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Explicit Hodge decomposition on Riemann surfaces

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Abstract We present a construction of an explicit Hodge decomposition for $\bar{\partial}$-operator on Riemann surfaces.

Keywords $\bar{\partial}$-Operator · Riemann surface · Hodge decomposition

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1 Introduction

Classical Hodge decomposition on the space $Z^{(0,1)}(V) \subset \mathcal{E}^{(0,1)}(V)$ of smooth $\bar{\partial}$-closed $(0,1)$-forms on a smooth algebraic curve $V \subset \mathbb{CP}^n$, $(n \geq 2)$ with metric induced by Fubini-Study metric of $\mathbb{CP}^n$, has the following form

Hodge Theorem [22,24,38,39] *For any form $\phi \in Z^{(0,1)}(V)$ there exists a unique Hodge decomposition:*

$$\phi = \bar{\partial} R_1[\phi] + H_1[\phi], \tag{1.1}$$

where H_1 is the orthogonal projection operator from $Z^{(0,1)}(V)$ onto the subspace $\mathcal{H}^{(0,1)}(V)$ of antiholomorphic $(0,1)$-forms on V, $R_1 = \bar{\partial}^* G_1$, $\bar{\partial}^* : \mathcal{E}^{(0,1)}(V) \to \mathcal{E}^{(0,0)}(V)$, $\bar{\partial}^* = -*\bar{\partial}*$

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is the Hodge dual operator for $\bar{\partial}$, $*$ *is the Hodge operator, and* G_1 *is the Hodge-Green operator for Laplacian* $\triangle = \bar{\partial}\bar{\partial}^* + \bar{\partial}^*\bar{\partial}$ *on* V.

Hodge Theorem was proved by Hodge in [22] using the Fredholm's theory of integral equations. Weyl used his method of orthogonal projection from [38] to correct and simplify the Hodge's proof in [39], and was followed by Kodaira in [24], who also used Weyl's method of orthogonal projection. However, the Hodge Theorem, as it is formulated above and in [3,6,7], is not explicit enough for some applications. This disadvantage was pointed out by Griffiths and Harris in [12, §0.6], where the authors remarked that "the Hilbert space method has the disadvantage of not giving us the Green's operator" in the form of an integral operator with "a beautiful kernel on $M \times M$ with certain singularities along the diagonal".

A specific problem that we have in mind is an explicit solution of the inverse conductivity problem on a bordered Riemann surface, in which the conductivity function has to be reconstructed from the Dirichlet-to-Neumann map on its boundary (see [4,16]), in more general setting going back to. We notice that article [16] of Henkin and Novikov on this subject was motivated by article [17] by the authors of the present article. In [20] we made the first step toward explicit solution of the inverse conductivity problem by constructing an explicit Hodge-type decomposition for $\bar{\partial}$-*closed residual currents* of homogeneity zero on reduced complete intersections in $\mathbb{CP}^n$. In the present article using Theorem 1 from [20] we construct an explicit formula for operator R_1 in (1.1) assuming the knowledge of operator H_1. A problem that is definitely worth considering is the construction of an explicit form of H_1. Our choice of Riemann surfaces is motivated by the application mentioned above, though we consider the generalization of Theorem 1 from [20] to locally complete intersections as another interesting and important task.

The main result of the present article is the construction in Theorem 2 of an explicit Hodge decomposition for $\bar{\partial}$-closed forms on an arbitrary Riemann surface assuming the knowledge of the Hodge projection. This construction is based on a generalized version of Theorem 1 from [20], which is presented in Sect. 2 and covers the case of arbitrary homogeneity. The decompositions in Theorem 1 and in the propositions below are explicit in the sense that they depend only on the equations from (2.1) describing V as a subvariety of $\mathbb{CP}^n$, and are defined by explicit integral operators with singular kernels of the Coleff–Herrera [5] and of the Cauchy–Weil–Leray types [27,37].

Construction of integral formulas on $\mathbb{CP}^n$ with application to complex Radon transform was initiated in [17] and [2]. Application of such formulas to solution of $\bar{\partial}$-equation on singular analytic spaces was initiated in [18] and motivated further work in this direction (see for example [1,9,34]). The development of specific residual formulas in [20] and in Theorem 1 above has a long history going back to Poincaré [31], Leray [27], Grothendieck [13], Herrera-Lieberman [21], Dolbeault [8], and Coleff and Herrera [5]. As it is pointed out in [5], the authors' work was conceived on the one hand as a generalization of the theory of residues of meromorphic forms and of the Grothendieck's theory of residues presented by Hartshorne [14], and on the other hand of the work of Ramis and Ruget [33] on dualizing complex in analytic geometry.

The modern development of the formulas of Cauchy–Weil–Leray type was initiated in [26,28,29,32].

As a preliminary step in the construction of the explicit Hodge decomposition of Theorem 2 we use Theorem 1 and construct in Proposition 3.2 an intermediate explicit Hodge-type decomposition on an arbitrary compact Riemann surface $\mathcal{X}$, not necessarily embeddable into $\mathbb{CP}^2$. According to a classical result (see [25]), going back to Gauss and Riemann, such a surface admits an embedding as an algebraic submanifold in $\mathbb{CP}^3$. This embedding can be composed with a generic projection on $\mathbb{CP}^2$ to produce an immersion into $\mathbb{CP}^2$ with only nodes as singularities (see [12,15]). Then we use the Hodge-type decomposition of Theorem 1 on the image $\mathcal{C}$ of this immersion, which we lift and appropriately modify on $\mathcal{X}$. An important role in this construction is played by Proposition 3.1, in which we establish an isomorphism between the residual cohomologies on $\mathcal{C}$ and the cohomologies of the structural sheaf of $\mathcal{C}$. This proposition might be considered as a step in constructing a Hodge-type decomposition of cohomologies on curves with singularities, following direction of [6,7,13,15]. We notice that the proof of Theorem 2 generalizes verbatim to the case of a nonsingular projective complete intersection.

In Sect. 4 using Theorems 1 and 2 we obtain explicit formulas for solutions of $\bar{\partial}$-equation and present two explicit versions of the *Hodge-Kodaira Vanishing Theorem* for open Riemann surfaces.

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2 Generalized version of Theorem 1 from [20]

Below we present a generalized version of Theorem 1 from [20], which gives a Hodge-type decomposition of residual currents of arbitrary homogeneity on reduced complete intersections in $\mathbb{CP}^n$. Before formulating this version we introduce definitions from [19] and [20].

Let V be a complete intersection subvariety

$$V = \left\{z \in \mathbb{CP}^n : \ P_1(z) = \cdots = P_m(z) = 0\right\} \tag{2.1}$$

of dimension $n - m$ in $\mathbb{CP}^n$ defined by a collection $\{P_k\}_{k=1}^m$ of homogeneous polynomials. Let

$$\left\{U_\alpha = \left\{z \in \mathbb{CP}^n : \ z_\alpha \neq 0\right\}\right\}_{\alpha=0}^n$$

be the standard covering of $\mathbb{CP}^n$, and let

$$\mathbf{F}^{(\alpha)}(z) = \begin{bmatrix} F_1^{(\alpha)}(z) \\ \vdots \\ F_m^{(\alpha)}(z) \end{bmatrix} = \begin{bmatrix} P_1(z)/z_\alpha^{\deg P_1} \\ \vdots \\ P_m(z)/z_\alpha^{\deg P_m} \end{bmatrix}$$

be collections of nonhomogeneous polynomials satisfying

$$\mathbf{F}^{(\alpha)}(z) = A_{\alpha\beta}(z) \cdot \mathbf{F}^{(\beta)}(z) = \begin{bmatrix} \left(z_\beta/z_\alpha\right)^{\deg P_1} & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \left(z_\beta/z_\alpha\right)^{\deg P_m} \end{bmatrix} \cdot \mathbf{F}^{(\beta)}(z)$$

on $U_{\alpha\beta} = U_\alpha \cap U_\beta$.

