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Local Langlands correspondence for classical groups and affine Hecke algebras

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Abstract Using the results of Colette Moeglin on the representations of p-adic classical groups (based on methods of James Arthur) and its relation with representations of affine Hecke algebras established by the author, we show that the category of smooth complex representations of a quasi-split p-adic classical group and its pure inner forms is naturally decomposed into subcategories which are equivalent to a tensor product of categories of unipotent representations of classical groups. A statement of this kind had been conjectured by G. Lusztig. All classical groups (general linear, orthogonal, symplectic and unitary groups) appear in this context. We get also parameterizations of representations of affine Hecke algebras, which seem not all to be in the literature yet. All this sheds some light on what is known as the stable Bernstein center.

Let F be a non-Archimedean local field of characteristic 0—this assumption on the characteristic is used in [25] but not in [15]—, and $n \geq 1$ an integer. The symbol G will denote the group of F-rational points of a quasi-split classical group $\underline{G}$ of semi-simple rank n defined over F. We will mean by that either a symplectic group or a (at least in the even rank case, the full, i.e. non connected) orthogonal group. (The case of unitary groups will be treated in the appendix.) If G is orthogonal, we will denote by G^- its unique pure inner form [V, GGP]. If G is symplectic, we will leave G^- undefined (there is no pure inner form $\neq G$). We will often write G^+ for G and denote by $Rep(G^{\pm})$ the category of smooth complex representations of $G^{\pm}$.

Using methods of Arthur [1], Moeglin has determined in [25] the Langlands-Deligne parameters which correspond to supercuspidal representations (for both G and G^-), including information on reducibility points. The author has used this information in [14] to deduce

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the parameters of the affine Hecke algebras that have been shown in [15] to correspond to the Bernstein components of the category $Rep(G^{\pm})$.

The aim of the present work is to show that, putting together different Bernstein components, one obtains a natural decomposition of $Rep(G^{\pm})$ into subcategories $\mathcal{R}_F^{\varphi_0}(\underline{G})$ which are equivalent to a tensor product of categories of unipotent representations of classical groups (in the sense of [21]). (Because of the categorical nature of the tensor product, we have in fact to restrict to the full subcategory of finitely generated representations.) A statement of this kind had been conjectured by Lusztig [23, section 19]. The φ_0 can be seen as inertial classes of Langlands parameters for G (i.e. modulo restriction to the inertial subgroup). When ${}^LG$ is orthogonal, a quasi-split outer form (equal to G if G is symplectic) will be added to obtain uniform statements. All classical groups (general linear, orthogonal, symplectic and unitary groups) appear in this context. Once the results in [21] appropriately generalized to symplectic, unitary and the (full) even orthogonal group, one should be able to compute multiplicities in standard modules from intersection cohomology as described in [23, section 19].

Taking into account the local Langlands correspondence (which can be considered as established now, see remarks in **1.7** for further remarks, although no final account has be written yet), we get from this parameterizations of representations of affine Hecke algebras, which seem not all to be in the literature yet. In addition, we explain, how to get a direct correspondence for the irreducible representations in $\mathcal{R}_F^{\varphi_0}(\underline{G})$ by conjugacy classes of parameters (s, u, Ξ) in a given complex reductive group (where s is a semi-simple element, u a unipotent element such that $sus^{-1} = u^q$ and Ξ an irreducible representation of the group of components of the common centralizer of s and u).

The plan of this paper is the following: in section **1.**, we summarize the results of Moeglin on the Langlands correspondence for supercuspidal representations of G. We recall the author's results relating the Bernstein components of $Rep(G^{\pm})$ to representations of affine Hecke algebras and give the definition of the categories $\mathcal{R}_F^{\varphi_0}(\underline{G})$. In section **2.**, we explain how to get a direct correspondence for the irreducible representations in $\mathcal{R}_F^{\varphi_0}(\underline{G})$ by parameters (s, u, Ξ) in a given complex reductive group. The last section **3.** is devoted to the parametrization of representations of affine Hecke algebras, taking into account the local Langlands correspondence. At the end, corollary **3.5**, we give the final decomposition result (which does not depend on a final written account of the local Langlands correspondence). There are three Appendices **A**, **B** and **C**. In Appendix **A**, it is explained how results in [15] generalize to the full orthogonal group which is not connected. In the Appendix **B**, we give an account of the notion of tensor product in the context of linear abelian categories and show that it applies to the categories that we are considering. Unitary groups are treated in Appendix **C**, although the results are used progressively in the main body of the paper.

Remark that those results of this paper which apply to non quasi-split inner forms of G are slightly conditional as some argument needed for orthogonal groups is not written in the literature, although this should be easy (cf.[25, p. 346, l.-22-20]). Remark that this *does not* depend on the generalization of the work of Arthur [1] to inner forms or an ultimate version of the fundamental. A similar remark applies to the pure inner form of the even unitary group treated in [24,26].

One may expect that a similar pattern holds for a general quasi-split p-adic reductive group and its pure inner forms according to Lusztig's conjecture. The present work is much based on an equivalence with the category of representations of an appropriate affine Hecke algebra proved in [15] and the knowledge of the Langlands parameters of supercuspidal representations subsequent to the work of [1]. But, [15] has a big potential to generalize. The method of [1] is not known or expected to generalize, but, as for example shown in [12,13], it should be possible to get a clear idea what are the affine Hecke algebras appearing in the case of a generic supercuspidal support. The results of Opdam [28,29] on representations of affine Hecke algebra should give then a clear idea which generic discrete series representations have non-generic supercuspidal representations in their L-packet. In this context, a generalization of [21,22] to groups which are not adjoint would also be helpful. (In the present work, this generalization was not necessary because the results could be deduced from what is known on the Langlands correspondence for classical groups.)

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1. We will denote by ${}^LG$ the "L-group" of G, which means that its connected component is the Langlands dual group of the connected component of G and that it is either a symplectic or a full (disconnected) orthogonal group. We will write $Z_{{}^LG}$ for the center of the connected component of ${}^LG$ (which is trivial if and only if G is symplectic and of order two otherwise) and denote by $\iota : {}^LG \rightarrow GL_N(\mathbb{C})$ the canonical embedding, i.e. N equals $2n$ if G is orthogonal and N equals $2n+1$ if G is symplectic. If l is an integer between 1 and n, H_l will denote (the group of F rational points) of a split classical group of semi-simple rank l of the same type (symplectic, even or odd orthogonal) as G. The symbols H_l^+ and H_l^- will have the appropriate meaning. We will also denote by ι the canonical embedding ${}^LH_l \rightarrow GL_L(\mathbb{C})$, hoping that this will not be a source of confusion.

The connected component of ${}^LG$, the Langlands dual group, will be denoted $\widehat{G}$.

Let W_F be the Weil group of F. It's the semi-direct product of the inertial subgroup I_F with the cyclic subgroup generated by a Frobénius automorphism Fr, $W_F = \langle Fr \rangle \ltimes I_F$. A character of W_F is called unramified, if it is trivial on I_F. By local class field theory, such a character is identified with a character of $F^\times$, trivial on the units of its ring of integers O_F, the character sending Fr^{-1} to q being identified with the absolute value $|\cdot|_F$.

We will call Langlands parameter for G a continuous homomorphism ρ of W_F into ${}^LG$ which sends Fr to a semi-simple element and assume in addition in the even-orthogonal case that the kernel of $det \circ \rho$ equals the Weil group of the splitting field of G. (It follows from the continuity that the image of I_F is finite.) In the case, where ${}^LG$ is the odd orthogonal group, parameters with $det(\rho) \neq 1$ just correspond to representations of an identical copy of the symplectic group. A homomorphism $\rho : W_F \times SL_2(\mathbb{C}) \rightarrow {}^LG$ will be called a Langlands–Deligne parameter, if its restriction to the first factor is a Langlands parameter and the restriction to the second factor a morphism of algebraic groups. A Langlands or Langlands–Deligne parameter for ${}^LG$ will be called *discrete*, if its image is not included in a proper Levi subgroup. Two Langlands or Langlands–Deligne parameters are said *equivalent* if they are conjugated by an element of ${}^LG$. (Usually, one considers only conjugation by an element of $\widehat{G}$, but, as we take here for G the full (non-connected) even orthogonal group, one has to take conjugation in ${}^LG$. For the other groups, this does not matter [10, 8.1 (ii)].)

If ρ is an irreducible representation of W_F, the set of equivalence classes of representations of the form $\rho^s := \rho|\cdot|_F^s$, $s \in \mathbb{C}$, will be called the *inertial class* of ρ. The group of unramified characters of W_F acts on the inertial class of ρ by torsion. We will denote by t_ρ the order of the stabilizer of the equivalence class of ρ. If ρ and ρ' are in the same inertial class, then $t_\rho = t_{\rho'}$, and the definition of t_ρ does not depend neither on the choice of Fr.

If ρ is a self-dual representation, we will say that it is of type ${}^LG$, if it factors through a group of type ${}^LG$ (meaning that the image of ρ is contained in an orthogonal group if ${}^LG$ is orthogonal and in a symplectic group if ${}^LG$ is symplectic). Otherwise, we will say that ρ is not of type ${}^LG$. We stress that the use of either of these notions will presume that ρ is self-dual.

If a is an integer ≥ 1, $sp(a)$ will denote the unique irreducible representation of $SL_2(\mathbb{C})$ of dimension a.

1.1 Theorem [25, 2.5.1]

(1) A Langlands–Deligne parameter $\varphi : W_F \times SL_2(\mathbb{C}) \to {}^LG$ corresponds to a supercuspidal representation of G^+ or G^-, if and only if

$$\iota \circ \varphi = \bigoplus_{\rho \text{ not of type } {}^LG} \left(\bigoplus_{k=1}^{a_\rho} (\rho \otimes sp(2k)) \right) \oplus \bigoplus_{\rho \text{ of type } {}^LG} \left(\bigoplus_{k=1}^{a_\rho} (\rho \otimes sp(2k-1)) \right),$$

where the ρ are irreducible representations and the a_ρ non-negative integers.

(2) Given φ as in (1), denote by $z_{\varphi,\rho,k}$ the diagonal matrix in ${}^LG$ that acts by -1 on the space of the direct summand $\rho \otimes sp(2k)$ (resp. $\rho \otimes sp(2k-1)$) of $\iota \circ \varphi$ and by 1 elsewhere. Put $S_\varphi = C_{{}^LG}(Im(\varphi))/C_{{}^LG}(Im(\varphi))^\circ$. The elements $z_{\varphi,\rho,k}$ lie in $C_{{}^LG}(Im(\varphi))$ and their images $\overline{z}_{\varphi,\rho,k}$ generate the commutative group S_φ.
A pair (φ, ϵ) formed by a discrete Langlands–Deligne parameter as in (1) and a character ϵ of S_φ corresponds to a supercuspidal representation of either G^+ or G^-, if and only if ϵ is alternating, i.e. $\epsilon(\overline{z}_{\varphi,\rho,k}) = (-1)^{k-1}\epsilon(\overline{z}_{\varphi,\rho,1})$ with $\epsilon(\overline{z}_{\varphi,\rho,1}) = -1$ for ρ not of type ${}^LG$ and $\epsilon(\overline{z}_{\varphi,\rho,1}) \in \{1, -1\}$ for ρ of type ${}^LG$. It corresponds to a supercuspidal representation of G^+ if $\epsilon_{|Z_{{}^LG}} = 1$ and to a supercuspidal representation of G^- otherwise.

