. \tag{1.3}$$

For the definition of BMO_L, see Definition 2.1 and Lemma 1.5, which is in turn motivated by definitions in [8] (see also [1,5] and [12]). As a corollary, we get that

Corollary 1.2

$$\varphi(\sqrt{-L}): L^p \longrightarrow L^p, \quad \forall\, p \in (1, \infty). \tag{1.4}$$

In a compact setting, and in the absence of a boundary, results like Theorem 1.1 are well-known (see [17], for example). Also well-known are sufficient conditions for such results in the general setting of a metric measure space, for example, see Theorem 1.6 below, which is due to [6]. However, except for well-behaved scalar elliptic operators (like the Laplace–Beltrami operator associated with the metric on M) with nice boundary conditions and global "heat kernel" bounds, such sufficient conditions are not easy to verify. As mentioned in the abstract, our main technical lemma for proving Theorem 1.1 is to check the following variant of the sufficiency condition proved in Theorem 3 of [6] (see also [8]):

Lemma 1.3 *With $\varphi^\#(\sqrt{-L})$ as in (1.14) and (1.15), denote by $k^\#(x, y)$ and $k_t(x, y)$ the integral kernels of $\varphi^\#(\sqrt{-L})$ and the composite operator $\varphi^\#(\sqrt{-L})e^{tL}$ respectively. We have for some $\varepsilon > 0$,*

$$\sup_{t\in(0,\varepsilon]} \sup_{y\in\overline{M}} \int_{M\setminus B_{\sqrt{t}}(y)} |k^\#(x, y) - k_t(x, y)|dx < \infty. \tag{1.5}$$

Observe that the due to the generality of the elliptic operator L (we do not assume, for instance, that L can be written in the form D^*D, where D is a first order differential operator), and due to the generality of the boundary conditions, Gaussian bounds on the integral kernel of e^{tL} are rather non-trivial to derive. Short time bounds are known from the work in [10]:

Theorem 1.4 (Greiner) *Let $\overline{M}$ be a compact manifold with boundary, and L be a second order self-adjoint negative semidefinite elliptic system defined by regular elliptic boundary conditions. If $p(t, x, y)$ denotes the integral kernel for e^{tL}, for some $\kappa \in (0, \infty)$ we have,*

$$|p(t, x, y)| \lesssim t^{-n/2} e^{-\kappa d(x,y)^2/t}, \quad t \in (0, 1],\ x, y \in \overline{M}, \tag{1.6}$$

and

$$|\nabla_x p(t, x, y)| \lesssim t^{-n/2-1/2} e^{-\kappa d(x,y)^2/t}, \quad t \in (0, 1],\ x, y \in \overline{M}. \tag{1.7}$$

Since the "heat kernel" bounds here are only known for short time, we prove Theorem 1.1 in two main steps. Firstly, we define a concept of local BMO_L spaces, denoted by BMO_L^ϵ (see Definition 2.1 below). Then, Lemma 1.3 proves that $\varphi^{\#}(\sqrt{-L})\colon L^\infty \to \mathrm{BMO}_L^\epsilon$. This we supplement by the following lemma, which proves that BMO_L^ϵ is in fact independent of ϵ.

Lemma 1.5

$$\|f\|_{BMO_L^{\sqrt{2R}}} \cong \|f\|_{BMO_L^{\sqrt{R}}} \tag{1.8}$$

where $R > 0$.

1.1 Tools, preliminaries and motivation

Our main tool will be the functional calculus used in [2], namely

$$\varphi(\sqrt{-L}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t) e^{it\sqrt{-L}} dt. \tag{1.9}$$

We see that $\sqrt{-L}$ is a first order elliptic self-adjoint operator with compact resolvent. So, $\mathrm{Spec}(\sqrt{-L})$ is a discrete subset of $[0, \infty)$, and it is no loss of generality to assume that $\varphi(\lambda)$ is an even function of λ. This reduces (1.9) to

$$\varphi(\sqrt{-L}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}(t) \cos t\sqrt{-L}\, dt, \tag{1.10}$$

where $\cos t\sqrt{-L}$ is the solution operator to the "wave equation" with zero initial velocity, i.e.,

$$u(t, x) = \cos t\sqrt{-L}\, f(x) \tag{1.11}$$

where

$$\partial_t^2 u - Lu = 0, \quad u(0, x) = f(x), \quad \partial_t u(0, x) = 0 \tag{1.12}$$

together with the coercive boundary conditions mentioned above.

We split (1.10) into two parts in the following way: let, for $a > 0$ small, $\theta(t)$ be an even function, such that

$$\theta \in C_c^\infty((-a, a)), \qquad \theta(t) \equiv 1 \quad \text{on } [-a/2, a/2]. \tag{1.13}$$

Now, denote

$$\hat{\varphi}^{\#}(t) = \theta(t)\hat{\varphi}(t), \quad \hat{\varphi}^{b}(t) = (1 - \theta(t))\hat{\varphi}(t). \tag{1.14}$$

Then, we can write

$$\begin{aligned} \varphi(\sqrt{-L}) &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{\varphi}^{\#}(t) \cos t\sqrt{-L} dt + \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty}, \hat{\varphi}^{b}(t) \cos t\sqrt{-L} dt \\ &= \varphi^{\#}(\sqrt{-L}) + \varphi^{b}(\sqrt{-L}), \end{aligned} \tag{1.15}$$

where both $\varphi^{\#}(\sqrt{-L})$ and $\varphi^{b}(\sqrt{-L})$ are self-adjoint.

Now, (1.2) implies that φ^b is smooth and rapidly decreasing. So we have

$$|\varphi^b(\lambda)| \lesssim (1 + |\lambda|)^{-m}, \quad \text{for some } m > \frac{n}{2}.$$

Then, the ellipticity of L implies

$$\varphi^b(\sqrt{-L})\colon L^2(M) \longrightarrow H^m(M) \subset C(\overline{M}). \tag{1.16}$$

This is because, we can write $\varphi^b(\lambda) = (1+|\lambda|^2)^{-s/2}\psi^b(\lambda),\ s \in \mathbb{N}$, where $\psi^b(\lambda)$ is a bounded function. That implies

$$\varphi^b(\sqrt{-L}) = (I-L)^{-s/2}\psi^b(\sqrt{-L}).$$

By the spectral theorem, $\psi^b(\sqrt{-L})$ is bounded on L^2, and

$$(I-L)^{-s/2}\colon L^2(M) \to \mathcal{D}((-L)^{s/2}) \subset H^s(M).$$

We can see that $L^1(M) \subset C(\overline{M})^*$ and the inclusion is continuous. Also, when $m > n/2$, $H^m(M) \subset C(\overline{M})$ by Sobolev embedding. This gives, via duality on (1.16), and self-adjointness of $\varphi^b(\sqrt{-L})$, that

$$\varphi^b(\sqrt{-L})\colon L^1(M) \longrightarrow L^2(M). \tag{1.17}$$

This interpolates with (1.1) to give

$$\varphi^b(\sqrt{-L})\colon L^p(M) \longrightarrow L^p(M) \quad \forall\, p \in (1,\infty). \tag{1.18}$$

Results of the type (1.4) have been well-studied for complete Riemannian manifolds M without boundary and $L = \Delta$, the Laplace–Beltrami operator. When M is compact, we refer to [19], particularly the combination of Theorem 1.3 of Chapter XII and Theorem 2.5 of Chapter XI. For results in the non-compact setting, refer to [2,14,21–23], etc. In the papers which deal with a non-compact setting, an additional difficulty is in analyzing $\varphi^b(\sqrt{-\Delta})$, because of the failure of the compact Sobolev embedding. Particularly for manifolds like the hyperbolic space, where the volume growth is exponential with respect to the distance, one requires more stringent restrictions on φ, namely, φ being holomorphic on a strip around the x-axis, satisfying bounds of the form (1.2). This condition was first introduced in the paper [3]. This motivated some research on the optimal width of said strip; to the best of our knowledge, the sharpest results in this direction appear in [22]. Also, in [2], $\varphi^\#(\sqrt{-\Delta})$ was analyzed as a pseudodifferential operator, something we cannot do in our present setting because of the presence of a boundary. It is well-known that the scalar square root of the Laplacian fails the "transmission condition" (see, for example, equation (18.2.20) of [11]) to be a pseudodifferential operator (in this context, see also [25], [26]). Our analysis will be a combination of methods from [6,9], and the approach of [23], which follows and refines the results in [14].

