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Log canonical thresholds in positive characteristic

Zhixian Zhu[1]

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Abstract In this paper, we study the singularities of pairs in arbitrary characteristic via jet schemes. For a smooth variety X in characteristic 0, Ein, Lazarsfeld and Mustaţă showed that there is a correspondence between irreducible closed cylinders and divisorial valuations on X. Via this correspondence, one can relate the codimension of a cylinder to the log discrepancy of the corresponding divisorial valuation. We now extend this result to positive characteristic. In particular, we prove Mustaţă's log canonical threshold formula avoiding the use of log resolutions, making the formula available also in positive characteristic. As a consequence, we get a comparison theorem via reduction modulo p and a version of inversion of adjunction in positive characteristic.

Introduction

Let k be a perfect field of arbitrary characteristic. Given $m \geq 0$ and a scheme X over k, we denote by X_m the mth order jet scheme of X. The set of k-points of X_m is

$$X_m(k) = \mathrm{Hom}(\mathrm{Spec}\, k[t]/(t^{m+1}), X).$$

If X is a smooth integral variety of dimension n, then X_m is a smooth variety of dimension $n(m+1)$ and the truncation morphism $\rho_m^{m+1} : X_{m+1} \to X_m$ is locally trivial with fiber $\mathbf{A}^n$.

The space of arcs X_∞ is the projective limit of the jet schemes X_m and thus parameterizes all formal arcs on X. One writes $\psi_m : X_\infty \to X_m$ for the natural map. The inverse images of constructible subsets by the canonical projections $\psi_m : X_\infty \to X_m$ are called *cylinders*. If C is the cylinder $\psi_m^{-1}(S)$ defined by a constructible subset $S \subset X_m$, then the codimension of C in X_∞ is the codimension of S in X_m. Interesting examples of cylinders arise as follows. Consider a non-zero ideal sheaf $\mathfrak{a} \subset \mathcal{O}_X$ defining a subscheme $Y \subset X$. For every $p \geq 0$, the

✉ Zhixian Zhu
zhixian@umich.edu

1 Department of Mathematics, University of Michigan, Ann Arbor, MI 48109, USA

contact locus of order p of $\mathfrak{a}$ is the locally closed cylinder

$$\mathrm{Cont}^p(Y) = \mathrm{Cont}^p(\mathfrak{a}) := \left\{ \gamma \in X_\infty \;\middle|\; \mathrm{ord}_\gamma(\mathfrak{a}) = p \right\}.$$

Similarly, we define the closed cylinder

$$\mathrm{Cont}^{\geq p}(Y) = \mathrm{Cont}^{\geq p}(\mathfrak{a}) := \left\{ \gamma \in X_\infty \;\middle|\; \mathrm{ord}_\gamma(\mathfrak{a}) \geq p \right\}.$$

Jet schemes and arc spaces are fundamental objects for the theory of motivic integration, due to Kontsevich [12] and Denef and Loeser [3]. Furthermore, in characteristic 0, using the central result of this theory, the Change of Variable formula, one can show that there is a close link between the log discrepancy defined in terms of divisorial valuations and the geometry of the contact loci in arc spaces. This link was first explored by Mustaţă in [13,14], and then further studied in [2,5,7,9].

The main purpose of this paper is to show that the correspondence between irreducible closed cylinders and divisorial valuations in [5] holds for smooth varieties of arbitrary characteristic. We now explain it as follows. Let X be a smooth integral variety of dimension n over k. An important class of valuations of the function field $k(X)$ of X consists of *divisorial valuations*. These are the valuations of the form

$$\nu = q \cdot \mathrm{ord}_E : k(X)^* \to \mathbf{Z}$$

where E is a divisor over X, (that is, a prime divisor on a normal variety X', having a birational morphism to X) and q is a positive integer number. One can associate an integer number to a divisorial valuation $\nu = q \cdot \mathrm{ord}_E$, called the log discrepancy of ν, equal to $q \cdot (1 + \mathrm{ord}_E(K_{X'/X}))$, where $K_{X'/X}$ is the relative canonical divisor. These numbers determine the log canonical threshold $\mathrm{lct}(X, Y)$ of a pair (X, Y), where Y is a closed subscheme of X.

For every closed irreducible nonempty cylinder $C \subset X_\infty$ which does not dominate X, one defines

$$\mathrm{ord}_C : k(X)^* \to \mathbf{Z}$$

by taking the order of vanishing along the generic point of C. These valuations are called *cylinder valuations*. If C is an irreducible component of $\mathrm{Cont}^{\geq p}(Y)$ for some subscheme Y of X, then C is a cylinder. The valuation ord_C is called a *contact valuation*. It is easy to see that every divisorial valuation is a cylinder valuation. When the ground field is of characteristic zero, Ein, Lazarsfeld and Mustaţă showed the above classes of valuations coincide, by showing that:

(a) Every contact valuation is a divisorial valuation [5, Theorem A];
(b) Every cylinder valuation is a contact valuation [5, Theorem C].

We thus have a correspondence between irreducible closed cylinders that do not dominate X and divisorial valuations. Via this correspondence, one can relate the codimension of the cylinder to the log discrepancy of the divisorial valuation. This yields a quick proof of Mustaţă's log canonical threshold formula.

Theorem 0.1 ([14, Corollary 3.6], [5, Corollary B]) *If X is a smooth complex variety and $Y \subset X$ is a closed subscheme, then the log canonical threshold of the pair (X, Y) is given by*

$$\mathrm{lct}(X, Y) = \min_m \left\{ \frac{\mathrm{codim}(Y_m, X_m)}{m+1} \right\}.$$

The key ingredients in the proofs of the above theorems in [5] are the Change of Variable formula developed in the theory of motivic integration and the existence of log resolutions. While a version of the Change of Variable formula also holds in positive characteristic, the use of log resolutions in the proofs in [5,14] restricted the result to ground fields of characteristic zero. In this paper, we show by induction on the codimensions of cylinders and only using the Change of Variable formula for blow-ups along smooth centers that the above correspondence between divisorial valuations and cylinders holds in arbitrary characteristic.

Theorem A *Let X be a smooth variety of dimension n over a perfect field k. There is a correspondence between irreducible closed cylinders $C \subset X_\infty$ that do not dominate X and divisorial valuations as follows:*

(1) *If C is an irreducible closed cylinder which does not dominate X, then there is a divisor E over X and a positive integer q such that*

$$\mathrm{ord}_C = q \cdot \mathrm{ord}_E .$$

Furthermore, we have $\operatorname{codim} C \geq q \cdot (1 + \mathrm{ord}_E(K_{-/X}))$.

(2) *To every divisor E over X and every positive integer q, we can associate an irreducible closed cylinder C which does not dominate X such that*

$$\mathrm{ord}_C = q \cdot \mathrm{ord}_E \quad \textit{and} \quad \operatorname{codim} C = q \cdot (1 + \mathrm{ord}_E(K_{-/X})).$$

Given E and q, the cylinder C we construct in the proof of Theorem A has the following maximality property: any cylinder C' with $\mathrm{ord}_{C'} = q \cdot \mathrm{ord}_E$ is contained in our C, as in [2]. We are able to prove the log canonical threshold formula avoiding use the log resolutions.

Theorem B *Let X be a smooth variety of dimension n defined over a perfect field k, and Y be a closed subscheme. Then*

$$\mathrm{lct}(X, Y) = \inf_{C \subset X_\infty} \frac{\operatorname{codim} C}{\mathrm{ord}_C(Y)} = \inf_{m \geq 0} \frac{\operatorname{codim}(Y_m, X_m)}{m+1}$$

where C varies over the irreducible closed cylinders which do not dominate X, and $\mathrm{ord}_C(Y) := \mathrm{ord}_\gamma(Y)$ *where γ is the generic point of C.*

Moreover, we will show that log canonical threshold only depends on the asymptotic behavior of jet schemes, i.e., $\mathrm{lct}(X, Y) = \liminf_{m \to \infty} \frac{\operatorname{codim}(Y_m, X_m)}{m+1}$.

The paper is organized as follows. In the first section, we review some basic definitions and notations concerning jet schemes, cylinders and valuations. In Sect. 2 we prove a version of the Change of Variable formula and construct the correspondence between cylinders and divisorial valuations. The proofs of Theorems A and B are also given in Sect. 2. In Sect. 3, we apply the log canonical threshold formula and obtain a semicontinuity result which leads to a comparison theorem via reduction modulo p, as well as a version of inversion of adjunction in positive characteristic.

1 Introduction to jet schemes and log canonical threshold

In this section, we first recall the definition and some basic properties of jet schemes and arc spaces. For a more detailed discussion of jet schemes, see [6,13].

We start with the absolute setting and explain the relative version of jet schemes later. Let k be a field of arbitrary characteristic. In this paper, a variety is a separated scheme of finite type over k. Given a scheme X of finite type over k and an integer $m \geq 0$, the mth order jet scheme X_m of X is a scheme of finite type over k satisfying the following adjunction

$$\mathrm{Hom}_{\mathrm{Sch}/k}(Y, X_m) \cong \mathrm{Hom}_{\mathrm{Sch}/k}(Y \times \mathrm{Spec}\, k[t]/(t^{m+1}), X) \tag{1}$$

for every scheme Y of finite type over k. It follows that if X_m exists, then it is unique up to a canonical isomorphism. We will show the existence in Proposition 1.2.