Following [13] and [15] we consider a line bundle $\mathcal{L}$ on V with transition functions

$$l_{\alpha\beta}(z) = \det A_{\alpha\beta} = \left(\frac{z_\beta}{z_\alpha}\right)^{\sum_{k=1}^m \deg P_k}$$

on $U_{\alpha\beta}$ and the *dualizing bundle* on the complete intersection subvariety V

$$\omega_V^\circ = \omega_{\mathbb{CP}^n} \otimes \mathcal{L}, \tag{2.2}$$

where $\omega_{\mathbb{CP}^n}$ is the canonical bundle on $\mathbb{CP}^n$.

For $q = 1, \dots, n-m$ we denote by $\mathcal{E}^{(n,n-m-q)}(V, \mathcal{L}(-\ell)) = \mathcal{E}^{(0,n-m-q)}\left(V, \omega_V^{\circ}(-\ell)\right)$ the space of C^∞ differential forms of bidegree $(n, n-m-q)$ with coefficients in $\mathcal{L} \otimes \mathcal{O}(-\ell)$, i.e. the space of collections of forms

$$\left\{\gamma_\alpha \in \mathcal{E}^{(n,n-m-q)}(U_\alpha)\right\}_{\alpha=0}^{n}$$

satisfying

$$\gamma_\alpha = l_{\alpha\beta} \cdot \left(\frac{z_\beta}{z_\alpha}\right)^{-\ell} \gamma_\beta + \sum_{k=1}^{m} F_k^{(\alpha)} \cdot \gamma_k^{\alpha\beta} \quad \text{on } U_\alpha \cap U_\beta. \tag{2.3}$$

Then, following [5,19,30] we define residual currents and $\bar\partial$-closed residual currents on V. By a residual current $\phi \in C_R^{(0,q)}(V, \mathcal{O}(\ell))$ of homogeneity ℓ we call a collection $\left\{\Phi_\alpha^{(0,q)}\right\}_{\alpha=0}^{n}$ of C^∞ differential forms satisfying equalities

$$\Phi_\alpha = \left(\frac{z_\beta}{z_\alpha}\right)^{\ell} \Phi_\beta + \sum_{k=1}^{m} F_k^{(\alpha)} \cdot \Omega_k^{(\alpha\beta)} \text{ on } U_\alpha \cap U_\beta, \tag{2.4}$$

acting on $\gamma \in \mathcal{E}^{(n,n-m-q)}(V, \mathcal{L}(-\ell))$ by the formula

$$\langle \phi, \gamma \rangle = \sum_\alpha \int_{U_\alpha} \vartheta_\alpha \gamma_\alpha \wedge \Phi_\alpha \bigwedge_{k=1}^{m} \bar\partial \frac{1}{F_k^{(\alpha)}} \overset{\text{def}}{=} \lim_{t\to 0} \sum_\alpha \int_{T_\alpha^{\epsilon(t)}} \vartheta_\alpha \frac{\gamma_\alpha \wedge \Phi_\alpha}{\prod_{k=1}^m F_k^{(\alpha)}}, \tag{2.5}$$

where $\{\vartheta_\alpha\}_{\alpha=0}^n$ is a partition of unity subordinate to the covering $\{U_\alpha\}_{\alpha=0}^n$, and the limit in the right-hand side of (2.5) is taken along an admissible path in the sense of Coleff–Herrera [5], i.e. an analytic map $\epsilon : [0,1] \to \mathbb{R}^m$ satisfying conditions

$$\begin{cases} \lim_{t\to 0} \epsilon_m(t) = 0, \\ \lim\limits_{t\to 0} \dfrac{\epsilon_j(t)}{\epsilon_{j+1}^l(t)} = 0, \text{ for any } l \in \mathbb{N} \text{ and } j = 1, \dots, m-1, \end{cases} \tag{2.6}$$

and

$$T_\alpha^{\epsilon(t)} = \left\{z \in U_\alpha : \left|F_k^{(\alpha)}(z)\right| = \epsilon_k(t) \text{ for } k = 1, \dots, m\right\}. \tag{2.7}$$

Therefore, a residual current ϕ can be considered as an equivalence class of collections $\left\{\Phi_\alpha^{(0,q)}\right\}$, where two collections represent the same residual current if their actions in (2.5) coincide.

From definition (2.5) we obtain the following definition of $\bar\partial$-operator on residual currents. Namely, we define for a differential form $\psi \in \mathcal{E}^{(n,n-m-q-1)}(V, \mathcal{L}(-\ell))$

$$\langle \bar\partial\phi, \psi \rangle = \lim_{t\to 0} \sum_\alpha \int_{T_\alpha^{\epsilon(t)}} \vartheta_\alpha \frac{\psi_\alpha \wedge \bar\partial \Phi_\alpha}{\prod_{k=1}^m F_k^{(\alpha)}}. \tag{2.8}$$

A residual current ϕ of homogeneity ℓ we call $\bar\partial$-closed $\left(\text{denoted } \phi \in Z_R^{(0,q)}(V, \mathcal{O}(\ell))\right)$ if there exist a representative $\left\{\Phi_\alpha^{(0,q)}\right\}$ and smooth forms $\Omega_k^{(\alpha)}$, such that the following equality holds

$$\bar\partial \Phi_\alpha = \sum_{k=1}^{m} F_k^{(\alpha)} \cdot \Omega_k^{(\alpha)} \text{ on } U_\alpha. \tag{2.9}$$

We notice that from the definitions above it follows that if a residual current ϕ is defined by a $\bar\partial$-closed form, then $\bar\partial\phi = 0$ in the sense of (2.8). The converse, however, is not true in general (for a counterexample, see for instance Remark 12.1 in Andersson–Larkang, arXiv:1703.01861).

Below we present an extended version of Theorem 1 from [20] that is used in this article.

Theorem 1 *Let $V \subset \mathbb{CP}^n$ be a reduced complete intersection subvariety as in (2.1). Then*

(i) *for an arbitrary $\phi \in Z_R^{(0,q)}(V, \mathcal{O}(\ell))$ the following representation holds*

$$\phi = \bar\partial I_q[\phi] + L_q[\phi], \tag{2.10}$$

where $L_q[\phi] = 0$ if $1 \le q < n-m$, $L_{n-m}[\phi] \in Z_R^{(0,n-m)}(V, \mathcal{O}(\ell))$ is defined by formula

$$L_{n-m}[\phi] = \sum_{0\le r\le d-n-1-\ell} C(n,m,d,r) \lim_{t\to 0} \int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar z \cdot \zeta\rangle^r \cdot \frac{\phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$
$$\bigwedge \det\left[\bar z \; \overbrace{Q(\zeta,z)}^{m} \; \overbrace{d\bar z}^{n-m}\right] \wedge \omega(\zeta) \tag{2.11}$$

with $d = \sum_{k=1}^m \deg P_k$, and the current $I_q[\phi] \in C^{(0,q-1)}(V, \mathcal{O}(\ell))$ is defined by formula

$$I_q[\phi] = C(n,q,m) \lim_{t\to 0} \int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$
$$\bigwedge \det\left[\frac{\bar z}{B^*(\zeta,z)} \; \frac{\bar\zeta}{B(\zeta,z)} \; \overbrace{Q(\zeta,z)}^{m} \; \overbrace{\frac{d\bar z}{B^*(\zeta,z)}}^{q-1} \; \overbrace{\frac{d\bar\zeta}{B(\zeta,z)}}^{n-m-q}\right] \wedge \omega(\zeta), \tag{2.12}$$

where the functions $\{Q_k^i(\zeta,z)\}$ for $k = 1,\dots,m$, $i = 0,\dots,n$ satisfy

$$\begin{cases} P_k(\zeta) - P_k(z) = \sum_{i=0}^n Q_k^i(\zeta,z)\cdot(\zeta_i - z_i), \\ Q_k^i(\lambda\zeta,\lambda z) = \lambda^{\deg P_k - 1}\cdot Q_k^i(\zeta,z) \text{ for } \lambda\in\mathbb{C}, \end{cases} \tag{2.13}$$

and

$$B^*(\zeta,z) = \sum_{j=0}^n \bar z_j\cdot(\zeta_j - z_j), \qquad B(\zeta,z) = \sum_{j=0}^n \bar\zeta_j\cdot(\zeta_j - z_j);$$

