



Transcendental p -adic continued fractions

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Abstract We establish a new transcendence criterion of p -adic continued fractions which are called Ruban continued fractions. By this result, we give explicit transcendental Ruban continued fractions with bounded p -adic absolute value of partial quotients. This is p -adic analogy of Baker's result. We also prove that p -adic analogy of Lagrange theorem for Ruban continued fractions is not true.

Keywords Transcendental numbers · Continued fractions · Diophantine approximations · p -Adic numbers

Mathematics Subject Classification 11J81 · 11J70 · 11J61

1 Introduction

Maillet [7] is the first person who gave explicit transcendental continued fractions with bounded partial quotients. After that, Baker [1] extended Maillet's results with LeVeque Theorem [5] which is Roth Theorem for algebraic number fields.

There exist continued fraction expansions in p -adic number field \mathbb{Q}_p , not just in \mathbb{R} . Schneider [9] was motivated by Mahler's work [6] and gave an algorithm of p -adic continued fraction expansion. In the same year, Ruban [8] also gave an different algorithm of p -adic continued fraction expansion. Ubolsri, Laohakosol, Deze, and Wang gave several transcendence criteria for Ruban continued fractions (see [3, 12, 14, 15]). The proofs are mainly based on the theory of p -adic Diophantine approximations. However, they only studied Ruban continued fractions with unbounded p -adic absolute value of partial quotients. In this paper, we study analogy of Baker's transcendence criterion for Ruban continued fractions with bounded p -adic absolute value of partial quotients.

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Let p be a prime. We denote by $|\cdot|_p$ the valuation normalized to satisfy $|p|_p = 1/p$. A function $[\cdot]_p$ is given by the following:

$$[\cdot]_p : \mathbb{Q}_p \rightarrow \mathbb{Q}; [\alpha]_p = \begin{cases} \sum_{n=m}^0 c_n p^n & (m \leq 0), \\ 0 & (m > 0), \end{cases}$$

where $\alpha = \sum_{n=m}^{\infty} c_n p^n$, $c_n \in \{0, 1, \dots, p - 1\}$, $m \in \mathbb{Z}$, $c_m \neq 0$. The function is called a *p-adic floor function*. If $\alpha \neq [\alpha]_p$, then we can write α in the form

$$\alpha = [\alpha]_p + \frac{1}{\alpha_1}$$

with $\alpha_1 \in \mathbb{Q}_p$. Note that $|\alpha_1|_p \geq p$ and $[\alpha_1]_p \neq 0$. Similarly, if $\alpha_1 \neq [\alpha_1]_p$, then we have

$$\alpha_1 = [\alpha_1]_p + \frac{1}{\alpha_2}$$

with $\alpha_2 \in \mathbb{Q}_p$. We continue the above process provided $\alpha_n \neq [\alpha_n]_p$. In this way, it follows that α can be written in the form

$$\alpha = [\alpha]_p + \frac{1}{[\alpha_1]_p + \frac{1}{[\alpha_2]_p + \frac{1}{\ddots [\alpha_{n-1}]_p + \frac{1}{\alpha_n}}}}$$

For simplicity of notation, we write the continued fraction

$$[[\alpha]_p, [\alpha_1]_p, [\alpha_2]_p, \dots, [\alpha_{n-1}]_p, \alpha_n].$$

α_n is called the *n-th complete quotient* and either $[\alpha]_p$ or $[\alpha_n]_p$ is called a *partial quotient*. If the above process stopped in a certain step, then

$$\alpha = [[\alpha]_p, [\alpha_1]_p, [\alpha_2]_p, \dots, [\alpha_{n-1}]_p, [\alpha_n]_p]$$

is called a *finite Ruban continued fraction*. Otherwise, in the same way, we have

$$\alpha = [[\alpha]_p, [\alpha_1]_p, [\alpha_2]_p, \dots, [\alpha_{n-1}]_p, [\alpha_n]_p, \dots]$$

which is called an *infinite Ruban continued fraction*. As a remark, according to the fact that the Hensel expansion of a *p*-adic number is unique, we have the uniqueness of Ruban continued fraction expansions.