Let L be a field extension of k. A morphism $\mathrm{Spec}\, L[t]/(t^{m+1}) \to X$ is called an L-valued m-jet of X. If γ_m is a point in X_m, we call it an m-jet of X. If κ is the residue field of γ_m, then γ_m induces a morphism $(\gamma_m)_\kappa : \mathrm{Spec}\, \kappa[t]/(t^{m+1}) \to X$.

It is easy to check that $X_0 = X$. For every $j \leq m$, the natural ring homomorphism $k[t]/(t^{m+1}) \to k[t]/(t^{j+1})$ induces a closed embedding

$$\mathrm{Spec}\, k[t]/(t^{j+1}) \to \mathrm{Spec}\, k[t]/(t^{m+1})$$

and the adjunction (1) induces a truncation map $\rho^m_j : X_m \to X_j$. For simplicity, we usually write π^X_m or simply π_m for the projection $\rho^m_0 : X_m \to X = X_0$. A morphism of schemes $f : X \to Y$ induces morphisms $f_m : X_m \to Y_m$ for every m. At the level of L-valued points, this takes an $L[t]/(t^{m+1})$-valued point γ of X_m to $f \circ \gamma$. For every point $x \in X$, we write $X_{m,x}$ for the fiber of π_m over x, the m-jets of X centered at x.

In Sect. 3, we will use the relative version of jet schemes. We now recall some basic facts about this context.

We work over a fixed separated scheme S of finite type over a noetherian ring R. Let $f : W \to S$ be a scheme of finite type over S. If s is a point in S, we denote by W_s the fiber of f over s.

Definition 1.1 The mth relative jet scheme $(W/S)_m$ satisfies the following adjunction

$$\mathrm{Hom}_{\mathrm{Sch}/S}(Y \times_R \mathrm{Spec}\, R[t]/(t^{m+1}), W) \cong \mathrm{Hom}_{\mathrm{Sch}/S}(Y, (W/S)_m), \tag{2}$$

for every scheme of finite type Y over S.

As in the absolute setting, we have $(W/S)_0 \cong W$. If $(W/S)_m$ and $(W/S)_j$ exist with $m \geq j$, then there is a canonical projection $\rho^m_j : (W/S)_m \to (W/S)_j$. For simplicity, we usually write π_m for the projection $\rho^m_0 : (W/S)_m \to W$.

The proof of the existence of the relative jet schemes is similar to that of the absolute case. For details, see [14].

Proposition 1.2 *If $f : W \to S$ is a scheme of finite type over S, then the mth order relative jet scheme $(W/S)_m$ exists for every $m \in \mathbf{N}$.*

For every scheme morphism $S' \to S$ and every W/S as above, we denote by W' the fiber product $W \times_S S'$. For every point $s \in S$, we denote by W_s the fiber of W over s. By the functorial definition of relative jet schemes, we can check that

$$(W'/S')_m \cong (W/S)_m \times_S S'$$

for every m. In particular, for every $s \in S$, we conclude that the fiber of $(W/S)_m \to S$ over s is isomorphic to $(W_s)_m$.

Recall that $\pi_m : (W/S)_m \to W$ is the canonical projection. We now show that there is an S-morphism, called the zero-section map, $\sigma_m : W \to (W/S)_m$ such that $\pi_m \circ \sigma_m = \mathrm{id}_W$. We have a natural map $g_m : W \times \mathrm{Spec}\, R[t]/(t^{m+1}) \to W$, the projection onto the first factor. By (2), g_m induces a morphism $\sigma^W_m : W \to (W/S)_m$, the *zero-section* of π_m. For simplicity, we usually write σ_m for σ^W_m. One can check that $\pi_m \circ \sigma_m = \mathrm{id}_W$.

Note that for every m and every scheme W over S, there is a natural action:

$$\Gamma_m : \mathbf{A}^1_S \times_S (W/S)_m \to (W/S)_m$$

of the monoid scheme $\mathbf{A}^1_S$ on the jet schemes $(W/S)_m$ defined as follows. For an A-valued point (a, γ_m) of $\mathbf{A}^1_S \times_S (W/S)_m$ where $a \in A$ and $\gamma_m : \mathrm{Spec}\, A[t]/(t^{m+1}) \to W$, we define $\Gamma_m(a, \gamma_m)$ as the composition map $\mathrm{Spec}\, A[t]/(t^{m+1}) \xrightarrow{a^*} \mathrm{Spec}\, A[t]/(t^{m+1}) \xrightarrow{\gamma_m} W$, where a^* corresponds to the A-algebra homomorphism $A[t]/(t^{m+1}) \to A[t]/(t^{m+1})$ mapping t to at. One can check that the image of the zero section σ_m is equal to $\Gamma_m(\{0\} \times (W/S)_m)$.

Lemma 1.3 *Let $f : W \to S$ be a family of schemes and $\tau : S \to W$ a section of f. For every $m \geq 1$, the function*

$$d(s) = \dim \left(\pi_m^{W_s}\right)^{-1} (\tau(s))$$

is upper semi-continuous on S.

Proof Due to the local nature of the assertion, we may assume that $S = \mathrm{Spec}\, A$ is an affine scheme. Given a point $s \in S$, we denote by $w = \tau(s)$ in W. Let W' be an open affine neighborhood of w in W. Consider the restriction map $f' : W' \to S$ of f, one can show that there is an nonzero element $h \in A$ such that τ maps the affine neighborhood $S' \cong \mathrm{Spec}\, A_h$ of s into W'. Let W'' be the affine neighborhood $(f')^{-1}(S')$ of w and $f'' : W'' \to S'$ the induced map. The restriction of τ defines a section $\tau' : S' \to W''$. Replacing f by f'' and τ by τ', we may and will assume that both W and S are affine schemes. Let $W = \mathrm{Spec}\, B$, where B is a finitely generated A-algebra. The section τ induces a ring homomorphism $\tau^* : B \twoheadrightarrow A$. Choose a set of A-algebra generators $u_1, \ldots, u_n$ of B such that $\tau^*(u_i) = 0$. Let C be the polynomial ring $A[x_1, \ldots, x_n]$. We define a ring homomorphism $\varphi : C \to B$ which maps x_i to u_i for every i. Let $I = (f_1, \ldots, f_r)$ be the kernel of φ. One can check that $f_l \in (x_1, \ldots, x_n)$ for every l with $1 \leq l \leq r$. Hence W is a closed subscheme of $\mathbf{A}^n_S = \mathrm{Spec}\, A[x_1, \ldots, x_n]$ defined by the system of polynomials (f_l) and the zero section $o : S \to \mathbf{A}^n_S$ factors through τ. It is clear that $(\mathbf{A}^n_S)_m = \mathrm{Spec}\, A[a_{i,j}] \cong \mathbf{A}^{n(m+1)}_S$ for $1 \leq i \leq n$ and $0 \leq j \leq m$ and $\sigma_m^{\mathbf{A}^n_S} \circ o : S \to \mathbf{A}^{n(m+1)}_S$ is the zero-section.

We thus obtain an embedding $(W/S)_m \subset \mathbf{A}^{(m+1)n}_S$ which induces an embedding $(\pi_m^W)^{-1}(\tau(S)) \subset (\pi_m^{\mathbf{A}^n_S})^{-1}(o(S)) \cong \mathbf{A}^{mn}_S = \mathrm{Spec}\, A[a_{i,j}]$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Given any $v_i = \sum_{j=0}^m a_{i,j} t^j$ in $A[t]/(t^{m+1})$ for $1 \leq i \leq n$, we can write

$$f_l(v_1, \ldots, v_n) = \sum_{p=0}^{m} g_{l,p}(a_{i,j}) t^p, \tag{3}$$

for some polynomials $g_{l,p}$ in $A[a_{i,j}]$ with $1 \leq i \leq n$ and $0 \leq j \leq m$. Recall that $(W/S)_m$ as a subscheme of $\mathbf{A}^{n(m+1)}_S$ is defined by the polynomials $g_{l,p}$ in Eq. (3). Let $\deg a_{i,j} = j$ for $1 \leq i \leq n$ and $1 \leq j \leq m$. Since f_l has no constant terms, we can check that each $g_{l,p}$ is homogenous of degree p. We deduce that the coordinate ring of $(\pi_m^W)^{-1}(\tau(S))$, denoted by T, is isomorphic to $A[a_{i,j}]/(g_{l,p})$. Hence T is a graded A-algebra.

For every $s \in S$ corresponding to a prime ideal $\mathfrak{p}$ of A, we obtain that

$$d(s) = \dim \left(\pi_m^{W_s}\right)^{-1} (\tau(s)) = \dim(T \otimes_A A/\mathfrak{p}).$$

Our assertion follows from a semi-continuity result on the dimension of fibers of a projective morphism (see [4, Theorem 14.8]). □

Remark 1.4 Let X be a smooth variety over a field k and Y a closed subscheme of X. If T is an irreducible component of Y_m for some m, then T is invariant under the action of $\mathbf{A}^1$. Since $\pi_m(T) = \sigma_m^{-1}(T \cap \sigma_m(X))$, it follows that $\pi_m(T)$ is closed in X.