Let us discuss some of the main lines of investigation in [6,9]. The results therein are rather general, set in the context of an open subset X of a metric measure space of homogeneous type. If L is a negative semi-definite, self-adjoint operator on $L^2(X)$, they assume that the integral kernel of L satisfies

$$|p(t,x,y)| \lesssim t^{-n/m} e^{-\kappa \operatorname{dist}(x,y)^{m/(m-1)}/t^{1/(m-1)}}, \quad 0 < t \le 1, \kappa \in (0,\infty). \tag{1.19}$$

Theorem 3.1 of [9] establishes L^p-continuity of $\varphi((-L)^{1/m})$ via proving that it is weak type $(1,1)$. What mainly concerns us with [9] is the fact that they proved L^p continuity of $\varphi((-L)^{1/m})$ via using the following result from [6]:

Theorem 1.6 (Duong–McIntosh) *Under the hypothesis on L outlined above, let $k_t(x,y)$ denote the integral kernel of $\varphi((-L)^{1/m})(I - e^{tL})$, where $\varphi\colon \mathbb{R} \to \mathbb{R}$ is bounded and continuous. Also assume*

$$\sup_{t>0}\sup_{y\in X}\int_{X\setminus B_{t^{1/m}}(y)}|k_t(x,y)|dx<\infty. \tag{1.20}$$

Then $\varphi((-L)^{1/m})$ *is of weak type* $(1,1)$.

It is naturally interesting to investigate when conditions like (1.20) are satisfied. In this paper, our aim is to check that (1.20) holds for a large class of operators L, as mentioned before, in the setting of a smooth compact manifold with boundary.

We also note here that in the proof of L^p-boundedness of $\varphi(\sqrt{-L})$ in [20], the main approach is to prove the following

Lemma 1.7 (Taylor) *There exists* $C<\infty$*, independent of* $s\in(0,1]$ *and of* $y,y'\in\overline{M}$*, such that*

$$dist(y,y')\le\frac{s}{2}\implies\|K^{\#}(.,y)-K^{\#}(.,y')\|_{L^1(B_1(y)\setminus B_s(y))}\le C,$$

where $K^{\#}(x,y)$ *is the integral kernel of* $\varphi^{\#}(\sqrt{-L})$.

With that in place, as is noted in [20], the weak type (1, 1) property of $\varphi^{\#}(\sqrt{-L})$ is a consequence of Proposition 3.1 of [14], which is a variant of Theorem 2.4 in Chapter III of [4].

Note that we are yet to argue the $L^\infty-\mathrm{BMO}_L$ boundedness of $\varphi^b(\sqrt{-L})$; this we will do at the beginning of Sect. 3. So, for all continuity related aspects, for the rest of our investigation, we will mainly be concerned with just $\varphi^{\#}(\sqrt{-L})$. These boundedness considerations will largely be addressed in Sects. 2 and 3. For those sections, our standing assumption will be the following:

Assumption $\cos t\sqrt{-L}$ has finite speed of propagation, which, by scaling L if necessary, we will assume to be ≤ 1.

It seems a challenging question to determine when $\cos t\sqrt{-L}$ has finite speed of propagation. However, for a reasonable class of operators L, we can prove the following "Davies–Gaffney" type estimates:

Proposition 1.8 *Let* $-L=D^*D+H$ *with the generalized Dirichlet or Neumann boundary conditions on* D*, as defined in* (5.3) *and* (5.4)*, and with* $H\ge 0$ *in* $L^2(M)$*. Let* U,V *be two open balls such that* $dist(U,V)=r$*. With* $t>0$ *fixed, let* $\phi(x)=\frac{r}{t}dist\,(x,U)$ *and* $P=[D,e^{\phi}]$ *denote the usual commutator operator. Then* $\cos t\sqrt{-L}$ *has finite speed of propagation if it satisfies for all* $v\in L^2(V)$ *the following*:

$$\|e^{-\phi/2}Pv\|_{L^2}\le\frac{r}{t}\|e^{\phi/2}v\|_{L^2}. \tag{1.21}$$

Equation (1.21) follows when $|D\phi|$ is bounded. As a special case, (1.21) follows trivially when $L=\Delta$ with the Dirichlet or Neumann boundary conditions.

One last comment: for the purposes of proving Lemma 1.3, we will also a modification of (1.9), following [14,20,23]. For details on this, see Sect. 2.2.

1.2 Outline of the paper

In Sect. 2, we prove that Definition 2.1 of $\mathrm{BMO}_L^{\epsilon}$ is independent of ϵ, as long as we are on a compact setting. This is the content of Lemma 1.5. Then we proceed to prove our main technical lemma of the paper, Lemma 1.3. We begin Sect. 3 by arguing the $L^\infty - \mathrm{BMO}_L$ continuity of $\varphi^b(\sqrt{-L})$, and then prove in Proposition 3.1 (using Lemma 1.3) the $L^\infty - \mathrm{BMO}_L$ continuity of $\varphi^\#(\sqrt{-L})$, which finally proves Theorem 1.1. In Appendix 1, we collect together some useful information about the integral kernels of the operators e^{tL} and $e^{-t\sqrt{-L}}$. The properties we establish are quite parallel to their usual scalar Laplacian counterparts, and are at the background of some of the estimates we derive in the main body of the paper. In Appendix 2, we prove some partial results towards establishing sufficient criteria for $\cos t\sqrt{-L}$ to have finite propagation speed. To wit, we prove that for those operators L which can be written in the specific form (5.1), under generalized Dirichlet or Neumann boundary conditions (see (5.3) and (5.4) below), and under the assumption (1.21), $\cos t\sqrt{-L}$ has finite speed of propagation. This is the content of Proposition 1.8.

2 Proof of Lemma 1.3

2.1 BMO_L and its variants

We combine the definition of BMO_L in [7] with the definition of local BMO spaces in [21] to give the following

Definition 2.1 $f \in L^1_{loc}(M)$ is in $\mathrm{BMO}^\epsilon_L(M)$ if

$$\frac{1}{|B|}\int_B |f(x) - e^{tL}f(x)|dx \leq C, \tag{2.1}$$

where $\sqrt{t}$ is the radius of the ball B, and B ranges over all balls in M of radius $\leq \epsilon$. Let

$$\|f\|_{BMO^\epsilon_L} = \sup_{B\in\mathcal{B}} \frac{1}{|B|}\int_B |f(x) - e^{tL}f(x)|dx, \tag{2.2}$$

where $\sqrt{t}$ is the radius of the ball B, and $\mathcal{B}$ contains all balls of radius $\leq \epsilon$.

We now make the observation that our definition of BMO^ϵ_L is actually independent of the ϵ chosen.

Proof Clearly,

$$\|f\|_{\mathrm{BMO}_L^{\sqrt{2R}}} \geq \|f\|_{\mathrm{BMO}_L^{\sqrt{R}}}.$$

For the reverse inequality, let us fix a point $y \in \overline{M}$. Then we have, for $r \leq R$,

$$\begin{aligned}
&\frac{1}{|B_{\sqrt{2r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |f(x) - e^{2rL}f(x)|dx - \frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{r}}(y)} |f(x) - e^{rL}f(x)|dx \\
&\lesssim \frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |f(x) - e^{2rL}f(x) - \chi_{B_{\sqrt{r}}(y)}(x)f(x) + \chi_{B_{\sqrt{r}}(y)}(x)e^{rL}f(x)|dx \\
&= \frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |\chi_A(x)f(x) - e^{2rL}f(x) + \chi_{B_{\sqrt{r}}(y)}(x)e^{rL}f(x)|dx,
\end{aligned}$$

where A denotes the "annulus" $B_{\sqrt{2r}}(y) \setminus B_{\sqrt{r}}(y)$, which can be covered by at most K balls of radius $\sqrt{r}$, where K is a positive number independent of r, because $\overline{M}$ is compact. Also, in the ensuing calculation, we tacitly use the fact that the volume of a ball of radius r is uniformly bounded.

