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Characterization of projective spaces by Seshadri constants

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Abstract We prove that an n-dimensional complex projective variety is isomorphic to $\mathbb{P}^n$ if the Seshadri constant of the anti-canonical divisor at some smooth point is greater than n. We also classify complex projective varieties with Seshadri constants equal to n.

Keywords Fano varieties · Projective space · Seshadri constants · Classification

Mathematics Subject Classification Primary 14J45; Secondary 14M99

1 Introduction

It is believed that the projective space $\mathbb{P}^n$ has the most positive anti-canonical divisor among complex projective varieties. Various characterizations of $\mathbb{P}^n$ have been found corresponding to different explanations of the "positivity" of the anti-canonical divisor. Using Kodaira vanishing theorem, Kobayashi and Ochiai [14] proved that if an n-dimensional projective manifold X with an ample line bundle H satisfies $-K_X \equiv (n+1)H$, then $(X, H) \cong (\mathbb{P}^n, \mathcal{O}(1))$. Kobayashi–Ochiai's characterization was generalized by Ionescu [11] (in the smooth case) and Fujita [8] (allowing Gorenstein rational singularities) assuming the weaker condition that $K_X + (n+1)H$ is not ample. Later, Cho, Miyaoka and Shepherd-Barron [5] (simplified by Kebekus in [13]) showed that a Fano manifold is isomorphic to $\mathbb{P}^n$ if the anti-canonical degree of every curve is at least $n+1$. Their proofs rely on deformation of rational curves which still work if we allow isolated local complete intersection quotient singularities (see [4]). Besides, Kachi and Kollár [12] gave characterizations of $\mathbb{P}^n$ in arbitrary characteristic that generalized [5,13,14] with a volume lower bound assumption.

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The purpose of this paper is to provide a characterization of $\mathbb{P}^n$ among complex $\mathbb{Q}$-Fano varieties by the local positivity of the anti-canonical divisor, namely the *Seshadri constants*. Recall that a complex projective variety X is said to be *$\mathbb{Q}$-Fano* if X has klt singularities and $-K_X$ is an ample $\mathbb{Q}$-Cartier divisor.

Definition 1 Let X be a normal projective variety and L an ample $\mathbb{Q}$-Cartier divisor on X. Let $p \in X$ be a smooth point. The *Seshadri constant* of L at p, denoted by $\epsilon(L, p)$, is defined as

$$\epsilon(L, p) := \sup\{x \in \mathbb{R}_{>0} \mid \sigma^*L - xE \text{ is ample}\},$$

where $\sigma : \mathrm{Bl}_p X \to X$ is the blow-up of X at p, and E is the exceptional divisor of σ.

It is clear that $\epsilon(-K_{\mathbb{P}^n}, p) = n + 1$ for any point $p \in \mathbb{P}^n$. Our main result characterizes $\mathbb{P}^n$ as the only $\mathbb{Q}$-Fano variety with Seshadri constant greater than n:

Theorem 2 *Let X be a complex $\mathbb{Q}$-Fano variety of dimension n. If there exists a smooth point $p \in X$ such that $\epsilon(-K_X, p) > n$, then $X \cong \mathbb{P}^n$.*

Note that Theorem 2 only assumes that $\epsilon(-K_X, p) > n$ for *some* smooth point p rather than *any* smooth point (although the existence of such p immediately implies the same inequality for a general smooth point). We also remark here that when X is smooth, Theorem 2 was obtained by Bauer and Szemberg in [1, Theorem 1.7] using different methods.

Since the Seshadri constant of a quadric hypersurface in $\mathbb{P}^{n+1}$ is equal to n, the lower bound on the Seshadri constant in Theorem 2 is sharp. It turns out that this is not the only $\mathbb{Q}$-Fano varieties achieving such lower bound, and the full list is given by the following theorem.

Theorem 3 *Let X be a n-dimensional complex $\mathbb{Q}$-Fano variety. Then there exists a smooth point $p \in X$ with $\epsilon(-K_X, p) = n$ if and only if X is one of the following:*

1. *a degree $d+1$ weighted hypersurface $X_{d+1} = (x_0 x_{n+1} = f(x_1, \ldots, x_n)) \subset \mathbb{P}(1^{n+1}, d)$,*
2. *a quartic weighted hypersurface $X_4 = (x_{n+1}^2 + x_n h(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1}))$ $(h \neq 0)$ or $(x_n x_{n+1} = f(x_0, \ldots, x_{n-1})) \subseteq \mathbb{P}(1^n, 2, 2)$,*
3. *the blow-up of $\mathbb{P}^n$ along the complete intersection of a hyperplane and a hypersurface of degree $d \le n$,*
4. *the quotient of the quadric $Q_k = (\sum_{i=0}^k x_i^2 = 0) \subseteq \mathbb{P}^{n+1}$ $(2 \le k \le n+1)$ by an involution $\tau(x_i) = \delta_i x_i$ $(\delta_i = \pm 1)$ that is fixed point free in codimension 1 and such that not all the δ_i $(i = 0, \ldots, k)$ are the same,*
5. *a Gorenstein log del Pezzo surface of degree ≥ 4 (for the classification of such surfaces, see [10, §3]).*

When X is smooth, the condition $\epsilon(-K_X, p) = n$ implies that $(-K_X \cdot C) \ge n$ for any curve $C \subset X$ passing through a very general point p. If in addition X has dimension at least 3, then by [3,19] X is either a quadric hypersurface or the blow-up of $\mathbb{P}^n$ along a smooth subvariety of codimension 2 and degree $d \le n$ contained in a hyperplane. On the other hand, in the surface case some of our results have been proved by [21, Theorem 1.8] under the somewhat restrictive assumption that $(K_X^2) \in \{4, 5, 6, 7, 8, 9\}$. Hence the above theorem is a natural generalization of their results to the singular and higher dimensional case, although our proof uses a completely different strategy.

Finally we show that in general the Seshadri constant $\epsilon(-K_X, p)$ can be any rational number between 0 and n. This is in sharp contrast with Theorem 2 where we have seen that there is a gap between n and $n + 1$ for the possible values of $\epsilon(-K_X, p)$.

Theorem 4 *For any rational number $0 < c \leq n$, there exists an n-dimensional $\mathbb{Q}$-Fano variety X with a smooth point p such that $\epsilon(-K_X, p) = c$.*

The paper is organized as follows. In Sect. 2, we prove Theorem 2. Denote the blow up of X at p by $\sigma : \hat{X} = \mathrm{Bl}_p X \to X$, then the divisor $D := \sigma^*(-K_X) - \epsilon(-K_X, p)E$ is nef by the definition of the Seshadri constant. Under the assumption that $\epsilon(-K_X, p) > n$, we use Kawamata–Viehweg vanishing theorem to show that D is semiample and $g = g_{|mD|} : \hat{X} \to Y$ maps E isomorphically onto its image for sufficiently divisible k. A simple computation yields that $(-K_{\hat{X}} \cdot C) = \epsilon(-K_X, p) - (n-1) > 1$ for any curve C contracted by g. We show in Lemma 8 that g cannot be birational under these assumptions and therefore has to be a morphism of fiber type with target $Y = g(E) \cong \mathbb{P}^{n-1}$. Then Lemma 6 implies that $\hat{X}$ is a $\mathbb{P}^1$-bundle over $\mathbb{P}^{n-1}$, thus $X \cong \mathbb{P}^n$. The proof of Lemma 8 relies on a dimension reduction argument and Lemma 5. As an application of Theorem 2, we show that $\mathbb{P}^n$ is the only Ding-semistable $\mathbb{Q}$-Fano variety of volume at least $(n+1)^n$ (see Theorem 10). This improves the equality case of [7, Theorem 1.1] where Fujita proved for Ding-semistable Fano manifolds.