We now turn to the projective limit of jet schemes in the absolute setting. It follows from the description in the proof of Proposition 1.2 that the projective system

$$\cdots \to X_m \to X_{m-1} \to \cdots \to X_0$$

consists of affine morphisms. Hence the projective limit exists in the category of schemes over k. This is called the *space of arcs* of X, denoted by X_∞. Note that in general, it is not of finite type over k. There are natural projection morphisms $\psi_m : X_\infty \to X_m$. It follows from the projective limit definition and the functorial description of the jet schemes that for every field extension L of k, we have

$$\operatorname{Hom}(\operatorname{Spec}(L), X_\infty) \simeq \varprojlim \operatorname{Hom}(\operatorname{Spec} L[t]/(t^{m+1}), X) \simeq \operatorname{Hom}(\operatorname{Spec} L[[t]], X)$$

An L-valued point of X_∞ which corresponds to a morphism from Spec $L[[t]]$ to X is called an *L-valued arc*. We denote the closed point of Spec $L[[t]]$ by 0 and the generic point by η. A point in X_∞ is called an *arc in* X. If γ is a point in X_∞ with residue field κ, γ induces a κ-valued arc, i.e., a morphism γ_κ : Spec $\kappa[[t]] \to X$. If $f : X \to Y$ is a morphism of schemes of finite type, by taking the projective limit of the morphisms $f_m : X_m \to Y_m$ we get a morphism $f_\infty : X_\infty \to Y_\infty$. If X is a smooth variety of pure dimension n over k, then all truncation maps ρ^m_{m-1} are locally trivial with fiber $\mathbf{A}^n$. In particular, all projections $\psi_m : X_\infty \to X_m$ are surjective and $\dim X_m = (m+1)n$.

For every scheme X, a *cylinder* in X_∞ is a subset of the form $C = \psi_m^{-1}(S)$, for some m and some constructible subset $S \subseteq X_m$. From now on, we will assume that X is smooth and of pure dimension n. We say that a cylinder $C = \psi_m^{-1}(S)$ is *irreducible (closed, open, locally closed)* if so is S. It is clear that all these properties of C do not depend on the particular choice of m and S. We define the *codimension of* C by

$$\operatorname{codim} C := \operatorname{codim}(S, X_m) = (m+1)n - \dim S.$$

Since the truncation maps are locally trivial, codim C is independent of the particular choice of m and S.

Let $\mathfrak{a}$ be the defining ideal sheaf of a closed subscheme Z of X. Given an L-valued arc γ : Spec $L[[t]] \to X$, the inverse image of Z by γ is defined by a principal ideal in $L[[t]]$. If this ideal is generated by t^e with $e \geq 0$, then we define the vanishing order of γ along Z to be $\operatorname{ord}_\gamma(Z) = \operatorname{ord}_\gamma(\mathfrak{a}) := e$. On the other hand, if this is the zero ideal, we put $\operatorname{ord}_\gamma(Z) = \operatorname{ord}_\gamma(\mathfrak{a}) = \infty$. If γ is a point in X_∞ having the residue field L, then we define $\operatorname{ord}_\gamma(Z)$ by considering the corresponding morphism Spec $L[[t]] \to X$. The *contact locus of order e* with Z is the subset of X_∞

$$\operatorname{Cont}^e(Z) = \operatorname{Cont}^e(\mathfrak{a}) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma(Z) = e\}.$$

We similarly define

$$\operatorname{Cont}^{\geq e}(Z) = \operatorname{Cont}^{\geq e}(\mathfrak{a}) := \{\gamma \in X_\infty \mid \operatorname{ord}_\gamma(Z) \geq e\}.$$

For $m \geq e$, we can define constructible subsets $\operatorname{Cont}^e(Z)_m$ and $\operatorname{Cont}^{\geq e}(Z)_m$ of X_m in the obvious way. (In fact, the former one is locally closed, while the latter one is closed.) By definition, we have

$$\operatorname{Cont}^e(Z) = \psi_m^{-1}\left(\operatorname{Cont}^e(Z)_m\right) \text{ and } \operatorname{Cont}^{\geq e}(Z) = \psi_m^{-1}\left(\operatorname{Cont}^{\geq e}(Z)_m\right).$$

This implies that $\mathrm{Cont}^{\geq e}(Z)$ is a closed cylinder and $\mathrm{Cont}^{e}(Z)$ is a locally closed cylinder in X_∞.

Let k be a perfect field. In the rest of this section, we review some definitions in the theory of singularities of pairs (X, Y) over k. We refer the reader to [10, Section 2.3] for a more detailed introduction. For varieties over a non-perfect field, a more general definition of log canonical threshold will be introduced in Sect. 3. From now on, we always assume varieties are $\mathbf{Q}$-Gorenstein. Suppose X' is a normal variety over k and $f : X' \to X$ is a birational (not necessarily proper) map. Let E be a prime divisor on X'. Any such E is called a divisor *over* X. The local ring $\mathcal{O}_{X',E} \subset k(X')$ is a DVR which corresponds to a divisorial valuation ord_E on $k(X) = k(X')$. The closure of $f(E)$ in X is called the *center* of E, denoted by $c_X(E)$. If $f' : X'' \to X$ is another birational morphism and $F \subset X''$ is a prime divisor such that $\mathrm{ord}_E = \mathrm{ord}_F$ as valuations of $k(X)$, then we consider E and F to define the same divisor over X.

Let E be a prime divisor over X as above. If Z is a closed subscheme of X, then we define $\mathrm{ord}_E(Z)$ as follows. We may assume that E is a divisor on X' and that the scheme-theoretic inverse image $f^{-1}(Z)$ is an effective Cartier divisor on X'. Then $\mathrm{ord}_E(Z)$ is the coefficient of E in $f^{-1}(Z)$. Recall that the *relative canonical divisor* $K_{X'/X}$ is the unique $\mathbf{Q}$-divisor supported on the exceptional locus of f such that $mK_{X'/X}$ is linearly equivalent with $mK_{X'} - f^*(mK_X)$ for some positive integer m. When X is smooth, we can alternatively describe $K_{X'/X}$ as follows. Let U be a smooth open subset of X' such that $\mathrm{codim}(X' \backslash U, X') \geq 2$. The restriction of f to U is a birational morphism of smooth varieties, we denote it by g. In this case, the relative canonical divisor $K_{U/X}$ is the effective Cartier divisor defined by $\det(dg)$ on U. Since $\mathrm{codim}(X' \backslash U, X') \geq 2$, $K_{U/X}$ uniquely determines a divisor $K_{X'/X}$ on X'.

We also define $\mathrm{ord}_E(K_{-/X})$ as the coefficient of E in $K_{U/X}$. Note that both $\mathrm{ord}_E(Y)$ and $\mathrm{ord}_E(K_{-/X})$ do not depend on the particular choice of f, X' and U.

For every real number $c > 0$, the *log discrepancy* of the pair (X, cY) with respect to E is

$$a(E; X, cY) := \mathrm{ord}_E(K_{-/X}) + 1 - c \cdot \mathrm{ord}_E Y.$$

Let Z be a closed subset of X. A pair (X, cY) is *Kawamata log terminal* (klt for short) around Z if $a(E; X, cY) > 0$ for every divisor E over X such that $c_X(E) \cap Z \neq \emptyset$.

The log canonical threshold of (X, Y) at Z, denoted by $\mathrm{lct}_Z(X, Y)$, is defined as follows: if $Y = X$, we set $\mathrm{lct}_Z(X, Y) = 0$, otherwise

$$\mathrm{lct}_Z(X, Y) = \sup\{c \in \mathbf{R}_{\geq 0} \mid (X, cY) \text{ is klt around } Z\}.$$

In particular, $\mathrm{lct}_Z(X, Y) = \infty$ if and only if $Z \cap Y = \emptyset$. If $Z = X$, we simply write $\mathrm{lct}(X, Y)$ for $\mathrm{lct}_Z(X, Y)$. Given a closed point $x \in X$, we write $\mathrm{lct}_x(X, Y)$ for $\mathrm{lct}_{\{x\}}(X, Y)$.

By the definition of $a(E; X, Y)$, we obtain that

$$\begin{aligned}\mathrm{lct}_Z(X, Y) &= \sup\left\{c \in \mathbf{R} \mid c \cdot \mathrm{ord}_E(Y) < \mathrm{ord}_E(K_{-/X}) + 1 \text{ for all } E \text{ with } c_X(E) \cap Z \neq \emptyset\right\} \\ &= \inf_{E/X} \frac{\mathrm{ord}_E(K_{-/X}) + 1}{\mathrm{ord}_E Y}\end{aligned}$$

where E varies over all divisors over X such that $c_X(E) \cap Z \neq \emptyset$. This also implies that $\mathrm{lct}_Z(X, Y) = \inf_{x \in Z} \mathrm{lct}_x(X, Y)$ where x varies over all closed points of X in Z.