(ii) *a $\bar\partial$-closed residual current $\phi \in Z_R^{(0,n-m)}(V, \mathcal{O}(\ell))$ is $\bar\partial$-exact, i.e. there exists a current $\psi \in C^{(0,n-m-1)}(V, \mathcal{O}(\ell))$ such that $\phi = \bar\partial\psi$, iff*

$$L_{n-m}[\phi] = 0. \tag{2.14}$$

Remark Current $I_q[\phi]$ in Theorem 1 might not be a residual current in general. Estimates proved in [20] show that it is a residual current on the set of regular points of V, and, therefore, in general, on manifolds. However, $I_q[\phi] \in C^{(0,q-1)}(V, \mathcal{O}(\ell))$, i.e. it is a current on reduced complete intersections, which can be defined with the use of equality

$$I_q[\phi](\bar\partial g) = \bar\partial I_q[\phi](g) = \phi[g] - L[g],$$

(that is equivalent to equality (5.4) in [20]), where the right hand side is represented by residual currents.

We present here a sketch of proof of the theorem. In this sketch we are concerned with extending the validity of all lemmas and propositions comprising the proof of Theorem 1 in [20] to the case of nonzero homogeneity.

In the lemma below we prove that the operators L and I defined in [20] preserve homogeneity, validating equalities (2.6), (2.11) and Proposition 2.3 from [20] in the case of nonzero homogeneity.

Lemma 2.1 *If $\phi \in C_R^{(0,q)}(V, \mathcal{O}(\ell))$ is a residual current defined by a differential form Φ in a neighborhood of V satisfying*

$$\Phi(\lambda\zeta) = \lambda^{\ell} \cdot \Phi(\zeta), \tag{2.15}$$

then for the forms $L_{n-m}[\Phi]$ and $I_q[\Phi]$ defined by operators L_{n-m} and I_q in formulas (3.24) *and* (4.22) *of* [20] *respectively, the following equalities hold*

$$\begin{aligned} L_{n-m}[\Phi](\lambda z) &= \lambda^{\ell} \cdot L_{n-m}[\Phi](z), \\ I_q[\Phi](\lambda z) &= \lambda^{\ell} \cdot I_q[\Phi](z). \end{aligned} \tag{2.16}$$

Proof To prove the preservation of homogeneity for operator L_{n-m} we use the following equality for $|\lambda| = 1$, after changing variables to $\zeta = \lambda w$

$$\int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar{\lambda}\bar{z} \cdot \zeta\rangle^r \cdot \frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)} \bigwedge \det \left[\bar{\lambda}\bar{z}\ \overbrace{Q(\zeta, \lambda z)}^{m}\ \overbrace{\bar{\lambda}d\bar{z}}^{n-m} \right] \wedge \omega(\zeta)$$

$$= \int_{\{|\zeta|=1,\{|P_k(w)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar{\lambda}\bar{z} \cdot \lambda w\rangle^r \cdot \frac{\Phi(\lambda w)}{\prod_{k=1}^m P_k(\lambda w)} \bigwedge \det \left[\bar{\lambda}\bar{z}\ \overbrace{Q(\lambda w, \lambda z)}^{m}\ \overbrace{\bar{\lambda}d\bar{z}}^{n-m} \right] \wedge \omega(\lambda w)$$

$$= \lambda^{\ell-d}\bar{\lambda}^{n-m+1}\lambda^{d-m}\lambda^{n+1} \int_{\{|\zeta|=1,\{|P_k(w)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar{z} \cdot w\rangle^r \cdot \frac{\Phi(w)}{\prod_{k=1}^m P_k(w)}$$

$$\bigwedge \det \left[\bar{\lambda}\bar{z}\ \overbrace{Q(w, z)}^{m}\ \overbrace{d\bar{z}}^{n-m} \right] \wedge \omega(w)$$

$$= \lambda^{\ell} \int_{\{|\zeta|=1,\{|P_k(w)|=\epsilon_k(t)\}_{k=1}^m\}} \langle \bar{z} \cdot w\rangle^r \cdot \frac{\Phi(w)}{\prod_{k=1}^m P_k(w)} \bigwedge \det \left[\bar{\lambda}\bar{z}\ \overbrace{Q(w, z)}^{m}\ \overbrace{d\bar{z}}^{n-m} \right] \wedge \omega(w),$$

where $d = \sum_{k=1}^m \deg P_k$.

Similarly, we obtain equality

$$\int_{\{|\zeta|=1,\{|P_k(w)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\Phi(\lambda w)}{\prod_{k=1}^m P_k(\lambda w)}$$

$$\bigwedge \det \left[\frac{\bar{\lambda}\bar{z}}{B^*(\lambda w, \lambda z)}\ \frac{\bar{\lambda}\bar{w}}{B(\lambda w, \lambda z)}\ \overbrace{Q(\lambda w, \lambda z)}^{m}\ \overbrace{\frac{\bar{\lambda}d\bar{z}}{B^*(\lambda w, \lambda z)}}^{q-1}\ \overbrace{\frac{\bar{\lambda}d\bar{w}}{B(\lambda w, \lambda z)}}^{n-m-q} \right] \wedge \omega(\lambda w)$$

$$= \lambda^{\ell-d}\lambda^{d-m}\bar{\lambda}^{n-m+1}\lambda^{n+1}\int_{\{|\zeta|=1,\{|P_k(w)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\Phi(w)}{\prod_{k=1}^m P_k(w)}$$

$$\bigwedge \det\left[\frac{\bar{z}}{B^*(w,z)}\ \frac{\bar{w}}{B(w,z)}\ \overbrace{Q(w,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(w,z)}}^{q-1}\ \overbrace{\frac{d\bar{w}}{B(w,z)}}^{n-m-q}\right]\wedge\omega(w)$$

$$= \lambda^{\ell}\int_{\{|\zeta|=1,\{|P_k(w)|=\epsilon_k(t)\}_{k=1}^m\}} \frac{\Phi(w)}{\prod_{k=1}^m P_k(w)}$$

$$\bigwedge \det\left[\frac{\bar{z}}{B^*(w,z)}\ \frac{\bar{w}}{B(w,z)}\ \overbrace{Q(w,z)}^{m}\ \overbrace{\frac{d\bar{z}}{B^*(w,z)}}^{q-1}\ \overbrace{\frac{d\bar{w}}{B(w,z)}}^{n-m-q}\right]\wedge\omega(w).$$

□

To include the case of nonzero homogeneity Lemmas 3.2 and 3.3 in [20] have to be reformulated. In particular, Lemma 3.2 has to be replaced by the following Lemma.