In Sect. 3, we classify all $\mathbb{Q}$-Fano varieties with Seshadri constants equal to n. By the same argument as in the proof of Theorem 2, we still have that D is semiample. We divide the classification into two parts. In Sect. 3.1, we study the case when g is birational. We show that $g|_E$ is a closed embedding, $-(K_Y + g(E))$ is ample, $g(E)$ is nef (see Lemmas 11). We classify such pairs $(Y, g(E))$ in Lemma 13. Then we obtain the partial classification after a detailed study of the structure of the birational morphism g (see Lemmas 12 and 14). In Sect. 3.2, we study the case when g is of fiber type. It is not hard to see that every fiber of g has dimension 1, the general fiber of g is isomorphic to $\mathbb{P}^1$, $g|_E : E \to Y$ is a double cover, and $-K_{\hat{X}}$ is g-ample. After pulling back g to E and taking the normalization, we obtain a conic bundle $\widetilde{g} : \widetilde{X} \to E \cong \mathbb{P}^{n-1}$ with two sections (see Lemmas 16, 18 and Corollary 17). From the classification of the conic bundle $\widetilde{g}$ and the quotient map $g|_E$ (see Lemmas 19 and 20), we finish the classification of X and hence prove Theorem 3. Finally in Sect. 4, we provide examples showing that the Seshadri constant of a $\mathbb{Q}$-Fano variety can be any positive rational number less than n.

2 Proof of Theorem 2

Lemma 5 *Let $\pi : S \to T$ be a proper birational morphism between normal surfaces. Let $C \subset S$ be a K_S-negative π-exceptional curve. Then $(-K_S \cdot C) \leq 1$, with equality if and only if S has only Du Val singularities along C. (Since K_S is not necessarily $\mathbb{Q}$-Cartier, we use the intersection theory of Weil divisors on surfaces by Mumford [20].)*

Proof Let $\phi : \tilde{S} \to S$ be the minimal resolution of S. Denote the exceptional curves of ϕ by E_i. Then we have

$$K_{\tilde{S}} + \sum_i a_i E_i \equiv \phi^* K_S, \quad \text{where } a_i \geq 0.$$

Let $\tilde{C}$ be the birational transform of C under ϕ. Since $\pi \circ \phi$ contracts $\tilde{C}$, we have $(\tilde{C}^2) < 0$. By the assumption that C is K_S-negative, we have

$$(K_{\tilde{S}} \cdot \tilde{C}) = (\phi^* K_S \cdot \tilde{C}) - \sum_i a_i (E_i \cdot \tilde{C}) \leq (K_S \cdot C) < 0.$$

Hence $\tilde{C}$ is a (-1)-curve on $\tilde{S}$ and $(-K_S \cdot C) \leq (-K_{\tilde{S}} \cdot \tilde{C}) = 1$.

It is clear that $(-K_S \cdot C) = 1$ if and only if $\sum_i a_i(E_i \cdot \tilde{C}) = 0$, i.e. $a_i = 0$ whenever $\tilde{C}$ intersects E_i. By the negativity lemma (cf. [16, Lemma 3.41]), this is equivalent to saying that $a_i = 0$ whenever E_i is connected to $\tilde{C}$ through a chain of ϕ-exceptional curves. Thus the equality holds if and only if S has Du Val singularities along C.

Lemma 6 *Let $\pi : S \to T$ be a proper surjective morphism from a normal surface S to a smooth curve T. Assume that the general fiber of π is isomorphic to $\mathbb{P}^1$, and that all fibers of π are generically reduced and irreducible. Then π is a smooth $\mathbb{P}^1$-fibration, i.e. S is a geometrically ruled surface over T.*

Proof For any closed point $t \in T$, denote by S_t the scheme-theoretic fiber of π at t. It is clear that π is flat, so $\chi(S_t, \mathcal{O}_{S_t}) = \chi(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}) = 1$. Besides, S being normal implies that the Cartier divisor S_t on S has no embedded points. Then S_t being generically reduced and irreducible yields that S_t is an integral curve. Therefore, $S_t \cong \mathbb{P}^1$.

7 (*Proof of Theorem 2*) Denote by $\sigma : \hat{X} = \mathrm{Bl}_p X \to X$ the blow up of X at p with exceptional divisor E. Let $D := \sigma^*(-K_X) - \epsilon(-K_X, p)E$ be the nef divisor. Since $-K_{\hat{X}} = \sigma^*(-K_X) - (n-1)E$, we know that $D - K_{\hat{X}}$ is ample. Hence Shokurov's basepoint-free theorem [16, Theorem 3.3] implies that D is semiample.

Let $g : \hat{X} \to Y$ be the ample model of D (i.e. g is the morphism determined by the complete linear system $|kD|$ for some $k \gg 0$). Let m be a positive integer such that mD is Cartier. Notice that $mD - E - K_{\hat{X}}$ is ample by $\epsilon(-K_X, p) > n$, so Kawamata–Viehweg vanishing implies that $H^1(\hat{X}, mD - E) = 0$. Hence $H^0(\hat{X}, mD) \to H^0(E, mD|_E)$ is surjective for all $m \in \mathbb{Z}_{>0}$ such that mD is Cartier. As a result, $g|_E : E \to Y$ is a closed embedding. Thus any curve C contracted by g is not contained in E, which implies that $(C \cdot \sigma^*(-K_X)) > 0$. Since $0 = (C \cdot D) = (C \cdot \sigma^*(-K_X)) - \epsilon(-K_X, p)(C \cdot E)$, we know that $(C \cdot E) > 0$.

Suppose g contracts C to a point $y \in Y$. Consider the scheme-theoretic fiber $g^{-1}(y)$ of g. Since $g|_E$ is a closed embedding, the scheme-theoretic intersection $E \cap g^{-1}(y)$ is a reduced closed point, say q. If there is another curve $C' \neq C$ contained in $g^{-1}(y)$, then $E \cap g^{-1}(y)$ has multiplicity at least 2 at q, a contradiciton! So $\mathrm{Supp}\, g^{-1}(y) = C$ and $g^{-1}(y)$ is smooth and transversal to E at q. In particular, we have $(C \cdot E) = 1$ for any curve C contracted by g. Since $\hat{X}$ has klt singularities, it is Cohen–Macaulay by [16, Theorem 5.22]. In addition we have $-K_{\hat{X}} \sim_{g,\mathbb{Q}} \lambda E$ where $\lambda = \epsilon(-K_X, p) - n + 1 > 1$. Hence by the following lemma, g cannot be birational.

Lemma 8 *Let $g : \hat{X} \to Y$ be a proper birational morphism between quasi-projective normal varieties and E a smooth g-ample Cartier divisor on $\hat{X}$ such that $-K_{\hat{X}} \sim_{g,\mathbb{Q}} \lambda E$ for some $\lambda \geq 1$. Assume that $\hat{X}$ is Cohen–Macaulay and $g|_E : E \to G = g(E)$ is an isomorphism, then $\lambda = 1$ and Y is smooth along G.*

Proof Let H be a very ample divisor on Y such that $H^0(Y, \mathcal{O}_Y(H)) \to H^0(G, \mathcal{O}_G(H))$ is surjectve. Let $y \in Y$ be a closed point in the exceptional locus of g and let $H_1, \ldots, H_{n-2}$ be general members of $|H|$ containing y. Let $C = g^{-1}(y)$ and $S = g^*H_1 \cap \cdots \cap g^*H_{n-2}$. We claim that S is a normal surface. Since $E|_C$ is ample and $g|_E$ is an isomorphism, it is easy to see as above that C is an irreducible curve and $E \cap C$ is supported at a single point q. As $\hat{X}$ is Cohen–Macaulay, S is S_2. By Bertini's theorem $S \backslash C$ is smooth in codimension one and $G \cap H_1 \cap \cdots \cap H_{n-2}$ (scheme-theoretic intersection) is smooth at y. It follows that $E|_S$ is smooth at q. Since E is Cartier, we see that S is also smooth at $q \in C$, hence S is smooth in codimension one and it is normal.

It is clear that $g|_S$ is a birational morphism that contracts C. By adjunction $K_S = (K_X + g^*H_1 + \cdots + g^*H_{n-2})|_S$, thus $(-K_S \cdot C) = (-K_{\hat{X}} \cdot C) = \lambda(E \cdot C) = \lambda \geq 1$. On the other hand by Lemma 5 we have $(-K_S \cdot C) \leq 1$. Hence $\lambda = (-K_S \cdot C) = 1$ and S has only Du Val singularities along C. Since contracting a (-1)-curve (i.e. a curve that has anti-canonical degree 1) from a surface with Du Val singularities produces a smooth point, $g(S)$ and hence Y is smooth at y. Note that y is arbitrary in the exceptional locus, so Y is smooth along G.