(3) Suppose that φ satisfies the property in (1). *Let t_o be the number of ρ of type ${}^LG$ with a_ρ odd, put $t_o = 1$ if there are none of them, and let t_e be the number of the remaining ρ of type ${}^LG$ for which a_ρ is even.*
If G is symplectic, there are $2^{t_o-1}2^{t_e}$ non isomorphic supercuspidal representations of G^+ with Langlands–Deligne parameter φ.
If G is orthogonal, put $\epsilon_{\varphi,\rho} = (-1)^{\frac{a_\rho(a_\rho+1)}{2}}$, if ρ is not of type ${}^LG$, and put $\epsilon_{\varphi,\rho} = (-1)^{\frac{a_\rho}{2}}$, if ρ is of type ${}^LG$ and a_ρ even. There exists a supercuspidal representation of G^+ with Langlands–Deligne parameter φ if and only if either there is a ρ of type ${}^LG$ with a_ρ odd or $\prod_\rho \epsilon_{\varphi,\rho} = 1$.
If the above existence condition is satisfied, the number of supercuspidal representations with Langlands–Deligne parameter φ equals $2^{t_o-1}2^{t_e}$ and all these representations of G^+ are non isomorphic.
The remaining alternating characters correspond to representations of G^-, remarking that there are $2^{t_0+t_e}$ alternating characters for orthogonal G.

Proof (1) and (2) are stated as this in the paper of Moeglin. Concerning (3), if G is an orthogonal group, the theorem in the paper of Moeglin says that there is a supercuspidal representation of G^+ associated to φ, if and only if there exists an alternating character ϵ_φ corresponding to φ which takes value 1 on -1. The number of non isomorphic supercuspidal representations corresponding to φ equals the number of alternating characters with this property.

For ρ not of type ${}^LG$, there is a unique choice of an alternating character and its value on -1 is $\prod_{k=1}^{a_\rho}(-1)^k$. For ρ of type ${}^LG$, there are always two choices of an alternating character. If a_ρ is even and not divisible by 4 the value taken on -1 is always -1. If a_ρ is divisible by 4, the value taken on -1 is always 1. If a_ρ is odd, there is one alternating character which takes value 1 on -1 and another one which takes value -1 on -1.

One concludes by remarking that, if there are alternating characters attached to a ρ which take respectively value 1 and -1 on -1, then one can of course always find an alternating character for φ with value 1 on -1.

If G is symplectic, one can conclude as above, after having observed that there is then always a ρ of type ${}^LG$ with a_ρ odd. □

1.2 Definition We will fix for the rest of the paper in each inertial class $\mathcal{O}$ of an irreducible representation of W_F a base point $\rho_{\mathcal{O}}$. It will always be assumed to be the equivalence class of a unitary representation, which is in addition self-dual if $\mathcal{O}$ contains such an element. In this last case, we take ρ of the same type as ${}^LG$ if there is such a representation in $\mathcal{O}$. This base point will be called in the sequel a *normed* representation (w.r.t. ${}^LG$).

A discrete Langlands parameter $\tau : W_F \rightarrow {}^LG$ will be called *normed*,if $\iota \circ \tau$ is the direct sum of inequivalent normed representations of W_F. If ${}^LM \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_r}(\mathbb{C}) \times {}^LH_l$ is a Levi subgroup of ${}^LG$, then a discrete Langlands parameter $\varphi : W_F \rightarrow {}^LM$ is called normed if it is of the form $\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_r(\gamma), \tau(\gamma))$, where the ρ_i are irreducible normed representations and τ is a discrete normed Langlands parameter. A Langlands parameter $\varphi : W_F \rightarrow {}^LG$ will be called normed, if there is a minimal Levi subgroup ${}^LM$ of ${}^LG$ containing the image of φ, such that φ is a discrete normed Langlands parameter with respect to ${}^LM$.

If s is a semi-simple element in $GL_N(\mathbb{C})$, then χ_s will denote the unramified character of W_F, such that $\chi_s(Fr) = s$. If φ is a normed Langlands parameter and s is a semi-simple element in $C_{GL_N(\mathbb{C})}(Im(\varphi))$ such that $(\iota \circ \varphi)\chi_s$ is a self-dual representation of W_F of the same type as ${}^LG$, then we will denote by φ_s the Langlands parameter $W_F \rightarrow {}^LG$ such that $\iota \circ \varphi_s$ is equivalent to the representation $\gamma \mapsto \varphi_s(\gamma) = \varphi(\gamma)\chi_s(\gamma)$ of W_F.

The set of equivalence classes of Langlands parameters of the form φ_s with s as above will be called the *inertial orbit* of φ.

If ρ' is an irreducible representation of W_F, $m(\rho'; \varphi)$ will denote the multiplicity of ρ' (up to equivalence) in the representation $\rho_1 \oplus \cdots \oplus \rho_r \oplus \rho_r^\vee \oplus \cdots \oplus \rho_1^\vee \oplus (\iota \circ \tau)$.

Remark Normed Langlands parameters will play a similar role as the trivial Langlands parameter for the set of unipotent representations. Unfortunately, it seems not possible to fix in general a "canonical" base point (see also remark after **1.4**). The choice of a base point has no influence on the essentially intrinsic nature of our results.

1.3 Proposition *Two Langlands parameters $W_F \rightarrow {}^LG$ are in the same inertial orbit, if and only if their restrictions to the inertial subgroup I_F are conjugate by an element of ${}^LG$.*

Proof For the proof, it is enough to consider the case where one of the two Langlands parameters is normed. So, let φ and φ' be two Langlands parameters.

Suppose first that φ and φ' are in the same inertial orbit as defined above. Then, as representations of I_F, $\iota \circ \varphi_{|I_F}$ and $\iota \circ \varphi'_{|I_F}$ are certainly isomorphic. One deduces from this, analog to [10, 8.1 (ii)] in the case of Langlands parameters, that $\varphi_{|I_F}$ and $\varphi'_{|I_F}$ are conjugate by an element of ${}^LG$. This proves one direction.

For the other direction suppose that $\varphi_{|I_F}$ and $\varphi'_{|I_F}$ are conjugate by an element of ${}^LG$. Then, as representations of I_F, $\iota \circ \varphi_{|I_F}$ and $\iota \circ \varphi'_{|I_F}$ are certainly isomorphic. Remark that an irreducible representation of W_F is determined, up to twist by an unramified character, by its restriction to I_F and more particular by an irreducible component of this restriction. It follows that for each irreducible component ρ of $\iota \circ \varphi$, there is an irreducible component ρ' of $\iota \circ \varphi'$ such that the two irreducible representations have a common irreducible component when restricted to I_F. In addition, ρ' is in the inertial class of ρ (as irreducible representation of W_F) and the restrictions $\rho_{|I_F}$ and $\rho'_{|I_F}$ are isomorphic. The same is true, if one starts from an irreducible component of $\iota \circ \varphi'$. This induces a bijection between irreducible components of $\iota \circ \varphi$ and irreducible components of $\iota \circ \varphi'$, sending an irreducible component of $\iota \circ \varphi$ to an irreducible representation in its inertial orbit. It follows that there is a semi-simple element s

in $C_{GL_N(\mathbb{C})}(Im(\varphi))$ such that $\iota \circ \varphi'$ is isomorphic to the representation $\gamma \mapsto \varphi(\gamma)\chi_s(\gamma)$. By definition, this means that φ' is in the inertial orbit of the normed Langlands parameter φ. □

1.4 Proposition *Let $\mathcal{O}$ be the inertial orbit of an irreducible representation of W_F and denote by $\rho_{\mathcal{O}}$ the normed representation in its inertial orbit. The map $\mathcal{O} \to \mathbb{C}$, defined by $\rho \mapsto f_\rho := \rho(Fr^{t_\rho})\rho_{\mathcal{O}}(Fr^{t_\rho})^{-1}$, where ρ denotes a representative of the equivalence class which is an unramified twist of $\rho_{\mathcal{O}}$, is a well-defined bijection. If $\rho_{\mathcal{O}}$ is self-dual, then ρ is self-dual, if and only if $f_\rho \in \{\pm 1\}$.*

Proof By definition, there is a complex number s such that $\rho = \rho_{\mathcal{O}} \otimes |\cdot|_F^s$. It follows from this that the map $\rho \mapsto f_\rho$ is well defined and that, for ρ as above, $f_\rho = q^{-st_\rho}$. One sees that the map is surjective. Put $\rho' = \rho_{\mathcal{O}} \otimes |\cdot|_F^{s'}$. Then $f_\rho = f_{\rho'}$ is equivalent to $q^{(s-s')t_\rho} = 1$. This implies that $|\cdot|_F^{s'-s}$ stabilizes the equivalence class of ρ. Consequently, $\rho' = \rho \otimes |\cdot|_F^{s'-s} \simeq \rho$. So, the map is also injective.

Assume now $\rho_{\mathcal{O}}$ self-dual and that χ is an unramified character such that $\rho := \rho_{\mathcal{O}} \otimes \chi$ is also self-dual. This implies that χ^2 stabilizes $\rho_{\mathcal{O}}$ and consequently one has $f_\rho^2 = 1$. One sees that it is enough to twist $\rho_{\mathcal{O}}$ by $|\cdot|_F^{\frac{i\pi}{t_\rho log(q)}}$ to get a representation ρ' that satisfies $f_{\rho'} = -1$. □

Remark In general, it is not possible to distinguish an element ρ of $\mathcal{O}$, such that $f_\rho = 1$, even if ρ is self-dual: although ρ is then induced from an irreducible representation of the Weil group of an unramified extension of degree t_ρ, there is no reason why $\rho(Fr^{t_\rho})$ should be a scalar. (For example, in the case $t_\rho = 1$, conjugation by Fr gives an isomorphic representation of the restriction to the inertia group, but it is not necessarily the same representation.) That is the reason, why we had to make a choice in our definition of a normed representation.

1.5 Definition If ρ is self-dual, we will denote by ρ_- the unique element in its inertial orbit such that $f_{\rho_-} = -1$. When H is an even orthogonal quasi-split group and ζ is the quadratic character of W_F whose kernel corresponds to the splitting field of H, we will denote by H_- the quasi-split outer form of H whose splitting field corresponds to the kernel of ζ_-. We will leave H_- undefined when H is odd-orthogonal and put $H_- = H$ if H is symplectic, although H_- will be distinguished from H. We will also write H_+ for H. The notation $H_\pm^\pm$ will then have the appropriate meaning.

Remark If ρ is self-dual, ρ is either orthogonal or symplectic. However, it happens that ρ_- is not of the same type (i.e. orthogonal or symplectic) as ρ, and both cases, $\det(\rho) = \det(\rho_-)$ and $\det(\rho) \neq \det(\rho_-)$, happen. (Examples can be easily deduced from [27, theorem 1].) By our convention of a normed representation, this can only appear if ρ_0 is of type ${}^LG$.

1.6 If $\varphi_0 : W_F \to {}^LG$ is a normed Langlands parameter, denote by $supp(\varphi_0)$ the set of irreducible representations ρ of W_F, up to isomorphism, with $m(\rho;\varphi_0) \neq 0$ and by $supp'(\varphi_0)$ the subset formed by those representations which are selfdual. We will put an equivalence $\sim$ on $supp(\varphi_0)$ defined by $\rho \sim \rho^\vee$.

Denote by $\mathcal{S}(\varphi_0)$ the set of families of pairs $(a_{\rho,+}, a_{\rho,-})_\rho$ of non-negative integers indexed by $supp'(\varphi_0)$, such that

$$m(\rho;\varphi_0) \geq \begin{cases} a_{\rho,+}(a_{\rho,+}+1) + a_{\rho,-}(a_{\rho,-}+1), & \text{if } \rho \text{ and } \rho_- \text{ not of type } {}^LG, \\ a_{\rho,+}^2 + a_{\rho,-}(a_{\rho,-}+1), & \text{if } \rho \text{ of type } {}^LG, \text{ but not } \rho_-, \\ a_{\rho,+}^2 + a_{\rho,-}^2, & \text{if } \rho \text{ and } \rho_- \text{ of type } {}^LG, \end{cases}$$

with the additional condition that the terms of both sides in the above inequalities have same parity (if ρ and ρ_- are both not of type ${}^LG$, this is always satisfied).