Remark 1.5 The definition of log canonical threshold involves all exceptional divisors over X. In characteristic zero, it is enough to only consider the divisors on a log resolution, see [11, Proposition 8.5]. In particular, we deduce that $\mathrm{lct}_Z(X, Y)$ is a positive rational number. In positive characteristics, it is not clear that $\mathrm{lct}_Z(X, Y) > 0$. We will see in §3 as a corollary of inversion of adjunction that we have, as in characteristic zero, $\mathrm{lct}_x(X, Y) \geq 1/\mathrm{ord}_x(Y) > 0$, for every point $x \in Y$. Here $\mathrm{ord}_x(Y)$ is the maximal integer value q such that the ideal $I_{Y,x} \subseteq m_{X,x}^q$.

2 Cylinder valuations and divisorial valuations

The main goal of this section is to establish the correspondence between cylinders and divisorial valuations as described in the introduction. Let X be a variety over a field k. Recall that a subset C of X_∞ is *thin* if there is a proper closed subscheme Z of X such that $C \subset Z_\infty$.

Lemma 2.1 *Let X be a smooth variety over k. If C is a nonempty cylinder in X_∞, then C is not thin.*

For the proof of Lemma 2.1, see [5, Proposition 1].

Lemma 2.2 *Let $f : X' \to X$ be a proper birational morphism of schemes over k. Let Z be a closed subset of X and $F = f^{-1}(Z)$. If f is an isomorphism over $X \backslash Z$, then the restriction map of f_∞*

$$\varphi : X'_\infty \backslash F_\infty \to X_\infty \backslash Z_\infty$$

is bijective on the L-valued points for every field extension L of k. In particular, φ is surjective.

Proof Since f is proper, the valuative criterion for properness implies that an arc γ : $\mathrm{Spec}\, L[\![t]\!] \to X$ lies in the image of f_∞ if and only if the induced morphism γ_η : $\mathrm{Spec}\, L(\!(t)\!) \to X$ can be lifted to X'. An arc γ is not contained in Z_∞ implies that γ_η factors through $X \backslash Z \hookrightarrow X$. Since f is an isomorphism over $X \backslash Z$, hence there is a unique lifting of γ_η to X'. This shows that φ is surjective. The injectivity of φ follows from the valuative criterion for separatedness of f. The last assertion follows from the fact that a morphism of schemes (not necessary to be of finite type) over k is surjective if the induced map on L-valued points is surjective for every field extension L. □

The Change of Variable Theorem due to Kontsevich [12] and Denef and Loeser [3] will play an important role in our arguments. We now state a special case of this theorem as Lemma 2.3.

Lemma 2.3 *Let X be a smooth variety of dimension n over k and Z a smooth irreducible closed subvariety of codimension $c \geq 2$. Let $f : X' \to X$ be the blow up of X along Z and E the exceptional divisor. Hence $K_{X'/X} = (c-1)E$.*

(a) *For every positive integer e and every $m \geq 2e$, the induced morphism*

$$\psi_m^{X'}\left(\mathrm{Cont}^e\left(K_{X'/X}\right)\right) \to f_m\left(\psi_m^{X'}\left(\mathrm{Cont}^e\left(K_{X'/X}\right)\right)\right)$$

is a piecewise trivial $\mathbf{A}^e$ fibration.

(b) *For every $m \geq 2e$, the fiber of f_m over a point $\gamma_m \in f_m(\psi_m^{X'}(\mathrm{Cont}^e(K_{X'/X})))$ is contained in a fiber of $X'_m \to X'_{m-e}$.*

For the proof of Lemma 2.3, see [1, Theorem 3.3].

Let X be a smooth variety of dimension n over k. For every irreducible cylinder C which does not dominate X, we define a discrete valuation as follows. Let γ be the generic point of C and L the residue field of γ. We thus have an induced ring homomorphism $\gamma^* : \mathcal{O}_{X,\gamma(0)} \to L[\![t]\!]$. Lemma 2.1 implies that $\ker \gamma^*$ is zero. Hence γ^* extends to an injective homomorphism $\gamma^* : k(X) \to L(\!(t)\!)$. We define a map

$$\mathrm{ord}_C : k(X)^* \to \mathbf{Z}$$

by $\mathrm{ord}_C(f) := \mathrm{ord}_\gamma(f) = \mathrm{ord}_t(\gamma^*(f))$. If C does not dominate X, then ord_C is a discrete valuation. If C' is a dense subcylinder of C, then they define the same valuation. Given an element $f \in k(X)^*$, we can check that $\mathrm{ord}_C(f) = \mathrm{ord}_{\gamma'}(f)$ for general point γ' in C. Let Y be a subscheme of X defined by an ideal sheaf $\mathfrak{a}$, we define

$$\mathrm{ord}_C(Y) = \mathrm{ord}_C(\mathfrak{a}) := \mathrm{ord}_\gamma(Y)$$

where γ is the generic point of C. Similarly, we have $\mathrm{ord}_C(Y) = \mathrm{ord}_{\gamma'}(Y)$ for general points γ' of C.

In the rest of this section, we assume that k is a perfect field. We first prove that every valuation defined by a cylinder is a divisorial valuation.

Lemma 2.4 *If C is an irreducible closed cylinder in X_∞ which does not dominate X, then there exist a divisor E over X and a positive integer q such that*

$$\mathrm{ord}_C = q \cdot \mathrm{ord}_E\,. \tag{4}$$

Furthermore, we have $\mathrm{codim}(C) \geq q \cdot (1 + \mathrm{ord}_E(K_{-/X}))$.

Proof We will prove that such divisor E can be reached by a sequence of blow ups of smooth centers after shrinking to suitable open subsets. Let (R, m) be the valuation ring associated to the valuation ord_C. Suppose that C is $\psi_m^{-1}(S)$ for some closed irreducible subset S in X_m. Since X is smooth, we have $\psi_m(C) = S$. Chevalley's Theorem implies that the image of the cylinder C by the projection $\psi_0(C) = \pi_m(S)$ is a constructible set. We denote its closure in X by Z. This is *the center of* ord_C. If C is irreducible and does not dominate X, then Z is a proper reduced irreducible subvariety of X. The generic smoothness theorem implies that there is a nonempty open subset U of X such that $U \cap Z$ is smooth. Since U contains the the generic point of Z, then $C \cap U_\infty$ is an open dense subcylinder of C. Note that U_∞ is an open subset of X_∞, we have

$$\mathrm{codim}(C, X_\infty) = \mathrm{codim}(C \cap U_\infty, U_\infty)$$

and the cylinders $C \subseteq X_\infty$ and $C \cap U_\infty \subseteq U_\infty$ define the same valuation of $k(X) = k(U)$. This implies that we can replace X by U and C by $C \cap U_\infty$. As a consequence, we may and will assume that Z is a smooth subvariety of X.

If Z is a prime divisor on X, then the local ring $\mathcal{O}_{X,Z}$ is a discrete valuation ring of $k(X)$ with maximal ideal $m_{X,Z}$. Given two local rings (A, p) and (B, q) of $k(X)$, we denote by $(A, p) \preceq (B, q)$ if $A \subseteq B$ is a local inclusion, i.e., $p = q \cap A$. This defines a partial order on the set of local rings of $k(X)$. By the definition of Z, we deduce that

$$(\mathcal{O}_{X,Z}, m_{X,Z}) \preceq (R, m).$$

Since every valuation ring is maximal with respect to the partial order $\preceq$, it follows that $\mathcal{O}_{X,Z}$ is equal to the valuation ring R of ord_C, and $\mathrm{ord}_C = q \cdot \mathrm{ord}_Z$ for some integer $q > 0$. Therefore we may take $X' = X$ and $E = Z$, in which case $\mathrm{ord}_E(K_{-/X}) = 0$. The equality $\mathrm{ord}_C Z = q \cdot \mathrm{ord}_Z Z = q$ implies that C is a subcylinder of $\mathrm{Cont}^{\geq q}(E)$. Since E is a smooth divisor, we obtain that $\mathrm{codim}\,\mathrm{Cont}^{\geq q}(E) = q$. This proves the inequality

$$\mathrm{codim}(C) \geq q \cdot (1 + \mathrm{ord}_E(K_{-/X})) = q.$$

We now assume that Z is not a divisor, i.e., codim $Z \geq 2$. Let $f : X' \to X$ be the blow up of X along Z. We claim that there exists an irreducible closed cylinder C' in X'_∞ such that the morphism f_∞ maps C' into C dominantly.