Now, the last quantity in the above equation is

$$\begin{aligned}
&\leq \frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |\chi_A(x)f(x)-\chi_A(x)e^{2rL}f(x)|dx\\
&\quad+\frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |\chi_{B_{\sqrt{r}}(y)}(x)e^{2rL}f(x)-\chi_{B_{\sqrt{r}}(y)}(x)e^{rL}f(x)|dx\\
&\leq \frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |\chi_A(x)f(x)-\chi_A(x)e^{rL}f(x)|dx\\
&\quad+\frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |\chi_A(x)e^{2rL}f(x)-\chi_A(x)e^{rL}f(x)|dx\\
&\quad+\frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |\chi_{B_{\sqrt{r}}(y)}(x)e^{2rL}f(x)-\chi_{B_{\sqrt{r}}(y)}(x)e^{rL}f(x)|dx\\
&=\frac{1}{|B_{\sqrt{r}}(y)|}\int_A |f(x)-e^{rL}f(x)|dx+\frac{1}{|B_{\sqrt{r}}(y)|}\int_A |e^{2rL}f(x)-e^{rL}f(x)|dx\\
&\quad+\frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{r}}(y)} |e^{2rL}f(x)-e^{rL}f(x)|dx\\
&=A+B+C\ (\text{say})\,.
\end{aligned}$$

Putting everything together, we have that

$$\begin{aligned}
&\frac{1}{|B_{\sqrt{2r}}(y)|}\int_{B_{\sqrt{2r}}(y)} |f(x)-e^{2rL}f(x)|dx-\frac{1}{|B_{\sqrt{r}}(y)|}\int_{B_{\sqrt{r}}(y)} |f(x)-e^{rL}f(x)|dx\\
&\lesssim A+B+C. \qquad (2.3)
\end{aligned}$$

Now, if we let $e^{rL}f(x)=g(x)$, and can prove that

$$\|g\|_{\mathrm{BMO}_L^{\sqrt{r}}}\lesssim\|f\|_{\mathrm{BMO}_L^{\sqrt{r}}}, \qquad (2.4)$$

then we have that each of A, B, C is $\lesssim\|f\|_{\mathrm{BMO}_L^{\sqrt{R}}}$, giving us our result from (2.3).

Now we justify (2.4). Choose $\varepsilon>0$ and let $\mathcal{B}$ consist of all balls in M whose radii are $\leq\varepsilon$. Choose $s>0$ such that $\sqrt{s}\leq\varepsilon$. As per the notation above, if $u(x)=e^{sL}f(x)-f(x)$, observe that it suffices to prove that

$$\sup_{B_{\sqrt{s}}\in\mathcal{B}}\frac{1}{|B_{\sqrt{s}}|}\int_{B_{\sqrt{s}}} |e^{sL}u(x)|dx\lesssim\sup_{B_{\sqrt{s}}\in\mathcal{B}}\frac{1}{|B_{\sqrt{s}}|}\int_{B_{\sqrt{s}}} |u(x)|dx.$$

Now, we have,

$$\begin{aligned}
\sup_{B_{\sqrt{s}}\in\mathcal{B}}\frac{1}{|B_{\sqrt{s}}|}\int_{B_{\sqrt{s}}} |e^{sL}u(x)|dx&\lesssim\sup_{B_{\sqrt{s}}\in\mathcal{B}}\frac{1}{|B_{\sqrt{s}}|}\int_{B_{\sqrt{s}}}\int_M |p(s,z,x)u(z)|dzdx\\
&\lesssim\sup_{B_{\sqrt{s}}\in\mathcal{B}}\frac{1}{|B_{\sqrt{s}}|}s^{-n/2}\int_{B_{\sqrt{s}}}\|u\|_{L^1(M)}\ (\text{from (1.6)})\\
&=s^{-n/2}\|u\|_{L^1(M)}.
\end{aligned}$$

So we are done if we can prove that

$$\|u\|_{L^1(M)} \lesssim \sup_{B_{\sqrt{s}} \in \mathcal{B}} \frac{1}{|B_{\sqrt{s}}|} \|u\|_{L^1(B_{\sqrt{s}})}.$$

Consider a partition of $\overline{M}$ into balls coming from $\mathcal{B}$. Let $\overline{M} = \coprod_n B_n$, where $B_n \in \mathcal{B}$. Then,

$$\begin{aligned}\frac{1}{|M|}\|u\|_{L^1(M)} &= \frac{1}{|M|}\sum_n \|u\|_{L^1(B_n)} = \frac{1}{|M|}\sum_n |B_n| \frac{1}{|B_n|}\|u\|_{L^1(B_n)} \\ &\leq \frac{1}{|M|}\sum_n |B_n| \sup_{B \in \mathcal{B}} \frac{1}{|B|}\|u\|_{L^1(B)} = \sup_{B\in\mathcal{B}} \frac{1}{|B|}\|u\|_{L^1(B)}.\end{aligned}$$

This finishes the proof. □

2.2 A modification of the [2] functional calculus

At this point, let us recall the main approach of [14,20,23]. The analysis in these papers avoided producing a parametrix for (1.10). Instead, they replaced (1.10) by the following

$$\varphi(\sqrt{-L}) = \frac{1}{2}\int_{-\infty}^{\infty} \varphi_k(t)\mathcal{J}_{k-1/2}(t\sqrt{-L})\,dt, \tag{2.5}$$

where

$$\mathcal{J}_\nu(\lambda) = \lambda^{-\nu} J_\nu(\lambda),$$

$J_\nu(\lambda)$ denoting the standard Bessel function (see [23], equation (3.1) and [24], Chapter 3, Section 6 for more details on Bessel functions), and

$$\varphi_k(t) = \prod_{j=1}^{k}\left(-t\frac{d}{dt} + 2j - 2\right)\hat{\varphi}(t).$$

Taylor [23] derives (2.5) from (1.10) by an integration by parts argument (see (3.7)–(3.9) of [23]). Similarly, from [23], (3.14), we have

$$\varphi^{\#}(\sqrt{-L}) = \frac{1}{2}\int_{-\infty}^{\infty} \psi_k(t)\mathcal{J}_{k-1/2}(t\sqrt{-L})\,dt \tag{2.6}$$

with

$$\psi_k(t) = \prod_{j=1}^{k}\left(-t\frac{d}{dt} + 2j - 2\right)\hat{\varphi}^{\#}(t), \quad \text{where supp } \psi_k \subset [-a, a]. \tag{2.7}$$

Also, (1.2) implies

$$|(t\partial_t)^j\hat{\varphi}(t)| \leq C_j|t|^{-1}, \quad \forall j \in \left\{0, 1, \ldots, \left[\frac{n}{2}\right] + 2\right\},$$

which in turn implies

$$|\psi_k(t)| \leq C_k|t|^{-1}, \quad 0 \leq k \leq \left[\frac{n}{2}\right] + 2. \tag{2.8}$$

Now, let $k^{\#}(x, y)$ denote the integral kernel of $\varphi^{\#}(\sqrt{-L})$, that is,

$$\varphi^{\#}(\sqrt{-L})f(x) = \int_M k^{\#}(x, y)f(y)dy. \tag{2.9}$$

Without any loss of generality, we can scale L so that the speed of propagation of $\cos t\sqrt{-L}$ is ≤ 1, which has been stated as an assumption on page 4. Also, let us select $a = 1$ in (2.7) and (1.13).