Remark 9 In fact more is true. Under the same assumptions of the lemma, $\hat{X}$ is indeed the blowup of Y along a divisor in G. We postpone its proof to the next section.

Returning to the proof of Theorem 2, we see that g has to be a fiber type contraction. Since $g|_E$ is a closed embedding, we know that $g|_E : E \to Y$ is in fact an isomorphism. In particular, $E \cong Y \cong \mathbb{P}^{n-1}$. Let us define S, H_i as in the proof of Lemma 8. By the same argument there, S is a normal surface. Since the singular set of $\hat{X}$ has codimension at least 2, by generic smoothness we know that the generic fiber of $g : \hat{X} \to Y$ is smooth. So the contraction g being $K_{\hat{X}}$-negative implies that the general fiber of g is a smooth rational curve. In particular, the generic fiber of $g|_S : S \to g(S)$ is isomorphic to $\mathbb{P}^1$. Hence applying Lemma 6 yields that $C \cong \mathbb{P}^1$, which means that $g : \hat{X} \to Y$ is a smooth $\mathbb{P}^1$-fibration.

It is clear that $s = g|_E^{-1} : Y \to E$ gives a section of g, thus $\hat{X} = \mathbb{P}_Y(\mathcal{E})$ is a $\mathbb{P}^1$-bundle where $\mathcal{E}$ is a rank 2 vector bundle over Y. Then the section E corresponds to a surjection $\mathcal{E} \twoheadrightarrow \mathcal{N}$ for some line bundle $\mathcal{N}$ on Y. Denote the kernel of this surjection by $\mathcal{M}$. By the adjunction formula on $\mathbb{P}^1$-bundles, we know that $\mathcal{O}_Y(-1) \cong s^*N_{E/\hat{X}} \cong \mathcal{M}^{-1} \otimes \mathcal{N}$. For simplicity we may assume $\mathcal{M} \cong \mathcal{O}_Y$, then we get $\mathcal{N} \cong \mathcal{O}_Y(-1)$ and hence a short exact sequence

$$0 \to \mathcal{O}_Y \to \mathcal{E} \to \mathcal{O}_Y(-1) \to 0.$$

Since $\operatorname{Ext}^1(\mathcal{O}_Y(-1), \mathcal{O}_Y) \cong H^1(\mathbb{P}^{n-1}, \mathcal{O}(1)) = 0$, the above exact sequence splits. So $\mathcal{E} \cong \mathcal{O}_Y \oplus \mathcal{O}_Y(-1)$ and E corresponds to the second projection $\mathcal{O}_Y \oplus \mathcal{O}_Y(-1) \twoheadrightarrow \mathcal{O}_Y(-1)$. As a result, $\hat{X}$ is isomorphic to the blow up of $\mathbb{P}^n$ at one point with E corresponding to the exceptional divisor. Therefore, $X \cong \mathbb{P}^n$. □

The following is an application of Theorem 2 to Ding-semistable $\mathbb{Q}$-Fano varieties with maximal volume (see [7] or [18] for backgrounds). This improves Fujita's result on the equality case in [7, Theorem 5.1]. We remark that a different proof is presented in [18, Proof 2 of Theorem 36].

Theorem 10 *Let X be a Ding-semistable $\mathbb{Q}$-Fano variety of dimension n. If $((-K_X)^n) \geq (n+1)^n$, then $X \cong \mathbb{P}^n$.*

Proof Notice that $((-K_X)^n) \leq (n+1)^n$ by [7, Corollary 1.3]. Thus we have $((-K_X)^n) = (n+1)^n$. Let $p \in X$ be a smooth point. From [7, Proof of 5.1], we see that $\epsilon(-K_X, p) = n+1$. Hence $X \cong \mathbb{P}^n$ by Theorem 2. □

3 Equality case

In this section we prove Theorem 3. Let X be an n-dimensional $\mathbb{Q}$-Fano variety with a smooth point $p \in X$. Assume $\epsilon(-K_X, p) = n$. Following the proof of Theorem 2, we have that $D = \sigma^*(-K_X) - nE$ is semiample on $\hat{X}$ and induces the morphism $g : \hat{X} \to Y$. We now separate into two cases based on different behavior of g.

3.1 g is birational

Lemma 11 *If $g:\hat{X}\to Y$ is birational, then $g|_E$ is a closed embedding, $-(K_Y+g(E))$ is ample and $g(E)\cong\mathbb{P}^{n-1}$ is a nef divisor in the smooth locus of Y. Moreover, Y is a $\mathbb{Q}$-Fano variety.*

Proof We see that $mD-E-K_{\hat{X}}=(m-1)D$ is nef and big, so Kawamata–Viehweg vanishing implies that $g|_E : E\to Y$ is a closed embedding as in the proof of Theorem 2. Hence $g(E)\cong E\cong\mathbb{P}^{n-1}$. By Lemma 8, it lies in the smooth locus of Y.

Since g is induced by D, $-(K_Y+g(E))=\pi_* D$ is ample. To show the nefness of $g(E)$ we only need to show that $(L\cdot g(E))\geq 0$ for a line L in $g(E)$. We may assume L intersects the the exceptional locus of g. Denote by L' the strict transform of L in $\hat{X}$. Let $W=g^*g(E)-E$, then it is an effective Cartier divisor supported on $\mathrm{Ex}(g)$. Since $-W\sim_{g,\mathbb{Q}}-K_{\hat{X}}$ is g-ample, we have $\mathrm{Ex}(g)\subseteq W$, hence $(L'\cdot W)\geq 1$ and $(L\cdot g(E))=(L'\cdot(E+W))=-1+(L'\cdot W)\geq 0$.

According to Lemma 11, we are now in the situation of Lemma 8 with $\lambda=1$. In order to classify X, we first need to study the structure of the birational map $g:\hat{X}\to Y$ in greater detail. This is accomplished by the following lemma.

Lemma 12 *Under the same notations and assumptions as in Lemma 8, $\hat{X}$ is the blowup of Y along a divisor in G.*

Proof First note that by Lemma 8 and its proof, $\hat{X}$ has only compound Du Val singularities along $\mathrm{Ex}(g)$, hence after shrinking $\hat{X}$ and Y we may assume that $\hat{X}$ has only klt singularities.

Let $W=g^*G-E$ as above, then W is g-exceptional and $-W$ is a g-ample Cartier divisor on $\hat{X}$, hence we have $\hat{X}\cong\mathrm{Proj}\oplus_{m=0}^{\infty}\mathcal{J}_m$ where $\mathcal{J}_m=g_*\mathcal{O}_{\hat{X}}(-mW)(m=0,1,\ldots)$. It is clear that each $\mathcal{J}_m$ is an ideal sheaf on Y. Let $\mathcal{J}=\mathcal{J}_1$, we claim that $\mathcal{J}$ is the ideal sheaf of a hypersurface in g_*E and $\mathcal{J}_m=\mathcal{J}^m$.

To see this, note that since $-mW-K_{\hat{X}}\sim_{g,\mathbb{Q}}(m+1)E$ is g-ample and $\hat{X}$ is klt, we have $R^1g_*\mathcal{O}_{\hat{X}}(-mW)=0$ for all $m\geq 0$. Hence from the pushforward g_* of

$$0\to\mathcal{O}_{\hat{X}}(-g^*G-mW)\to\mathcal{O}_{\hat{X}}(-(m+1)W)\to\mathcal{O}_E(-(m+1)W)\to 0$$

we obtain an exact sequence

$$0\to\mathcal{J}_m(-G)\to\mathcal{J}_{m+1}\to\mathcal{O}_E(-(m+1)W)\to 0$$

Taking $m=0$, by Nakayama lemma we see that locally $\mathcal{J}=(a,b)$ is the ideal sheaf of $g(W)$ where $a=0$ (resp. $a=b=0$) is the local defining equation of G (resp. $g(W)$). Note that the restriction of g to E is an isomorphism, so $g(W)\cong W\cap E$ is a divisor (not necessarily irreducible or reduced) in G. Suppose we have shown $\mathcal{J}_m=\mathcal{J}^m$ for some $m\geq 1$ (the case $m=1$ being clear), then the above exact sequence tells us that $\mathcal{J}_{m+1}$ is generated by $a\cdot\mathcal{J}_m$ and b^{m+1}, hence $\mathcal{J}_{m+1}=\mathcal{J}^{m+1}$ as well. The claim then follows by induction on m and the lemma follows immediately from the claim.