We define $S_p = \{[\alpha]_p \mid \alpha \in \mathbb{Q}_p\}$, $S'_p = \{[\alpha]_p \mid |\alpha|_p \geq p \text{ for } \alpha \in \mathbb{Q}_p\}$. Let $(a_i)_{i \geq 0}$ be a sequence with $a_0 \in S_p$ and $a_i \in S'_p$ for all $i \geq 1$, and $(n_i)_{i \geq 0}$ be an increasing sequence of positive integers. Let $(\lambda_i)_{i \geq 0}$ and $(k_i)_{i \geq 0}$ be sequences of positive integers. Assume that for all i ,

$$\begin{aligned} n_{i+1} &\geq n_i + \lambda_i k_i \\ a_{m+k_i} &= a_m \text{ for } n_i \leq m \leq n_i + (\lambda_i - 1)k_i - 1. \end{aligned}$$

Consider a *p*-adic number α defined by

$$\alpha = [a_0, a_1, a_2, \dots, a_n, \dots].$$

Then α is called a *quasi-periodic Ruban continued fraction*.

The main theorem is the following.

Theorem 1 Let $(a_i)_{i \geq 0}, (n_i)_{i \geq 0}, (\lambda_i)_{i \geq 0}$, and $(k_i)_{i \geq 0}$ be as in the above, and $A \geq p$ be a real number. Assume that $(a_i)_{i \geq 0}$ is a non-ultimately periodic sequence such that $|a_i|_p \leq A$ for each i . If $a_{n_i} = a_{n_i+1} = \dots = a_{n_i+k_i-1} = (p-1) + (p-1)p^{-1} = p - p^{-1}$ for infinitely many i and

$$\liminf_{i \rightarrow \infty} \frac{\lambda_i}{n_i} > B = B(A),$$

where B is defined by

$$B = \frac{2 \log A}{\log p} - 1,$$

then α is transcendental.

As a remark, a sequence $(a_n)_{n \geq 0}$ is said to be *ultimately periodic* if there exist integers $k \geq 0$ and $l \geq 1$ such that $a_{n+l} = a_n$ for all $n \geq k$.

For example, the following p -adic numbers are transcendental:

$$\left[0, \overline{p - p^{-1} \cdot 2 \cdot 3^0}, \overline{p^{-1} \cdot 2 \cdot 3^1}, \overline{p - p^{-1} \cdot 2 \cdot 3^2}, \overline{p^{-1} \cdot 2 \cdot 3^3}, \dots, \overline{p - p^{-1} \cdot 2 \cdot 3^{2m}}, \overline{p^{-1} \cdot 2 \cdot 3^{2m+1}}, \dots \right], \tag{1}$$

$$\left[0, \overline{p^{-1}, p^{-2} \cdot 8 \cdot 17^0}, \overline{p - p^{-1}, p - p^{-1} \cdot 8 \cdot 17^1}, \overline{p^{-1}, p^{-2} \cdot 8 \cdot 17^2}, \overline{p - p^{-1}, p - p^{-1} \cdot 8 \cdot 17^3}, \dots, \overline{p^{-1}, p^{-2} \cdot 8 \cdot 17^{2m}}, \overline{p - p^{-1}, p - p^{-1} \cdot 8 \cdot 17^{2m+1}}, \dots \right], \tag{2}$$

where $2 \cdot 3^i$ and $8 \cdot 17^i$ indicate the number of times a block of partial quotients is repeated. (1) is the case that for $i \geq 0$, $a_{n_{2i}} = p - p^{-1}$, $a_{n_{2i-1}} = p^{-1}$, $n_i = 3^i$, $\lambda_i = 2 \cdot 3^i$, $k_i = 1$, $A = p$ in Theorem 1. (2) is the case that for $i \geq 0$, $a_{n_{2i}} = p^{-1}$, $a_{n_{2i+1}} = p^{-2}$, $a_{n_{2i+1}} = a_{n_{2i+1}+1} = p - p^{-1}$, $n_i = 17^i$, $\lambda_i = 8 \cdot 17^i$, $k_i = 2$, $A = p^2$ in Theorem 1.

A well-known Lagrange’s theorem states that the continued fraction expansion for a real number is ultimately periodic if and only if the number is quadratic irrational. For Schneider continued fractions, p -adic analogy of Lagrange’s theorem is not true, that is, there exists a quadratic irrational number whose Schneider continued fraction is not ultimately periodic (See e.g. Weger [2], Tilborghs [11], van der Poorten [13]). This paper deals with analogy of Lagrange’s theorem for Ruban continued fractions.