Put $\kappa'_\rho = 1$ if ρ is of type ${}^LG$ and $\kappa'_\rho = 0$ otherwise. If $S = (a_{\rho,+}, a_{\rho,-})_\rho$ lies in $\mathcal{S}(\varphi_0)$, then the dimension L_S of the representation

$$\bigoplus_{\rho \in supp'(\varphi_0)} \left(\bigoplus_{k=1}^{a_{\rho,+}} (\rho \otimes sp(2k - \kappa'_\rho)) \oplus \left(\bigoplus_{k=1}^{a_{\rho,-}} \rho_- \otimes sp(2k - \kappa'_{\rho_-}) \right) \right)$$

has the same parity as N. If we denote by l_S the semi-simple rank of the group H_{l_S} whose Langlands dual embeds canonically into $GL_{L_S}(\mathbb{C})$, then there is, up to equivalence, a unique discrete Langlands–Deligne parameter $\varphi^S : W_F \to {}^LH_{l_S}$ [10, 8.1.ii] (as we consider the full orthogonal group, the restriction for the even orthogonal group does not apply), such that the above representation is equivalent to $\iota \circ \varphi^S$. Denote by $\widehat{S}$ the set of alternating characters of S_{φ^S} (see theorem **1.1**, 2) for the definition of this group) and, for $\epsilon \in \widehat{S}$, by ϵ_Z its restriction to $Z_{{}^LG}$ (which can be 1 or -1). Put $d_S = +$ or $d_S = -1$ according to whether $det(\varphi^S) = det(\varphi_0)$ or not. (Remark that in the latter case necessarily $det(\varphi^S) = det(\varphi_0)_-$ seen as representation of W_F.) Then φ^S defines a Langlands–Deligne parameter for the quasi-split group H_{l_S,d_S}. Write $\tau_{S,\epsilon}$ for the irreducible supercuspidal representation of $H^{\epsilon_Z}_{l_S,d_S}$ which corresponds to the Langlands–Deligne parameter φ^S and the alternating character ϵ of S_{φ^S}.

Denote by k_ρ the dimension of ρ and put $m_\pm(\rho; \varphi^S) = m(\rho; \varphi^S) + m(\rho_-; \varphi^S)$. Define ${}^LM_S$ to be the Levi subgroup of ${}^LG$ that is isomorphic to

$$\prod_{\rho \in (supp(\varphi_0) - supp'(\varphi_0))/\sim} GL_{k_\rho}(\mathbb{C})^{m(\rho;\varphi_0)}$$
$$\times \prod_{\rho \in supp'(\varphi_0)} GL_{k_\rho}(\mathbb{C})^{[(m(\rho;\varphi_0) - m_\pm(\rho;\varphi^S))]/2} \times {}^LH_{l_S}.$$

(Here $/\sim$ stands for the equivalence classes w.r.t. the relation $\rho \sim \rho^\vee$ defined above.) Let φ_S be the discrete Langlands–Deligne parameter $W_F \times SL_2(\mathbb{C}) \to {}^LM_S$ such that

$$\iota \circ \varphi_S = \bigoplus_{\rho \in supp(\varphi_0)} [m(\rho; \varphi_0) - m(\rho; \varphi^S)]\rho \oplus (\iota \circ \varphi^S).$$

Denote by $M_S^{\epsilon_Z}$ the standard Levi subgroup of $G_{d_S}^{\epsilon_Z}$ with L-group ${}^LM$. It is isomorphic to a product of general linear groups with one factor isomorphic to $H^{\epsilon_Z}_{l_S,d_S}$. For $S \in \mathcal{S}(\varphi_0)$, $\epsilon \in \widehat{S}$, denote by $\sigma_{S,\epsilon}$ the supercuspidal representation of $M_S^{\epsilon_Z}$ which corresponds to φ_S and ϵ (i.e. the factor $H^{\epsilon_Z}_{l_S,d_S}$ acts by $\tau_{S,\epsilon}$) and by $\mathcal{O}_{S,\epsilon}$ the corresponding inertial orbit, i.e. $\mathcal{O}_{S,\epsilon}$ is the set of equivalence classes of representations of $M_S^{\epsilon_Z}$ which are unramified twists of $\sigma_{S,\epsilon}$.

In general, if σ' is an irreducible supercuspidal representation of M', we will denote by $\varphi_{\sigma'}$ the Langlands–Deligne parameter of σ' obtained by applying **1.1** and the local Langlands correspondence to the GL_k.

Theorem *The family $(M_S^{\epsilon_Z}, \mathcal{O}_{\sigma_{S,\epsilon}})_{S,\epsilon}$ exhausts the set of inertial orbits of supercuspidal representations σ' of standard Levi subgroups M' of G, G_-, G^- and G^-_- with $\widehat{M'} \supseteq \widehat{M}$, such that $(\varphi_{\sigma'})_{|W_F}$ lies in the inertial orbit of φ_0.*

One has $(M_S, \mathcal{O}_{\sigma_{S,\epsilon}}) = (M_{S'}, \mathcal{O}_{\sigma_{S',\epsilon'}})$, if and only if $(S, \epsilon) = (S', \epsilon')$.

Proof The first part follows directly from the constructions and theorem **1.1**. For the second part: to have $(M_S, \mathcal{O}_{\sigma_{S,\epsilon}}) = (M_{S'}, \mathcal{O}_{\sigma_{S',\epsilon'}})$, one needs $l_S = l_{S'}$ and $\tau_{S,\epsilon} = \tau_{S',\epsilon'}$, but then the other factors of $\sigma_{S,\epsilon}$ and $\sigma_{S',\epsilon'}$ must be unramified twists of each other. □

1.7 We summarize below the properties of the local Langlands correspondence for G, which is known now (see remarks below after the statement), although no final account has been written yet. If $\mathcal{O}$ is the inertial orbit of a supercuspidal representation of a Levi subgroup of $G_{\pm}^{\pm}$, we will denote by $Rep_{\mathcal{O}}(G_{\pm}^{\pm})$ the corresponding Bernstein component of $Rep(G_{\pm}^{\pm})$ [2].

If $\varphi_0 : W_F \rightarrow {}^LG$ is a normed Langlands parameter for G, put $\mathcal{S}_+(\varphi_0) = \{S \in \mathcal{S}(\varphi_0)|\ det(\varphi_S) = det(\varphi_0)\}$, $\mathcal{S}_-(\varphi_0) = \{S \in \mathcal{S}(\varphi_0)|\ det(\varphi_S) \neq det(\varphi_0)\}$, and, for $S \in \mathcal{S}(\varphi_0)$, $\widehat{S}^{\pm} = \{\epsilon \in \widehat{S} | \epsilon_Z = \pm 1\}$,

$$\mathcal{R}_{F,\pm}^{\varphi_0,\pm}(\underline{G}) = \sum_{S \in \mathcal{S}_{\pm}(\varphi_0), \epsilon \in \widehat{S}^{\pm}} Rep_{\mathcal{O}_{S,\epsilon}}(G_{\pm}^{\pm}).$$

Denote by $\mathcal{R}_F^{\varphi_0,\cdot}(\underline{G})$ the direct sum of $\mathcal{R}_{F,+}^{\varphi_0,\cdot}(\underline{G})$ and $\mathcal{R}_{F,-}^{\varphi_0,\cdot}(\underline{G})$ and by $\mathcal{R}_F^{\varphi_0}(\underline{G})$ the direct sum of $\mathcal{R}_F^{\varphi_0,+}(\underline{G})$ and $\mathcal{R}_F^{\varphi_0,-}(\underline{G})$.

Local Langlands correspondence

For a fixed normed Langlands parameter $\varphi_0 : W_F \rightarrow {}^LG$ for G, the set of equivalence classes of pairs (φ, Ξ) with $\varphi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^LG$ a Deligne-Langlands parameter with $\varphi_{|W_F}$ in the inertial orbit of φ_0, and Ξ an irreducible representation of $C_{\widehat{G}}(Im(\varphi))/(C_{\widehat{G}}(Im(\varphi)))^0$, is in natural bijection with $\mathcal{R}_F^{\varphi_0}(\underline{G})$.

Pairs (φ, Ξ) with $\Xi_{|Z_{L_G}} = 1$ (resp. $\Xi_{|Z_{L_G}} = -1$) correspond to representations of $G_{\pm}^{+}$ (*resp.* $G_{\pm}^{-}$), the ones with $det(\varphi) = det(\varphi_0)$ (resp. $det(\varphi) \neq det(\varphi_0)$) to representations of $G_{+}^{\pm}$ (resp. $G_{-}^{\pm}$), those with φ discrete to square integrable representations and the ones with $\varphi(W_F)$ bounded to tempered representations.

All smooth irreducible representations of G^+ and G^- appear when φ_0 varies.

In addition, the following equalities of local constants hold: if ${}^LM$ is the standard Levi subgroup of a maximal standard parabolic subgroup ${}^LP$ of ${}^LG$, denote by r_1, r_2 the irreducible components of the regular representation of ${}^LM$ on the Lie algebra of the unipotent radical of ${}^LP$. Let π be an irreducible smooth representation of the corresponding maximal Levi subgroup M of G and $\varphi_\pi : W_F \times SL_2(\mathbb{C}) \rightarrow {}^LM$ its Langlands–Deligne parameter. Then, the local factors defined by the Langlands-Shahidi method satisfy, for $i = 1, 2$,

$$\gamma(r_i \circ \varphi_\pi, s) = \gamma(\pi, r_i, s),\ \epsilon(r_i \circ \varphi_\pi, s) = \epsilon(\pi, r_i, s) \textit{ and } L(r_i \circ \varphi_\pi, s) = L(\pi, r_i, s).$$

Remark (i) The representations Ξ have to be taken relative to the group of connected components defined by the centralizer of $Im(\varphi)$ in $\widehat{G}$, although the image of φ may not lie in $\widehat{G}$. Remark that this difference did not matter for **1.1**.

(ii) It is explained in [30, 8] how to define the local factors for non generic representations and also for representations of inner forms (see also [13, section 4]).

(iii) If G is quasi-split and ${}^LM = GL_k(\mathbb{C}) \times {}^LH_l$, then r_1 is the standard representation $id_{GL_k(\mathbb{C})} \otimes \iota$ and $r_2 = Sym^2 \circ id_{GL_k(\mathbb{C})}$ or $\wedge^2 \circ id_{GL_k(\mathbb{C})}$, depending if ${}^LG$ is symplectic or orthogonal.

(iv) The local Langlands correspondence for orthogonal and symplectic groups can be considered as known now in consequence of the work of Moeglin [25] and what is known on the R-groups [3]. (There is only a slight restriction on some argument which has not been written for the inner forms in [25], but this is not a real problem as remarked in the introduction of this paper.)
Results on the preservation of local factors for symplectic and orthogonal Galois representations have been established by Cogdell–Shahidi–Tsai [5], completing work of Henniart [18]. No final account has been written on all this yet.

Remark that, for $G = SO_{2n+1}(F)$, the case of $\varphi_0 = 1$ has been solved in [21] with additional work in [32]. As the constructions in [32] are compatible with [13] and the local Langlands correspondence for quasi-split tori is known to preserve local factors, it follows from [13] that local factors are preserved for this correspondence (which coincides, at least on the level of Langlands–Deligne parameters, with the one in **1.7**).