Now we will classify the pairs $(Y,g(E))$ satisfying the statement of Lemma 11. By abuse of notation, we will simply denote the divisor by E instead of $g(E)$. We remark that Bonavero, Campana and Wiśniewski classified such pairs in [2] when Y is smooth.

Lemma 13 *Let Y be an n-dimensional $\mathbb{Q}$-Fano variety containing a prime divisor $E\cong\mathbb{P}^{n-1}$ in its smooth locus.*

1. *If $\rho(Y) = 1$, then either Y is a weighted projective space $\mathbb{P}(1^n, d)$ for some $d \in \mathbb{Z}_{>0}$ and E is the hyperplane defined by the vanishing of the last coordinate, or $n = 2$, $Y \cong \mathbb{P}^2$ and E is a smooth conic;*
2. *If $\rho(Y) \geq 2$ and $-(K_Y + E)$ is ample, then Y is a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-d))$ over $\mathbb{P}^{n-1}$ for some $d \in \mathbb{Z}_{\geq 0}$ and E is a section. If $n \geq 3$ and $d \geq n$ then E is the only section with negative normal bundle.*

Proof Note that in the case $\rho(Y) = 1$, E is necessarily an ample divisor on Y. As E does not intersect the singular locus of Y, Y has only isolated singularities. By adjunction $-(K_Y + E)|_E = -K_E$ is ample, hence $-(K_Y + E)$ is ample as well. Let Y° be the smooth locus of Y and $i : E \to Y^\circ$ the inclusion.

First assume $\rho(Y) = 1$ and $n \geq 3$. By the generalized version of Lefschetz hyperplane theorem [9, Theorem II.1.1], $H_i(Y^\circ, E, \mathbb{Z}) = H^i(Y^\circ, E, \mathbb{Z}) = 0$ for $i < n$, hence by the universal coefficient theorem, $H^n(Y^\circ, E, \mathbb{Z})$ is torsion free. As $n \geq 3$, this implies the restriction map $i^* : H^2(Y^\circ, \mathbb{Z}) \to H^2(E, \mathbb{Z})$ is injective and has torsion free cokernel. But $H^2(E, \mathbb{Z}) \cong \mathbb{Z}$ since $E \cong \mathbb{P}^{n-1}$, so i^* is in fact an isomorphism. As Y is $\mathbb{Q}$-Fano we have $H^1(Y, \mathcal{O}_Y) = 0$ by Kawamata–Viehweg vanishing and Y is Cohen–Macaulay by [16, Theorem 5.22]. Since $Z = \mathrm{Sing} Y$ consists of isolated points and $n \geq 3$, by the long exact sequence of cohomology with support

$$\cdots \to H^1_Z(Y, \mathcal{O}_Y) \to H^1(Y, \mathcal{O}_Y) \to H^1(Y^\circ, \mathcal{O}_{Y^\circ}) \to H^2_Z(Y, \mathcal{O}_Y) \to \cdots$$

we get $H^1(Y^\circ, \mathcal{O}_{Y^\circ}) = 0$. Combining this with the exponential sequence $0 \to \mathbb{Z} \to \mathcal{O}_{Y^\circ} \to \mathcal{O}^*_{Y^\circ} \to 0$, we see that the restriction $i^* : \mathrm{Cl}(Y) = \mathrm{Pic}(Y^\circ) \to \mathrm{Pic}(E) \cong \mathbb{Z}$ is also an isomorphism.

Let H be the ample generator of $\mathrm{Cl}(Y)$, then $E \sim dH$ for some $d \in \mathbb{Z}_{>0}$. Let $\pi : Y' \to Y$ be the (normalization of the) cyclic cover of degree d of Y ramified at E and $E' = \pi^{-1}(E)_{\mathrm{red}}$. Then $K_{Y'} + E' = \pi^*(K_Y + E)$ as E is the only branched divisor, hence Y' is also $\mathbb{Q}$-Fano and E' satisfies the same assumptions of the lemma. We also have $\mathcal{O}_{E'}(dE') \cong \mathcal{O}_{E'}(\pi^* E) = \pi^* N_{E/Y} \cong \mathcal{O}_{E'}(d)$, hence $N_{E'/Y'} \cong \mathcal{O}_{E'}(1)$ is the hyperplane class. Note that E' is ample since it's the preimage of the ample divisor E. It now follows from the long exact sequence

$$0 \to H^0(Y', \mathcal{O}_{Y'}) \to H^0(Y', \mathcal{O}_{Y'}(E')) \to H^0(E', N_{E'/Y'}) \to H^1(Y', \mathcal{O}_{Y'}) = 0$$

that the linear system $|E'|$ is base point free, has dimension n and defines an isomorphism $Y' \cong \mathbb{P}^n$ such that E' is mapped to a hyperplane. Our original pair (Y, E) is then obtained by taking a cyclic quotient of degree d ramified at E', and is easily seen to be as claimed in the statement of the lemma.

Next assume $\rho(Y) = 1$ and $n = 2$. Then Y has quotient singularity and is $\mathbb{Q}$-factorial, hence $\mathrm{Cl}(Y)$ has rank one. As E is ample, $\pi_1(E) \to \pi_1(Y^\circ)$ is surjective by [9, Theorem II.1.1], but $\pi_1(E) = \pi_1(\mathbb{P}^1) = 0$, so Y° is simply connected as well. In particular, $\mathrm{Cl}(Y) = \mathrm{Pic}(Y^\circ)$ is torsion-free and thus $\cong \mathbb{Z}$. Let r be the index of $i^*\mathrm{Cl}(Y)$ in $\mathrm{Pic}(E)$. As $-(K_Y + E)|_E = -K_E$ has degree 2, $r = 1$ or 2. Let H be the ample generator of $\mathrm{Cl}(Y)$, then $(H.E) = r$ and $E \sim dH$ for some $d \in \mathbb{Z}_{>0}$. Let $\pi : Y' \to Y$ be the corresponding cyclic cover of degree d and define E' as before. By the same argument as the $n \geq 3$ case, we have $N_{E'/Y'} \cong \mathcal{O}_{E'}(r)$, and if $r = 1$, the linear system $|E'|$ defines an isomorphism $(Y', E') \cong (\mathbb{P}^2, \text{hyperplane})$, while if $r = 2$, the linear system $|E'|$ embeds Y' into $\mathbb{P}^3$ as a quadric surface. Taking cyclic quotients, we see that the original (Y, E) is again as claimed.

Finally assume $\rho(Y) \geq 2$ and $-(K_Y + E)$ is ample. Let l be a line in E. We claim that there is an extremal ray $\mathbb{R}_{\geq 0}[\Gamma]$ in $\overline{NE}(Y)$ generated by an integral curve Γ on Y such that

$[\Gamma] \notin \mathbb{R}_{\geq 0}[l]$ and $(E \cdot \Gamma) > 0$. If $(E \cdot l) = 0$, then such Γ exists since E is not numerically trivial. If $(E \cdot l) > 0$, consider the exact sequence

$$0 \to T_E|_l \to T_Y|_l \to N_{E/Y}|_l \to 0.$$

It is clear that $T_E|_l$ is ample because $E \cong \mathbb{P}^{n-1}$. On the other hand, $\deg N_{E/Y}|_l = (E \cdot l) > 0$ hence $N_{E/Y}|_l$ is ample. Therefore, $T_Y|_l$ is also ample which implies that l is a very free rational curve in Y. Since $\rho(Y) \geq 2$, $\mathbb{R}_{\geq 0}[l]$ cannot be an extremal ray of $\overline{NE}(Y)$ (otherwise the contraction of l will contract Y to a single point), which means that such Γ exists.