We prove that analogy of Lagrange’s theorem for Ruban continued fractions is not true in Sect. 2. Auxiliary results for main results are gathered in Sect. 3. In Sect. 4, we prove Theorem 1 and give criteria of quadratic or transcendental in a certain class of Ruban continued fractions. These proofs are mainly based on the proof of Baker’s results and the non-Archimedean version of Roth’s theorem for an algebraic number field [10].

2 Rational and quadratic irrational numbers

Wang [14] and Laohakosol [4] characterized rational numbers with Ruban continued fractions as follows.

Proposition 2 Let α be a p -adic number. Then α is rational if and only if its Ruban continued fraction expansion is finite or ultimately periodic with the period $p - p^{-1}$.

Proof See [14] or [4]. □

Next, we prove that analogy of Lagrange’s theorem for Ruban continued fractions is not true by the similar method as in [2]. We consider a Ruban continued fraction for $\alpha = \sqrt{D}$ where $D \in \mathbb{Z}$ not a square, but a quadratic residue modulo p , if p is odd, 1 modulo 8, if $p = 2$, so that $\alpha \in \mathbb{Q}_p$. If the Ruban continued fraction of α is $[a_0, a_1, a_2, \dots]$, then there exist rational numbers R_n, Q_n such that

$$\alpha_n = \frac{R_n + \sqrt{D}}{Q_n}$$

for $n \in \mathbb{Z}_{\geq 0}$. Obviously, $R_0 = 0, Q_0 = 1$, and for all n we have the recursion formula

$$R_{n+1} = -(R_n - a_n Q_n), \quad Q_{n+1} = \frac{D - R_{n+1}^2}{Q_n}$$

by induction on n .

Proposition 3 *If $R_m Q_m \leq 0$, and $R_{m+1}^2 > D$ for some m , then the Ruban continued fraction expansion of α is not ultimately periodic.*

Proof We show $R_{m+1} Q_{m+1} < 0, R_{m+2}^2 > D$, and $|R_{m+2}| > |R_{m+1}|$. Let us assume $R_m Q_m < 0$. Then we have $R_m R_{m+1} < 0$ by the recursion formula for R_{m+1} . We also obtain $Q_m Q_{m+1} < 0$ by the recursion formula for Q_{m+1} and $R_{m+1}^2 > D$. Hence, we get $R_{m+1} Q_{m+1} < 0$. Furthermore, by $a_{m+1} \neq 0$, we have

$$|R_{m+2}| = |R_{m+1}| + a_{m+1} |Q_{m+1}| > |R_{m+1}|,$$

so that $R_{m+2}^2 > D$. Next, let us assume $R_m Q_m = 0$. By $R_m = 0$, we have $R_{m+1} = a_m Q_m$. By the recursion formula for Q_{m+1} , we have $Q_m Q_{m+1} < 0$. Thus, we obtain $R_{m+1} Q_{m+1} < 0$. In the same way, we see $|R_{m+2}| > |R_{m+1}|$ and $R_{m+2}^2 > D$. Since $(|R_n|)_{n \geq m}$ is strictly increasing, the Ruban continued fraction expansion for \sqrt{D} is not ultimately periodic. □

Corollary 4 *If $D < 0$, then the Ruban continued fraction expansion of p -adic number \sqrt{D} is not ultimately periodic.*

Proof Since $R_0 Q_0 = 0$, and $R_1^2 \geq 0$, the corollary follows. □

3 Auxiliary results

For an infinite Ruban continued fraction $\alpha = [a_0, a_1, a_2, \dots]$, we define nonnegative rational numbers q_n, r_n by using recurrence equations:

$$\begin{cases} q_{-1} = 0, \quad q_0 = 1, \quad q_n = a_n q_{n-1} + q_{n-2}, & n \geq 1, \\ r_{-1} = 1, \quad r_0 = a_0, \quad r_n = a_n r_{n-1} + r_{n-2}, & n \geq 1. \end{cases}$$

Let λ be a variable. Then the Ruban continued fraction has the following properties which are the same properties as the continued fraction expansions for real numbers: For all $n \geq 0$,

$$[a_0, a_1, \dots, a_n] = \frac{r_n}{q_n}, \tag{3}$$

$$[a_0, a_1, \dots, a_n, \lambda] = \frac{\lambda r_n + r_{n-1}}{\lambda q_n + q_{n-1}}, \tag{4}$$

$$r_{n-1} q_n - r_n q_{n-1} = (-1)^n. \tag{5}$$

Those are easily seen by induction on n .