Let e be the vanishing order $\mathrm{ord}_C(K_{X'/X})$. We can assume that $C = (\psi_m^X)^{-1}(S)$ for some closed irreducible subset S in X_m with $m \geq 2e$. The smoothness of X implies that $C \backslash Z_\infty$ is a dense subset of C. Let $F = f^{-1}(Z)$ be the exceptional divisor on X'. It is clear that $f_\infty^{-1}(Z_\infty) = F_\infty$. We denote by

$$\varphi : X'_\infty \backslash F_\infty \to X_\infty \backslash Z_\infty$$

the restriction of f_∞. Let γ be the generic point of C and L the residue field of γ. Hence γ induces a morphism

$$\gamma_L : \mathrm{Spec}\, L[[t]] \to X.$$

Lemma 2.1 implies that $\gamma \in X_\infty \backslash Z_\infty$. By Lemma 2.2, we deduce that φ is bijective on the L-valued points, hence there is a unique L-valued point of X'_∞ mapping to γ_L via φ. We denote by γ' its underlying point in X'_∞. It is clear that $f_\infty(\gamma') = \gamma$. For simplicity we write γ_m for $\psi_m^X(\gamma)$ and γ'_m for $\psi_m^{X'}(\gamma')$. By Lemma 2.3 part (a), we deduce that $f_m^{-1}(\gamma_m)$ is an affine space of dimension e over the residue field of γ_m. Hence the image of $f_m^{-1}(\gamma_m)$ in X'_m, denoted by T, is irreducible. Since γ_m is the generic point of S, there is a unique component of $f_m^{-1}(S)$ which contains T. Let S' be this component and C' the cylinder $(\psi_m^{X'})^{-1}(S')$ in X'_∞. We now check that the closed irreducible cylinder C' satisfies the above conditions. The fact

$$f_m\left(\gamma'_m\right) = f_m \circ \psi_m^{X'}(\gamma') = \psi_m^X(\gamma) = \gamma_m$$

implies that $\gamma'_m \in T$. We deduce that $\gamma' \in C'$. It follows that f_∞ maps C' into C dominantly.

The fact that the center of ord_C on X is Z implies that $\mathrm{ord}_C(F) > 0$, hence $e = \mathrm{ord}_C(K_{X'/X}) > 0$. Lemma 2.3 implies that $f_m : S' \to S$ is a dominant morphism with general fibers of dimension e. We thus have $\dim S' = \dim S + e$, hence

$$\mathrm{codim}\, C' = \dim X'_m - \dim S' = \dim X_m - (\dim S + e) = \mathrm{codim}\, C - e$$

We now set $X^{(0)} = X$, $X^{(1)} = X'$, $C^{(0)} = C$ and $C^{(1)} = C'$. By the construction of C', we deduce that ord_C and $\mathrm{ord}_{C'}$ are equal as valuations of $k(X)$. If the center of $\mathrm{ord}_{C'}$ on X' is not a divisor, then we blow up this center again (we may need to shrink X' to make the center smooth). We now run the above argument for the variety $X^{(1)}$ and $C^{(1)}$ and obtain $X^{(2)}$ and $C^{(2)}$. Since every such blow up decreases the codimension of the cylinder, which is an non-negative integer, we deduce that after s blow ups, the center of the valuation $\mathrm{ord}_{C^{(s)}}$ on $X^{(s)}$ is a divisor, denoted by E. We have

$$\mathrm{ord}_C = \mathrm{ord}_{C^{(1)}} = \cdots = \mathrm{ord}_{C^{(s)}} = q \cdot \mathrm{ord}_E\,.$$

We now check the inequality $\mathrm{codim}\, C \geq q \cdot (1 + \mathrm{ord}_E(K_{-/X}))$. At each step, we have

$$\mathrm{codim}(C) = \mathrm{codim}(C^{(1)}) + \mathrm{ord}_C\left(K_{X^{(1)}/X}\right)$$
$$\mathrm{codim}(C^{(1)}) = \mathrm{codim}(C^{(2)}) + \mathrm{ord}_C\left(K_{X^{(2)}/X^{(1)}}\right)$$
$$\cdots$$
$$\mathrm{codim}(C^{(s-1)}) = \mathrm{codim}(C^{(s)}) + \mathrm{ord}_C\left(K_{X^{(s)}/X^{(s-1)}}\right)$$

We thus obtain that

$$\begin{aligned}\operatorname{codim}(C) &= \operatorname{codim}(C^{(s)}) + \sum_{i=1}^{s} \operatorname{ord}_C\left(K_{X^{(i)}/X^{(i-1)}}\right)\\ &= \operatorname{codim}(C^{(s)}) + \operatorname{ord}_C\left(K_{X^{(s)}/X}\right)\end{aligned}$$

It is clear that $\operatorname{ord}_C(E) = q \cdot \operatorname{ord}_E(E) = q$, hence $C^{(s)} \subseteq \operatorname{Cont}^{\geq q}(E)$, and therefore $\operatorname{codim} C^{(s)} \geq \operatorname{codim} \operatorname{Cont}^{\geq q}(E) = q$. This completes the proof. □

Lemma 2.5 *Let X be a smooth variety and S a constructible subset of X_m for some m.*

(a) $\overline{\psi_m^{-1}(S)} = \psi_m^{-1}(\overline{S})$.
(b) *If U is an open subset of X and C is a cylinder in U_∞, then the closure $\overline{C}$ in X_∞ is a closed cylinder in X_∞.*

Proof We first prove part (a). Since ψ_m is continuous with respect to the Zariski topologies, we deduce that $\psi_m^{-1}(\overline{S})$ is closed. We thus have $\overline{\psi_m^{-1}(S)} \subseteq \psi_m^{-1}(\overline{S})$. If $\overline{\psi_m^{-1}(S)} \neq \psi_m^{-1}(\overline{S})$, then there is an arc $\gamma \in \psi_m^{-1}(\overline{S}) \backslash \overline{\psi_m^{-1}(S)}$. Let U be an affine neighborhood of $\psi_0(\gamma)$ in X and $W = S \cap U_m$. It is clear that

$$\gamma \in \left(\psi_m^U\right)^{-1}(\overline{W}) \backslash \overline{\left(\psi_m^U\right)^{-1}(W)}.$$

In order to get a contradiction, we can replace X by U and S by W. We thus may assume that X is an affine variety. It follows from the construction of jet schemes that X_m are smooth affine varieties. Let $X_m = \operatorname{Spec} A_m$ for every $m \geq 0$. Hence $X_\infty = \operatorname{Spec} A$ where A is the inductive limit $\varinjlim A_m$. We claim that if $\overline{\psi_m^{-1}(S)} \neq \psi_m^{-1}(\overline{S})$, then there is an integer $n \geq m$ such that

$$\overline{\psi_n\left(\overline{\psi_m^{-1}(S)}\right)} \neq \psi_n\left(\psi_m^{-1}(\overline{S})\right).$$

Since $\psi_n(\psi_m^{-1}(\overline{S})) = (\rho_m^n)^{-1}(\overline{S})$ and $\psi_n(\psi_m^{-1}(S)) = (\rho_m^n)^{-1}(S)$, we deduce that

$$\overline{\left(\rho_m^n\right)^{-1}(S)} = \overline{\psi_n\left(\psi_m^{-1}(S)\right)} \subseteq \overline{\psi_n\left(\overline{\psi_m^{-1}(S)}\right)} \subsetneq \left(\rho_m^n\right)^{-1}(\overline{S}).$$

On the other hand, since ρ_m^n is a locally trivial affine bundle with fiber $\mathbf{A}^{\dim X(n-m)}$, we have $\overline{(\rho_m^n)^{-1}(S)} = (\rho_m^n)^{-1}(\overline{S})$. We thus get an contraction.

We now prove the claim. Let I be the radical ideal defining $\overline{\psi_m^{-1}(S)}$ in X_∞ and J the radical ideal defining $\psi_m^{-1}(\overline{S})$. If $\overline{\psi_m^{-1}(S)} \neq \psi_m^{-1}(\overline{S})$, then there is an element $f \in I \backslash J$. There exist an integer $n \geq m$ such that $f \in A_n$. Let $I_n = I \cap A_n$ and $J_n = J \cap A_n$. It is clear that $\psi_n(\psi_m^{-1}(\overline{S}))$ is the closed subset of X_n defined by J_n. Similarly $\overline{\psi_n(\psi_m^{-1}(S))} = (\rho_m^n)^{-1}(\overline{S})$ is the closed subset of X_n defined by the ideal I_n. Since $f \in I_n \backslash J_n$, we thus have the assertion of the claim. This completes the proof of part (a).