Now let $h : Y \to Z$ be the contraction of Γ. As we argued in the proof of Theorem 2, $h|_E : E \to Y$ is a closed embedding, hence $(E \cdot \Gamma) = 1$. Since $-(K_Y + E)$ is ample, we have $(-K_Y \cdot \Gamma) > 1$. Then by the same reasoning as in the proof of Theorem 2, we conclude that h has to be a fiber type contraction. Hence Y is a $\mathbb{P}^1$-fibration over $Z \cong \mathbb{P}^{n-1}$ admitting a section $h|_E^{-1} : Z \to E$, so $Y \cong \mathbb{P}_Z(\mathcal{O} \oplus \mathcal{O}(-d))$ with $d \geq 0$. If $n \geq 3$, then E corresponds to either a surjection $\mathcal{O} \oplus \mathcal{O}(-d) \twoheadrightarrow \mathcal{O}$ or a surjection $\mathcal{O} \oplus \mathcal{O}(-d) \twoheadrightarrow \mathcal{O}(-d)$. If in addition $d \geq n$, then $-(K_Y + E)$ being ample implies that E is the unique section corresponding to the second projection $\mathcal{O} \oplus \mathcal{O}(-d) \twoheadrightarrow \mathcal{O}(-d)$.

Combining the last two lemmas we can give a partial classification of X:

Lemma 14 *If g is birational then X is one of the following:*

1. *a degree $d+1$ weighted hypersurface $X_{d+1} = (x_0 x_{n+1} = f(x_1, \ldots, x_n)) \subset \mathbb{P}(1^{n+1}, d)$;*
2. *the blow-up of $\mathbb{P}^n$ along the complete intersection of a hyperplane and a hypersurface of degree $d \leq n$;*
3. *a Gorenstein log del Pezzo surface of degree ≥ 5.*

Proof By Lemma 13, we have the following cases:

(1) $Y \cong \mathbb{P}(1^n, d)$ with homogeneous coordinate $[y_0 : \cdots : y_n]$ and $g(E) = (y_n = 0)$. We have $N_{g(E)/Y} \cong \mathcal{O}_E(d)$. By Lemma 12, $\hat{X}$ is obtained by blowing up a hypersurface $S = (f = 0)$ in $g(E)$ where f is a homogeneous polynomial in $y_0, \ldots, y_{n-1}$. As $N_{E/\hat{X}} \cong \mathcal{O}_E(-1)$ we see that $\deg f = d+1$. Consider the rational map $\phi : Y \dashrightarrow \mathbb{P}(1^{n+1}, d)$ given by

$$[y_0 : \cdots : y_n] \mapsto [x_0 : \cdots : x_{n+1}] = \left[\frac{f(y_0, \ldots, y_{n-1})}{y_n} : y_n : y_0 : \cdots : y_{n-1} \right]$$

whose image lies in the weighted hypersurface X_{d+1} define by $x_0 x_{n+1} = f(x_1, \ldots, x_n)$. It is clear that ϕ is contracts $g(E)$ to the point $[1 : 0 : \cdots : 0]$ and the indeterminacy locus of ϕ is exactly S. By inspecting each affine chart $(x_i \neq 0) \subset Y$ it is easy to see that after blowing up S, ϕ extends to a birational morphism $\hat{X} \to X_{d+1}$ that contracts E, hence $X \cong X_{d+1}$ as in the first case in the statement of the lemma.

(2) Y is a $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-d))$ over $\mathbb{P}^{n-1}$ $(n \geq 3)$ and $g(E)$ is a section. Since $g(E)$ is nef by Lemma 11, we have $d < n$ by Lemma 13. Going back to the last part of the proof of Lemma 13 we see that the section $g(E)$ corresponds to a surjection $\mathcal{O} \oplus \mathcal{O}(-d) \twoheadrightarrow \mathcal{O}$ and hence $N_{g(E)/Y} \cong \mathcal{O}_E(d)$. By Lemma 12 as in previous case, $\hat{X}$ is obtained by blowing up a hypersurface S of degree $d+1$ in $g(E)$. It is straightforward to see that the elementary transformation of Y with center S is the $\mathbb{P}^1$-bundle $\mathbb{P}(\mathcal{O} \oplus \mathcal{O}(-1))$ over $\mathbb{P}^{n-1}$, which is isomorphic to the blowup of a point R on $\mathbb{P}^n$, such that the strict transform E' (resp. H) of $g(E)$ (resp. the negative section on Y) becomes the exceptional divisor over R (resp. a hyperplane in $\mathbb{P}^n$ that is disjoint from R). Contracting E' and reversing this procedure we see that X is the blowup of $\mathbb{P}^n$ along a hypersurface of degree $d+1 \leq n$ in a hyperplane.

(3) $Y \cong \mathbb{P}^2$ and $g(E)$ is a smooth conic, or Y is a ruled surface over $\mathbb{P}^1$ and $g(E)$ is a section. In either case Y is smooth and $\hat{X}$ is obtained by blowing up subschemes of $g(E)$. Locally on Y, such a subscheme is defined by $(a = b^k = 0)$ where a, b are local coordinates such that $g(E) = (a = 0)$. $\hat{X}$ then has local equation $at = b^k$ or $a = b^k t$ and it follows that both $\hat{X}$ and X have only Du Val singularities of type A. As $D = \sigma^*(-K_X) - 2E$ is big and nef and Cartier in this case we have $(K_X^2) = (D^2) - 4(E^2) = (D^2) + 4 \geq 5$, so X is as described in the third case of the statement of the lemma. □

3.2 g is of fiber type

Lemma 15 *If g is of fiber type, then every fiber has dimension 1, $g|_E : E \to Y$ is a double cover and $-K_{\hat{X}} \sim_{g.\mathbb{Q}.} E$ is g-ample.*

Proof Since $\epsilon(-K_X, p) > n-1$, $\hat{X}$ is $\mathbb{Q}$-Fano, so $-K_{\hat{X}} \sim_{g.\mathbb{Q}.} E$ is g-ample. $D|_E$ is ample, so $E \to Y$ is finite and every fiber of g has dimension one. Let l be a general fiber, then $l \cong \mathbb{P}^1$ and $(-K_{\hat{X}} \cdot l) = 2 = (E \cdot l)$, so E is a double section.

Similar to the previous case, we first analyze the local structure of g in a slightly more general setting. For ease of notations, we call $g : \hat{X} \to Y$ (where $\hat{X}$ and Y are normal quasi-projective varieties) a *rational conic bundle* if g is proper, every fiber of g has dimension 1 and the general fiber is isomorphic to $\mathbb{P}^1$. If in addition $\hat{X}$ is Cohen–Macaulay and there exists a Cartier divisor E on $\hat{X}$ such that $-K_{\hat{X}} \sim_{g.\mathbb{Q}.} E$ is g-ample, then we say that the rational conic bundle is *Gorenstein*. It is clear that a conic bundle is automatically a Gorenstein rational conic bundle.

Lemma 16 *Let $g : S \to C$ be a Gorenstein rational conic bundle. Assume* $\dim S = 2$*, then S is a conic bundle and in particular has only Du Val singularities.*

Proof Let l be an irreducible component of a fiber of g, then $(-K_S \cdot l) = (E \cdot l)$ is a positive integer since E is Cartier and $-K_S$ is g-ample. On the other hand, if F is a general fiber of g then $(-K_S \cdot F) = 2$. Hence every fiber of g has at most two irreducible components (counting multiplicities), so on the minimal resolution of S (which is a birationally ruled surface over C), every fiber over C has one of the following as its dual graph:

$$(-2) - (-1) - (-2),$$
$$(-1) - (-2) - (-2) - \cdots - (-2) - (-1),$$

or

$$\begin{matrix} (-2) & & \\ & \searrow & \\ & & (-2) - (-2) - \cdots - (-2) - (-1) \\ & \nearrow & \\ (-2) & & \end{matrix}$$

As S is obtained by contracting those (-2)-curves, it has only Du Val singularities and is a conic bundle. □

Corollary 17 *If $g : \hat{X} \to Y$ is a Gorenstein rational conic bundle such that Y is smooth, then $\hat{X}$ is a conic bundle over Y. In particular, $\hat{X}$ is a hypersurface in $\mathbb{P}(\mathcal{E})$ for some rank* 3 *vector bundle $\mathcal{E}$ on Y.*

Proof Let $y \in Y$ and C a general complete intersection curve on Y passing through y. Let $S = \hat{X} \times_Y C$. Since $\hat{X}$ is Cohen–Macaulay, S is S_2. From the proof of Lemma 16 we know that the fiber $g^{-1}(y)$ has at most 2 irreducible components (counting multiplicities), hence S is smooth at every generic point of $g^{-1}(y)$, for otherwise $g^{-1}(y)$ contains a component of multiplicity $\geq 2^2 = 4$. It follows that S is normal. By adjunction it is easy to see that S is a Gorenstein rational conic bundle over C, so by Lemma 16, S has only Du Val singularities and is a conic budle, hence every fiber of g is isomorphic to a conic and $\hat{X}$ has cDV singularities which is Gorenstein. The lemma then follows from standard arguments (see e.g. [6, Theorem 7]).