Now we address one fundamental question: why this choice of a and why is the finite propagation speed of $\cos t\sqrt{-L}$ so important? This has to do with the support of the integral kernel $k^{\#}(x, y)$. If the speed of propagation of $\cos t\sqrt{-L}$ is ≤ 1, then $k^{\#}(x, y)$ is supported within a distance ≤ 1 from the diagonal. Let us justify this: we have

$$\varphi^{\#}(\sqrt{-L})f(x) = \int_M k^{\#}(x, y)f(y)dy = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} \hat{\varphi}^{\#}(t)\cos t\sqrt{-L}f(x)dt.$$

Suppose the propagation speed of $\cos t\sqrt{-L}$ is ≤ 1. Then, when $|t| \leq 1$,

$$\text{supp} \cos t\sqrt{-L}f(x) \subset \{x \in \overline{M} : \text{dist}\,(x, \text{supp}\, f) \leq t\}.$$

When $|t| > 1$, $\hat{\varphi}^{\#}(t) = 0$. So, for all $t \in \mathbb{R}$, $\varphi^{\#}(\sqrt{-L})f(x)$ will be zero for all $x \in \overline{M}$ such that $\text{dist}(x, \text{supp}\, f) > 1$. Since this happens for all $f \in L^2(M)$, we have that $k^{\#}(x, y)$ is supported within a distance of 1 from the diagonal. This property will be crucially used in the sequel.

Using (2.6), (2.7) and the fact that ψ_k is an even function, we can write

$$k^{\#}(x, z) = \int_0^1 \psi_k(s)B_k(s, x, z)ds \tag{2.10}$$

where $B_k(t, x, y)$ is the integral kernel of $\mathcal{J}_{k-1/2}(t\sqrt{-L})$, that is,

$$\mathcal{J}_{k-1/2}(t\sqrt{-L})f(x) = \int_M B_k(t, x, y)f(y)dy.$$

Let us also record the following formula:

$$\mathcal{J}_{k-1/2}(t\sqrt{-L}) \approx \int_{-1}^1 (1 - s^2)^{k-1}\cos st\sqrt{-L}ds. \tag{2.11}$$

We have assumed that the speed of propagation of $\cos t\sqrt{-L}$ is ≤ 1. Observe that, written in symbols, this means,

$$\text{supp}\, f \subset K \Rightarrow \text{supp} \cos t\sqrt{-L}f \subset K_{|t|},$$

where $K_{|t|} = \{x \in \overline{M} : \text{dist}(x, K) \leq |t|\}$. This gives, in conjunction with (2.11),

$$\text{supp}\, f \subset K \Rightarrow \text{supp}\, \mathcal{J}_{k-1/2}(t\sqrt{-L})f \subset K_{|t|}. \tag{2.12}$$

We now derive a technical estimate on $\|B_k(s, x, .)\|_{L^2(B_1(x))}$ which we will find essential in the sequel. These estimates are variants of Lemma 2.2 in [20].

Lemma 2.2 *If* $G : \mathbb{R} \longrightarrow \mathbb{R}$ *satisfies*

$$|G(\lambda)| \lesssim (1 + |\lambda|)^{-\gamma-1}, \quad \gamma > n/2, \tag{2.13}$$

then

$$\|G(s\sqrt{-L})\|_{\mathcal{L}(L^2, L^\infty)} \lesssim s^{-n/2}, \quad s \in (0, 1]. \tag{2.14}$$

This implies, in particular, the following

$$\|B_k(s,x,.)\|_{L^2(B_1(x))} \lesssim s^{-n/2}, \quad s \in (0,1],\ x \in \overline{M}. \tag{2.15}$$

Proof We use the following estimate:

$$|\mathcal{J}_{k-1/2}(\lambda)| \lesssim (1+|\lambda|)^{-k}, \quad k > 0.$$

We write

$$G(s\sqrt{-L}) = (I - s^2L)^{-\sigma} G(s\sqrt{-L})(I - s^2L)^{\sigma}, \quad 2\sigma = \gamma + 1. \tag{2.16}$$

Now, let $F(\lambda) = G(\lambda)(1-\lambda^2)^{\sigma}$. Then, using $\gamma > n/2$ and $2\sigma = \gamma + 1$, we see that

$$\begin{aligned}|F(\lambda)| &= |G(\lambda)(1-\lambda^2)^{\sigma}| \lesssim (1+|\lambda|)^{-\gamma-1}|1-\lambda^2|^{\sigma}\\ &\lesssim (1+|\lambda|)^{2\sigma-\gamma-1} \le C.\end{aligned}$$

Since F is bounded, by the spectral theorem, $F(s\sqrt{-L})\colon L^2 \longrightarrow L^2$ is continuous. So by virtue of (2.16), our task is reduced to proving that

$$\|(I - s^2L)^{-\sigma}\|_{\mathcal{L}(L^2,L^\infty)} \lesssim s^{-n/2}, \quad \sigma > n/4. \tag{2.17}$$

Now, we use the following identity from that can be derived from the definition of the gamma function:

$$(I - s^2L)^{-\sigma} \approx \int_0^\infty e^{-r} e^{rs^2L} r^{\sigma-1} dr,$$

which gives

$$\begin{aligned}\|(I - s^2L)^{-\sigma}\|_{\mathcal{L}(L^2,L^\infty)} &\lesssim \int_0^{s^{-2}} e^{-r}\|e^{rs^2L}\|_{\mathcal{L}(L^2,L^\infty)} r^{\sigma-1}dr\\ &\quad + \int_{s^{-2}}^\infty e^{-r}\|e^{rs^2L}\|_{\mathcal{L}(L^2,L^\infty)} r^{\sigma-1}dr\\ &\lesssim \int_0^{s^{-2}} e^{-r}(rs^2)^{-n/4} r^{\sigma-1}dr + \int_{s^{-2}}^\infty e^{-r} r^{\sigma-1}dr\\ &\lesssim (s^{-n/2}+1) \lesssim s^{-n/2}, \quad s \in (0,1],\end{aligned}$$

where in going from the second to the third step, we have used that

$$\|e^{tL}\|_{\mathcal{L}(L^2,L^\infty)} \lesssim t^{-n/4}, \quad t \in (0,1]$$

and

$$\|e^{tL}\|_{\mathcal{L}(L^2,L^\infty)} \lesssim 1, \quad t > 1.$$

This establishes (2.14). That is,

$$|G(s\sqrt{-L})f(x)| \lesssim s^{-n/2}\|f\|_{L^2}, \quad s \in (0,1].$$

In particular, with $f = \delta_x$ and using the compactness of $\overline{M}$, we have

$$\|g(s,x,.)\|_{L^2} \lesssim s^{-n/2}, \quad s \in (0,1].$$

□

We are in a position to prove Lemma 1.3.

Proof We have

$$\varphi^{\#}(\sqrt{-L})e^{tL}f(x) = \int_M k_t(x, y)f(y)dy. \tag{2.18}$$

Also,

$$\varphi^{\#}(\sqrt{-L})e^{tL}f(x) = \int_M k^{\#}(x, y)e^{tL}f(y)dy = \int_M k^{\#}(x, y)\int_M p(t, y, z)f(z)dzdy$$
$$= \int_M\int_M k^{\#}(x, y)p(t, y, z)f(z)dzdy.$$

Interchanging the variables y and z, we get

$$\varphi^{\#}(\sqrt{-L})e^{tL}f(x) = \int_M\int_M k^{\#}(x, z)p(t, z, y)f(y)dydz. \tag{2.19}$$

Comparing (2.18) and (2.19), we get

$$k_t(x, y) = \int_M k^{\#}(x, z)p(t, z, y)dz = e^{tL_y}k^{\#}(x, y).$$

Interchanging x and y and using the symmetry of the integral kernels, we get

$$k_t(x, y) = k_t(y, x) = e^{tL_x}k^{\#}(y, x) = e^{tL_x}k^{\#}(x, y).$$

Henceforth, we shall drop the subscript x in L_x and L will refer to a differential operator in the x-variable, unless otherwise mentioned explicitly. To show (1.5), all we want is a uniform bound on

$$\|e^{tL}k^{\#}(., y) - k^{\#}(., y)\|_{L^1(M\setminus B_{\sqrt{t}}(y))}. \tag{2.20}$$