For the proof of part (b), let $C = (\psi_m^U)^{-1}(S)$ for some integer $m \geq 0$ and some constructible subset S of U_m. We now consider S as a constructible subset of X_m and apply part (a), we thus obtain $\overline{C} = \overline{\psi_m^{-1}(S)} = (\psi_m^X)^{-1}(\overline{S})$. This completes the proof. □

Lemma 2.6 *Let X and X' be smooth varieties over a field k, and $f : X' \to X$ a blow up with smooth center. If C' is an irreducible closed cylinder of X', then the closure of the image $f_\infty(C')$, denoted by C, is an irreducible cylinder in X'. We also have*

$$\operatorname{ord}_C = \operatorname{ord}_{C'};\ \ \operatorname{codim} C = \operatorname{codim} C' + \operatorname{ord}_{C'}\left(K_{X'/X}\right).$$

Proof Let $e = \operatorname{ord}_{C'}(K_{X'/X})$. For simplicity, we write ψ'_m for $\psi_m^{X'}$ and ψ_m for ψ_m^X for every $m \geq 0$. We first show that C is a closed cylinder. We choose an integer $p \geq e$ and a constructible subset T' of X'_p such that $C' = (\psi'_p)^{-1}(T')$. Let $m = e + p$. We denote by S' the inverse image of T' by the canonical projection $\rho_p^m : X'_m \to X'_p$. Let $S = f_m(S')$. Lemma 2.3 part (b) implies that $f_m^{-1}(f_m(S')) \subseteq (\rho_p^m)^{-1}(T') = S'$. We thus have $f_m^{-1}(f_m(S')) = S'$. It follows that $f_\infty(C') = \psi_m^{-1}(S)$. This implies that

$$C = \overline{f_\infty(C')} = \overline{\psi_m^{-1}(S)} = \psi_m^{-1}(\overline{S})$$

is an irreducible closed cylinder in X_∞. Here the last equality follows from Lemma 2.5 part (a). Since C' dominates C, we have $\operatorname{ord}_C = \operatorname{ord}_{C'}$. The codimension equality follows from the fact that $\dim S' = \dim S + e$ by Lemma 2.3. □

Lemma 2.7 *Let X be a smooth variety over a perfect field k. If $f : Y \to X$ is a birational morphism from a normal variety Y and E is a prime divisor on Y, then for every positive integer q, there exist an irreducible cylinder $C \subset X_\infty$ such that* $\operatorname{ord}_C = q \cdot \operatorname{ord}_E$ *and*

$$\operatorname{codim}(C) = q \cdot (1 + \operatorname{ord}_E(K_{Y/X})) \tag{5}$$

Proof Let ν be the divisorial valuation $q \cdot \operatorname{ord}_E$ on the function field $k(X)$. We define a sequence of varieties and maps as follows. Let $Z^{(0)}$ be the center of ν on X and $X^{(0)} = X$. We choose an open subset $U^{(0)}$ of $X^{(0)}$ such that $Z^{(0)} \cap U^{(0)}$ is a nonempty smooth subvariety of $U^{(0)}$. If $Z^{(0)} \cap U^{(0)}$ is not a divisor, then let $f_1 : X^{(1)} \to U^{(0)}$ be the blow up of $U^{(0)}$ along $Z^{(0)} \cap U^{(0)}$ and $h_1 : X^{(1)} \to X$ the composition of f_1 with the embedding $U^{(0)} \hookrightarrow X$. If $f_i : X^{(i)} \to U^{(i-1)}$ and $h_i : X^{(i)} \to X^{(i-1)}$ are already defined, then we denote by $Z^{(i)}$ the center of ν on $X^{(i)}$. We pick an open subset $U^{(i)} \subset X^{(i)}$ such that $Z^{(i)} \cap U^{(i)}$ is a smooth subvariety of $U^{(i)}$. If $Z^{(i)}$ is not a divisor, then we denote by $f_{i+1} : X^{(i+1)} \to U^{(i)}$ the blow up of U_i along $Z^{(i)} \cap U^{(i)}$ and $h_{i+1} : X^{(i+1)} \to X^{(i)}$ the composition of f_{i+1} with the embedding $U^{(i)} \to X^{(i)}$. By [10, Lemma 2.45], we know there is an integer $s \geq 0$ such that $Z^{(s)}$ is a prime divisor on $U^{(s)}$ and $\operatorname{ord}_{Z^{(s)}} = \operatorname{ord}_E$. Hence we can replace Y by a smooth variety $U^{(s)}$ and $E = Z^{(s)} \cap U^{(s)}$. We write $g_i : Y \to X^{(i)}$ for the composition of morphisms h_j for j with $i < j \leq s$ and the embedding $U^{(s)} \subset X^{(s)}$.

Let C_s be the locally closed cylinder $\operatorname{Cont}^q(E)$ in Y_∞ and C_0 the closure of its image $(g_0)_\infty(C_s)$ in X_∞. It is clear that $\operatorname{codim} C_s = q$. We now show that $C = C_0$ is a cylinder that satisfies our conditions. For every i with $1 \leq i \leq s$, we denote by C_i the closure of the image of C_s in $X_\infty^{(i)}$ under the map $(g_i)_\infty : Y_\infty \to X_\infty^{(i)}$. Similarly, we denote by D_i the closure of the image of C_s in $U_\infty^{(i)}$. It is clear that D_i is the closure of the image of C_{i+1} in $U_\infty^{(i)}$ under the map $(f_{i+1})_\infty : X_\infty^{(i+1)} \to U_\infty^{(i)}$ and C_i is the closure of D_i in $X_\infty^{(i)}$. By Lemmas 2.6 and 2.5 part (b), using descending induction on $i < s$, we deduce that D_i is a cylinder in $U_\infty^{(i)}$ and C_i is a cylinder in $(X^{(i)})_\infty$. We also deduce that $\operatorname{ord}_{C_i} = \operatorname{ord}_{D_i} = \operatorname{ord}_{C_{i+1}}$ and

$$\operatorname{codim} C_i = \operatorname{codim} D_i = \operatorname{codim} C_{i+1} + \operatorname{ord}_{C_i}\left(K_{X^{(i+1)}/X^{(i)}}\right).$$

We thus obtain $\mathrm{ord}_C = \mathrm{ord}_{C_1} = \cdots = \mathrm{ord}_{C_s} = q \cdot \mathrm{ord}_E$ and

$$\begin{aligned}\mathrm{codim}\, C &= \mathrm{codim}\, C_1 + \mathrm{ord}_C \left(K_{X^{(1)}/X} \right) \\ &\cdots \\ &= \mathrm{codim}\, C_s + \sum_{i=0}^{s-1} \mathrm{ord}_C \left(K_{X^{(i+1)}/X^{(i)}} \right) = q + q \cdot \mathrm{ord}_E(K_{Y/X}).\end{aligned}$$

□

It is clear that Theorem A follows from Lemmas 2.4 and 2.7. We now prove Theorem B.

Proof If $Y = X$, the assertion is trivial. Hence we may and will assume Y is a proper closed subscheme of X. By Theorem A, we deduce that

$$\mathrm{lct}(X, Y) := \inf_E \frac{1 + \mathrm{ord}_E(K_{-/X})}{\mathrm{ord}_E(Y)} = \inf_C \frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)}.$$

where C varies over the irreducible closed cylinders which do not dominate X.

We first show that

$$\mathrm{lct}(X, Y) \leq \inf_{m \geq 0} \frac{\mathrm{codim}(Y_m, X_m)}{m+1} \tag{†}$$

For every $m \geq 0$, let S_m be an irreducible component of Y_m which computes the codimension of Y_m in X_m and C_m the closed irreducible cylinder $\psi_m^{-1}(S_m)$ in X_∞. We thus obtain

$$\mathrm{codim}(C_m) = \mathrm{codim}(S_m, X_m) = \mathrm{codim}(Y_m, X_m).$$

The image $\psi_0(C_m) = \rho_0^m(S_m)$ is contained in Y, which implies that C_m does not dominate X. By the definition of contact loci, we know that $Y_m = \mathrm{Cont}^{\geq m+1}(Y)_m$ in X_m. This implies that $\mathrm{ord}_{C_m}(Y) \geq m + 1$. We conclude that

$$\mathrm{lct}(X, Y) \leq \frac{\mathrm{codim}(C_m)}{\mathrm{ord}_{C_m}(Y)} = \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

Taking infimum over all integers $m \geq 0$, we now have the inequality (†).

We now prove the reverse inequality. Let C be an irreducible closed cylinder which does not dominate X. If $\mathrm{ord}_C(Y) = 0$, then $\frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} = \infty$. Hence

$$\frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} \geq \inf_{m \geq 0} \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

From now on, we may and will assume that $\mathrm{ord}_C(Y) > 0$. Let $m = \mathrm{ord}_C(Y) - 1$. Since C is a subcylinder of the contact locus $\mathrm{Cont}^{\geq m+1}(Y) = \psi_m^{-1}(Y_m)$, we have

$$\frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} \geq \frac{\mathrm{codim}(Y_m, X_m)}{m+1} \geq \inf_{m \geq 0} \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

We now take infimum over all cylinders C which do not dominate X and obtain

$$\mathrm{lct}(X, Y) = \inf_C \frac{\mathrm{codim}\, C}{\mathrm{ord}_C(Y)} \geq \inf_{m \geq 0} \frac{\mathrm{codim}(Y_m, X_m)}{m+1}.$$

□

Let X be a smooth variety over a perfect field k, Y a closed subscheme of X, and Z a closed subset of X. Recall that

$$\operatorname{lct}_Z(X, Y) = \inf_{E/X} \frac{\operatorname{ord}_E(K_{-/X}) + 1}{\operatorname{ord}_E Y}$$

where E varies over all divisors over X whose center in X intersects Z. By the correspondence in Theorem A(2), we deduce that for every such divisor E over X, the corresponding irreducible closed cylinder C satisfies $\overline{\psi_0^X(C)} \cap Z \neq \emptyset$. Applying the argument in the proof of Theorem B, we can show the following generalized log canonical threshold formula in terms of jet schemes.