Unfortunately in our classification problem, the Gorenstein rational conic bundle $g : \hat{X} \to Y$ does not have a smooth base. Nevertheless, there is a smooth double section E. Hence we would like to apply Corollary 17 to $\widetilde{g} : \widetilde{X} \to \widetilde{Y}$, where $\widetilde{Y} \cong E$ and $\widetilde{X}$ is the normalization of $\hat{X} \times_Y \widetilde{Y}$. For this purpose, we need to show that $\widetilde{X}$ is Gorenstein rational conic bundle over $\widetilde{Y}$. This is given by the following lemma.

Lemma 18 *Let $g : \hat{X} \to Y$ be a Gorenstein rational conic bundle and $\phi : \widetilde{Y} \to Y$ a finite morphism between normal varieties. Let $\widetilde{X}$ be the normalization of $\hat{X} \times_Y \widetilde{Y}$. Assume that $\hat{X}$ has klt singularities and the branch divisor of ϕ is disjoint from the singular locus of $\widetilde{Y}$ and Y. Then $\widetilde{g} : \widetilde{X} \to \widetilde{Y}$ is also a Gorenstein rational conic bundle.*

Proof By shrinking Y we may assume either ϕ is unramified in codimension one or both Y and $\widetilde{Y}$ are smooth. In the first case $\widetilde{X}$ is also klt by [16, Proposition 5.20] hence is CM, and the other properties of Gorenstein rational conic bundles are preserved by a finite base change that is étale in codimension one. In the second case g is a conic bundle by Lemma 17, hence the same holds for $\widetilde{g}$. □

The pullback E' of E to $\widetilde{X}$ is then a union of two sections E_1 and E_2. If they are disjoint, we have a simple description of the conic bundle $\widetilde{g}$:

Lemma 19 *Let $\widetilde{g} : \widetilde{X} \to \widetilde{Y}$ be a conic bundle with smooth base. Assume that there are two disjoint sections E_1 and E_2 that are Cartier as divisors on $\widetilde{X}$ and such that $-K_{\widetilde{X}} \sim_{g,\mathbb{Q}} E_1 + E_2$. Then there is a birational morphism $u : \widetilde{X} \to Z = \mathbb{P}_{\widetilde{Y}}(\mathcal{O} \oplus \mathcal{L})$ (where $\mathcal{L} \cong N_{E_1/\widetilde{X}}$) sending E_1, E_2 to two disjoint sections E_1', E_2' of Z such that $\widetilde{X}$ is the blow up of Z along a divisor in E_2.*

Proof If every fiber of $\widetilde{g}$ is an irreducible $\mathbb{P}^1$ then $\widetilde{X} \cong \mathbb{P}_{\widetilde{Y}}(\mathcal{O} \oplus \mathcal{L})$ and there is nothing to prove. So we may assume $l = l_1 + l_2$ is a reducible fiber. We have $(E_1 + E_2 \cdot l_j) = (-K_{\widetilde{X}} \cdot l_j) = 1$ $(j = 1, 2)$. Since the section E_i is Cartier, we have $(E_i \cdot l_j) = \delta_{ij}$ after rearranging indices. Let $u : \widetilde{X} \to Z$ be the contraction of the extremal ray $\mathbb{R}_+[l_2]$ and let E_1', E_2' be strict transform of E_1, E_2. As E_i is a section of $\widetilde{g}$ and $E_i \to \widetilde{Y}$ factors through E_i', the restriction $u|_{E_i}$ is an isomorphism. In addition we have $-(K_{\widetilde{X}} + E_2) \sim_{u,\mathbb{Q}} 0$ since its intersection number with l_2 is zero. Hence the lemma follows by a direct application of Lemma 12. □

Putting everything together and specializing to $E \cong \mathbb{P}^{n-1}$, we now finish the second part of the classification of X with $\epsilon(-K_X, p) = n$.

Lemma 20 *If g is of fiber type then X is one of the following:*

1. *A Gorenstein log del Pezzo surface of degree 4;*

2. *Quotient of a quadric hypersurface in $\mathbb{P}^{n+1}$ by an involution that is fixed point free in codimension 1;*
3. *A quartic weighted hypersurface in $\mathbb{P}(1^n, 2^2)$.*

Proof If $n = \dim X = 2$ then by Lemma 16, $\hat{X}$ and hence X has only Du Val singularities. We have $\sigma^*(-K_X) - 2E \sim_{g.\mathbb{Q}.} 0$, so $(K_X^2) = -4(E^2) = 4$ and we are in case (1). Hence in the remaining part of the proof we assume that $n \geq 3$.

We keep using the notations introduced in this subsection. Let $\widetilde{X} \to \bar{X}$ be the Stein factorization of the composition $\widetilde{X} \to \hat{X} \to X$, then $\bar{X} \to X$ is a double cover. The double cover $E \to Y$ is either unramified in codimension one or the quotient $\mathbb{P}^{n-1} \to \mathbb{P}(1^{n-1}, 2)$ in which case the branch divisor is a hyperplane on $\mathbb{P}^{n-1}$, so the conditions and conclusions of Lemma 18 are satisfied and we see that $\widetilde{g} : \widetilde{X} \to \widetilde{Y}$ is a conic bundle over $\widetilde{Y} \cong \mathbb{P}^{n-1}$ by Corollary 17.

If $h : \widetilde{X} \to \hat{X}$ is unramified in codimension one, so is $\bar{X} \to X$ and we have $\operatorname{codim}_{E_1 \cap E_2} E_i \geq 2$. But since $\widetilde{X}$ is Cohen–Macaulay and $E' = E_1 + E_2$ is a Cartier divisor, $E_1 \cup E_2$ is S_2. It follows that E_1 and E_2 do not intersect at all, hence they are disjoint smooth Cartier divisors in $\widetilde{X}$ with normal bundle $\mathcal{O}_{\mathbb{P}^{n-1}}(-1)$. As $K_{\widetilde{X}} + E_1 + E_2 = h^*(K_{\hat{X}} + E) \sim_{g.\mathbb{Q}.} 0$, it follows from Lemma 19 that $\widetilde{X}$ is a blowup of $Z \cong \mathbb{P}_{\widetilde{Y}}(\mathcal{O} \oplus \mathcal{O}(-1)) \cong \mathrm{Bl}_z\mathbb{P}^n$ along a hypersurface in the strict transform of a hyperplane. For the normal bundle to match, it is the blowup of a quadric hypersurface. As $\bar{X}$ is obtained by contracting $E_1 \cup E_2$ from $\widetilde{X}$, it is a quadric hypersurface in $\mathbb{P}^{n+1}$, and X is the quotient of $\bar{X}$ by an involution that acts fixed point free in codimension one as in case (2).