Lemma 2.2 *Let $V \subset \mathbb{CP}^n$ be a reduced complete intersection subvariety as in (2.1), let $U \supset V$ be an open neighborhood of V in $\mathbb{CP}^n$, and let $\Phi \in \mathcal{E}_c^{(0,n-m)}(U \cap U_\alpha)$ be a differential form of homogeneity ℓ on $U \cap U_\alpha$ for some $\alpha \in (0,\dots,n)$.*

Then formula

$$\lim_{t\to 0}\int_{\{|\zeta|=\tau,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \langle\bar{z}\cdot\zeta\rangle^r\cdot\frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\wedge\det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]\wedge\omega(\zeta), \quad (2.17)$$

where $\{\epsilon_k(t)\}_{k=1}^m$ is an admissible path, defines a differential form of homogeneity ℓ on U, real analytic with respect to z. If $\Phi(\zeta) = \sum_{k=1}^m F_k^{(\alpha)}(\zeta)\Omega_k(\zeta)$ with $\Omega_k \in \mathcal{E}_c^{(0,n-m)}(U \cap U_\alpha)$, then the limit in (2.17) is equal to zero.

Proof Preservation of homogeneity follows from the first equality in (2.16) in Lemma 2.1. As in Lemma 3.2 in [20] without loss of generality we can consider only the case $\alpha = 0$, i.e. $\Phi \in \mathcal{E}_c^{(0,n-m)}(U \cap \{\zeta_0 \neq 0\})$. Then formula (3.15) in [20] for a form Φ satisfying equality (2.15) has to be replaced by the following formula

$$\int_{\{|\zeta|=\tau,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0+\sum_{j=1}^n \bar{z}_j\cdot w_j\right)^r\cdot\frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$

$$\wedge\det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]\wedge\left(\zeta_0^{n+r}d\zeta_0\right)\bigwedge_{j=1}^n dw_j$$

$$= \int_{\{|\zeta|=\tau,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0+\sum_{j=1}^n \bar{z}_j\cdot w_j\right)^r\cdot\frac{\Phi(\zeta)}{\prod_{k=1}^m P_k(\zeta)}$$

$$\wedge\det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]\wedge\left(\zeta_0^{n+r}d\zeta_0\right)\bigwedge_{j=1}^n dw_j$$

$$= \int_{\{|\zeta|=\tau,\{|F_k(w)|\cdot\chi_k(w)=\epsilon_k(t)\}_{k=1}^m\}} \left(\bar{z}_0+\sum_{j=1}^n \bar{z}_j\cdot w_j\right)^r\left(\zeta_0^{n+r+\ell-\sum_{k=1}^m \deg P_k}d\zeta_0\right)$$

$$\times \frac{\Phi(w)}{\prod_{k=1}^{m} F_k(w)} \wedge \det\left[\bar{z}\ \overbrace{Q(e^{i\phi_0}, w, z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \bigwedge_{j=1}^{n} dw_j$$

$$= i \int_0^{2\pi} e^{i\left(n+r+\ell-\sum_{k=1}^{m} \deg P_k+1\right)\phi_0} d\phi_0 \int_{\{w\in U_0, \{|F_k(w)|\cdot\chi_k(w)=\epsilon_k(t)\}_{k=1}^{m}\}} \left(\bar{z}_0 + \sum_{j=1}^{n} \bar{z}_j \cdot w_j\right)^r$$

$$\times \rho_0(w)^{n+r+\ell-\sum_{k=1}^{m} \deg P_k+1} \cdot \frac{\Phi(w)}{\prod_{k=1}^{m} F_k(w)} \wedge \det\left[\bar{z}\ \overbrace{Q(e^{i\phi_0}, w, z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \bigwedge_{j=1}^{n} dw_j, \quad (2.18)$$

where we used nonhomogeneous coordinates $\{w_i = \zeta_i/\zeta_0\}_{i=1}^{n}$ in the subset $\mathbb{S}^{2n+1}(\tau)\setminus\{\zeta_0 = 0\}$ of the sphere $\mathbb{S}^{2n+1}(\tau)$ of radius τ in $\mathbb{C}^{n+1}$, notation $\zeta_0 = \rho_0(w)\cdot e^{i\phi_0}$ with

$$\rho_0(w) = \frac{\tau}{\sqrt{1 + \sum_{i=1}^{n} |w_i|^2}}$$

on $\mathbb{S}^{2n+1}(\tau)$, nonhomogeneous polynomials

$$F_k(w) = P_k(\zeta)/\zeta_0^{\deg P_k}$$

in $\mathbb{S}^{2n+1}(\tau)\setminus\{\zeta_0 = 0\}$, and equality (2.15).

The last statement of the Lemma follows as in Lemma 3.2 of [20] from application of Theorem 1.7.6(2) in [5] to the interior integral in the right-hand side of (2.18). □

Lemma 3.3 has to be replaced by the following Lemma.

Lemma 2.3 *Let $\phi \in Z_R^{(0,q)}(V, \mathcal{O}(\ell))$ be a $\bar\partial$-closed residual current defined by a collection of forms $\left\{\Phi_\alpha^{(0,n-m)}\right\}_{\alpha=0}^{n}$ of homogeneity ℓ on a neighborhood U of the reduced subvariety V as in (2.1) satisfying (2.4) and (2.9), and let $\Phi(\zeta) = \sum_{\alpha=0}^{n} \vartheta_\alpha(\zeta)\Phi_\alpha(\zeta)$ be a differential form of homogeneity ℓ on U.*

Then for an arbitrary $\gamma \in \mathcal{E}^{(n,0)}(V, \mathcal{L}(-\ell))$ the equality

$$\lim_{\tau\to 0} \int_{T_\beta^{\delta(\tau)}} \frac{\gamma(z)}{\prod_{k=1}^{m} F_k^{(\beta)}(z)} \bigwedge \left(\lim_{t\to 0} \int_{\{|\zeta|=1, \{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^{m}\}} \langle \bar{z}\cdot\zeta\rangle^r \right.$$

$$\left. \times \frac{\Phi(\zeta)}{\prod_{k=1}^{m} P_k(\zeta)} \wedge \det\left[\bar{z}\ \overbrace{Q(\zeta, z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right] \wedge \omega(\zeta)\right) = 0 \quad (2.19)$$

holds unless

$$r \le \sum_{k=1}^{m} \deg P_k - \ell - n - 1. \quad (2.20)$$

Proof We notice that for all values of $a > 0$ and $\epsilon = (\epsilon_1, \ldots, \epsilon_k)$ the sets

$$S(a) = \left\{\zeta \in \mathbb{S}^{2n+1}(a) : \left\{|P_k(\zeta)| = \epsilon_k \cdot a^{\deg P_k}\right\}_{k=1}^{m}\right\}$$

are real analytic subvarieties of $\mathbb{S}^{2n+1}(a)$ of real dimension $2n+1-m$ satisfying

$$c\cdot a^{2n+1-m}\cdot \text{Volume}\,(S(1)) < \text{Volume}_{2n+1-m}\,(S(a)) < C\cdot a^{2n+1-m}\cdot \text{Volume}\,(S(1))\,. \quad (2.21)$$