(v) For the group $Sp_4(F)$, Gan and Takeda gave in [9] properties for the local Langlands correspondence, which makes it unique.

1.8 The following result of [14] is obtained by linking the results of Moeglin to [15] (see the remark after theorem **A.7** in the appendix for the case of the non connected orthogonal group). Recall that it can well happen that ρ is orthogonal and ρ_- symplectic or vice versa [27]. The terminology for affine Hecke algebras with parameters used below is the one from [20], after evaluation in $q^{1/2}$ as done in [21,22].

Recall the equivalence relation on $supp(\varphi_0)$ given by $\rho \sim \rho^\vee$ introduced in **1.6**.

Theorem [14, Theorem 5.2 and remark thereafter] *Let φ_0 be a normed Langlands parameter, $S \in \mathcal{S}(\varphi_0)$, $S = (a_{\rho,+}, a_{\rho,-})_\rho$ and $\epsilon \in \widehat{S}$.*

The category $Rep_{\mathcal{O}_{S,\epsilon}}(G_{d_S}^{\epsilon Z})$ is equivalent to the category of right modules over the tensor product $\otimes_{\rho \in supp(\varphi_0)/\sim} \mathcal{H}_{\varphi_0,S,\rho}$ (taken in the category of $\mathbb{C}$-algebras) where $\mathcal{H}_{\varphi_0,S,\rho}$ are extended affine Hecke algebras of the following type:

- *if ρ is not self-dual, $\mathcal{H}_{\varphi_0,S,\rho}$ is an affine Hecke algebra with root datum equal to the one of $GL_{m(\rho;\varphi_0)}$ and equal parameters q^{t_ρ};*
- *if ρ and ρ_- are both of the same type as ${}^LG$ and $a_{\rho,+} = a_{\rho,-} = 0$, then $\mathcal{H}_{\varphi_0,S,\rho}$ is the semi-direct product of an affine Hecke algebra with root datum equal to the one of $SO_{m(\rho;\varphi_0)}$ and equal parameters q^{t_ρ} by the group algebra of a finite cyclic group of order 2, which acts by the outer automorphism of the root system;*
- *otherwise, putting $\kappa_{\rho,\pm 1} = 0$ (resp. $= 1$) if $\rho_\pm$ is of type ${}^LG$ (resp. not of type ${}^LG$), $\mathcal{H}_{\varphi_0,S,\rho}$ is an affine Hecke algebra with root datum equal to the one of $SO_{m(\rho;\varphi_0)-m_\pm(\rho;\varphi^S)+1}$ and unequal parameters $q^{t_\rho}, \ldots, q^{t_\rho}, q^{t_\rho(a_{\rho,+}+a_{\rho,-}+(\frac{\kappa_{\rho,+}+\kappa_{\rho,-}}{2}))}; q^{t_\rho|a_{\rho,+}-a_{\rho,-}+(\frac{\kappa_{\rho,+}-\kappa_{\rho,-}}{2})|}$, remarking that $m(\rho;\varphi_0) - m_\pm(\rho;\varphi^S) + 1$ is necessarily an odd number.*

Remark (i) If ρ is not of type ${}^LG$ and $a_{\rho,+} = a_{\rho,-} = 0$, then it is well known that the affine Hecke algebra $\mathcal{H}_\rho$ expressed above is isomorphic to an affine Hecke algebra with root datum equal to the one of $Sp_{m(\rho;\varphi_0)}$ and equal parameter q^{t_ρ}.

(ii) The $\kappa_{\rho,\cdot}$ in the above theorem is related to the κ'_ρ in **1.6** by the relation $\kappa'_\rho = 1 - \kappa_{\rho,+}$ and $\kappa'_{\rho_-} = 1 - \kappa_{\rho,-}$.

(iii) By [16], the above equivalence of categories preserves temperedness and discreteness (in the definition for square integrability modulo the "center" for Hecke algebra-representations, the "center" has of course to be taken trivial, so that there are no discrete series representations if the based root system which defines the Hecke algebra has a factor of type A_n). Unitarity is conjectured.

1.9 The notion of a tensor product of linear abelian categories is treated in [8] and recalled in the Appendix **B**. It applies to the category of modules of finite presentation over a coherent $\mathbb{C}$-algebra and in particular to the category of finitely generated modules over a noetherian $\mathbb{C}$-algebra (for ex. an extended affine Hecke algebra or a finite tensor product of such algebras (cf. **B.3**)). Denote by $\mathcal{R}_F^{\varphi_0}(\underline{G})_f$ the full subcategory of $\mathcal{R}_F^{\varphi_0}(\underline{G})$ whose objects are the finitely generated representations (by [2, 3.10], these are precisely the representations which are admissible relative to the action of the Bernstein center) and recall the equivalence relation on $supp(\varphi_0)$ given by $\rho \sim \rho^\vee$ introduced in **1.6**.

Corollary *The category $\mathcal{R}_F^{\varphi_0}(\underline{G})_f$ is equivalent to the category*

$$\bigoplus_{S\in\mathcal{S}(\varphi_0),\epsilon\in\widehat{S}^{\pm}} \left(\bigotimes_{\rho\in supp(\varphi_0)/\sim} (right - \mathcal{H}_{\varphi_0,S,\rho} - modules)_f \right).$$

Proof It follows from **B.2** and **B.3** that the tensor product exists and can be applied by **1.8** to the category $\mathcal{R}_F^{\varphi_0}(\underline{G})$ (defined in **1.7**) to give the statement of the corollary. □

1.10 The above results generalize to pure inner forms of quasi-split unitary groups over F, as remarked in Appendix **C**, **C.1–C.5**.

2. The object of this section is to relate, for a given normed Langlands parameter φ_0, the Deligne–Langlands–Lusztig parameters for the irreducible representations in $\mathcal{R}_F^{\varphi_0}(G)$ to data given by semi-simple and unipotent elements in a given complex group. Recall that we have put in **1.6** an equivalence relation on the set of normed representations of W_F (up to isomorphism) defined by $\rho\sim\rho^\vee$.

2.1 Proposition ([10, section 4]) *Let ${}^LM = GL_{k_1}(\mathbb{C})\times\cdots\times GL_{k_r}(\mathbb{C})\times {}^LH_l$ be a standard Levi subgroup of ${}^LG$, $\varphi : W_F \to {}^LM$ a discrete normed Langlands parameter, $\iota\circ\varphi = \rho_1\oplus\cdots\oplus\rho_r\oplus(\iota\circ\tau)$.*

Then, one has $C_{{}^LG}(Im(\varphi)) = \prod_\rho H_{\rho;\varphi}(m(\rho;\varphi))$, the product going over representatives of the equivalence classes of irreducible normed representations ρ of W_F, while the $H_{\rho;\varphi}(m)$ are complex classical groups with $H_{\rho;\varphi}(m)$ isomorphic to $GL_m(\mathbb{C})$ if ρ is not self-dual, to $Sp_m(\mathbb{C})$ if ρ is not of type ${}^LG$ and to $O_m(\mathbb{C})$ if ρ is of type ${}^LG$ (with the convention $O_1(\mathbb{C}) = \{\pm 1\}$ if $m = 1$).

Finally, $C_{GL_N(\mathbb{C})}(Im(\varphi)) = \prod_\rho G_{\rho;\varphi}(m(\rho;\varphi))$, the product going over representatives of the equivalence classes of irreducible representations ρ of W_F, while $G_{\rho;\varphi}(m)$ is isomorphic to $GL_m(\mathbb{C})$, if ρ is self-dual, to $GL_m(\mathbb{C})\times GL_m(\mathbb{C})$ if ρ is not self-dual, and the group $G_{\rho;\varphi}(m)$ contains $H_{\rho;\varphi}(m)$ in each case.

On the other side, $C_{GL_N(\mathbb{C})}(Im(\varphi_s)) \subseteq \prod_\rho G_{\rho;\varphi}(m(\rho;\varphi))$ for every unramified twist φ_s of φ with s in the centralizer of $Im(\varphi)$.

2.2 Recall that the invariant f_ρ of an irreducible representation ρ of W_F has been defined in **1.4**.

Lemma *Let ${}^LM \simeq GL_{k_1}(\mathbb{C})\times\cdots\times \mathrm{GL}_{k_d}(\mathbb{C})\times {}^LH_l$ be a standard Levi subgroup of ${}^LG$ and let $\varphi : W_F \to {}^LM$ be a discrete Langlands parameter, $\gamma\mapsto(\rho_1(\gamma),\ldots,\rho_d(\gamma),\tau(\gamma))$. Denote by φ_0 the normed Langlands parameter associated to φ. Write $\iota\circ\tau = \tau_1\oplus\cdots\oplus\tau_r$ for the decomposition of $\iota\circ\tau$ into irreducible representations. Denote by s_τ the element of $C_{{}^LH_l}(Im(\tau))$ which corresponds to the diagonal matrix $(f_{\tau_1},\ldots,f_{\tau_r})$ and by s_φ the element of $C_{{}^LM}(Im(\varphi))$ that corresponds to $(f_{\rho_1},\ldots,f_{\rho_d},s_\tau)$.*

The element s_φ lies in $C_{{}^LG}(Im(\varphi_0))$ and in $C_{{}^LG}(Im(\varphi))$.

Suppose: if ρ_i is self-dual, then it is of the same type as the normed representation in its inertial orbit. Then, $C_{{}^LG}(Im(\varphi))$ and $C_{C_{{}^LG}(Im(\varphi_0))}(s_\varphi)$ are canonically isomorphic.

Remark As φ and consequently τ are discrete, the representations τ_i are all of type ${}^LG$ and non isomorphic. In addition, $f_{\tau_i}\in\{\pm 1\}$ for $i = 1,\ldots,r$.

Proof By the proposition **2.1**, one has $C_{{}^LG}(Im(\varphi_0)) = \prod_i GL_{l_i}(\mathbb{C})\times\prod_j Sp_{m_j}(\mathbb{C})\times\prod_k O_{n_k}(\mathbb{C})$, where the first product goes over the ρ_i which are not self-dual, the second one over the ρ_j which are not of the same type as ${}^LG$ and the third one over the ρ_k

which are of the same type as ${}^LG$. The centralizer of $Im(\varphi_0)$ is determined by the partition of the summands of $\iota\circ\varphi_0$ obtained by putting together representations which are either isomorphic or isomorphic to the dual of the other one. The different parts of this partition of the summands of $\iota\circ\varphi_0$ give then rise to factors which are respectively isomorphic to $GL_{l_i}(\mathbb{C})$, $Sp_{m_j}(\mathbb{C})$ or $O_{n_k}(\mathbb{C})$ depending if the representations in the part are not self-dual, orthogonal or symplectic, where l_i denotes half the number of elements in the corresponding part and m_j and n_k the total number of elements in the part. The analog result holds for the centralizer of $Im(\varphi)$. As φ_0 is normed, it is clear that the partition of $\iota\circ\varphi$ is finer than the partition of $\iota\circ\varphi_0$.

Writing $C_{{}^LG}(Im(\varphi_0))$ as above as a product, one sees that the centralizer of s_φ in $C_{{}^LG}(Im(\varphi_0))$ is the product of the centralizers of the components of s_φ in the different factors. So, one can reduce to consider the following three cases:

(i) All summands of $\iota\circ\varphi$ are in the same inertial orbit and the normed representation in this orbit is not self-dual. In particular, τ is trivial.
(ii) All summands of $\iota\circ\varphi$ are in the same inertial orbit and the normed representation in this orbit is not of type ${}^LG$. In particular, τ is trivial.
(iii) All summands of $\iota\circ\varphi$ are in the same inertial orbit and the normed representation in this orbit is of type ${}^LG$. In particular, $\iota\circ\tau$ is either trivial or equal to an element of the inertial orbit of this normed representation of type ${}^LG$.