To derive (2.20), it is clear (by the Mean value theorem) that a uniform bound on

$$\|e^{t'L}tLk^{\#}(., y)\|_{L^1(M\setminus B_{\sqrt{t}}(y))},$$

where $t' \in (0, t]$, will suffice. Now, since $k^{\#}(x, y)$ is a fixed kernel, we can choose t small enough such that

$$\|e^{t'L}tLk^{\#}(., y)\|_{L^1(M\setminus B_{\sqrt{t}}(y))} \leq C\|tLk^{\#}(., y)\|_{L^1(M\setminus B_{\sqrt{t}}(y))},$$

where C does not depend on t. This latter quantity, using the relation between $k^{\#}(x, y)$ and $B_k(s, x, y)$ given by (2.10), is equal to

$$\left\|tL\int_0^1 \psi_k(s)B_k(s, ., y)ds\right\|_{L^1(M\setminus B_{\sqrt{t}}(y))}. \tag{2.21}$$

Now, when $s' \leq \sqrt{t}$, we have by (2.12) that $\int_0^{s'}\psi_k(s)B(s, x, y)ds$ is supported on $\{(x, y) \in \overline{M}\times\overline{M}\colon \operatorname{dist}(x, y) \leq s'\}$. So, (2.21) gives via (2.8) that,

$$\left\|tL\int_0^1 \psi_k(s)B_k(s, ., y)ds\right\|_{L^1(M\setminus B_{\sqrt{t}}(y))} \lesssim t\int_{\sqrt{t}}^1 \frac{1}{s}\|LB_k(s, ., y)\|_{L^1(M\setminus B_{\sqrt{t}}(y))}ds,$$

which we must prove to be uniformly bounded. If we can prove

$$\|LB_k(s, ., y)\|_{L^1(M\setminus B_{\sqrt{t}}(y))} \lesssim s^{-2},$$

then we are done. Observe that this will be implied by

$$\|LB_k(s,.,y)\|_{L^2(M\setminus B_{\sqrt{t}}(y))} \lesssim s^{-n/2-2}. \tag{2.22}$$

This is because, from (2.12), we see that $B_k(s,.,y)$ is supported on the ball $B_s(y) \subset \overline{M}$, so

$$\begin{aligned}\|LB_k(s,.,y)\|_{L^1(M\setminus B_{\sqrt{t}}(y))} &\lesssim |B_s(y)|^{1/2}\|LB_k(s,.,y)\|_{L^2(M\setminus B_{\sqrt{t}}(y))}\\ &\lesssim |B_s(y)|^{1/2}s^{-n/2-2} \lesssim s^{-2}.\end{aligned}$$

So, we are done if we can prove that,

$$\|LB_k(s,.,y)\|_{L^2} \lesssim s^{-n/2-2}. \tag{2.23}$$

We observe that (2.23) is another variant of Lemma 2.2 of [20] and proceeds along absolutely similar lines. See the lemmas below, which finish the proof. □

Lemma 2.3

$$\|Le^{-s\sqrt{-L}}\|_{\mathcal{L}(L^2,L^\infty)} \lesssim s^{-n/2-2}, \quad s \in (0,1]. \tag{2.24}$$

Proof For $f \in L^2(M)$, call

$$u_s(x) = e^{-s\sqrt{-L}} f(x), \quad s > 0,\ x \in \overline{M}. \tag{2.25}$$

Then u is a solution of

$$(\partial_s^2 + L)u = 0, \quad \text{on } (0,\infty)\times M \tag{2.26}$$

$$B(x,\partial_x)u = 0, \quad \text{on } (0,\infty)\times \partial M, \tag{2.27}$$

where B represents the coercive boundary condition defining $\mathcal{D}(L)$. We have, by the Hille–Yosida theorem,

$$\|u_s\|_{L^2(M)} = \|e^{-s\sqrt{-L}} f\|_{L^2} \le \|f\|_{L^2(M)}, \quad \forall s > 0.$$

Let us pick $\delta \in (0,1]$, $s_0 = \delta$, and $x_0 \in \overline{M}$. Let $U = \{x \in \overline{M}\colon \operatorname{dist}(x,x_0) < 2\delta\}$. We now scale the s and the x variables by a factor of $1/\delta$, and let $v_s(x)$ denote the new function corresponding to u_s in the scaled variables. Then v solves

$$(\partial_s^2 + \tilde{L})v = 0, \quad \text{on } (1/2,3/2)\times \tilde{U}, \tag{2.28}$$

$$\tilde{B}(x,\partial_x)v = 0, \quad \text{on } (1/2,3/2)\times (\tilde{U}\cap \partial M), \tag{2.29}$$

which is a coercive boundary valued elliptic system with uniformly smooth coefficients and uniform ellipticity bounds. On calculation,

$$\|v\|_{L^2((1/2,3/2)\times\tilde{U})} \approx \delta^{-n/2}\|u\|_{L^2((\delta/2,3\delta/2)\times U)} \lesssim \delta^{-n/2}\|f\|_{L^2}.$$

From elliptic regularity estimates we get that

$$\|Lv_1(.)\|_{L^\infty(\tilde{U}_0)} \lesssim \|v\|_{L^2((1/2,3/2)\times\tilde{U})} \lesssim \delta^{-n/2}\|f\|_{L^2}. \tag{2.30}$$

This can be obtained by iterating the estimate (11.29) of Chapter 5 of [24] to prove that $\|u\|_{H^k(I\times\tilde{U}_0)} \lesssim \|u\|_{L^2((1/2,3/2)\times\tilde{U})}$, where $1 \in I \subset (1/2,3/2)$, and taking k high enough such that $H^k \hookrightarrow C^{2,\alpha}$ for some α. This implies (2.30). Scaling back gives our result

$$|Lu_{s_0}(x_0)| \lesssim \delta^{-n/2-2}\|f\|_{L^2}.$$

□

Lemma 2.4 *If* $G\colon \mathbb{R} \to \mathbb{R}$ *satisfies*

$$|G(\lambda)| \lesssim (1+|\lambda|)^{-\gamma-1}, \quad \gamma > \frac{n}{2},$$

then

$$\|LG(s\sqrt{-L})\|_{\mathcal{L}(L^2,L^\infty)} \lesssim s^{-n/2-2}, \quad s \in (0,1]. \tag{2.31}$$

This implies (2.23).

Proof We start by using the formula

$$(I+s\sqrt{-L})^{-\sigma} = \frac{1}{\Gamma(\sigma)} \int_0^\infty e^{-t} e^{-ts\sqrt{-L}} t^{\sigma-1} dt.$$

That implies, in conjunction with (2.24),

$$\begin{aligned} \|L(I+s\sqrt{-L})^{-\sigma}\|_{\mathcal{L}(L^2,L^\infty)} &\lesssim \int_0^{1/s} e^{-t}(st)^{-n/2-2} t^{\sigma-1} dt \\ &\quad + \int_{1/s}^\infty e^{-t} \|Le^{-ts\sqrt{-L}}\|_{\mathcal{L}(L^2,L^\infty)} t^{\sigma-1} dt \\ &\lesssim s^{-n/2-2} + 1, \end{aligned} \tag{2.32}$$

where $\sigma > n/2+2$. Also, in the above calculation, we have used that when $r \geq 1$,

$$\begin{aligned} \|Le^{-r\sqrt{-L}}\|_{\mathcal{L}(L^2,L^\infty)} &= \|Le^{-\sqrt{-L}} e^{-(r-1)\sqrt{-L}}\|_{\mathcal{L}(L^2,L^\infty)} \\ &\lesssim \|e^{-(r-1)\sqrt{-L}}\|_{\mathcal{L}(L^2,L^2)} \leq 1. \end{aligned}$$

The facts that (2.32) implies (2.31) and (2.31) implies (2.23) are absolutely similar to the proof of Lemma 2.2. □

3 L^∞ – BMO$_L$ continuity

We see that (1.17) gives, by duality,

$$\varphi^b(\sqrt{-L})\colon L^2 \longrightarrow L^\infty. \tag{3.1}$$

Now, we can prove the following inclusion on a compact manifold:

$$\|.\|_{\mathrm{BMO}_L} \lesssim \|.\|_{L^\infty}. \tag{3.2}$$

This is because, for small enough $t > 0$, we have

$$\frac{1}{|B_{\sqrt{t}}(y)|} \int_{B_{\sqrt{t}}(y)} |f(x) - e^{tL} f(x)| dx \leq \|f - e^{tL} f\|_{L^\infty} \lesssim \|f\|_{L^\infty}.$$