We denote

$$\Phi_\alpha(\zeta,z)=\langle\bar{z}\cdot\zeta\rangle^r\cdot\frac{\Phi_\alpha(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]\wedge\omega(\zeta),$$

and apply the Stokes' formula to the differential form

$$\beta(\zeta,z)=\sum_{\alpha=0}^n\vartheta_\alpha(\zeta)\Phi_\alpha(\zeta,z)=\sum_{\alpha=0}^n\langle\bar{z}\cdot\zeta\rangle^r\cdot\frac{\vartheta_\alpha(\zeta)\Phi_\alpha(\zeta)}{\prod_{k=1}^m P_k(\zeta)}\det\left[\bar{z}\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar{z}}^{n-m}\right]\wedge\omega(\zeta) \tag{2.22}$$

on the variety

$$\left\{\zeta\in\mathbb{C}^{n+1}:\left\{|P_k(\zeta)|=\epsilon_k\cdot|\zeta|^{\deg P_k}\right\}_{k=1}^m,\ a<|\zeta|<1\right\}$$

with the boundary

$$\left\{\zeta:|\zeta|=a,\ \left\{|P_k(\zeta)|=\epsilon_k\cdot a^{\deg P_k}\right\}_{k=1}^m\right\}\bigcup\left\{\zeta:|\zeta|=1,\ \{|P_k(\zeta)|=\epsilon_k\}_{k=1}^m\right\}.$$

Then using equality (2.9) we obtain the equality

$$\begin{aligned}&\int_{\{|\zeta|=1,\{|P_k(\zeta)|=\epsilon_k(t)\}_{k=1}^m\}}\beta(\zeta,z)-\int_{\left\{|\zeta|=a,\ \{|P_k(\zeta)|=\epsilon_k(t)\cdot a^{\deg P_k}\}_{k=1}^m\right\}}\beta(\zeta,z)\\&=\sum_{\alpha=0}^n\sum_{j=1}^m\int_a^1 d\tau\int_{\left\{|\zeta|=\tau,\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\}_{k=1}^m\right\}}F_j^{(\alpha)}(\zeta)\cdot\beta_j^{(\alpha)}(\zeta,z)\\&\quad+\sum_{\alpha=0}^n\int_a^1 d\tau\int_{\left\{|\zeta|=\tau,\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\}_{k=1}^m\right\}}d|\zeta|\rfloor\left(\bar\partial\vartheta_\alpha(\zeta)\wedge\Phi_\alpha(\zeta,z)\right)\end{aligned} \tag{2.23}$$

for arbitrary t and $0<a<1$.

Using estimate (2.21) and the homogeneity property

$$\Phi_\beta(t\cdot\zeta)=\bar{t}^{-(n-m)}\cdot t^\ell\cdot\Phi_\beta(\zeta) \tag{2.24}$$

of the coefficients of $\Phi^{(0,n-m)}$ from Proposition 1.1 in [17] we obtain that if

$$r+n+1+\ell-\sum_{k=1}^m\deg P_k>0, \tag{2.25}$$

then

$$\left|\int_{\left\{|\zeta|=a,\ \{|P_k(\zeta)|=\epsilon_k(t)\cdot a^{\deg P_k}\}_{k=1}^m\right\}}\beta(\zeta,z)\right|<C'(\epsilon)\cdot a^{r+2n+1-m-(n-m)+\ell-\sum_{k=1}^m\deg P_k}\longrightarrow 0$$

as t is fixed and $a\to 0$.

For the first sum of integrals in the right-hand side of (2.23) we have

$$\begin{aligned}&\left|\int_a^1 d\tau\int_{\left\{|\zeta|=\tau,\ \{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\}_{k=1}^m\right\}}F_j^{(\alpha)}(\zeta)\cdot\beta_j^{(\alpha)}(\zeta,z)\right|\\&<C\int_a^1 d\tau\cdot\tau^{r+2n+1-m-(n-m)+\ell-\sum_{k=1}^m\deg P_k}\end{aligned}$$

$$\times \int_{\left\{|\zeta|=\tau,\ \left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}} F_j^{(\alpha)}(\zeta)\cdot\left(d|\zeta|\rfloor\beta_j^{(\alpha)}(\zeta,z)\right)$$

$$< C'(\epsilon)\frac{\left(1-a^{r+n+2+\ell-\sum_{k=1}^m \deg P_k}\right)}{r+n+2+\ell-\sum_{k=1}^m \deg P_k} \longrightarrow 0 \tag{2.26}$$

as $t\to 0$, since $a<1$, condition (2.25) is satisfied, and $C'(\epsilon)\to 0$ as $t\to 0$ by Lemma 2.2.

For the second sum of integrals in the right-hand side of (2.23) we use equality

$$\begin{aligned}
&\lim_{t\to 0}\sum_{\{\beta:\ U_\beta\cap U_\alpha\neq\emptyset\}}\int_a^1 d\tau\int_{\left\{\zeta_\alpha\neq 0,|\zeta|=\tau,\left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}} d|\zeta|\rfloor\left(\bar\partial\vartheta_\beta(\zeta)\wedge\Phi_\beta(\zeta,z)\right)\\
&=\lim_{t\to 0}\int_a^1 d\tau\sum_{\{\beta:\ U_\beta\cap U_\alpha\neq\emptyset\}}\int_{\left\{\zeta_\alpha\neq 0,|\zeta|=\tau,\left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}}\langle\bar z\cdot\zeta\rangle^r\\
&\times d|\zeta|\rfloor\left(\bar\partial\vartheta_\beta(\zeta)\wedge\frac{\Phi_\beta(\zeta)}{\left[\prod_{k=1}^m P_k(\zeta)\right]\big|_\beta}\bigwedge\det\left[\bar z\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar z}^{n-m}\right]\wedge\omega(\zeta)\right)\\
&=\lim_{t\to 0}\int_a^1 d\tau\sum_{\{\beta:\ U_\beta\cap U_\alpha\neq\emptyset\}}\int_{\left\{\zeta_\alpha\neq 0,|\zeta|=\tau,\left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}}\langle\bar z\cdot\zeta\rangle^r\\
&\times d|\zeta|\rfloor\left(\bar\partial\vartheta_\beta(\zeta)\big|_{U_\alpha}\wedge\frac{\Phi_\alpha(\zeta)}{\left[\prod_{k=1}^m P_k(\zeta)\right]\big|_\alpha}\wedge\det\left[\bar z\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar z}^{n-m}\right]\wedge\omega(\zeta)\right)\\
&=\lim_{t\to 0}\int_a^1 d\tau\int_{\left\{\zeta_\alpha\neq 0,|\zeta|=\tau,\left\{|P_k(\zeta)|=\epsilon_k(t)\cdot\tau^{\deg P_k}\right\}_{k=1}^m\right\}}\langle\bar z\cdot\zeta\rangle^r\\
&\times d|\zeta|\rfloor\left(\left[\sum_{\{\beta:\ U_\beta\cap U_\alpha\neq\emptyset\}}\bar\partial\vartheta_\beta(\zeta)\big|_{U_\alpha}\right]\wedge\frac{\Phi_\alpha(\zeta)}{\left[\prod_{k=1}^m P_k(\zeta)\right]\big|_\alpha}\right.\\
&\left.\bigwedge\det\left[\bar z\ \overbrace{Q(\zeta,z)}^{m}\ \overbrace{d\bar z}^{n-m}\right]\wedge\omega(\zeta)\right)=0.
\end{aligned}$$

which follows from equality (2.4), Lemma 2.2, and the transformation formula for the Grothendieck's residue from Proposition 4.2 in [17] (for isolated singularities see [12,36]). □

This completes the sketch of proof of Theorem 1.

Below we formulate a corollary of Theorem 1, which will be used in what follows. This corollary gives an explicit form of the *Hodge-Kodaira Vanishing Theorem* for projective complete intersections (see III.7.15 in [15] and §2.1 of [12].