Lemma 5 *The following equalities hold:*

$$|q_n|_p = |a_1 a_2 \cdots a_n|_p, \quad n \geq 1, \tag{6}$$

$$\begin{cases} |r_n|_p = |a_0 a_1 \cdots a_n|_p, & n \geq 1, & (a_0 \neq 0) \\ |r_1|_p = 1, \quad |r_n|_p = |a_2 a_3 \cdots a_n|_p, & n \geq 2, & (\text{otherwise}) \end{cases} \tag{7}$$

$$\left| \alpha - \frac{r_n}{q_n} \right|_p = \frac{1}{|a_{n+1}|_p |q_n|_p^2}, \quad n \geq 0. \tag{8}$$

Proof See [14]. □

Lemma 6 *If α' is a Ruban continued fraction in which the first $n + 1$ partial quotients are the same as those of α , then*

$$|\alpha - \alpha'|_p \leq |q_n|_p^{-2}.$$

Proof Since r_n/q_n is a n -th convergent to both α and α' , and (8), the lemma follows. □

Lemma 7 *The following inequalities hold:*

$$q_n \leq |q_n|_p, \quad r_n \leq |r_n|_p, \quad \text{for all } n \geq -1.$$

Proof The proof is by induction on n . It is obvious that for $n = -1, 0$. By Lemma 5 and the definition of Ruban continued fraction expansions, we have

$$\begin{aligned} q_n &\leq a_n |q_{n-1}|_p + |q_{n-2}|_p \leq \left(p - \frac{1}{|a_n|_p} \right) |q_{n-1}|_p + |q_{n-2}|_p \\ &\leq |q_{n-1}|_p \left(p + \frac{1}{p} - \frac{1}{|a_n|_p} \right) \leq |q_n|_p. \end{aligned}$$

The proof for r_n is similar. □

For $\beta \in \overline{\mathbb{Q}}$, let $f_\beta(X) = \sum_{i=0}^n d_i X^i$ be a minimum polynomial of β in $\mathbb{Z}[X]$. Put

$$H(\beta) := \max_{0 \leq i \leq n} |d_i|.$$

$H(\beta)$ is called a *primitive height* of β .

Lemma 8 *Suppose $a_0 = 0$. Let h, k be positive integers and consider the Ruban continued fraction*

$$\eta = [0, a_1, \dots, a_{h-1}, \overline{a_h, \dots, a_{h+k-1}}].$$

Then η is rational or quadratic irrational. Furthermore, we have

$$H(\eta) \leq \begin{cases} p & (\text{if } \eta \text{ is rational and } h = 1) \\ |q_{h-1}|_p^2 & (\text{if } \eta \text{ is rational and } h \geq 2) \\ 2|q_{h+k-1}|_p^4 & (\text{if } \eta \text{ is quadratic irrational}). \end{cases}$$

Proof By $\eta_h = \eta_{h+k}$ and (4), we obtain

$$\eta = \frac{\eta_h r_{h-1} + r_{h-2}}{\eta_h q_{h-1} + q_{h-2}} = \frac{\eta_h r_{h+k-1} + r_{h+k-2}}{\eta_h q_{h+k-1} + q_{h+k-2}}.$$

Eliminating η_h , we have

$$A\eta^2 + B\eta + C = 0,$$

where

$$\begin{aligned} A &= q_{h-2}q_{h+k-1} - q_{h-1}q_{h+k-2}, \\ B &= q_{h-1}r_{h+k-2} + r_{h-1}q_{h+k-2} - r_{h-2}q_{h+k-1} - q_{h-2}r_{h+k-1}, \\ C &= r_{h-2}r_{h+k-1} - r_{h-1}r_{h+k-2}. \end{aligned}$$

Therefore, η is either rational or quadratic irrational. By the assumption $a_0 = 0$, it follows that $r_n \leq q_n$, $|r_n|_p \leq |q_n|_p$ for all $n \geq 0$. By induction on n , it is easy to check that $r_n|r_n|_p$, $q_n|q_n|_p \in \mathbb{Z}$ for all $n \geq 0$.