If $h : \widetilde{X} \to \hat{X}$ is ramified in codimension one, then it is ramified along $\widetilde{g}^*H$ where H is a hyperplane on $\widetilde{Y}$. As in the last paragragh $E_1 \cap E_2$ has pure codimension one, so E' is a union of two $\mathbb{P}^{n-1}$ intersecting transversally at a hyperplane. The conic bundle $\widetilde{X}$ is a hypersurface in some $\mathbb{P}(\mathcal{E})$ over $\widetilde{Y}$. To compute $\mathcal{E}$, first note that $-(K_{\widetilde{X}} + E') = \widetilde{g}^*M$ for some $M \in \operatorname{Pic}(E)$ since it restricts to a trivial bundle on every fiber of $\widetilde{g}$; we also have $-(K_{\widetilde{X}} + E')|_{E'} = -K_{E'} = (n-1)\widetilde{g}^*H$, so $M \sim (n-1)H$. Combining with $N_{E'/\widetilde{X}} \cong \widetilde{g}^*\mathcal{O}_{\widetilde{Y}}(-H)$ we have $-K_{\widetilde{X}}|_{E'} \cong \widetilde{g}^*(n-2)H$. Now apply $\widetilde{g}_*$ to the exact sequence

$$0 \to \mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}} - E) \to \mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}}) \to \mathcal{O}_{E'}(-K_{\widetilde{X}}) \to 0$$

we obtain another exact sequence

$$\begin{aligned} 0 \to \mathcal{O}_{\widetilde{Y}}((n-1)H) \to \widetilde{g}_*\mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}}) \to \mathcal{O}_{\widetilde{Y}}((n-2)H) \\ \oplus \mathcal{O}_{\widetilde{Y}}((n-3)H) \to R^1\widetilde{g}_*\mathcal{O}_{\widetilde{X}} \otimes M = 0 \end{aligned}$$

hence $\widetilde{g}_*\mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}}) \cong \oplus_{k=1}^3 \mathcal{O}_{\widetilde{Y}}((n-k)H)$ and we may choose $\mathcal{E} \cong \oplus_{k=0}^2 \mathcal{O}_{\widetilde{Y}}(kH)$. Let π be the projection $\mathbb{P}(\mathcal{E}) \to \widetilde{Y}$ and $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(1)$ the relative hyperplane class. $\widetilde{X}$ corresponds to section of $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2) \otimes \pi^*\mathcal{O}_{\widetilde{Y}}(mH)$ for some $m \in \mathbb{Z}$ and by adjunction formula we have $\mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}}) \cong \mathcal{O}_{\widetilde{X}}(1) \otimes \widetilde{g}^*\mathcal{O}_{\widetilde{Y}}((n-3-m)H)$, hence $\widetilde{g}_*\mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}}) \cong \mathcal{E} \otimes \mathcal{O}_{\widetilde{Y}}((n-3-m)H)$. Comparing this to the previous formula for $\widetilde{g}_*\mathcal{O}_{\widetilde{X}}(-K_{\widetilde{X}})$ we see that $m = 0$. The surjection $\mathcal{E} \twoheadrightarrow \mathcal{O}_{\widetilde{Y}}$ defines a section S of $\mathbb{P}(\mathcal{E}) \to \widetilde{Y}$ that is disjoint with $\widetilde{X}$ (since $\mathcal{O}_{\mathbb{P}(\mathcal{E})}(2)|_S \cong \mathcal{O}_S$) and the linear projection from S makes $\widetilde{X}$ into a double cover over the $\mathbb{P}^1$-bundle $\mathbb{P}_{\widetilde{Y}}(\mathcal{O}(H) \oplus \mathcal{O}(2H))$, which is also the blowup of a point on $\mathbb{P}^n$, such that E' is mapped to the exceptional divisor and $\widetilde{g}^*H$ to the strict transform of a hyperplane passing through the center of blowup. $\bar{X}$ is then a double cover of $\mathbb{P}^n$, and as $-(K_{\widetilde{X}} + E') \sim (n-1)\widetilde{g}^*H$ we have $-K_{\bar{X}} \sim (n-1)\tau^*H'$ where H' is a hyperplane on $\mathbb{P}^n$ and $\tau : \bar{X} \to \mathbb{P}^n$ the double cover. It follows that $\bar{X}$ is a weighted hypersurface of degree 4 in $\mathbb{P}(1^{n+1}, 2)$. The original X is then obtained as the quotient of $\bar{X}$ by an involution that fixes a hyperplane section (i.e. the strict transform of $\widetilde{g}^*H$), hence is a quartic weighted hypersurface in $\mathbb{P}(1^n, 2^2)$ as in case (3).

21 (*Proof of Theorem* 3) By Lemmas 14 and 20, we have the following five possibilities for X. Note that by Theorem 2 it suffices to show that $\epsilon(-K_X, p) \geq n$ in each case.

(1) $X \cong X_{d+1} = (x_0 x_{n+1} = f(x_1, \ldots, x_n)) \subseteq \mathbb{P}(1^{n+1}, d)$. If $d = 1$ then X is a quadric hypersurface and the result is clear (or see case (4)). Otherwise $d > 1$ and we have $q = [0 : \cdots : 0 : 1] \in X$. Let p be a smooth point on X and let $\sigma : Z \to \mathbb{P}(1^{n+1}, d)$ be the blowup of $\mathbb{P}(1^{n+1}, d)$ at p with exception divisor V. Let H be the divisor class $\mathcal{O}(1)$ on $\mathbb{P}(1^{n+1}, d)$, then we have $\sigma^*(-K_X) - nE = n(\sigma^* H - V)|_{\hat{X}}$. The base locus of the linear system $|\sigma^* H - V|$ on Z is the strict transform of the line l joining p and q. For general choice of p we have $l \not\subseteq X$, hence $\sigma^*(-K_X) - nE$ is nef on $\hat{X}$, yielding $\epsilon(-K_X, p) \geq n$.

(2) X is a quartic hypersurface in $\mathbb{P}(1^n, 2^2)$. Up to weighted projective isomorphism we may assume that X is defined by the equation $q(x_n, x_{n+1}) + x_n h(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1})$ where $\deg q = \deg h = 2$, $\deg f = 4$ and $h = 0$ if $q \neq a x_{n+1}^2$. Let $p \in X$ be a smooth point and define H, V in the similar way as in the first case. We have $\sigma^*(-K_X) - nE = n(\sigma^* H - V)|_{\hat{X}}$. The base locus of $|\sigma^* H - V|$ is the plane Σ spanned by p and the line $(x_0 = \cdots = x_{n-1} = 0)$, so D is nef (i.e. $\epsilon(-K_X, p) \geq n$) if and only if for every curve $C \subseteq \Sigma \cap X$ we have $(D \cdot C) \geq 0$. It is easy to see that $\frac{1}{n}(D \cdot C) = \frac{1}{4} \deg C - \mathrm{mult}_p C$. As $\deg(\Sigma \cap X) \leq 4$ we see that $(D.C) \geq 0$ if and only if $\Sigma \cap X$ is an irreducible curve that is smooth at p. Suppose $p = [c_0 : \cdots : c_{n+1}]$, then $\Sigma \cap X$ is given by the equation $q(y_1, y_2) + h(c_0, \ldots, c_{n-1}) y_1 y_0^2 = f(c_0, \ldots, c_{n-1}) y_0^4$ in $\Sigma \cong \mathbb{P}(1, 2, 2)$. From this it is clear that $\epsilon(-K_X, p) \geq n$ for general $p \in X$ if and only if q is not a square or $hq \neq 0$. After another change of variable we see that X is a quartic hypersurface of the form $x_n x_{n+1} = f(x_0, \ldots, x_{n-1})$ or $x_{n+1}^2 + x_n h(x_0, \ldots, x_{n-1}) = f(x_0, \ldots, x_{n-1})$ $(h \neq 0)$.

(3) X is the blowup of a hypersurface S of degree $d \leq n$ in a hyperplane of $\mathbb{P}^n$. Let V be the exceptional divisor over S, H the pullback of $\mathcal{O}_{\mathbb{P}^n}(1)$ on X and $H' \subset X$ the strict transform of the hyperplane containing S. Let $p \in X$ be a point outside $H' \cup V$. We have $D = \sigma^*(-K_X) - nE \sim \sigma^* H' + n(\sigma^* H - E)$. We want to show that D is nef. Since $\sigma^* H - E$ is already nef, it remains to show that $(D \cdot l) > 0$ where l is a line in $\sigma^* H'$. Then a direct computation shows that $(D \cdot l) = (-K_X \cdot l) = (((n+1)H - V) \cdot l) = n + 1 - d > 0$. Thus D is nef and $\epsilon(-K_X, p) \geq n$.