Corollary 2.4 *If homogeneity ℓ satisfies inequality*

$$\ell > d-n-1, \tag{2.27}$$

then the operator in the right-hand side of equality 2.11 is zero,

$$\dim H_R^{n-m}(V,\mathcal{O}(\ell)) = 0, \tag{2.28}$$

and for any $\phi \in Z_R^{(0,n-m)}(V, \mathcal{O}(\ell))$

$$\phi = \bar{\partial} I_{n-m}[\phi]. \tag{2.29}$$

3 Explicit Hodge decomposition on nonsingular algebraic curves.

In order to construct an explicit Hodge decomposition on an arbitrary nonsingular algebraic curve $\mathcal{X}$ we use the existence of an immersion (see [25], 1.4 in [12], IV.3 in [15])

$$\mathcal{X} \stackrel{\varrho}{\to} \mathbb{CP}^2, \tag{3.1}$$

such that $\mathcal{C} = \varrho(\mathcal{X})$ is a plane curve with at most nodes as singularities. Our construction of the sought decomposition on $\mathcal{X}$ will be based on a similar decomposition on $\mathcal{C}$, which exists according to Theorem 1.

We construct a linear map $\varrho_* : \mathcal{E}^{(0,1)}(\mathcal{X}) \to Z_R^{(0,1)}(\mathcal{C})$. Here and below we use notations ϱ_* and ϱ^* for induced direct and respectively inverse maps on functions and differential forms. Since ϱ is biholomorphic everywhere outside of nodal points, we have to describe ϱ_* only in the neighborhoods of those points. Let $p \in \mathcal{C}$ be such nodal point with $p_1, p_2 \in \mathcal{X}$ such that $\varrho(p_1) = \varrho(p_2) = p$. Let z_1, z_2 be local coordinates at $p \in U \subset \mathbb{CP}^2$, such that $z_1(p) = z_2(p) = 0$, and such that

$$\mathcal{C} \cap U = \mathcal{C}_1 \cup \mathcal{C}_2,$$

where $\mathcal{C}_1 = \{q \in U : z_1(q) = 0\}$, and $\mathcal{C}_2 = \{q \in U : z_2(q) = 0\}$.

Let $\phi \in \mathcal{E}^{(0,1)}(\mathcal{X})$ be a smooth $\bar{\partial}$-closed form on $\mathcal{X}$ with local representations

$$\phi\Big|_{p_1} = \phi_1(z)d\bar{z}_1, \qquad \phi\Big|_{p_2} = \phi_2(z)d\bar{z}_2$$

on $\mathcal{X}$. Let $\widetilde{\phi}_1, \widetilde{\phi}_2$ be extensions of ϕ_1 and ϕ_2 to U such that

$$\widetilde{\phi}_j\Big|_{\mathcal{C}_j} = \phi_j, \quad \bar{\partial}_{z_k}\widetilde{\phi}_j = 0 \quad \text{for } k \neq j.$$

Then the differential form

$$\phi_* = \widetilde{\phi}_1 d\bar{z}_1 + \widetilde{\phi}_2 d\bar{z}_2 \tag{3.2}$$

defines a $\bar{\partial}$-closed residual current on $\mathcal{C}$ in the neighborhood $U \ni p$. Current ϕ_* is $\bar{\partial}$-exact in U since for the functions $\widetilde{\psi}_1$ and $\widetilde{\psi}_2$ chosen so that

$$\bar{\partial}_{z_j}\widetilde{\psi}_j = \widetilde{\phi}_j, \quad \bar{\partial}_{z_k}\widetilde{\psi}_j = 0 \quad \text{for } k \neq j,$$

we will have

$$\bar{\partial}\left(\widetilde{\psi}_1 + \widetilde{\psi}_2\right) = \widetilde{\phi}_1 d\bar{z}_1 + \widetilde{\phi}_2 d\bar{z}_2,$$

and

$$(\phi_*)^* = \phi.$$

In the proposition below we identify the spaces of residual cohomologies $H_R^{(0,1)}(\mathcal{C})$ of curve $\mathcal{C}$ and the cohomologies $H^1(\mathcal{C}, \mathcal{O})$ of the structural sheaf $\mathcal{O}$ on $\mathcal{C}$.

Proposition 3.1 (compare with [13,15]) (Isomorphism of cohomologies) *Let*

$$\mathcal{C} = \left\{z \in \mathbb{CP}^2 : P(z) = 0\right\}$$

be a curve in $\mathbb{CP}^2$ with nodal points as the only singularities. Then there exists an isomorphism

$$\iota : H_R^{(0,1)}(\mathcal{C}) \to H^1(\mathcal{C}, \mathcal{O}). \tag{3.3}$$

Proof To define ι we consider an arbitrary $\phi \in Z_R^1(\mathcal{C}, \mathcal{O})$ represented locally on a cover $\{U_i\}_{i=1}^N$ by residual currents

$$\left\{ \frac{\Phi_i^{(0,1)}}{F^{(i)}} \right\}_{i=1}^N$$

satisfying $\bar\partial \Phi_i = F^{(i)} \Omega_i^{(0,2)}$. Then, using existence of $\Psi_i^{(0,1)} \in \mathcal{E}^{(0,1)}(U_i)$ such that

$$\bar\partial \Psi_i^{(0,1)} = \Omega_i^{(0,2)}$$

we obtain

$$\bar\partial \left(\Phi_i - F^{(i)} \Psi_i \right) = 0 \quad \text{in } U_i,$$

and therefore

$$\Phi_i = \bar\partial \Theta_i + F^{(i)} \Psi_i$$

for some $\Theta_i \in \mathcal{E}^{(0,0)}(U_i)$.

From the last equality we obtain that

$$\bar\partial \left(\Theta_i - \Theta_j \right) = F_i \Omega_{ij} \quad \text{on } U_{ij} = U_i \cap U_j,$$

and therefore, by defining

$$\iota \{\Phi_i\} = \left\{ \Theta_i - \Theta_j \right\} \tag{3.4}$$

we obtain a cocycle in $Z^1(\mathcal{C}, \mathcal{O})$.

To construct the inverse to ι we take $\left\{\Theta_{ij}\right\}$—a cocycle in $Z^1(\mathcal{C}, \mathcal{O})$. We consider a partition of unity $\{\vartheta_i\}_{i=1}^N$ subordinate to some open cover $\{U_i\}_{i=1}^N$ of an open neighborhood $U \supset V$. Without loss of generality we may assume that $\left\{\Theta_{ij}\right\}$ are restrictions of holomorphic functions on U_{ij}. We define the cochain

$$\Theta_i = \sum_{k \neq i,\ U_k \cap U_i \neq \emptyset} \vartheta_k \Theta_{ik},$$

and notice that on U_{ij} we have the equality

$$\begin{aligned} \Theta_i - \Theta_j &= \sum_{k \neq i,\ U_k \cap U_i \neq \emptyset} \vartheta_k \Theta_{ik} - \sum_{k \neq j,\ U_k \cap U_j \neq \emptyset} \vartheta_k \Theta_{jk} \\ &= \sum_{k \neq i,j\ U_k \cap U_i \neq \emptyset} \vartheta_k \left(\Theta_{ik} - \Theta_{jk} \right) + \vartheta_j \Theta_{ij} - \vartheta_i \Theta_{ji} \\ &= \sum_{k \neq i,j\ U_k \cap U_i \neq \emptyset} \vartheta_k \Theta_{ij} + \vartheta_j \Theta_{ij} + \vartheta_i \Theta_{ij} + F_i \Omega_{ij} = \Theta_{ij} + F_i \Omega_{ij} \end{aligned}$$

with some functions $\Omega_{ij} \in \mathcal{E}^{(0,0)}(U_{ij})$, and its corollary

$$\bar\partial \left(\Theta_i - \Theta_j \right) = F_i \bar\partial \Omega_{ij}. \tag{3.5}$$

Therefore, the collection

$$\left\{ \Phi_i = \bar{\partial}\Theta_i = \sum_{k \neq i,\ U_k \cap U_i \neq \emptyset} \Theta_{ik} \bar{\partial} \vartheta_k \right\}_{i=1}^{N} \in Z_R^{(0,1)}(\mathcal{C}) \tag{3.6}$$

defines a $\bar{\partial}$-closed residual current in a neighborhood of V.