In all three cases, the centralizer of s_φ is determined by the partition of the coefficients of s_φ, obtained by putting equal coefficients in the same part. By proposition **1.4**, equal coefficients correspond to equal summands of $\iota\circ\varphi$, so that the two partitions correspond canonically to each other and have the same number of elements. The factors of the centralizer of s_φ which correspond to the different parts of the partition of the coefficients of s_φ are all general linear groups of order equal to the length of the partition in case (i). In case (ii) and (iii) they are general linear groups if the coefficients are $\neq \pm 1$, and groups of the same type as the group in the other cases, as by our assumption the appearance of groups of another type is excluded. This proves the proposition. □

2.3 Lemma *With the same notations as in* **2.2**, *assume that ρ is an irreducible representation of W_F of type ${}^LG$, such that ρ_- is not of type ${}^LG$ and $\iota\circ\varphi_0\simeq m\rho$.*

Then, $C_{GL_N(\mathbb{C})}(Im(\varphi_0))$ is canonically isomorphic to $GL_m(\mathbb{C})$, while $C_{{}^LG}\ (Im(\varphi_0))$ is isomorphic to $O_m(\mathbb{C})$.

Define s_φ as in **2.2**. *The element s_φ lies in $C_{{}^LG}(Im(\varphi_0))$ and in $C_{{}^LG}(Im(\varphi))$.*

Write $s_\varphi = diag(x_1,\ldots,x_{[\frac{m}{2}]},\widehat{1},x_{[\frac{m}{2}]}^{-1},\ldots,x_1^{-1})\in GL_m(\mathbb{C})$ (with 1 appearing only when m is odd and $[\frac{m}{2}]$ denoting the integer part of $\frac{m}{2}$). For $x\in\{x_1,\ldots,x_{[\frac{m}{2}]}\}$, denote by $m(x;s_\varphi)$ the multiplicity of x as an entry of s_φ and put

$$G_x=\begin{cases} GL_{m(x;s_\varphi)}\times GL_{m(x^{-1};s_\varphi)}, & \text{if } x\notin\{\pm 1\},\\ GL_{m(\pm 1;s_\varphi)}, & \text{if } x=\pm 1.\end{cases}$$

The group $C_{GL_N(\mathbb{C})}(s_\varphi)$ is canonically isomorphic to $\prod_x G_x$, the product going over equivalence classes of elements in the set $\{x_1,\ldots,x_{[\frac{m}{2}]}\}$ with respect to the relation $x\sim x^{-1}$, and to $C_{GL_N(\mathbb{C})}(Im(\varphi))$.

Denote by H_x (resp. H'_x) the subgroup of G_x defined (with J an appropriate matrix which needs not to be made more precise here) by

$$\begin{cases} \{(h,Jh^{-1}J)\mid h\in GL_{m(x;s_\varphi)}\} & \text{if } x\notin\{\pm 1\},\\ O_{m(1,s_\varphi)} & \text{if } x=1,\\ Sp_{m(-1,s_\varphi)}\ (\text{resp.} O_{m(-1,s_\varphi)}), & \text{if } x=-1,\end{cases}$$

and by H (resp. H') the image of $\prod_x H_x$ (resp. $\prod_x H'_x$) in $C_{GL_N(\mathbb{C})}(s_\varphi)$ by the above isomorphism.

Then, $C_{{}^LG}(Im(\varphi))$ is isomorphic to H and $C_{{}^LG}(s_\varphi)$ to H'.

In particular, $C_{C_{{}^LG}(Im(\varphi_0))}(s_\varphi)$ and $C_{{}^LG}(Im(\varphi))$ are only isomorphic if $m(-1, s_\varphi) = 0$.

Proof This follows immediately from the arguments in the proof of lemma **2.2**. □

2.4 Remark that at the end of the following definition, we will use results from **C.6** and **C.7**. However, at a first reading, one may avoid to look in the Appendix **C**.

Definition Let ${}^LM \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_d}(\mathbb{C}) \times {}^LH_l$ be a standard Levi subgroup of ${}^LG$ and let $\varphi_0 : W_F \to {}^LM$ be a discrete normed Langlands parameter, $\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_d(\gamma), \tau(\gamma))$.

Recall that by **2.1**, $C_{{}^LG}(Im(\varphi_0)) = \prod_\rho H_{\rho;\varphi}(m(\rho;\varphi))$, the product going over representatives of the equivalence classes of irreducible normed representations ρ of W_F (w.r.t. the relation defined in **1.6**), while the $H_{\rho;\varphi}(m)$ are complex classical groups with $H_{\rho;\varphi}(m)$ isomorphic to $GL_m(\mathbb{C})$ if ρ is not self-dual, to $Sp_m(\mathbb{C})$ if ρ is not of type ${}^LG$, and to $O_m(\mathbb{C})$ if ρ is of type ${}^LG$ (with the convention $O_1(\mathbb{C}) = \{\pm 1\}$ if $m = 1$).

Let s be a semi-simple element in $C_{{}^LG}(Im(\varphi_0))$ and denote by s_ρ the projection of s on $H_{\rho;\varphi}(m(\rho;\varphi))$. Define $C'_{H_{\rho;\varphi}(m(\rho;\varphi))}(s_\rho) = C_{H_{\rho;\varphi}(m(\rho;\varphi))}(s_\rho)$ except if ρ and ρ_- are not of the same type. In that case, put $m = m(\rho;\varphi)$ and denote by $H'_{\rho;\varphi}(m)$ the connected component of the L-group of the unramified quasi-split unitary group U_m and define $C'_{H_{\rho;\varphi}(m)}(s_\rho) = C_{H'_{\rho;\varphi}(m)}((-1)^{m-1}_{s_\rho})$ (where $(-1)_{s_\rho}$ is the Langlands parameter for U_m defined in **C.7**).

Put $C'_{C_{{}^LG}(Im(\varphi_0))}(s) = \prod_\rho C'_{H_{\rho;\varphi}(m(\rho;\varphi))}(s_\rho)$. For a subset I of $C'_{C_{{}^LG}(Im(\varphi_0))}(s)$, denote its centralizer by $C'_{C_{{}^LG}(Im(\varphi_0))}(s, I)$ (there is some subtlety if ρ and ρ_- are self-dual, but not of the same type), and write $C'^{+}_{C_{{}^LG}(Im(\varphi_0))}(s, I)$ for the subgroup of elements with determinant 1.

2.5 Lemme: *Let ${}^LM \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_d}(\mathbb{C}) \times {}^LH_l$ be a standard Levi subgroup of ${}^LG$ and let $\varphi_0 : W_F \to {}^LM$ be a discrete normed Langlands parameter, $\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_d(\gamma), \tau(\gamma))$.*

The set of equivalence classes of Langlands–Deligne parameters $\varphi : W_F \times SL_2(\mathbb{C}) \to {}^LG$ with $\varphi_{|W_F}$ in the inertial orbit of φ_0 is in bijection with the set of equivalence classes of pairs (s, φ_{SL_2}) consisting of a semisimple element s and an algebraic homomorphism $SL_2(\mathbb{C}) \to C'_{C_{{}^LG}(Im(\varphi_0))}(s)$ by mapping φ to $(s_\varphi, \varphi_{|SL_2(\mathbb{C})})$, so that $C_{\widehat{G}}(Im(\varphi))/C_{\widehat{G}}(Im(\varphi))^0$ is canonically isomorphic to

$$C'^{+}_{C_{{}^LG}(Im(\varphi_0))}(s_\varphi, \varphi(SL_2(\mathbb{C})))/C'^{+}_{C_{{}^LG}(Im(\varphi_0))}(s_\varphi, \varphi(SL_2(\mathbb{C})))^0.$$

Proof This is straightforward by the definitions, the above lemmas and **C.7**, **C.8**, remarking that $\varphi_{0,s}$ is, as element of the inertial orbit of φ_0, determined by $s_{\varphi_{0,s}}$ and that the map $s \mapsto s_{\varphi_{0,s}}$ is surjective. □

2.6 Theorem *Let ${}^LM \simeq GL_{k_1}(\mathbb{C}) \times \cdots \times \mathrm{GL}_{k_d}(\mathbb{C}) \times {}^LH_l$ be a standard Levi subgroup of ${}^LG$ and let $\varphi_0 : W_F \to {}^LM$ be a discrete normed Langlands parameter, $\gamma \mapsto (\rho_1(\gamma), \ldots, \rho_d(\gamma), \tau(\gamma))$.*

The set of equivalence classes of pairs (s, u) consisting of a semisimple element s and a unipotent element u in $C'_{{}^LG}(Im(\varphi_0))$ such that $sus^{-1} = u^q$ is in bijection with the set of equivalence classes of Langlands–Deligne parameters $\varphi : W_F \times SL_2(\mathbb{C}) \to {}^LG$ with $\varphi_{|W_F}$

in the inertial orbit of φ_0*, so that one has a canonical isomorphism between the group of connected components of the centralizers of the images,*

$$C_{\widehat{G}}(Im(\varphi))/C_{\widehat{G}}(Im(\varphi))^0 \simeq {C'}^{+}_{C_{^LG}(\varphi_0(W_F))}(s,u)/{C'}^{+}_{C_{^LG}(\varphi_0(W_F))}(s,u)^0.$$

Remark Remark that $C' = C$, if $\rho_{i,-}$ is of type ${}^LG$ whenever ρ_i is.

Proof By the preceding lemma, it remains to show that the equivalence classes of pairs (s, u) in the (possibly non connected) complex reductive group $C'_{^LG}(Im(\varphi_0))$ such that $sus^{-1} = u^q$ is in bijection with the set of equivalence classes of pairs (s, φ_{SL_2}) with s in $C'_{^LG}(Im(\varphi_0))$ and $\varphi_{SL_2} : SL_2(\mathbb{C}) \to C'_{C_{^LG}(Im(\varphi_0))}(s)$ a morphism or algebraic groups. This is proved in [19, 2.4]. The general assumption of this paper being that the group is semi-simple and simply connected, it has been checked in [13, 3.5] that this is still valid for a connected reductive group. As Mostow's theorem is valid for possibly non-connected algebraic groups, the assumption "connected" can be relaxed, too. (The connected component of the group noted $M_{\varphi_{SL_2}}$ in [19, 2.4] being reductive, $M_{\varphi_{SL_2}}$ is reductive.) □

3. In this section we will show at the end that the previous results allow to relate the category $\mathcal{R}_F^{\varphi_0}(G)$ to categories of unipotent representations of p-adic classical groups. However, before that, we will state some parameterizations of representations of collections of (possibly extended) affine Hecke algebras that follow from section **1.** and from the additional remarks about unitary groups in Appendix **C**.

The parametrization is given by a set of conjugation classes of triples (s, u, Ξ) associated to a given complex group, where s is a semi-simple element, u a unipotent element such that $sus^{-1} = u^q$ and Ξ an irreducible representation of the group of components of the common centralizer of s and u.

3.1 The following is the special case of **1.7**, **1.8** for $G = SO_{2d+1}$, $M = T$ and ρ the trivial representation that is treated in [21] with modifications in [32].