(4) $X = Q/\tau$ where Q is a quadric hypersurface and $\tau \in \mathrm{Aut}(Q)$ an involution that is fixed point free in codimension one. Let p_1 be a smooth point of Q, let $p_2 = \tau(p_1)$ and p be their image in X. Let $\psi : \hat{Q} \to Q$ be the blowup of p_1 and p_2 with exceptional divisors E_1 and E_2. Since $h : Q \to X$ is étale in codimension one, the divisor $D = \sigma^*(-K_X) - nE$ pulls back to $D' = \psi^*(-K_Q) - nE_1 - nE_2 = n(\psi^* H - E_1 - E_2)$ where H is the hyperplane class on Q. Similar to case (1), D' is the restriction of a line bundle (also denoted by D') on blowup of $\mathbb{P}^{n+1}$ at p_1, p_2 whose base locus is the strict transform of the line l joining p_1 and p_2. We also have $(D' \cdot l) = -n < 0$. Hence D is nef and $\epsilon(-K_X, p) \geq n$ if and only if $l \not\subseteq Q$. We may diagonalize τ and choose homogeneous coordinate x_i so that $\tau(x_i) = \delta_i x_i$ where $\delta_i = \pm 1$. It is then not hard to verify that $l \not\subseteq Q$ for general choice of p if and only if Q is given by the equation $\sum_{i=0}^{k} x_i^2 = 0$ for some $2 \leq k \leq n+1$ such that δ_i take different values for $i = 0, \ldots, k$.

(5) X is a Gorenstein log del Pezzo surface of degree $(K_X^2) \geq 4$. We claim that if S is a Gorenstein log del Pezzo surface of degree $d \geq 3$, then there exists an irreducible curve $C \in |-K_S|$ with a double point p lying in the smooth locus of S. After blowing up $d - 3$ general points on S, it suffices to prove the claim when $d = 3$, in which case S is a nodal cubic surface in $\mathbb{P}^3$ by [10, Theorem 4.4]. But then there are only finitely many lines on S whereas by dimension count there exists $C \in |-K_X|$ that is singular at any given $p \in S$, hence the claim follows immediately. Using such $C \in |-K_X|$ and take $p = \mathrm{Sing}(C)$, we

have $\sigma^*(-K_X) - 2E \sim C'$ where C' is the strict transform of C and $(C'^2) = (K_X^2) - 4 \geq 0$, hence C' is nef and $\epsilon(-K_X, p) \geq n = 2$.

It remains to show that all Fano varieties listed in the statement of Theorem 3 have only klt singularities. From the equations there we see that the singularities of X are always quotients of cA-type singularities that are étale in codimension 1 (hence are klt by [15, 1.42] and [16, Proposition 5.20]) except when X is a quartic hypersurface $x_{n+1}^2 + x_n h = f$ in $\mathbb{P}(1^n, 2^2)$ and $x \in (x_n = x_{n+1} = 0) \cap X$ satisfies $\text{mult}_x h = 2$ and $\text{mult}_x f \geq 3$. In the latter case, we may assume $x = [1 : 0 : \cdots : 0]$ and locally X is a double cover of $\mathbb{C}^n$ ramified along $D = (x_n h = f)$. If h is not a perfect square, then the pair $(\mathbb{C}^n, D)$ degenerates to $(\mathbb{C}^n, D_0)$ where $D_0 = (x_n h = 0)$ (consider the $\mathbb{C}^*$-action $(x_1, \ldots, x_n) \mapsto (t^2 x_1, \ldots, t^2 x_{n-1}, t x_n)$ for $t \neq 0$). Clearly $(\mathbb{C}^n, \frac{1}{2} D_0)$ is klt, so it follows from adjunction that $(\mathbb{C}^n, \frac{1}{2} D)$ is also klt which implies that X is klt by [16, Proposition 5.20]. If h is a perfect square, then by [16, page 168] we know that X is a cDV singularity which is klt as well. □

4 Seshadri constants below *n*

In this section, we prove Theorem 4 using the following examples.

Example 22 Let X be the weighted projective space $\mathbb{P}(1, a_1, \ldots, a_n)$ where $a_1 \leq \cdots \leq a_n$ are positive integers satisfying $\gcd(a_1, \ldots, a_n) = 1$. Let $p \in X$ be the smooth point with coordinate $[1 : 0 : \cdots : 0]$. We claim that the Seshadri constant of $-K_X$ at p is $\epsilon(-K_X, p) = \frac{1}{a_n}(1 + \sum_{i=1}^n a_i)$. As before let $\sigma : \hat{X} \to X$ be the blowup of X at p and E the exceptional divisor. Since $\hat{X}$ is a toric variety, the torus invariant divisor $L_x = \sigma^*(-K_X) - xE$ is nef if and only if it has non-negative intersection number with all torus invariant lines, and as $-K_X$ is ample on X and E has ample conormal bundle, it suffices to check $(L_x \cdot l_i) \geq 0$ where l_i is the strict transform of the line on X joining p and the point whose only nonzero coordinate is at the i-th entry ($i > 0$). It is straightforward to compute $(L_x \cdot l_i) = \frac{1}{a_i}(1 + \sum_{i=1}^n a_i) - x$, so $\epsilon(-K_X, p) = \frac{1}{a_n}(1 + \sum_{i=1}^n a_i)$. Taking $a_1 = \cdots = a_{m-1} = 1$, $a_m = r - m$, $a_{m+1} = \cdots = a_n = s$ where $1 \leq m < n$ and $s \geq r > m$ we get $\epsilon(-K_X, p) = n - m + \frac{r}{s}$, hence the Seshadri constant $\epsilon(-K_X, p)$ can be any rational number in the interval $(1, n]$.

Example 23 More generally, let X be the weighted projective space $\mathbb{P}(a_0, \ldots, a_n)$ where $a_0 \leq \cdots \leq a_n$ have no common factor and $p \in X$ a smooth point on the line $l : x_2 = \cdots = x_n = 0$ (such p exists exactly when $\gcd(a_0, a_1) = 1$). We claim that $\epsilon(-K_X, p)$ is the smaller one of $\frac{1}{a_n} \sum_{i=0}^n a_i$ and $\frac{1}{a_0 a_1} \sum_{i=0}^n a_i$. Indeed, since X is toric and p is invariant under an $(n-1)$-dimensional subtorus T, the Mori cone of $\hat{X} = \text{Bl}_p X$ is generated by a line in E and the strict transform $\hat{C}$ of a curve $C \subseteq X$ containing p that is invariant under the action of T. Hence C is the line joining p and a T-invariant point. For $D = \sigma^*(-K_X) - \delta E$, we have $(D \cdot \hat{C}) = \frac{1}{a_0 a_1} \sum_{i=0}^n a_i - \delta$ if $C = l$, otherwise $(D \cdot \hat{C}) = \frac{1}{a_j} \sum_{i=0}^n a_i - \delta$ for some j. The claim then follows by setting $(D \cdot \hat{C}) \geq 0$. Taking $a_0 = s - 1$, $a_1 = \cdots = a_{n-1} = s$, $a_n = (r-1)(s-1) - (n-1)s$ with $s \geq r \gg 0$ we get $\epsilon(-K_X, p) = \frac{r}{s}$, hence the Seshadri constant $\epsilon(-K_X, p)$ can be any rational number in the interval $(0, 1]$ as well.

Remark 24 As the previous examples give some possible values of $\epsilon(-K_X, p)$, it is natural to ask whether these are all possible values. When $\epsilon(-K_X, p) \geq n - 1$, the Rationality Theorem [16, Theorem 3.5] implies that $\epsilon(-K_X, p)$ is necessarily a rational number. When $\epsilon(-K_X, p) < n - 1$, it is not clear to us whether $\epsilon(-K_X, p)$ is rational, although there are no known examples of irrational Seshadri constants according to [17, Remark 5.1.13].

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