Let us assume that η is a quadratic irrational. By $|q_{h+k-1}|_p^2 A, |q_{h+k-1}|_p^2 B, |q_{h+k-1}|_p^2 C \in \mathbb{Z}$ and Lemma 7, we obtain

$$\begin{aligned} H(\eta) &\leq |q_{h+k-1}|_p^2 \max(|A|, |B|, |C|) \\ &\leq 2q_{h+k-1}^2 |q_{h+k-1}|_p^2 \leq 2|q_{h+k-1}|_p^4. \end{aligned}$$

Now let us assume that η is rational. By Proposition 2, we have

$$\eta = [a_0, \dots, a_{h-1}, -1/p],$$

that is,

$$\eta = \frac{pr_{h-2} - r_{h-1}}{pq_{h-2} - q_{h-1}}.$$

When $h = 1$, we see that $H(\eta) = p$. Next we consider the case $h \geq 2$. Since $(pr_{h-2} - r_{h-1})|q_{h-1}|_p$ and $(pq_{h-2} - q_{h-1})|q_{h-1}|_p$ are integers, we have

$$\begin{aligned} H(\eta) &\leq \max(|pr_{h-2} - r_{h-1}| |q_{h-1}|_p, |pq_{h-2} - q_{h-1}| |q_{h-1}|_p) \\ &\leq |q_{h-1}|_p^2, \end{aligned}$$

and the lemma follows. □

We recall a height of algebraic numbers which is different from the primitive height. Let K be an algebraic number field and \mathcal{O}_K be the integer ring of K , and $M(K)$ be the set of places of K . For $x \in K$ and $v \in M(K)$, we define the absolute value $|x|_v$ by

- (i) $|x|_v = |\sigma(x)|$ if v corresponds the embedding $\sigma : K \hookrightarrow \mathbb{R}$
- (ii) $|x|_v = |\sigma(x)|^2 = |\bar{\sigma}(x)|^2$ if v corresponds the pair of conjugate embeddings $\sigma, \bar{\sigma} : K \hookrightarrow \mathbb{C}$
- (iii) $|x|_v = (N(\mathfrak{p}))^{-\text{ord}_{\mathfrak{p}}(x)}$ if v corresponds to the prime ideal \mathfrak{p} of \mathcal{O}_K .

Set

$$\bar{H}_K(\beta) := \prod_{v \in M(K)} \max\{1, |\beta|_v\}$$

for $\beta \in K$. $\bar{H}_K(\beta)$ is called an *absolute height* of β . Then there are the following relations between primitive and absolute height.

Proposition 9 For $b \in \mathbb{Q}$ and $\beta \in \bar{\mathbb{Q}}$ with $[\mathbb{Q}(\beta), \mathbb{Q}] = D$, we have

$$\begin{aligned} H(b) &= \bar{H}_{\mathbb{Q}}(b), \\ \bar{H}_{\mathbb{Q}(\beta)}(\beta) &\leq (D + 1)^{1/2} H(\beta), \quad H(\beta) \leq 2^D \bar{H}_{\mathbb{Q}(\beta)}(\beta). \end{aligned}$$

Proof See Part B of [10]. □

The main tool for the proof of main results is the non-Archimedean version of Roth’s theorem for algebraic number fields.

Theorem 10 (Roth Theorem) *Let K be an algebraic number field, and v be in $M(K)$ with it extended in some way to \overline{K} . Let $\beta \in \overline{K} \setminus K$ and $\delta, C > 0$ be given. Then there are only finite many $\gamma \in K$ with the solution of the following inequality:*

$$|\beta - \gamma|_v \leq \frac{C}{H_K(\gamma)^{2+\delta}}.$$

Proof See Part D of [10]. □

4 Main results

Proof of Theorem 1 We may assume that $a_0 = 0$. By the assumption, there are infinitely many positive integers j which satisfy

$$a_{n_j} = a_{n_j+1} = \dots = a_{n_j+k_j-1} = p - p^{-1}. \tag{9}$$

Let Λ be an infinite set of j which satisfy (9).

For $i \in \Lambda$, we put

$$\eta^{(i)} := \left[0, a_1, \dots, a_{n_i-1}, \overline{p - p^{-1}} \right].$$

By Proposition 2, α is not rational. Suppose that α is an algebraic number of degree at least two. We show that if $\chi > 2$, then we have

$$|\alpha - \eta^{(i)}|_p > |q_{n_i-1}|_p^{-2\chi} \tag{10}$$

for all sufficiently large $i \in \Lambda$. Suppose the claim is false. By Proposition 2, $\eta^{(i)}$ is rational for each $i \in \Lambda$. By Lemma 8 and Proposition 9, we have

$$|\alpha - \eta^{(i)}|_p \leq |q_{n_i-1}|_p^{-2\chi} \leq \overline{H}_{\mathbb{Q}}(\eta^{(i)})^{-\chi}$$

for infinitely many i , which contradicts Theorem 10.