Theorem *Fix an integer* $d \geq 1$*. If* $(d_+, d_-) \neq 0$ *is a pair of integers which are each one products of two consecutive integers,* $d_+ = a_+(a_+ + 1)$ *and* $d_- = a_-(a_- + 1)$*, with* $d_+ + d_- \leq 2d + 1$*, denote by* $\mathcal{H}(d_+, d_-)$ *the affine Hecke algebra with root datum equal to the one of* $SO_{2d+1-d_+-d_-}$ *and unequal parameters* $q, \ldots, q, q^{a_++a_-+1}; q^{|a_+-a_-|}$*. Denote by* $\mathcal{H}(0, 0)$ *the affine Hecke algebra with root datum equal to the one of* $Sp_{2d}(\mathbb{C})$ *and equal parameters* $q, \ldots, q$*.*

Then, the set of triples (s, u, Ξ) *associated to the group* $Sp_{2d}(\mathbb{C})$ *with* $\Xi(-1) = 1$ *is in natural bijection with the set*

$$\bigcup_{(d_+,d_-),\ \frac{d_++d_-}{2}\ even} (irreducible\ right - \mathcal{H}(d_+, d_-) - modules)$$

and the one with $\Xi(-1) = -1$ *is in nature bijection with the set*

$$\bigcup_{(d_+,d_-),\ \frac{d_++d_-}{2}\ odd} (irreducible\ right - \mathcal{H}(d_+, d_-) - modules).$$

Remark By natural bijection, we mean what is implied by the properties of the local Langlands correspondence stated in **1.7**. We will not explain this here more, except that compact s should correspond to tempered representations, discrete (s, u) (i.e. those which do not lie in a proper parabolic subgroup) to discrete series representations and that Langlands-Shahidi local factors (defined for Hecke-algebra representations by equivalence of categories) equal the Artinian ones (deduced from (s, u)). The bijection associates implicitly to a triple (s, u, Ξ) a "supercuspidal support" according to **1.6**, but we will not give a precise construction here, neither in the subsequent cases. (For the above case, it can be found in [21] and [32].)

3.2 The following follows by combining **1.7** and **1.8** to the special case $G = Sp_{2d}(F)$ (for (i)) and $G = O_{2d}(F)$ (for (ii)) with ρ the trivial representation and M the maximal split torus. Remark that the result below cannot be deduced from the work of Lusztig [21] that applies only to p-adic groups of adjoint type.

Theorem *Fix an integer $d \geq 1$.*

If $(d_+, d_-) \neq 0$ is a pair of integers, which are squares, such that $d_+ + d_- \leq 2d+1$, denote by $\mathcal{H}(d_+, d_-)$ the affine Hecke algebra with root datum equal to the one of $SO_{2d+1-d_+-d_-}$, if $d_+ + d_-$ is even, and equal to the one of $SO_{2d+2-d_+-d_-}$, if $d_+ + d_-$ is odd,and unequal parameters $q, \dots, q, q^{\sqrt{d_+}+\sqrt{d_-}}; q^{|\sqrt{d_+}-\sqrt{d_-}|}$.

In addition, denote by $\mathcal{H}(0, 0)$ the semi-direct product of an affine Hecke algebra with equal parameter q and root datum equal to the one of SO_{2d} with the group algebra of a finite cyclic group of order 2, which acts by the outer automorphism of the root system. Define

$$\epsilon^+_{d_+,d_-} = \begin{cases} 4 & \text{if } d_+ \text{even}, d_+ + d_- \in 8\mathbb{Z}, d_+ \cdot d_- \neq 0, \\ 1 & \text{if } d_+ = d_- = 0, \\ 0 & \text{if } d_+ \text{even}, d_+ + d_- \in 4\mathbb{Z} \setminus 8\mathbb{Z}, \\ 2 & \text{otherwise.} \end{cases}$$

and put $\epsilon^-_{d_+,d_-} = 4 - \epsilon^+_{d_+,d_-}$ if $d_+ \cdot d_- \neq 0$, $\epsilon^-_{d_+,d_-} = 2 - \epsilon^+_{d_+,d_-}$ if exactly one of d_+ and d_- is 0, and $\epsilon^-_{0,0} = 0$.

(i) Denote by S_o the set of pairs of integers (d_+, d_-) such that d_+ and d_- are squares, $d_+ + d_-$ is odd and $\leq 2d + 1$. (Consequently $\epsilon^\pm_{d_+,d_-} = 2$. The set of triples (s, u, Ξ) associated to the group $SO_{2d+1}(\mathbb{C})$ is in natural bijection with the multiset

$$\bigcup_{(d_+,d_-)\in S_o} 2\ (irreducible\ right - \mathcal{H}(d_+, d_-) - modules).$$

(ii) Denote by S_e the set of pairs of integers (d_+, d_-) such that d_+ and d_- are squares, $d_+ + d_-$ is even and $\leq 2d + 1$. The set of triples (s, u, Ξ) associated to the group $O_{2d}(\mathbb{C})$ with $\Xi_{|Z_{\widehat{G}}} = \pm 1$ is in natural bijection with the multiset

$$\bigcup_{(d_+,d_-)\in S_e} \epsilon^\pm_{d_+,d_-}\ (irreducible\ right - \mathcal{H}(d_+, d_-) - modules).$$

3.3 The following follows by combining **1.7** and **1.8** as generalized to quasi-split unitary groups in Appendix **C**. Here U_m will denote the unramified quasi-split unitary group of semi-simple rank m.

Fix m. A triple (s, u, Ξ) as above relative to ${}^LU_m$, will be said associated to ${}^LU_m$, if s is not an element of the connected component $({}^LU_m)^0$ of ${}^LU_m$.

Theorem *If d_+ is a square integer and d_- the product of two consecutive integers, $d_- = a_-(a_- + 1)$, denote by $\mathcal{H}(d_+, d_-)$ the affine Hecke algebra with root datum equal to the one of $SO_{2d+1-d_+-d_-}$, if $d_+ + d_-$ is even, and to the one of $SO_{2d+2-d_+-d_-}$, if $d_+ + d_-$ is odd, and unequal parameters $q, \dots, q, q^{\sqrt{d_+}+a_-+\frac{1}{2}}; q^{|\sqrt{d_+}-a_--\frac{1}{2}|}$.*

In addition, denote by $\mathcal{H}(0,0)$ *the affine Hecke algebra with root datum equal to the one of* SO_{2d+1} *and unequal parameters* $q, q, \ldots, q, q^{1/2}$.

Denote by S_e *the set of pairs* (d_+, d_-) *with* d_+ *an even square,* d_- *the product of two consecutive integers,* $d_+ + d_- \leq 2d+1$*, and by* S_o *the set of pairs* (d_+, d_-) *with* d_+ *an odd square,* d_- *the product of two consecutive integers,* $d_+ + d_- \leq 2d+1$.

(i) *If* m *is an odd integer,* $m = 2d+1$*, then the set of triples* (s, u, Ξ) *associated to* ${}^L U_m$ *with* $\Xi_{|\{\pm 1\}}$ *fixed is in natural bijection with the multiset*

$$\bigcup_{(d_+,d_-)\in S_o} (irreducible\ right - \mathcal{H}(d_+,d_-) - modules).$$

(ii) *If* m *is an even integer,* $m = 2d$*, put* $\epsilon_{d_+,d_-} = 2$*, if* $d_+ \neq 0$*, and* $\epsilon_{d_+,d_-} = 1$ *otherwise. Then the set of triples* (s, u, Ξ) *associated to* ${}^L U_m$ *with* $\Xi(-1) = 1$ *is in natural bijection with the multiset*

$$\bigcup_{(d_+,d_-)\in S_e,\ \frac{d_+}{2}\ even} \epsilon_{d_+,d_-}\ (irreducible\ right - \mathcal{H}(d_+,d_-) - modules)$$

and the one with $\Xi(-1) = -1$ *is in natural bijection with the multiset*

$$\bigcup_{(d_+,d_-)\in S_e,\ \frac{d_+}{2}\ odd} 2\ (irreducible\ right - \mathcal{H}(d_+,d_-) - modules).$$

3.4 If t_ρ is an integer ≥ 1, denote by F_{t_ρ} the unramified extension of F of degree t_ρ, which is unique in a given algebraic closure of F. If φ is a Langlands parameter which is not normed and φ_0 is the normed Langlands parameter in its orbit, we put $\mathcal{R}^\varphi = \mathcal{R}^{\varphi_0}$. We also put $\varphi_0 = 1$, if φ_0 is the Langlands parameter relative to the minimal standard Levi subgroup that corresponds to the trivial representation.

Theorem *Assume that there is an irreducible representation* ρ *such that all irreducible components of* $\iota \circ \varphi_0$ *are either isomorphic to* ρ *or to* $\rho^\vee$*. Then, with* $m = m(\rho;\varphi)$*, one has*

(i) *if* ρ *is not self-dual, then the category* $\mathcal{R}_F^{\varphi_0}(\underline{G})$ *is equivalent to* $\mathcal{R}^1_{F_{t_\rho}}(GL_m)$.

(ii) *if* ρ *is self-dual and not of type* ${}^L G$*, then the category* $\mathcal{R}_F^{\varphi_0}(\underline{G})$ *is equivalent to* $\mathcal{R}^1_{F_{t_\rho}}(SO_{m+1})$.

(iii) *if* ρ *and* ρ_- *are both of type* ${}^L G$*, then the category* $\mathcal{R}_F^{\varphi_0}(\underline{G})$ *is equivalent to* $\mathcal{R}^1_{F_{t_\rho}}(Sp_{m-1})$ *if* m *is odd, and to* $\mathcal{R}^1_{F_{t_\rho}}(O_m)$ *otherwise.*

(iv) *if* ρ *and* ρ_- *are self-dual but not of the same type, then, with* U_m *equal to the unramified quasi-split unitary group of absolute rank* m*, the category* $\mathcal{R}_F^{\varphi_0}(\underline{G})$ *is equivalent to* $\mathcal{R}^1_{F_{t_\rho}}(U_m)$.

The same holds, if one replaces $\mathcal{R}^\cdot$ *by* $\mathcal{R}^{\cdot,+}$ *or* $\mathcal{R}^{\cdot,-}$.

Proof This follows from theorem **1.9** (and its generalization to unitary groups in **C.5** together with proposition **C.6**) applied to the above cases for φ_0, after remarking that in each of the cases the sets $\mathcal{S}^\cdot(\varphi_0)$ and $\mathcal{S}^\cdot(1)$ are equal, while the alternating characters associated to their elements are the same. There is no need here to restrict to finitely generated representations, as the tensor product is not involved. □

3.5 Recall the equivalence relation on $supp(\varphi_0)$ given by $\rho \sim \rho^\vee$ introduced in **1.6** and that the index f denotes the full subcategory of finitely generated representations.

Corollary *The category $\mathcal{R}_F^{\varphi_0}(\underline{G})_f$ is equivalent to*

$$\bigotimes_{\rho\in supp(\varphi_0)/\sim} \mathcal{R}^1_{F_{t_\rho}}(H_\rho(m(\rho;\varphi_0)))_f$$

with $H_\rho(m)$ equal to GL_m, SO_{m+1}, Sp_{m-1}, O_m or the unramified quasi-split unitary group U_m, if respectively ρ is not self-dual, not of type ${}^LG$, ρ and ρ_- are both of type ${}^LG$ with m odd, with m even, or ρ and ρ_- are self-dual but not of the same type.

The same holds, if one replaces $\mathcal{R}^{\cdot}$ by $\mathcal{R}^{\cdot,+}$ or $\mathcal{R}^{\cdot,-}$.

Remark As follows from remark (iii) after theorem **1.8**, this equivalence of category preserves temperedness. Discreteness is preserved if none of the H_ρ is a general linear group - otherwise there are no discrete series representations in $\mathcal{R}_F^{\varphi_0}(\underline{G})_f$. Unitarity is conjectured.