If we apply the map $\imath$ from (3.4) to the current in (3.6), then using equality (3.5) we obtain

$$\imath\{\Phi_i\} = \left\{\Theta_i - \Theta_j\right\} = \left\{\Theta_{ij}\right\} \in Z^1(\mathcal{C}, \mathcal{O}).$$

□

Let now $\left\{p^{(i)}\right\}_{i=1}^{r}$ be the nodal points in $\mathcal{C}$, and let points $p_1^{(i)}, p_2^{(i)} \in \mathcal{X}$ be such that $\varrho(p_1^{(i)}) = \varrho(p_2^{(i)}) = p^{(i)} \in \mathcal{C}$. We consider a collection of paths $\{\gamma_i\}_{i=1}^{r}$ in $\mathcal{X}$ such that $\gamma_i(0) = p_1^{(i)}$ and $\gamma_i(1) = p_2^{(i)}$ and functions $\{f_i\}_{i=1}^{r} \in \mathcal{E}(\mathcal{X})$ with supports in some neighborhoods $U_i \supset \gamma_i$ such that

$$\begin{cases} f_i(p_1^{(i)}) = 0, \ \ f_i(p_2^{(i)}) = 1, \\ \partial f_i = 0 \text{ in } V_i \Subset U_i. \end{cases} \tag{3.7}$$

Then for a path $\gamma_i \subset \mathcal{X}$ we have

$$\int_{\gamma_i} \bar{\partial} f_i = f_i(p_2^{(i)}) - f_i(p_1^{(i)}) = 1.$$

For an arbitrary form $\phi \in \mathcal{E}^{(0,1)}(\mathcal{X})$ and the corresponding residual current ϕ_* on $\mathcal{C}$ we consider the decomposition on $\mathcal{C}$ that follows from Theorem 1:

$$\phi_* = \bar{\partial} I[\phi_*] + L[\phi_*],$$

and the lift of this decomposition on $\mathcal{X}$:

$$\phi = (\phi_*)^* = \bar{\partial}(I[\phi_*])^* + (L[\phi_*])^*. \tag{3.8}$$

From Proposition 4.4 in [20] it follows that if $\phi \in \mathcal{E}^{(0,1)}(\mathcal{X})$, then $I[\phi_*] \in \mathcal{E}(\mathcal{C})$, and therefore

$$(I[\phi_*])^* \in \mathcal{E}(\mathcal{X}).$$

We consider the scalar product on $\mathcal{E}^{(0,1)}(\mathcal{X})$

$$\langle \phi, \psi \rangle = \int_{\mathcal{X}} \phi \wedge \overline{\psi} \tag{3.9}$$

and assume without loss of generality that the collection of forms $\left\{(L[(\bar{\partial} f_i)_*])^*\right\}_{i=1}^{r}$ is orthonormalized with respect to scalar product in (3.9). Then for $\phi \in \mathcal{E}^{(0,1)}(\mathcal{X})$ we define for $i = 1, \ldots, r$,

$$a_i[\phi] = \int_{\mathcal{X}} \phi \wedge \overline{(L[(\bar{\partial} f_i)_*])^*}, \tag{3.10}$$

and consider operators $\mathcal{L} : \mathcal{E}^{(0,1)}(\mathcal{X}) \to \mathcal{E}^{(0,1)}(\mathcal{X})$ and $\mathcal{I} : \mathcal{E}^{(0,1)}(\mathcal{X}) \to \mathcal{E}(\mathcal{X})$, defined as

$$\mathcal{L}[\phi] = (L[\phi_*])^* - \sum_{i=1}^{r} a_i[\phi](L[(\bar{\partial} f_i)_*])^*, \tag{3.11}$$

and

$$\mathcal{I}[\phi] = \left(I\left[\phi_* - \sum_{i=1}^{r} a_i[\phi](\bar\partial f_i)_*\right]\right)^* + \sum_{i=1}^{r} a_i[\phi] f_i. \tag{3.12}$$

Proposition 3.2 (Hodge-type decomposition on a nonsingular curve) *Let $\mathcal{X}$ be a nonsingular curve, and let $\mathcal{X} \overset{\varrho}{\to} \mathbb{CP}^2$ be an immersion of $\mathcal{X}$ into $\mathbb{CP}^2$ with r nodal points. Let $\{f_i\}_{i=1}^r$ satisfy (3.7) and let operators $\mathcal{L}$ and $\mathcal{I}$ be defined respectively in (3.11) and (3.12).*

Then for the space of $(0,1)$-forms $\mathcal{E}^{(0,1)}(\mathcal{X})$

(i) *the following decomposition holds*

$$\phi = \bar\partial\mathcal{I}[\phi] + \mathcal{L}[\phi], \tag{3.13}$$

(ii) *a $(0,1)$-form $\phi \in \mathcal{E}^{(0,1)}(\mathcal{X})$ is $\bar\partial$-exact, iff $\mathcal{L}[\phi] = 0$.*

Proof To prove equality (3.13) we use equality (3.8) to obtain for an arbitrary $\bar\partial$-closed form $\phi \in \mathcal{E}^{(0,1)}(\mathcal{X})$ the equality

$$\phi - \sum_{i=1}^{r} a_i[\phi]\bar\partial f_i = \bar\partial\left(I\left[\phi_* - \sum_{i=1}^{r} a_i[\phi](\bar\partial f_i)_*\right]\right)^* + \left(L\left[\phi_* - \sum_{i=1}^{r} a_i[\phi](\bar\partial f_i)_*\right]\right)^*,$$

which we can rewrite as

$$\phi = \bar\partial\left[\left(I\left[\phi_* - \sum_{i=1}^{r} a_i[\phi](\bar\partial f_i)_*\right]\right)^* + \sum_{i=1}^{r} a_i[\phi] f_i\right] + \mathcal{L}[\phi] = \bar\partial\mathcal{I}[\phi] + \mathcal{L}[\phi].$$

According to Theorem 1 and Proposition 3.1 the image of L^* is a subspace of the space of $\bar\partial$-closed forms—$\mathcal{E}^{(0,1)}(\mathcal{X})$ of dimension $\dim H^1_R(\mathcal{C}, \mathcal{O}) = p_a(\mathcal{C})$—the arithmetic genus of $\mathcal{C}$. We notice that for any collection $\{c_i\}_{i=1}^r$ there is no $g \in \mathcal{E}(\mathcal{C})$ such that

$$\bar\partial g^* = \sum_{i=1}^{r} c_i \bar\partial f_i,$$

because otherwise we would have

$$g^* - \sum_{i=1}^{r} c_i f_i = \text{const},$$

which contradicts g^* taking the same values at $p_1^{(i)}$ and $p_2^{(i)}$ for all $i = 1, \ldots, r$. Therefore,

$$\dim \operatorname{Ker}\mathcal{L} = \dim \operatorname{Span}\left\{(L[(\bar\partial f_i)_*])^*\right\} = r,$$

and

$$\dim \operatorname{Im}\{\mathcal{L}\} = \dim \operatorname{Im}\left\{L^*\right\} - r = p_a(\mathcal{C}) - r.$$

To prove item (ii) of Proposition 3.2 we consider a $\bar\partial$-exact form $\phi = \bar\partial g$ for $g \in \mathcal{E}(\mathcal{X})$ and assume that

$$\left(L\left[(\bar\partial g)_* - \sum_{i=1}^{r} a_i[\bar\partial g](\bar\partial f_i)_*\right]\right)^* \neq 0.$$

Then from the inequality above it follows that

$$\dim\left\{\mathcal{L}\left\{\bar\partial g, g \in \mathcal{E}(\mathcal{X})\right\}\right\} > r,$$

which, according to Proposition 3.1, contradicts the statement from IV.1 in [15], that

$$\dim H^{(0,1)}(\mathcal{X}) = p_a(\mathcal{C}) - r.$$

□

Using Theorem 1 and Proposition 3.2 we prove the following version of the Hodge Theorem for smooth algebraic curves, which gives a formula for solution $R_1[\phi]$ of equation

$$\bar{\partial} R_1[\phi] = \phi - H_1[\phi]$$

in terms of explicit integral operator $\mathcal{I}$ and the Hodge projection H_1. In a sense it addresses the deficiency of implicit definition of R_1 mentioned in Introduction.