By Lemma 6, we obtain $|\alpha - \eta^{(i)}|_p \leq |q_{m_i}|_p^{-2}$ for $i \in \Lambda$, where $m_i = n_i + k_i \lambda_i - 1$. Therefore, we get

$$|q_{m_i}|_p < |q_{n_i-1}|_p^{\chi}$$

for sufficiently large $i \in \Lambda$. By Lemma 5, we see $p^i \leq |q_i|_p \leq A^i$ for $i \geq 1$. Thus, for all sufficiently large $i \in \Lambda$, it follows that

$$\frac{\lambda_i}{n_i} < B + (\chi - 2) \frac{\log A}{\log p}.$$

Since there exists $\delta > 0$ such that $\lambda_i > (B + \delta)n_i$ for all sufficiently large i , we have for all sufficiently large $i \in \Lambda$,

$$2 + \frac{\log p}{\log A} \delta < \chi.$$

This inequality holds for each $\chi > 2$, a contradiction. □

We also obtain the following results.

Theorem 11 *Let α be a quasi-periodic Ruban continued fraction, and $A \geq p$ be a real number. Assume that $(a_i)_{i \geq 0}$ is a non-ultimately periodic sequence such that $|a_i|_p \leq A$ for each i , and $(k_i)_{i \geq 0}$ is bounded. If*

$$\limsup_{i \rightarrow \infty} \frac{\lambda_i}{n_i} > B' = B'(A),$$

where B' is defined by

$$B' = \frac{4 \log A}{\log p} - 1,$$

then α is quadratic irrational or transcendental.

Theorem 12 *Consider a quasi-periodic Ruban continued fraction*

$$\alpha = [a_0, \dots, a_{n_0-1}, \overline{a_{n_0}, \dots, a_{n_0+k_0-1}}^{\lambda_0}, a_{n_1}, \dots, a_{n_1+k_1-1}^{\lambda_1}, \dots],$$

where the notation means that $n_i = n_{i-1} + \lambda_{i-1}k_{i-1}$. Assume that $(a_i)_{i \geq 0}$ is not an ultimately periodic sequence, the sequences $(|a_i|_p)_{i \geq 0}$ and $(k_i)_{i \geq 0}$ are bounded, and that

$$\liminf_{i \rightarrow \infty} \frac{\lambda_i}{\lambda_{i-1}} > 4.$$

Then α is quadratic irrational or transcendental.

Remark 13 There exist quadratic irrational numbers whose Ruban continued fraction expansions are not ultimately periodic by Corollary 4. Therefore, it is difficult to determine whether a given Ruban continued fraction is quadratic irrational or transcendental. However, we see that there exist a transcendental number in the set of Ruban continued fractions which satisfy the assumption of Theorem 11 and 12. For example, (2) satisfies the assumption of Theorem 11 and 12.

In the following, c_1, c_2, \dots, c_6 denote positive real numbers which depend only on α , and we may assume that $a_0 = 0$.

Proof of Theorem 11 By the assumption, there exists $\delta > 0$ such that $\lambda_i > (B' + \delta)n_i$ for infinitely many i . For each positive integer i , there are only finitely many possibilities for k_i and for

$$a_{n_i}, a_{n_i+1}, \dots, a_{n_i+k_i-1}.$$

Therefore, there exist a positive integer k and $b_1, b_2, \dots, b_k \in S'_p$ such that there are infinitely many j which satisfy

$$k_j = k, a_{n_j} = b_1, \dots, a_{n_j+k_j-1} = b_k, \lambda_j > (B' + \delta)n_j. \tag{11}$$

Let Λ be an infinite set of j which satisfy (11).

For $i \in \Lambda$, we put

$$\eta^{(i)} := [0, a_1, \dots, a_{n_i-1}, \overline{b_1, \dots, b_k}].$$

By Proposition 2, α is not rational. Suppose that α is an algebraic number of degree at least three. We show that if $\chi > 2$, then we have