Proof Denote by $\mathcal{S}(\varphi_0)_\rho$ the projection of $\mathcal{S}(\varphi_0)$ on the ρ's component and by $\varphi_{0,\rho}$ the discrete Langlands parameter (unique up to equivalence) into some L-group ${}^LH_\rho$ of the same type as ${}^LG$ that satisfies the following condition (which determines also the semi-simple rank of ${}^LH_\rho$)

$$\iota\circ\varphi_{0,\rho} = \begin{cases} m(\rho;\varphi_0)\rho, & \text{if } \rho\simeq\rho^\vee;\\ m(\rho;\varphi_0)\rho\oplus m(\rho^\vee;\varphi_0)\rho^\vee, & \text{otherwise.}\end{cases}$$

Then one has $\mathcal{S}(\varphi_0)=\prod_\rho \mathcal{S}(\varphi_0)_\rho$, $\mathcal{S}(\varphi_0)_\rho=\mathcal{S}(\varphi_{0,\rho})$. In addition, if $S=(S_\rho)_\rho\in\mathcal{S}(\varphi_0)$, then $\mathcal{H}_{\varphi_0,S,\rho}=\mathcal{H}_{\varphi_{0,\rho},S_\rho,\rho}$ and the group of alternating characters $\widehat{S}$ is the product of the groups $\widehat{S_\rho}$, where S_ρ is seen as element of $\mathcal{S}(\varphi_{0,\rho})$.

Applying **B.4** to **1.9**, one gets from this

$$\begin{aligned}\mathcal{R}_F^{\varphi_0}(\underline{G})_f &\simeq \bigoplus_{S\in\mathcal{S}(\varphi_0),\epsilon\in\widehat{S}^\pm}\left(\bigotimes_{\rho\in supp(\varphi_0)/\sim}(right-\mathcal{H}_{\varphi_0,S,\rho}-modules)_f\right)\\ &\simeq \bigotimes_{\rho\in supp(\varphi_0)/\sim}\left(\bigoplus_{S_\rho\in\mathcal{S}(\varphi_{0,\rho}),\epsilon\in\widehat{S}_\rho^\pm}(right-\mathcal{H}_{\varphi_{0,\rho},S_\rho,\rho}-modules)_f\right).\end{aligned}$$

Using **3.4** and **B.5**, the statement of the corollary follows. □

Remark (1) A statement of this kind had been conjectured by Lusztig [23, section 19]. Once the results in [21] appropriately generalized to symplectic, unitary and the (full) even orthogonal group, one should be able to describe multiplicities in standard modules from intersection cohomology as described in [23, section 19].

(2) In general, of course all unramified quasi-split groups may appear in this conjecture of G. Lusztig. With some additional work, as described at the end of the introduction, it should be possible to figure out, which group should appear for a given quasi-split reductive p-adic group and a given generic supercuspidal support. It is quite clear how to define ρ and ρ_- in this context (as Langlands parameters or as representations) and one may expect that non-split quasi-split groups appear in general for H_ρ if the reducibility points of the representations associated to ρ and ρ_- are not the same.

(3) In [7], the appearance of the field extensions in **3.4, 3.5** has been worked out in more detail in terms of restriction of scalars for the general linear group. Some ideas for the general case are also given.

Appendix A: Equivalence of categories for the full orthogonal group

A.1 The aim of this appendix is to show how the results of [14,15] generalize to the full orthogonal group, which is not connected. So, in this appendix, H will denote a pure inner form of a full split orthogonal group, either split or not. The case when its connected component H^0 is an odd orthogonal group is quite easy. Then H is isomorphic to a direct product $H^0 \times \{\pm 1\}$. The Levi subgroups of H are of the form $M = M^0 \times \{\pm 1\}$, where M^0 is a Levi subgroup of H^0, so that the supercuspidal representations of M are of the form $\sigma^0 \eta$, where σ^0 is a supercuspidal representation of M^0 and η a character of $\{\pm 1\}$. One sees immediately that the restriction to $\{\pm 1\}$ of a representation in the supercuspidal support of an irreducible representation π of H is determined by the restriction of π to this group. So, one may decompose $Rep(H)$ as a direct sum of subcategories $Rep_{M^0,\mathcal{O}}(H^0) \oplus Rep_{M^0,\mathcal{O},-1}(H^0)$, where the $Rep_{M^0,\mathcal{O}}(H^0)$ denote the Bernstein components for H^0 and the $Rep_{M^0,\mathcal{O},-1}(H^0)$ denote the part with non-trivial restriction to $\{\pm 1\}$. As the results of [14,15] apply to $Rep(H^0)$, we are done.

A.2 Assume now for the rest of this appendix that n is even. Then, H is isomorphic to a semi-direct product $H^0 \rtimes \{1, r_0\}$, where H^0 is an even orthogonal group and r_0 is of order 2 and acts on H^0 by the outer isomorphism. We refer to [11] for results for the representation theory of a non connected reductive group. We consider only Levi subgroups which are *cuspidal* in the terminology of [11]. In particular, one deduces from this paper that the Bernstein decomposition is still valid and that, if M is a Levi subgroup of H and $\mathcal{O}$ denotes the inertial orbit of an irreducible supercuspidal representation of M, then, with the notations in [15], $i_P^H E_{B_\mathcal{O}}$ is a projective generator of $Rep_{(M,\mathcal{O})}(H)$, which implies that the category $Rep_{(M,\mathcal{O})}(H)$ is equivalent to the category of right-modules over $End_H(i_P^H E_{B_\mathcal{O}})$ by Morita theory.

The aim of this appendix is to show that $End_H(i_P^H E_{B_\mathcal{O}})$ has the form given in theorem **1.8**.

A.3 Denote by W^0 the Weyl group of H^0 and define $W := W^0 \rtimes \{1, r_0\}$, and similar for the Weyl group W^M of a Levi subgroup M of H. If M is a Levi subgroup of H, define $W(M) = W^M \backslash \{w \in W | w^{-1} M w = M\}$ and similarly $W^0(M^0)$, which will also be denoted (abusively) $W^0(M)$.

Lemma *One has $W(M) = W^0(M)$, except if M is isomorphic to a product of general linear groups and at least one of them has odd rank. In particular, $W(M) = W^0(M)$ if H is not quasi-split.*

Proof If M has a factor H_l with $l \geq 2$, then $r_0 \in M$. So, M has to be a product of linear groups if $W(M) \neq W^0(M)$. If M is a product of general linear groups of even rank, then every element of w that satisfies $w^{-1} M w = M$ must have an even number of sign changes $x \mapsto x^{-1}$ on the maximal torus. This means that it lies in W^0. If M is a product of general linear groups, one of them being of odd rank k, one sees that there is an element in W which induces the outer automorphism on GL_k and which does not lie in W^0. □

A.4 Let $\mathcal{O}$ be the inertial orbit of a supercuspidal representation of a Levi subgroup M of H. Its restriction to M^0 decomposes into one or two inertial orbits. Fix an orbit $\mathcal{O}^0$ in the restriction. Denote by $W(M, \mathcal{O})$ (resp. $W^0(M, \mathcal{O})$) the subset of elements of $W(M)$ (resp. $W^0(M)$) which stabilize $\mathcal{O}$ (resp. $\mathcal{O}^0$).

Lemma *One has $W(M, \mathcal{O}) = W^0(M, \mathcal{O})$ except if M is a product of general linear groups and at least one factor of $\mathcal{O}^0$ is the inertial orbit of a self-dual representation of a general linear group of odd rank.*

Proof The group $W(M, \mathcal{O})$ is a subgroup of $W(M)$. It follows that the equality $W(M, \mathcal{O}) = W^0(M, \mathcal{O})$ can only fail if M is a product of general linear groups and at least one of them has odd rank k. In addition, one sees that at least one factor of $\mathcal{O}$ corresponding to a $GL_k(F)$ with k odd must be the orbit of a self-dual representation. □

A.5 Denote by $R(\mathcal{O})$ the subgroup of elements r of $W(M, \mathcal{O})$ that send positive roots for M to positive roots. Define $R^0(\mathcal{O}) = R(\mathcal{O}) \cap W^0(M, \mathcal{O})$. Recall [15] that $W^0(M, \mathcal{O})$ is a semi-direct product $W^0_{\mathcal{O}} \rtimes R^0(\mathcal{O})$, so that one has $W(M, \mathcal{O}) = W^0_{\mathcal{O}} \rtimes R(\mathcal{O})$. As $ind_{M^0}^M E_{B_{\mathcal{O}^0}}$ is either equal to $E_{B_{\mathcal{O}}}$ or a direct sum $E_{B_{\mathcal{O}}} \oplus E_{B_{\mathcal{O}'}}$, one can define, for $w \in W^0(\mathcal{O}^0)$ and $r \in R^0(\mathcal{O})$, operators T_w and J_r in $End_H(i_P^H E_{B_{\mathcal{O}}})$ from the ones for $End_{G^0}(i_{P^0}^{H^0} E_{B_{\mathcal{O}^0}})$ by induction. If $r \in R(\mathcal{O}) \setminus R^0(\mathcal{O})$, note $\lambda(r)$ the action of r on $i_P^H E_{B_{\mathcal{O}}}$ by left-translation and by τ_r the one of r on $B_{\mathcal{O}}$ by right translation [15], and put $J_r = \tau_r \lambda(r)$. These operators J_r commute obviously with the other operators $J_{r'}$, $r' \in R(\mathcal{O})$, and satisfy the commuting relation $T_w J_r = J_r T_{r^{-1}wr}$ for $w \in W^0_{\mathcal{O}}$.

Lemma *The operators $sp_\chi J_r T_w$, $r \in R(\mathcal{O})$, $w \in W(M, \mathcal{O})$, are linearly independent for all $\chi \in \mathfrak{X}^{\mathrm{nr}}(M)$.*

Proof The proof of [15, 5.9] generalizes, as the commuting relations for the operators J_r, $r \in R(\mathcal{O})$ are still the same. □

A.6 Lemma *One has* $\mathrm{Hom}_H(i_P^H E_{B_{\mathcal{O}}}, i_P^G E_{K(B_{\mathcal{O}})}) = \bigoplus_{w,r} K(B_{\mathcal{O}}) J_r T_w$.

Proof This follows from the linear independence and the computation of the Jacquet module with help of the geometric lemma in the non connected case [6, 4.1], taking into account lemma **A.4**. □

A.7 Theorem *One has* $\mathrm{End}_H(i_P^H E_{B_{\mathcal{O}}}) = \bigoplus_{w,r} B_{\mathcal{O}} J_r T_w$.

Proof The proof of [15, 5.10] generalizes, as the commuting relations for the operators J_r, $r \in R(\mathcal{O})$ are still the same. □

Remark As the T_w satisfy the same relations as their restrictions to the space of the representation of the connected component, it follows that $End_H(i_H^G E_{B_{\mathcal{O}}})$ is an (possibly extended) affine Hecke algebra isomorphic to $End_{H^0}(i_{P^0}^{G^0} E_{B_{\mathcal{O}^0}})$, except if M is a product of general linear groups and at least one factor of $\mathcal{O}$ is the inertial orbit of a self-dual representation of a general linear group of odd rank. In this case, one has additional operators J_r with $r \in R(\mathcal{O}) \backslash R^0(\mathcal{O})$ as stated in **1.8**.