Equations (3.1) and (3.2) give

$$\|\varphi^b(\sqrt{-L}) f\|_{\mathrm{BMO}_L} \leq \|\varphi^b(\sqrt{-L}) f\|_{L^\infty} \lesssim \|f\|_{L^2}, \tag{3.3}$$

which means

$$\varphi^b(\sqrt{-L})\colon L^2 \longrightarrow \mathrm{BMO}_L. \tag{3.4}$$

Equations (3.2) and (3.4) give

$$\varphi^b(\sqrt{-L})\colon L^\infty \longrightarrow \mathrm{BMO}_L. \tag{3.5}$$

Finally, we have

Proposition 3.1

$$\varphi^\#(\sqrt{-L})\colon L^\infty \longrightarrow BMO_L.$$

Proof

$$\begin{aligned}
&\frac{1}{|B_{\sqrt{t}}(y)|}\int_{B_{\sqrt{t}}(y)} |\varphi^\#(\sqrt{-L})f(x) - e^{tL}\varphi^\#(\sqrt{-L})f(x)|dx\\
&\le \frac{1}{|B_{\sqrt{t}}(y)|}\int_{B_{\sqrt{t}}(y)} |\varphi^\#(\sqrt{-L})\psi(x)f(x) - e^{tL}\varphi^\#(\sqrt{-L})\psi(x)f(x)|dx\\
&\quad+\frac{1}{|B_{\sqrt{t}}(y)|}\int_{B_{\sqrt{t}}(y)} |\varphi^\#(\sqrt{-L})(1-\psi(x))f(x) - e^{tL}\varphi^\#(\sqrt{-L})(1-\psi(x))f(x)|dx\\
&= \Psi_1 + \Psi_2.
\end{aligned}$$

where ψ is a smooth cut-off function supported in $B_{\sqrt{t+\delta}}(y)$, and $\psi(x) \equiv 1$ on $B_{\sqrt{t}}(y)$. Clearly, by Hölder's inequality,

$$\begin{aligned}
\Psi_1 &= \frac{1}{|B_{\sqrt{t}}(y)|}\int_{B_{\sqrt{t}}(y)} |\varphi^\#(\sqrt{-L})\psi(x)f(x) - e^{tL}\varphi^\#(\sqrt{-L})\psi(x)f(x)|dx\\
&\le \frac{1}{\sqrt{|B_{\sqrt{t}}(y)|}}\|\varphi^\#(\sqrt{-L})\psi f - e^{tL}\varphi^\#(\sqrt{-L})\psi f\|_{L^2}\\
&\lesssim \frac{1}{\sqrt{|B_{\sqrt{t}}(y)|}}\|\varphi^\#(\sqrt{-L})\psi f\|_{L^2} \quad \text{(by contractivity of heat semigroup)}\\
&\lesssim \frac{1}{\sqrt{|B_{\sqrt{t}}(y)|}}\|\psi f\|_{L^2} \le \frac{1}{\sqrt{|B_{\sqrt{t}}(y)|}}\|\psi f\|_{L^\infty}\sqrt{|B_{\sqrt{t+\delta}}(y)|} \lesssim \|\psi f\|_{L^\infty} \le \|f\|_{L^\infty}.
\end{aligned}$$

Also

$$|\varphi^\#(\sqrt{-L})(I-e^{tL})(1-\psi(x))f(x)| \le \int_{M\setminus B_{\sqrt{t}}(x)} |(k^\#(x,z)-k_t(x,z))(1-\psi(z))f(z)|dz \tag{3.6}$$

$$\le \|f\|_{L^\infty}\int_{M\setminus B_{\sqrt{t}}(x)} |k^\#(x,z)-k_t(x,z)|dz \tag{3.7}$$

where k_t is the integral kernel of $\varphi^\#(\sqrt{-L})e^{tL}$. Now, choosing t small,[1] we are done by Lemmas 1.3 and 1.5. □

So, let us see the immediate consequence of Proposition 3.1. By virtue of this, we immediately have L^p-continuity of $\varphi(\sqrt{-L})$ upon application of Theorem 5.6 of [8]. This L^p-continuity result is not new of course. It is established in a more general context in [9]. Reference [20] has a different proof of this same result.

[1] This is the main reason for introducing BMO_L^ϵ instead of just the usual BMO_L.

Note also the $L^\infty - \mathrm{BMO}_L$ result in Theorem 6.2 of [8]. However, the conditions used to prove Theorem 6.2 in [8] are stronger than (1.6). So, in comparison, it can be said that we prove similar results in a more restricted setting, but with less assumptions on the heat semigroup e^{tL}.

It is also a natural question to ask what happens if we adopt the seemingly more natural definition of BMO spaces, as follows:

$$\|f\|_{\mathrm{BMO}^\epsilon} = \sup_{B\in\mathcal{B}} \frac{1}{|B|}\int_B |f(x) - A_t f(x)|dx, \tag{3.8}$$

where $\mathcal{B}$ contains all balls of radius less than or equal to ϵ, B is a ball of radius $\sqrt{t}$, and A_t is the operator whose integral kernel is given by

$$h(t,x,y) = \frac{1}{|B_{\sqrt{t}}(y)|}\chi_{B_{\sqrt{t}}(y)}(x).$$

On calculation, it can be seen that estimates on

$$\int_{d(x,y)\geq\sqrt{t},\, d(y,z)\geq\frac{\sqrt{t}}{2}} |k^\#(x,y) - k^\#(x,z)|dx$$

will imply that

$$\sup_{y\in\overline{M}}\sup_{t\in(0,\varepsilon]}\int_{d(x,y)\geq\sqrt{t}} |k^\#(x,y) - k_t(x,y)|dx \leq C, \tag{3.9}$$

where k_t represents the integral kernel of $\varphi^\#(\sqrt{-L})A_t$. The issue is, here A_t and $\varphi^\#(\sqrt{-L})$ do not necessarily commute.

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Appendix 1: Properties of heat and Poisson semigroups

In this Appendix, we include some essential facts about the semigroups e^{tL} and $e^{-t\sqrt{-L}}$. Henceforth, we will call them heat and Poisson semigroups respectively. Since $-L$ is a nonnegative semi-definite self-adjoint operator, by the Hille–Yosida theorem (see [16], Proposition 6.14), e^{tL} gives a contraction semigroup on $L^2(M)$. Our first lemma is the following

Lemma 4.1

$$\|e^{tL}\|_{\mathcal{L}(L^2,Lip)} \lesssim (1 + t^{-n/4-1/2}), \quad t > 0. \tag{4.1}$$

Proof We will first use the gradient estimate (1.7) to prove that

$$\int_M |\nabla_x p(t,x,y)|^2 dy \lesssim t^{-n/2-1}, \quad t\in(0,1]. \tag{4.2}$$

Using (1.7), we see that

$$\int_M |\nabla_x p(t,x,y)|^2 dy \lesssim t^{-n-1} \int_M e^{-2\kappa d(x,y)^2/t} dy.$$

Now we consider the identity mapping $i\colon (\overline{M}, g) \longrightarrow (\overline{M}, \frac{t}{2\kappa} g)$, where $(\overline{M}, \frac{t}{2\kappa} g)$ denotes the manifold $\overline{M}$ with a scaled metric. That gives,

$$\int_M e^{-2\kappa d(x,y)^2/t} dy = \int_M e^{-d(x,z)^2} |Ji| dz \approx t^{n/2} \int_M e^{-2d(x,z)^2} dz \approx t^{n/2},$$

where Ji denotes the Jacobian of the map i, which finally gives (4.2).