Proposition 2.8 *Let (X, Y) be a pair over a perfect field k and Z a closed subset of X. We have*

$$\operatorname{lct}_Z(X, Y) = \inf_{C \subset X_\infty} \frac{\operatorname{codim} C}{\operatorname{ord}_C(Y)} = \inf_{m \geq 0} \frac{\operatorname{codim}_Z(Y_m, X_m)}{m+1}$$

where C varies over all irreducible closed cylinders with $\overline{\psi_0(C)} \cap Z \neq \emptyset$, $\overline{\psi_0(C)} \neq X$, and $\operatorname{codim}_Z(Y_m, X_m)$ is the minimum codimension of an irreducible component T of Y_m such that $\overline{\pi_m(T)} \cap Z \neq \emptyset$.

Remark 2.9 We have seen that

$$\operatorname{lct}(X, Y) := \inf_E \frac{1 + \operatorname{ord}_E(K_{-/X})}{\operatorname{ord}_E(Y)} = \inf_C \frac{\operatorname{codim} C}{\operatorname{ord}_C(Y)} = \inf_{m \geq 0} \frac{\operatorname{codim}(Y_m, X_m)}{m+1}.$$

If one of the infimums can be achieved, then so are the other two. For example, when the base field k is of characteristic 0, the existence of log resolutions of (X, Y) implies that $\operatorname{lct}(X, Y)$ can be computed at some divisors E on a log resolution. Hence in characteristic zero, all the infimums can be replaced by minimums.

We now show that log canonical threshold only depends on the asymptotic behavior of jet schemes, i.e., $\operatorname{lct}(X, Y) = \liminf_{m \to \infty} \frac{\operatorname{codim}(Y_m, X_m)}{m+1}$. This is clear if there is no m computing $\operatorname{lct}(X, Y)$. Now we assume that there is one positive integer m such that

$$\operatorname{lct}(X, Y) = \frac{\operatorname{codim}(Y_m, X_m)}{m+1},$$

then there is a divisor E over X that computes the log canonical threshold. Theorem A implies that for every integer $q \geq 0$, there is a cylinder $C_q \subset X_\infty$ such that $\operatorname{ord}_{C_q} = q \operatorname{ord}_E$ and $\operatorname{codim} C_q = q \cdot (\operatorname{ord}_E(K_{-/X}) + 1)$. Let $m_q := \operatorname{ord}_{C_q}(Y) - 1 = q \operatorname{ord}_E(Y) - 1$. We obtain that $\operatorname{lct}(X, Y) = \frac{\operatorname{codim} C_q}{ord_{C_q}(Y)} \geq \frac{\operatorname{codim}(Y_{m_q}, X_{m_q})}{m_q+1}$ for every q. Hence there is a sequence $\{m_q\}$ such that $\operatorname{lct}(X, Y) = \frac{\operatorname{codim}(Y_{m_q}, X_{m_q})}{m_q+1}$. In particular, we deduce that

$$\operatorname{lct}(X, Y) = \liminf_{m \to \infty} \frac{\operatorname{codim}(Y_m, X_m)}{m+1}.$$

Similarly $\operatorname{lct}_Z(X, Y) = \liminf_{m \to \infty} \frac{\operatorname{codim}_Z(Y_m, X_m)}{m+1}$. We leave the proof to the reader.

Moreover, we can compute $\operatorname{lct}_x(X, Y)$ at a closed point x in terms of the asymptotic behavior of the jet schemes centered at x.

Proposition 2.10 *Let X be a smooth variety over a perfect field k and Y a closed subscheme of X. For every closed point x,*

$$\operatorname{lct}_x(X, Y) = \dim X - \limsup_{m\to\infty} \frac{\dim Y_{m,x}}{m+1}.$$

Proof Since x is a closed point, we have $\dim Y_{m,x} \le \dim T_m \le \dim Y_{m,x} + \dim Y$ for any component T_m of Y_m that computing $\operatorname{codim}_x(Y_m, X_m)$. We have

$$\begin{aligned}\operatorname{lct}_x(X, Y) &= \liminf_{m\to\infty} \frac{\operatorname{codim}(T_m, X_m)}{m+1}\\ &= \dim X - \limsup_{m\to\infty} \frac{\dim T_m}{m+1}\\ &= \dim X - \limsup_{m\to\infty} \frac{\dim Y_{m,x}}{m+1}.\end{aligned}$$

□

3 The log canonical threshold via jets

In this section, we apply Theorem B to deduce properties of log canonical threshold for pairs. We first assume that k is a perfect field. We denote by $\overline{k}$ the algebraic closure of k. For every scheme X over k, we write $\overline{X}$ for the fiber product $X \times_k \operatorname{Spec} \overline{k}$.

Corollary 3.1 *Let X be a smooth variety over a perfect field k and Y a closed subscheme of X. We have*

$$\operatorname{lct}(X, Y) = \operatorname{lct}(\overline{X}, \overline{Y}).$$

Proof For every scheme Z over the field k, we know that $\dim Z = \dim \overline{Z}$. We thus have for every $m \ge 0$,

$$\operatorname{codim}(Y_m, X_m) = \operatorname{codim}(\overline{Y}_m, \overline{X}_m).$$

Our assertion follows from Theorem B. □

We now generalize the notion of log canonical threshold for pairs over perfect fields to those defined over arbitrary fields. Let X be a smooth variety over a field k, Y a closed subscheme of X and Z a closed subset of X. Recall that when k is perfect, we have $\operatorname{lct}_Z(X, Y) = \inf_{x\in Z} \operatorname{lct}_x(X, Y)$ where x varies over the closed points in Z. We first define the log canonical threshold of (X, Y) at a closed point by passing to the algebraic closure. If x is a closed point of X, then the fiber of the map $\overline{X} \to X$ over x are a finite set $\{x_1, \ldots, x_l\}$. We define

$$\operatorname{lct}_x(X, Y) := \operatorname{lct}_z(\overline{X}, \overline{Y})$$

for some z in the fiber of $\overline{X}$ over x. Let G be the Galois group of the field extension $\overline{k}$ over k. It is clear that G acts on the fiber $\{x_1, \ldots, x_l\}$ transitively. For every $g \in G$ and every $z \in \{x_1, \ldots, x_l\}$, g induces an isomorphism between the jet schemes $\overline{Y}_{m,g(z)}$ and $\overline{Y}_{m,z}$. By

Proposition 2.10, we have

$$\begin{aligned}\mathrm{lct}_z(\overline{X},\overline{Y}) &= \dim \overline{X} - \limsup_{m\to\infty} \frac{\dim \overline{Y}_{m,z}}{m+1} \\ &= \dim \overline{X} - \limsup_{m\to\infty} \frac{\dim \overline{Y}_{m,g(z)}}{m+1} \\ &= \mathrm{lct}_{g(z)}(\overline{X},\overline{Y}).\end{aligned}$$

Hence our definition of $\mathrm{lct}_x(X, Y)$ does not depend on the choice of z. We define

$$\mathrm{lct}_Z(X,Y) := \inf_{x\in Z} \mathrm{lct}_x(X,Y)$$

where x varies over the closed points in Z. We can check that $\mathrm{lct}_Z(X, Y) = \mathrm{lct}_{\overline{Z}}(\overline{X}, \overline{Y})$. Since dimension does not change if we pass to the algebraic closure, the description of log canonical threshold in terms of jet schemes still holds. Hence we generalize Theorem B, Propositions 2.8, and 2.10 to an arbitrary field.

Remark 3.2 There is a naive definition of log canonical threshold of pairs over an arbitrary field, that is $\mathrm{lct}_Z(X, Y) := \inf_{E/X} \frac{\mathrm{ord}_E(K_{-/X})+1}{\mathrm{ord}_E(Y)}$ where E varies over all divisors over X with $C_X(E) \cap Z \neq \emptyset$. However this definition is not compatible with inseparable extensions. For instance, let k be an algebraically closed field of characteristic p and $K = k(s)$ the function field of $\mathbf{A}^1_k$. Let X be the affine space Spec $K[x]$ and Y the closed subscheme of X defined by $x^p - s$. We consider the pair (X, Y) over K. Since Y is a prime divisor on X and X is a smooth curve over K, we have $\mathrm{lct}(X, Y) = 1$. Let $\overline{K}$ be the algebraic closure of K. We thus have $X_{\overline{K}} = \mathbf{A}^1_{\overline{K}}$ and $Y_{\overline{K}}$ is a nonreduced subscheme of $X_{\overline{K}}$ defined by $(x - s^{1/p})^p$. One can check that $\mathrm{lct}(X_{\overline{K}}, Y_{\overline{K}}) = 1/p$.

Our next corollary of Theorem B is a semicontinuity result for log canonical thresholds. Let $f : X \to S$ be a smooth morphism and Y a closed subscheme. Let $\tau : S \to Y$ be a section of $f|_Y$, which is the restriction of f to Y. For every point $s \in S$, we denote by X_s the fiber of X over s and by Y_s the fiber of Y over s. We also denote by κ_s the residue field of s.