$$|\alpha - \eta^{(i)}|_p > |q_{n_i+k_i-1}|_p^{-4\chi} \tag{12}$$

for all sufficiently large $i \in \Lambda$. Suppose the claim is false. By Lemma 8, $\eta^{(i)}$ is rational or quadratic irrational for each $i \in \Lambda$. Let us assume that $\eta^{(i)}$ is quadratic irrational. Then there exists a quadratic field K such that $\eta^{(i)} \in K$ for all $i \in \Lambda$. Take a real number ε which satisfies $0 < \varepsilon < \chi - 2$. Then we have $2^{\chi-\varepsilon} < |q_{n_i+k_i-1}|_p^{4\varepsilon}$ for all sufficiently large $i \in \Lambda$. Put $v \in M(K)$ with $v \mid p$. We denote again by v one of the place extended to $K(\alpha)$. By $[K(\alpha)_v : \mathbb{Q}_p] = 1$, Lemma 8, and Proposition 9, we obtain

$$\begin{aligned} |\alpha - \eta^{(i)}|_v &= |\alpha - \eta^{(i)}|_p \leq |q_{n_i+k_i-1}|_p^{-4\chi} \\ &\leq (2|q_{n_i+k_i-1}|_p^4)^{\varepsilon-\chi} \leq H(\eta^{(i)})^{\varepsilon-\chi} \\ &\leq \frac{c_1}{\overline{H}_K(\eta^{(i)})^{\chi-\varepsilon}} \end{aligned}$$

for infinitely many i , which contradicts Theorem 10. In the same way, we see (12) in the case that $\eta^{(i)}$ is rational. By Lemma 6, we have $|\alpha - \eta^{(i)}|_p \leq |q_{m_i}|_p^{-2}$ for $i \in \Lambda$, where $m_i = n_i + k_i - 1$. Therefore, we obtain

$$|q_{m_i}|_p < |q_{n_i+k-1}|_p^{2\chi}$$

for sufficiently large $i \in \Lambda$. By Lemma 5, we see $p^i \leq |q_i|_p \leq A^i$ for $i \geq 1$. Thus, for all sufficiently large $i \in \Lambda$, we have

$$\lambda_i < c_2 + \left(\frac{1}{2}(B' + 1)\chi - 1\right),$$

so

$$\left(1 - \frac{\chi}{2}\right)(B' + 1) + \delta < \frac{c_2}{n_i}.$$

This inequality holds for each $\chi > 2$, and contradicts if i is sufficiently large in Λ . □

Proof of Theorem 12 Put

$$P_h^{(i)} := [a_{n_i+h-1}, a_{n_i+h-2}, \dots, a_{n_i}, \overline{a_{n_i+k_i-1}, a_{n_i+k_i-2}, \dots, a_{n_i}}]$$

for $i = 0, 1, 2, \dots$ and $h = 1, 2, \dots, k_i$. Put

$$P^{(i)} := \prod_{h=1}^{k_i} P_h^{(i)}.$$

For each positive integer i , there exist only finitely many possibilities for k_i and for

$$a_{n_i}, a_{n_i+1}, \dots, a_{n_i+k_i-1}.$$

$P^{(i)}$ is a function which depends only on $k_i, a_{n_i}, a_{n_i+1}, \dots, a_{n_i+k_i-1}$. Hence, there exists a real number P such that the greatest of those values $|P^{(i)}|_p$ which are attained for infinitely many i . Then there exists an integer l such that

$$|P^{(i)}|_p \leq P \text{ for all } i \geq l.$$

There exist a positive integer k and $b_1, b_2, \dots, b_k \in S'_p$ such that there are infinitely many j which satisfy

$$|P^{(j)}|_p = P, \quad k_j = k, \quad a_{n_j} = b_1, \dots, a_{n_j+k_j-1} = b_k. \tag{13}$$

Let Λ be an infinite set of j which satisfy (13). We may assume that $l = 0$.

Let us show that

$$|q_{n_{i+1}-1}|_p \leq c_3 P^{\lambda_i} |q_{n_i-1}|_p \quad \text{for all } i, \tag{14}$$

$$|q_{n_{i+1}-1}|_p \geq c_4 P^{\lambda_i} |q_{n_i-1}|_p \quad \text{for all } i \in \Lambda. \tag{15}$$