We have, as usual,

$$\nabla_x e^{tL} f(x) = \int_M \nabla_x p(t,x,y) f(y) dy \le \|\nabla_x p(t,x,.)\|_{L^2} \|f\|_{L^2},$$

which gives, by (4.2),

$$\|e^{tL}\|_{\mathcal{L}(L^2, Lip)} \le t^{-n/4-1/2}, \quad t \in (0,1]. \tag{4.3}$$

Now, if $\mathrm{Spec}(-L) \subset [\rho, \infty)$, then $\|e^{tL} f\|_{L^2} \le e^{-t\rho} \|f\|_{L^2}$, which, in conjunction with (4.3) means that for $t > 1$,

$$\begin{aligned}
|\nabla e^{tL} f(x)| &= |\nabla e^{L/2} e^{L/2} e^{(t-1)L} f(x)| \\
&\lesssim \|e^{L/2} e^{(t-1)L} f\|_{L^2} \quad \text{from (4.3)} \\
&\le \|e^{(t-1)L} f\|_{L^2} \quad \text{(contractivity of heat semigroup)} \\
&\le e^{-\rho(t-1)} \|f\|_{L^2} \lesssim \|f\|_{L^2}.
\end{aligned}$$

So, putting (4.3) and the last inequality together, we have

$$\|e^{tL}\|_{\mathcal{L}(L^2, \mathrm{Lip})} \lesssim (1 + t^{-n/4-1/2}), \quad t > 0.$$

□

Similarly, for the Poisson semigroup, we have

Lemma 4.2

$$\|e^{-t\sqrt{-L}}\|_{\mathcal{L}(L^2, L^\infty)} \lesssim (1 + t^{-n/2}), \quad t > 0, \tag{4.4}$$

and

$$\|e^{-t\sqrt{-L}}\|_{\mathcal{L}(L^2, Lip)} \lesssim (1 + t^{-n/2-1}), \quad t > 0. \tag{4.5}$$

Proof As in Lemma 4.1, starting from (1.6), we can establish that

$$\|e^{tL}\|_{\mathcal{L}(L^2, L^\infty)} \lesssim (t^{-n/4} + 1), \quad t > 0. \tag{4.6}$$

Now the estimate on $e^{-t\sqrt{-L}}$ can be obtained from the Subordination identity (see (5.22), Chapter 3 of [24]),

$$e^{-t\sqrt{-L}} \approx \int_0^\infty t e^{-t^2/4s} s^{-3/2} e^{sL} ds. \tag{4.7}$$

This gives,

$$\begin{aligned}\|e^{-t\sqrt{-L}}\|_{\mathcal{L}(L^2,L^\infty)} &\lesssim \int_0^\infty te^{-t^2/4s}s^{-3/2}\|e^{sL}\|_{\mathcal{L}(L^2,L^\infty)}ds\\ &\lesssim \int_0^\infty te^{-t^2/4s}s^{-3/2}(s^{-n/4}+1)ds\\ &\lesssim t\int_0^\infty e^{-t^2/4s}s^{-\frac{6+n}{4}}ds + t\int_0^\infty e^{-t^2/4s}s^{-3/2}ds. \end{aligned} \tag{4.8}$$

Calling the first integral above I_1 and the second one I_2, we get $I_2 \le c_0$, where c_n is a multiple of $\Gamma(\frac{n+1}{2})$ (see [24] for details, particularly pp. 247–248). Similarly,

$$I_1 = t\int_0^\infty e^{-t^2/4s}s^{-\frac{3+n/2}{2}}ds \le c_{n/2}\frac{t}{(t^2)^{\frac{n+2}{4}}} \le c_{n/2}t^{-n/2}.$$

This gives (4.4). Now, when $t \in (0, 1]$, we can write,

$$\begin{aligned}\|\nabla e^{-t\sqrt{-L}}f\|_{L^\infty} = \|\nabla e^{tL}e^{-t\sqrt{-L}}e^{-tL}f\|_{L^\infty} &\lesssim (t^{-n/4-1/2})\|e^{-t\sqrt{-L}}e^{-tL}f\|_{L^2} \quad \text{from (4.3)}\\ &\lesssim (t^{-n/4-1/2})\|e^{-tL}f\|_{L^2} \lesssim (t^{-n/4-1/2})\|f\|_{L^2}.\end{aligned}$$

Lastly, when $t \in [1, \infty)$,

$$\|\nabla e^{-t\sqrt{-L}}f\|_{L^\infty} = \|\nabla e^{-\sqrt{-L}}e^{-(t-1)\sqrt{-L}}f\|_{L^\infty} \lesssim \|e^{-(t-1)\sqrt{-L}}f\|_{L^2} \lesssim \|f\|_{L^2}.$$

This proves the lemma. □

Appendix 2: Finite propagation speed of $\cos t\sqrt{-L}$

In this section we investigate some sufficient criteria for $\cos t\sqrt{-L}$ to have finite propagation speed under special boundary conditions. Namely, we will establish finite speed of propagation for those L which can be written as

$$-L = D^*D + H \tag{5.1}$$

where D is a first order elliptic differential operator with either the generalized Dirichlet or Neumann boundary condition (see (5.3) and (5.4) below), and $H \in L^2(M)$ is nonnegative. To do this, we invoke the so-called Davies–Gaffney estimates:

Definition 5.1 An operator L satisfies the Davies–Gaffney estimates on a manifold M if

$$(e^{tL}u, v) \le e^{-\frac{r^2}{4t}}\|u\|_{L^2}\|v\|_{L^2} \tag{5.2}$$

for all $t > 0$, for all pairs of open subsets U, V of M, supp $u \subset U$, supp $v \subset V$, sections $u \in L^2(U)$, $v \in L^2(V)$ and $r = \text{dist}\,(U, V)$, the metric distance between U and V.

We also recall the following (see [18], Theorem 2).

Lemma 5.2 *For a self-adjoint negative semi-definite operator L on $L^2(M)$, satisfaction of the Davies–Gaffney estimates is equivalent to finite propagation speed property of $\cos t\sqrt{-L}$. Furthermore, it is enough to check the Davies–Gaffney estimates only for open sets U, V which are balls around some points.*

It might be pointed out that this method is eminently suited to establishing finite propagation speed type results particularly when the manifold has boundary or is less "nice" in some other way, as Lemma 5.2 above holds in the great generality of metric measure spaces (X, d, μ), where μ is a Borel measure with respect to the topology defined by d.

Now, let $-L = D^*D+H$, where $D\colon H^1(\overline{M}, E) \to H^2(\overline{M}, F)$ be a first-order differential operator between sections of vector bundles. Assume that the symbol $\sigma_D(x,\xi)\colon E_x \to F_x$ is injective for $x \in \overline{M}$, $\xi \in T_x^*\overline{M}\backslash\{0\}$. Following [20], consider the following generalization of the Dirichlet condition on $\mathcal{D}(D)$:

$$u \in \mathcal{D}(D) \Rightarrow \beta(x)u(x) = 0, \quad \forall x \in \partial M, \tag{5.3}$$

where $\beta(x)$ is an orthogonal projection on E_x for all $x \in \partial M$. We also consider the following generalization of the Neumann boundary condition:

$$u \in \mathcal{D}(D) \Rightarrow \gamma(x)\sigma_D(x,\nu)u(x) = 0, \quad \forall x \in \partial M, \tag{5.4}$$

where $\nu(x)$ is the outward unit normal to ∂M and $\gamma(x)$ is an orthogonal projection on E_x for all $x \in \partial M$.