Corollary 3.3 *Let $f : X \to S$ be a smooth morphism of relative dimension n and Y a closed subscheme of X. Let $\tau : S \to Y$ be a section of $f|_Y$. If t is a point in S, then for every point s in the closure $\overline{\{t\}}$, we have*

$$\mathrm{lct}_{\tau(t)}(X_t, Y_t) \geq \mathrm{lct}_{\tau(s)}(X_s, Y_s).$$

Proof The smoothness of f implies that for every $s \in S$, X_s is a smooth variety of dimension n over the field κ_s. In particular,

$$\dim(X_s)_m = (m+1)n = \dim(X_t)_m.$$

Applying the generalized Proposition 2.10, we obtain

$$\mathrm{lct}_{\tau(s)}(X_s, Y_s) = n - \limsup_{m\to\infty} \frac{\dim(Y_s)_{m,\tau(s)}}{m+1}.$$

Similarly, we have

$$\mathrm{lct}_{\tau(t)}(X_t, Y_t) = n - \limsup_{m\to\infty} \frac{\dim(Y_t)_{m,\tau(t)}}{m+1}.$$

In order to complete the proof, it is enough to show that for every $m \geq 0$,

$$\dim(Y_s)_{m,\tau(s)} \geq \dim(Y_t)_{m,\tau(t)}.$$

Recall that $(Y/S)_m$ is the mth relative jet scheme of Y/S. By Lemma 1.3, we deduce that for every $m \geq 0$, the function

$$d(s) = \dim(Y_s)_{m,\tau(s)}$$

is upper semi-continuous on S. This completes the proof. □

Remark 3.4 If every point in S has a residue field of characteristic 0, then Corollary 3.3 has a stronger version that the function defined by $g(s) = \mathrm{lct}_{\tau(s)}(X_s, Y_s)$ is lower-semicontinuous. We refer the reader to [14, Theorem 4.9] for a more detailed proof.

We now prove a comparison result in the setting of reduction to prime characteristic. Suppose that X is the affine variety $\mathbf{A}^n_{\mathbf{Z}}$ over the ring $\mathbf{Z}$ and $\mathfrak{a} \subset \mathbf{Z}[x_1, \ldots, x_n]$ is an ideal such that $\mathfrak{a} \subset (x_1, \ldots, x_n)$. Let Y be the subscheme of X defined by $\mathfrak{a}$ and

$$\tau : \operatorname{Spec} \mathbf{Z} \to Y$$

the section corresponding to the ring homomorphism $\mathbf{Z}[x_1, \ldots, x_n]/\mathfrak{a} \to \mathbf{Z}$ which maps the image of x_i to zero for each i. For every prime number p, let $X_{\mathfrak{p}}$ be the affine space $\mathbf{A}^n_{\mathbf{F}_p}$ and $Y_{\mathfrak{p}}$ the subscheme of $X_{\mathfrak{p}}$ defined by $\mathfrak{a} \cdot \mathbf{F}_p[x_1, \ldots, x_n]$. Note that a log resolution of $(X_{\mathbf{Q}}, Y_{\mathbf{Q}})$ induces a log resolution of the pair $(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ for p large enough. Let E be a divisor on a log resolution of (X, Y) which computes $\mathrm{lct}_0(X_{\mathbf{Q}}, Y_{\mathbf{Q}})$. When p is large enough, the divisor obtained by reduction modulo p of E also computes the log canonical threshold $\mathrm{lct}_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$. It follows that $\mathrm{lct}_0(X_{\mathbf{Q}}, Y_{\mathbf{Q}}) = \mathrm{lct}_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ for all but finitely many p. Applying Corollary 3.3, we obtain the following inequality for every prime p.

Corollary 3.5 *If (X, Y) is a pair as above, then for every prime integer p, we have*

$$\mathrm{lct}_0(X_{\mathbf{Q}}, Y_{\mathbf{Q}}) \geq \mathrm{lct}_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}}).$$

A similar corollary holds when the base scheme has dimension 1, with generic points of characteristic 0. Corollary 3.5 has an application to an open problem about the connection between log canonical thresholds and F-pure thresholds. Recall that in positive characteristic, Takagi and Watanabe [15] introduced an analogue of the log canonical threshold, the F-pure threshold. With the above notation, it follows from [8] that $\mathrm{lct}_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}}) \geq \mathrm{fpt}_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ for every prime p, where $\mathrm{fpt}_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ is the F-pure threshold of the pair $(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ at 0. By combining this with Corollary 3.5, we obtain the following result, which seems to have been an open question.

Corollary 3.6 *With the above notation, we have* $\mathrm{lct}_0(X_{\mathbf{Q}}, Y_{\mathbf{Q}})) \geq \mathrm{fpt}_0(X_{\mathfrak{p}}, Y_{\mathfrak{p}})$ *for every prime p.*

Corollary 3.7 *Let X be a smooth variety over a field k and Y a closed subscheme of X. If H is a smooth irreducible divisor on X which intersects Y and $Z \subset H$ is a nonempty closed subset, then*

$$\mathrm{lct}_Z(X, Y) \geq \mathrm{lct}_Z(H, H \cap Y).$$

Proof The case $H \cap Y = H$ is trivial since $\mathrm{lct}_Z(H, H \cap Y) = 0$. We may thus assume $Y \cap H \neq H$. Similarly, if $Z \cap Y = \emptyset$, then both $\mathrm{lct}_Z(X, Y)$ and $\mathrm{lct}_Z(H, H \cap Y)$ are equal to ∞. We will assume $Z \cap Y \neq \emptyset$ from now on.

By Proposition 2.8, we only have to prove that for every $m \geq 0$,

$$\mathrm{codim}_Z(Y_m, X_m) \geq \mathrm{codim}_Z((H \cap Y)_m, H_m).$$

Let T be an irreducible component of Y_m such that

$$\pi_m(T) \cap Z \neq \emptyset \quad \text{and} \quad \mathrm{codim}\, T = \mathrm{codim}_Z(Y_m, X_m).$$

Since H is a Cartier divisor on X, $H \cap Y$ is defined locally in Y by one equation. This implies that $(H \cap Y)_m = H_m \cap Y_m$ is defined locally in Y_m by $m + 1$ equations. If

$$\pi_m(T \cap H_m) \cap Z \neq \emptyset,$$

then there is a component of $T \cap H_m$, denoted by S, such that $\pi_m(S) \cap Z \neq \emptyset$ and $\dim S \geq \dim T - (m + 1)$. Note that $\dim X_m = \dim H_m + m + 1$ and we conclude that

$$\mathrm{codim}_Z((H \cap Y)_m, H_m) \leq \mathrm{codim}(S, H_m) \leq \mathrm{codim}(T \cap H_m, H_m) \leq \mathrm{codim}(T, X_m).$$

We now prove that $\pi_m(T \cap H_m) \cap Z \neq \emptyset$. Let $\gamma_m \in T$ such that $\pi_m(\gamma_m) \in Z$. Recall that $\sigma_m : Y \to Y_m$ is the zero section. Since T is invariant under the action of $\mathbf{A}^1$, the orbit of γ_m is a subset of T. In particular, $\sigma_m(\pi_m(\gamma_m)) \in T$. Since the zero section is functorial by its construction, we get $\sigma_m(Y \cap H) \subset Y_m \cap H_m$. In particular, $\sigma_m(\pi_m(\gamma_m))$ is in $T \cap H_m$ and its image under π_m is in Z. This completes our proof. □

Corollary 3.8 *If X is a smooth projective variety over a field k and $Y \subset X$ is a proper closed subscheme, then we have* $\mathrm{lct}(X, Y) > 0$.

Proof Since log canonical thresholds are computed after passing to an algebraic closure of k, we can assume k is algebraically closed. It follows from the definition that

$$\mathrm{lct}(X, Y) = \inf_{x \in Y} \mathrm{lct}_x(X, Y).$$

For every closed point $x \in Y$, we will show that

$$\mathrm{lct}_x(X, Y) \geq \frac{1}{\mathrm{ord}_x(Y)}. \tag{6}$$

We thus have $\mathrm{lct}_x(X, Y) \geq \frac{1}{d}$ where $d = \max_{x \in Y} \mathrm{ord}_x(Y)$. Here $\mathrm{ord}_x(Y)$ is the maximal integer q such that $I_{Y,x} \subseteq m_{X,x}^q$, where $m_{X,x}$ is the ideal sheaf defining x.

We prove the inequality (6) by induction on $\dim X$. If X is a smooth curve, then it follows from definition that $\mathrm{lct}_x(X, Y) = \frac{1}{\mathrm{ord}_x(Y)}$. We now assume that $\dim X \geq 2$. After replacing X by an open neighborhood of x, we may find H, a smooth divisor passing through x, such that $\mathrm{ord}_x(H \cap Y) = \mathrm{ord}_x(Y)$. By Corollary 3.7, we have

$$\mathrm{lct}_x(X, Y) \geq \mathrm{lct}_x(H, H \cap Y) \geq \frac{1}{\mathrm{ord}_x(H \cap Y)} = \frac{1}{\mathrm{ord}_x(Y)}.$$

This completes the proof. □

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