Firstly, an induction allows us to establish the mirror formula

$$\frac{q_m}{q_{m-1}} = [a_m, \dots, a_1], \quad \text{for all } m \geq 1.$$

Put

$$W_h^{(i)} := \frac{q_{n_i+h-1}}{q_{n_i+h-2}},$$

for $i = 0, 1, 2, \dots$ and $h = 1, 2, \dots, k_i \lambda_i$, and

$$W^{(i)} := \prod_{h=1}^{k_i \lambda_i} W_h^{(i)}.$$

Clearly, we have $q_{n_{i+1}-1} = W^{(i)} q_{n_i-1}$. It follows from Lemma 5 and 6 that for any i ,

$$\begin{aligned} |W^{(i)}|_p &= \prod_{h=1}^{k_i} \prod_{s=0}^{\lambda_i-1} |W_{h+sk_i}^{(i)}|_p \leq \prod_{h=1}^{k_i} \prod_{s=0}^{\lambda_i-1} \left(|P_h^{(i)}|_p + |U_{h,s}^{(i)}|_p^{-2} \right) \\ &\leq \prod_{h=1}^{k_i} \prod_{s=0}^{\lambda_i-1} |P_h^{(i)}|_p \left(1 + p^{-2(h+sk_i-1)} \right) \leq |P^{(i)}|_p^{\lambda_i} \prod_{h=1}^{k_i \lambda_i} \left(1 + p^{2-2h} \right) \\ &\leq c_3 P^{\lambda_i}, \end{aligned}$$

where $U_{1,0}^{(i)} = 1$ and otherwise $U_{h,s}^{(i)}$ is the denominator of $(h + sk_i - 1)$ -th convergent to $P_h^{(i)}$. Likewise, for all i , we have

$$\begin{aligned} |W^{(i)}|_p &\geq \prod_{h=1}^{k_i} \prod_{s=0}^{\lambda_i-1} \left(|P_h^{(i)}|_p - |U_{h,s}^{(i)}|_p^{-2} \right) \\ &\geq |P^{(i)}|_p^{\lambda_i} \left(1 - \frac{1}{|P_1^{(i)}|_p} \right) \prod_{h=2}^{k_i \lambda_i} (1 - p^{2-2h}). \end{aligned}$$

If $i \in \Lambda$, then $|P^{(i)}|_p = P$ and $P_1^{(i)}$ is independent of i . Therefore, we obtain

$$|W^{(i)}|_p \geq c_4 P^{\lambda_i} \quad \text{for all } i \in \Lambda.$$

If A and K are the upper bounds of the sequences $(|a_i|_p)_{i \geq 0}$ and $(k_i)_{i \geq 0}$, then for all i , we have

$$|q_{n_i+k_i-1}|_p \leq A^K |q_{n_i-1}|_p. \tag{16}$$

Now, there exist a real number $\delta > 0$ and an integer $N \geq 1$ such that $\lambda_i > (4 + \delta)\lambda_{i-1}$ for all $i > N$. Set $\chi := 2 + \delta/4$. For $i \in \Lambda$, we put

$$\eta^{(i)} := [0, a_1, \dots, a_{n_i-1}, \overline{b_1, \dots, b_k}].$$

By Proposition 2, α is not rational. Suppose that α is an algebraic number of degree at least three. Then we have

$$|\alpha - \eta^{(i)}|_p > |q_{n_i+k_i-1}|_p^{-4\chi}$$

for all sufficiently large $i \in \Lambda$. This follows by the same way as in the proof of Theorem 11. By Lemma 6, we see $|\alpha - \eta^{(i)}|_p \leq |q_{n_{i+1}-1}|_p^{-2}$ for all i . Therefore, we obtain

$$|q_{n_{i+1}-1}|_p < |q_{n_i+k_i-1}|_p^{2\chi} \tag{17}$$

for all sufficiently large $i \in \Lambda$. Applying (14), (15), (16), and (17), we have for all sufficiently large $i \in \Lambda$,

$$p^{\lambda_i} < c_5 c_6^i p^{(2\chi-1)(\lambda_{i-1}+\lambda_{i-2}+\dots+\lambda_N)}.$$

Taking logarithms, we see that for all sufficiently large $i \in \Lambda$,

$$\begin{aligned} \frac{\lambda_i}{\lambda_{i-1}} &< \frac{\log c_5 + i \log c_6}{\lambda_{i-1} \log P} + (2\chi - 1) \sum_{j=0}^{\infty} \left(\frac{1}{4 + \delta/2} \right)^j \\ &= \frac{\log c_5 + i \log c_6}{\lambda_{i-1} \log P} + 4 + \frac{\delta}{2}. \end{aligned}$$

Since $i/\lambda_i \rightarrow 0$ as $i \rightarrow \infty$, we have

$$\frac{\lambda_i}{\lambda_{i-1}} < \frac{\delta}{2} + 4 + \frac{\delta}{2} = 4 + \delta$$

for all sufficiently large $i \in \Lambda$. This contradicts, and the proof is complete. □

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