We first argue that both these boundary conditions have the consequence that

$$\langle \sigma_D(x,\nu)v, w\rangle = 0, \quad \forall x \in \partial M, \tag{5.5}$$

when $v \in \mathcal{D}(D)$, $w \in \mathcal{D}(D^*)$ and v, w are smooth. This is because

$$\int_M (\langle Dv, w\rangle - \langle v, D^*w\rangle)dV = \frac{1}{i}\int_{\partial M} \langle \sigma_D(x,\nu)v, w\rangle dS.$$

Now, $w \in \mathcal{D}(D^*)$ implies that the left hand side in the above equation vanishes. So, for the Dirichlet boundary condition, for smooth v, w we have

$$w \in \mathcal{D}(D^*) \Longrightarrow (I - \beta(x))\sigma_D(x,\nu)^*w(x) = 0, \quad x \in \partial M, \tag{5.6}$$

where ν is the outward unit normal to ∂M. This gives, for smooth v and w,

$$v \in \mathcal{D}(D), w \in \mathcal{D}(D^*) \Longrightarrow \langle \sigma_D(x,\nu)v, w\rangle = 0 \quad \text{on } \partial M.$$

For the Neumann boundary condition, (5.6) will be replaced by

$$w \in \mathcal{D}(D^*) \Longrightarrow (I - \gamma(x))w(x) = 0, \quad x \in \partial M \tag{5.7}$$

with the same conclusion (5.5). With that in place, we can now prove Proposition 1.8.

Proof We first observe that by the Cauchy–Schwarz inequality

$$(e^{tL}u, v) \le \|\chi_V e^{tL}u\|_{L^2}\|v\|_{L^2}, \tag{5.8}$$

where χ_V represents the characteristic function of V. So, to get finite speed of propagation, we want to establish (5.2), which will in turn be implied by

$$\|\chi_V e^{tL}u\|_{L^2} \le e^{-\frac{r^2}{4t}}\|u\|_{L^2}, \tag{5.9}$$

when supp $u \subset U$. Now, let $w = \chi_V e^{tL}u$ and call $\rho = \frac{r}{t}$. Then we have

$$\int_V |w|^2 dx \le e^{-\rho r}\int_V \langle w, w\rangle e^{\varphi(x)}dx \le e^{-\rho r}\int_M \langle e^{tL}u, e^{tL}u\rangle e^{\varphi(x)}dx. \tag{5.10}$$

Let us define

$$E(t) = \int_M \langle e^{tL}u, e^{tL}u\rangle e^{\varphi(x)}dx. \tag{5.11}$$

Differentiating (5.11) with respect to t, we get

$$\begin{aligned}
\frac{1}{2}E'(t) &= \text{Re}\int_M \langle \partial_t e^{tL}u, e^{tL}u\rangle e^{\varphi(x)}dx = \text{Re}\int_M \langle Le^{tL}u, e^{tL}u\rangle e^{\varphi(x)}dx \\
&= -\text{Re}\int_M \langle (D^*D+H)e^{tL}u, e^{tL}u\rangle e^{\varphi(x)}dx \\
&= -\text{Re}\int_M \langle De^{tL}u, D(e^{tL}ue^{\varphi(x)})\rangle dx - \text{Re}\int_M H\langle e^{tL}u, e^{tL}u\rangle e^{\varphi(x)}dx \\
&\quad + \text{Re}\frac{1}{i}\int_{\partial M}\langle \sigma(x,\nu)e^{tL}ue^{\varphi(x)}, De^{tL}u\rangle dS \\
&= -\text{Re}\int_M (\langle De^{tL}u, De^{tL}u\rangle e^{\varphi(x)} + \langle De^{tL}u, [D, e^{\varphi(x)}]e^{tL}u\rangle)dx \\
&\quad - \text{Re}\int_M H\langle e^{tL}u, e^{tL}u\rangle e^{\varphi(x)}dx + \text{Re}\frac{1}{i}\int_{\partial M}\langle \sigma(x,\nu)e^{tL}ue^{\varphi(x)}, De^{tL}u\rangle dS \\
&\le -\text{Re}\int_M (\langle De^{tL}u, De^{tL}u\rangle e^{\varphi(x)} + \langle De^{tL}u, [D, e^{\varphi(x)}]e^{tL}u\rangle)dx,
\end{aligned}$$

using the facts that $H \ge 0$ and that under the Dirichlet or the Neumann boundary condition, the last term $\int_{\partial M}\langle \sigma_D(x,\nu)e^{tL}ue^{\varphi(x)}, Du\rangle dS$ disappears. Now, if we can say that

$$\frac{1}{2}E'(t) \le \frac{\rho^2}{4}\int_M \langle e^{tL}u, e^{tL}u\rangle, e^{\varphi(x)}dx,$$

then we will be in a position to use Gronwall's inequality.

Now what is the condition that allows this? Let us define $P = [D, e^{\varphi}]$. Now we have

$$4(De^{tL}u, Pe^{tL}u) = 4(e^{\varphi/2}De^{tL}u, e^{-\varphi/2}Pe^{tL}u) \le 4\|e^{\varphi/2}De^{tL}u\|_{L^2}^2 + \|e^{-\varphi/2}Pe^{tL}u\|_{L^2}^2.$$

So it seems that the correct condition is to demand that

$$\|e^{-\varphi/2}Pe^{tL}u\|_{L^2}^2 \le \rho^2\|e^{\varphi/2}e^{tL}u\|_{L^2}^2$$

or,

$$\|e^{-\varphi/2}Pv\|_{L^2} \le \rho\|e^{\varphi/2}v\|_{L^2}. \tag{5.12}$$

Heuristically, we can say that a condition like this is expected, as the propagation phenomenon of $\cos t\sqrt{-L}$ will be dictated by the interaction of L, and hence of D with the distance function on $\overline{M}$.

So, now we can say

$$E'(t) \le \rho^2/2E(t). \tag{5.13}$$

This gives, by Gronwall's inequality, $E(t) \le e^{\rho^2 t/2}E(0)$. Plugging everything back, we have from (5.10),

$$\int_V |w|^2dx \le e^{\rho^2 t/2 - \rho r}\|u\|_{L^2}^2.$$

Using $\rho = r/t$, we have

$$\int_V |w|^2 dx \le e^{-\frac{r^2}{2t}} \|u\|_{L^2}^2. \tag{5.14}$$

This proves what we want. □

Remark 5.3 Though (1.21) does not seem to be much of an improvement over (5.2), in many practical situations (1.21) is easier to verify than (5.2). For example, if L is the Laplace Beltrami operator with Dirichlet or Neumann boundary condition, then (1.21) holds trivially, because $|\nabla\varphi(x)| \le \frac{r}{t}$ (as the gradient of the distance function to any set is known as a 1-Lipschitz function), which gives us back the special case of finite propagation speed of $\cos t\sqrt{-\Delta}$. Verifying (5.2) seems to be harder in this case.

Now, we extend the range of H in Proposition 1.8 to $H \in L^2(M)$. Towards that end, pick H_n continuous such that $H_n \longrightarrow H$ and consider $\mathcal{L}_n$ given by $-\mathcal{L}_n = D^*D + H_n$. Let $-\mathcal{L} = D^*D + H$.

We can see that $\mathcal{L}_n(u) \longrightarrow \mathcal{L}(u)$ for $u \in \mathcal{D}(D^*D)$ as $n \longrightarrow \infty$. That means, $\mathcal{L}_n \longrightarrow \mathcal{L}$ in the strong resolvent sense as $n \longrightarrow \infty$ (see [15], Theorem VIII.25(a)). Then, $\cos tx$ and e^{-tx} being bounded continuous functions on $\mathbb{R}$ for all $t > 0$, by Theorem VIII.20(b) of [15], we have $\forall u \in \mathcal{D}(D)$,

$$\cos t\sqrt{-\mathcal{L}_n}u \longrightarrow \cos t\sqrt{-\mathcal{L}}u,$$
$$e^{t\mathcal{L}_n}u \longrightarrow e^{t\mathcal{L}}u.$$

The finite propagation speed of $\cos t\sqrt{-\mathcal{L}}$ now follows by the Davies–Gaffney estimates: if for a fixed pair $U, V \subset \overline{M}$ of open sets, L^2 sections u, v such that supp $u \subset U$, supp $v \subset V$, $r = \text{dist}\,(U, V)$, we have

$$(e^{t\mathcal{L}_n}u, v) \le e^{-\frac{r^2}{4t}} \|u\|_{L^2}\|v\|_{L^2}, \tag{5.15}$$

then in the limit, we must have

$$(e^{t\mathcal{L}}u, v) \le e^{-\frac{r^2}{4t}} \|u\|_{L^2}\|v\|_{L^2}. \tag{5.16}$$

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