Theorem 2 (Explicit formulas in the Hodge decomposition) *Let $\mathcal{X}$ be a smooth algebraic curve, let $\mathcal{L}$ and $\mathcal{I}$ be operators from* (3.11) *and* (3.12) *respectively, and let $\left\{\omega_j\right\}_{j=1}^{g}$ be an orthonormal basis of holomorphic* $(1,0)$*-forms on $\mathcal{X}$, i.e.*

$$\int_{\mathcal{X}} \omega_j \wedge \bar{\omega}_k = \delta_{jk}, \;\; j,k = 1,\dots,g.$$

Then Hodge operators H_1 and R_1 in decomposition (1.1) *admit the following representations*

$$H_1[\phi] = \sum_{j=1}^{g} \left(\int_{\mathcal{C}} \phi \wedge \omega_j \right) \bar{\omega}_j,$$
$$R_1[\phi] = \mathcal{I}\big[\phi - H_1[\phi]\big] + const. \tag{3.14}$$

Proof From decomposition (1.1) we obtain that the form $\phi - H_1[\phi]$ is exact, and therefore, using item (ii) from Theorem 1 we obtain equality

$$\bar{\partial} R_1[\phi] - \bar{\partial}\mathcal{I}\big[\phi - H_1[\phi]\big] = 0,$$

which implies the second equality in (3.14). □

4 Explicit solution of $\bar{\partial}$-equation on affine curves.

In this section we prove solvability of the $\bar{\partial}$-equation on affine smooth algebraic curves. Let $\mathcal{X}$ be a nonsingular algebraic curve, and let $\varrho : \mathcal{X} \to \mathbb{CP}^2$ be an immersion of $\mathcal{X}$ as in (3.1) such that

$$\mathcal{C} = \varrho(\mathcal{X}) = \left\{ z \in \mathbb{CP}^2 : P(z) = 0 \right\} \tag{4.1}$$

is a plane curve of deg $P = d$ with at most nodes as singularities. Without loss of generality we may assume that the intersection of $\mathcal{C}$ with the line at infinity

$$\mathcal{C} \cap \left\{ (z_0, z_1, z_2) \in \mathbb{CP}^2 : z_0 = 0 \right\} = \left\{ z^{(1)}, \dots, z^{(r)} \right\} \tag{4.2}$$

consists of r points with multiplicities $m_1, \dots, m_r$ such that $\sum_{i=1}^{r} m_i = d$. We denote

$$\mathring{\mathcal{C}} = \mathcal{C} \setminus \left\{ z^{(1)}, \dots, z^{(r)} \right\},$$

and

$$\mathring{\mathcal{X}} = \mathcal{X} \setminus \left\{ \varrho^{-1}(z^{(1)}), \dots, \varrho^{-1}(z^{(r)}) \right\}.$$

Proposition 4.1 (Hodge-Kodaira Vanishing Theorem for affine curves) *Let $\mathcal{X}$ and $\mathcal{C} \subset \mathbb{CP}^2$ be curves as in (4.1), with $\mathcal{C}$ of degree d satisfying condition (4.2). Let $\phi^{(0,1)} \in \mathcal{E}(\mathring{\mathcal{X}})$ be a form on $\mathring{\mathcal{X}}$, and ϕ_* be its direct image on $\mathring{\mathcal{C}}$. If for some ℓ satisfying $\ell > d-3$ the form $\zeta_0^\ell \phi_*(\zeta)$ admits an extension $\psi(\zeta) \in C^{(0,1)}(\mathcal{C})$, then there exists a function $g \in \mathcal{E}(\mathring{\mathcal{X}})$ such that*

$$\begin{cases} \bar{\partial} g = \phi, \\ |g(\zeta)| \leq C|\zeta_0|^{-\ell}. \end{cases} \tag{4.3}$$

Proof Let $z^{(j)} \in \mathcal{C} \cap \left\{\mathbb{CP}^2 : z_0 = 0\right\}$ be one of the points of $\mathcal{C}$ at infinity. We consider a neighborhood $U^{(j)}$ of $z^{(j)}$ in $\mathbb{C}^2$ with coordinates ζ_1, ζ_2 such that

$$\mathcal{C} \cap U^{(j)} = \left\{\zeta \in U^{(j)} : \zeta_2 = 0\right\}.$$

Using Cauchy–Green formula we solve the $\bar{\partial}$-equation on $\mathcal{C} \cap U^{(j)}$ and obtain a function

$$g^{(j)}(\zeta_1) \in C(\mathcal{C}) \cap C^\infty(\mathring{\mathcal{C}})$$

with compact support in $\mathcal{C} \cap U^{(j)}$ satisfying equality

$$\bar{\partial} g^{(j)} = \psi\big|_{\mathcal{C} \cap V^{(j)}},$$

where $V^{(j)} \Subset U^{(j)}$.

Then the form

$$\tilde{\psi} = \psi - \sum_{j=1}^{d} \bar{\partial} g^{(j)}$$

defines a $\bar{\partial}$-closed residual current on $\mathcal{C}$ of degree $\ell > d-3$, and therefore, using Corollary 2.4 we obtain the existence of a function $\tilde{g} = I_1[\zeta_0^\ell \cdot \tilde{\psi}(\zeta)] \in C^{(0,0)}(\mathring{\mathcal{C}})$ satisfying the equation

$$\bar{\partial} \tilde{g} = \tilde{\psi}.$$

Therefore function $g_* = \tilde{g} + \sum_{j=1}^{d} g^{(j)}$ satisfies the equation

$$\bar{\partial} g_*(\zeta) = \psi(\zeta) = \zeta_0^\ell \phi_*(\zeta),$$

and the function $g(\zeta) = (\zeta_0^{-\ell} g_*)^*$ satisfies conditions (4.3). □

Combining results of Theorem 2 and Proposition 4.1 we obtain the following proposition.

Proposition 4.2 *Let $\mathcal{X}$ and $\mathcal{C} \subset \mathbb{CP}^2$ be curves as in (4.1), with $\mathcal{C}$ satisfying condition (4.2). Let $\phi^{(0,1)} \in \mathcal{E}(\mathring{\mathcal{X}})$ be a form on $\mathring{\mathcal{X}}$, such that its direct image ϕ_* has a compact support in $\mathring{\mathcal{C}}$. Then for arbitrary $\ell > d-3$ the function*

$$g(\zeta) = \left(\zeta_0^{-\ell} \cdot I_1[\zeta_0^\ell \phi_* - H_1[\zeta_0^\ell \phi_*]]\right)^*$$

satisfies equation

$$\bar{\partial} g = \phi.$$

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