In fact, we have an erratum to [14,15] w.r.t. the statements for the even dimensional special orthogonal group, the connected component of H: if $M = M^0$ (i.e. M is a product of general linear groups), $End_{H^0}(i_{P^0}^{H^0} E_{B_{\mathcal{O}^0}})$ is in general isomorphic to a tensor product $\bigotimes_\rho \mathcal{H}_\rho \otimes ((\otimes_{\rho'} \mathcal{H}^0_{\rho'}) \rtimes \mathbb{C}[R_{nq}])$, the first product going over elements ρ in the support of the normed Langlands parameter φ_0 associated to $\mathcal{O}$ which are not odd orthogonal and the second product over the odd orthogonal ones, R_{nq} being generated by Weyl group elements that send positive roots in $\Sigma_{\mathcal{O}}$ (in the notations of [15]) to positive roots and have sign changes

$x \mapsto x^{-1}$ on two factors $GL_k(F)$ and $GL_{k'}(F)$, on which odd orthogonal representations with distinct inertial orbits are defined. Here the $\mathcal{H}_\rho$ denote the (possibly extended) affine Hecke algebras from **1.8** and $\mathcal{H}_\rho^0$ the affine Hecke algebra part (i.e. omitting the finite group algebra part, if there is any). One remarks that the above semi-direct product is with a tensor product of affine Hecke algebras associated to odd orthogonal representations in the support, but *does not* decompose into a tensor product of semi-direct products of the different affine Hecke algebras with a group algebra.

Appendix B: Tensor product of abelian categories

B.1 Definition [8, 5.] Let k be a commutative ring and $(\mathcal{A}_i)_{i\in I}$ a finite family of k-linear abelian categories. A k-linear abelian category $\mathcal{A}$ equipped with a k-multilinear functor right exact in each variable

$$\otimes : \prod \mathcal{A}_i \to \mathcal{A}$$

is called *tensor product over k* of the categories $\mathcal{A}_i$ if and only if the following condition is satisfied: denote for a k-linear abelian category C by $\underline{Hom}_{k,e\ \grave{a}\ d}(\mathcal{A}, C)$ the category of right exact functors from $\mathcal{A}$ to C and by $\underline{Hom}_{k,e\ \grave{a}\ d}((\mathcal{A})_{i\in I}, C)$ the category of right exact functors multilinear in each variable from the product of the $\mathcal{A}_i$ to C.

One asks then that for every category C the composed functor with the above

$$\underline{Hom}_{k,e\ \grave{a}\ d}(\mathcal{A}, C) \to \underline{Hom}_{k,e\ \grave{a}\ d}((\mathcal{A}_i)_{i\in I}, C)$$

is an equivalence of categories.

B.2 Proposition [8, 5.3] *Let $(A_i)_{i\in I}$ be a finite family of coherent k-algebras that have a coherent tensor product over k. Denote by $(A_i)_{coh}$ (resp. $(\otimes A_i)_{coh}$) the corresponding abelian category of right modules of finite presentation. The tensor product over k*

$$\otimes : \prod (A_i)_{coh} \to (\otimes A_i)_{coh}$$

defines $(\otimes A_i)_{coh}$ as tensor product over k of the $(A_i)_{coh}$.

B.3 Proposition *(i) An extended affine Hecke algebra with unequal parameters is a coherent $\mathbb{C}$-algebra.*

(ii) Any finitely generated right module over an extended affine Hecke algebra with unequal parameters is coherent.

Proof (i) An affine Hecke algebra with unequal parameters is a free module of finite rank over the group ring of a finitely generated lattice. As the group ring of a finitely generated lattice is noetherian as quotient of a polynomial ring, the extended affine Hecke algebra is noetherian as a module. But every ideal of this algebra is a submodule. So, it is finitely generated. One concludes that an affine Hecke algebra is noetherian and in particular coherent. As an extended affine Hecke algebra is, as a module, the sum of an affine Hecke algebra with a finite dimensional $\mathbb{C}$-vector space, we are done.

(ii) A finitely generated right-module over a noetherian $\mathbb{C}$-algebra is coherent. □

B.4 Proposition *Let k be a commutative ring and $(\mathcal{A}_i)_{i\in I}$ a finite family of k-linear abelian categories. Assume that each $\mathcal{A}_i$ is a direct sum of k-linear categories $\mathcal{A}_{i,j}$, $j = 1, \dots, l_i$. Suppose that for each family of integers $\underline{j} = (j_i)_{i\in I}$, $1 \le j_i \le l_i$, the family $(\mathcal{A}_{i,j_i})_{i\in I}$ has a* *tensor product $\mathcal{A}_{\underline{j}}$. Then, the tensor product of the categories $\mathcal{A}_i$ is isomorphic to the direct sum of the categories $\mathcal{A}_{\underline{j}}$.*

Proof Write $\mathcal{J}$ for the set of the $\underline{j}$. One has an equivalence of categories between $\prod_i(\bigoplus_{j=1}^{l_i}\mathcal{A}_{j,i})$ and $\bigoplus_{\underline{j}\in\mathcal{J}}\prod_i\mathcal{A}_{j_i,i}$, and consequently between $\underline{Hom}_{k,e\ \grave{a}\ d}$ $((\bigoplus_{j=1}^{l_i}\mathcal{A}_{j,i})_{i\in I}, C)$ and $\bigoplus_{\underline{j}\in\mathcal{J}}\underline{Hom}_{k,e\ \grave{a}\ d}((\mathcal{A}_{j_i,i})_{i\in I}, C)$. Denote by $\mathcal{A}_{\underline{j}}$ the tensor product of $(\mathcal{A}_{j_i,i})_{i\in I}$. One sees immediately that $\bigoplus_{\underline{j}\in\mathcal{J}}\mathcal{A}_{\underline{j}}$ satisfies the universal property for the tensor product of $(\bigoplus_{j=1}^{l_i}\mathcal{A}_{j,i})_{i\in I}$. □

B.5 Proposition *Let $(\mathcal{H}_i)_{i\in I}$ be a finite family of extended affine Hecke algebras with parameters. Let $\mathcal{B}_i$ be a finite family of k-linear abelian categories with each $\mathcal{B}_i$ equivalent to the category $(\mathcal{H}_i)_f$ of finitely generated modules over $\mathcal{H}_i$. Then, the tensor product of the k-linear abelian categories $\mathcal{B}_i$ exists and is equivalent to the tensor product of the categories $(\mathcal{H}_i)_f$.*

Proof The equivalence of categories $\mathcal{B}_i \to (\mathcal{H}_i)_f$ gives equivalences of categories $\prod\mathcal{B}_i \to \prod(\mathcal{H}_i)_f$ and $\underline{Hom}_{k,e\ \grave{a}\ d}(((\mathcal{H}_i)_f)_{i\in I}, C) \to \underline{Hom}_{k,e\ \grave{a}\ d}((\mathcal{B}_i)_{i\in I}, C)$. With this, it is immediate that the $(\mathcal{B}_i)_{i\in I}$ satisfy the universal property with respect to the tensor product of the $\mathcal{H}_i$. □

Appendix C: The case of the unitary group

C.1 In this appendix, we will show that the results of the section **1.** and **2.** generalize to quasi-split unitary groups. We will give a few remarks, justifying that [15] generalizes to pure inner forms of unitary groups. To obtain the generalization of [14], the reference to [25] has to be replaced by [24] (see also [26]) and for the full Langlands correspondence one has to take into account the results on R-groups in [4].

To generalize section **2.**, we rely on [10] for appropriate results for Langlands parameters for unitary groups.

C.2 In this section, H will denote the group of F-rational points of a quasi-split unitary group $\underline{H}$ with respect to a quadratic extension E/F. As $\underline{H}$ is not split, the L-group of H is a semi-direct product $GL_n(\mathbb{C}) \rtimes Gal(E/F)$, where $GL_n(\mathbb{C})$ is the Langlands dual group of H.

According to the parity of n, we will say that H is an even or odd unitary group. We will denote by W_E the Weil group of E. The notion of a conjugate-orthogonal and a conjugate-symplectic representation of W_E is defined in [10]. A conjugate-dual representation ρ of W_E will be said of type ${}^LH$ if either n is even and ρ is conjugate-sympletic, or n is odd and ρ is conjugate-orthogonal. Otherwise, we will say that ρ is not of type ${}^LH$. We stress that the use of either these notions will presume that ρ is conjugate-dual. The same terminology will also be used when W_E is replaced by the Weil-Deligne group $W_E \times SL_2(\mathbb{C})$.

There is a unique pure inner form of H which we will denote by H^-. If n is odd, then H^- is isomorphic to H and if n is even then H^- is not quasi-split. We will write again sometimes H^+ for H.

C.3 A Langlands parameter for H is a morphism $W_F \to {}^LH$ such that the projection to the first factor is a Langlands parameter (as defined in **1.**) and the projection to the second factor is the projection $W_F \to Gal(E/F)$. The definition of a Langlands–Deligne parameter is straightforward.

It is explained in [10, section 8] that Langlands and Langlands–Deligne parameters for H are in bijective correspondence with conjugate-dual representations of type ${}^LH$ of W_E or $W_E \times SL_2(\mathbb{C})$ respectively. It follows from this also that it does not matter to define equivalence for Langlands parameters or Langlands–Deligne parameters by conjugation by an element of $\widehat{H}$ or ${}^LH$. If φ is a Langlands or a Langlands–Deligne parameter for H we will denote by φ_E the corresponding conjugate-dual representation of type ${}^LH$.

With this terminology, replacing $\iota \circ \varphi$ by φ_E, it is shown in [24, 8.4.4] (see also [26]) that the part of theorem **1.1** that applies to H^+ generalizes. As H is isomorphic to H^- in the odd case, one sees easily that this implies the whole theorem **1.1** in the odd case. For the pure inner form of the even quasi-split unitary group, the result is slightly conditional on an argument which has not been written in [24,26] in this case, but which is not crucial after the author of these papers.

C.4 The definition in **1.2** has to be modified to choose in each inertial class of an irreducible representation of W_E a base point that is conjugate-dual if there is such a representation in the inertial class and, if possible, even conjugate dual of the same type as ${}^LH$.

A standard Levi subgroup M of H has the form $GL_{k_1}(E) \times \cdots \times GL_{k_r}(E) \times H_l$, where H_l is a unitary group of the same type (even or odd) as H. One has the equality $n = 2(k_1 + \cdots + k_r) + L$, where L is defined by $\widehat{H_l} = GL_L(\mathbb{C})$. One has then ${}^LM = GL_{k_1}(\mathbb{C}) \times \cdots \times GL_{k_r}(\mathbb{C}) \times {}^LH_l$. If $\varphi = (\rho_1, \ldots, \rho_k, \tau) : W_F \to {}^LM$ is a discrete Langlands parameter, we will denote by $\rho_{i,E}$ the corresponding irreducible representation $W_E \to GL_{k_i}(\mathbb{C})$, by ${}^c\rho_{i,E}$ the conjugate representation and by φ_E the conjugate-dual representation $W_E \to GL_N(\mathbb{C})$ of type ${}^LH$ that is isomorphic to $\tau_E \oplus \bigoplus_{i=1}^k (\rho_{i,E} \oplus {}^c\rho_{i,E}^\vee)$. We will call φ normed, if φ_E is normed in the sense defined by **1.2**.

If s is an element in the centralizer of $\varphi_E(W_F)$ in $\widehat{G}$ such that the representation $\varphi_{E,s}$ in the inertial class of φ_E is conjugate-dual of type ${}^LG$, then we will denote by φ_s the corresponding Langlands parameter for ${}^LG$. The set of the φ_s will be the *inertial orbit* of φ. The proof of proposition **1.3** generalizes, after replacing $\iota \circ \varphi$ by φ_E and remarking that $W_E \cap I_F = I_E$, and one sees that two Langlands parameters relative to H lie in the same inertial orbit, if and only if their restriction to the inertia subgroup of I_F are conjugated by an element of ${}^LH$. One defines the multiplicity $m(\rho;\varphi)$ to be the multiplicity of ρ in φ_E.

The proposition **1.4** generalizes obviously to representations of W_E, replacing self-dual by dual-conjugate, remarking that $|\cdot|_E$ is self-conjugate. One defines then for a conjugate-dual representation ρ the representation ρ_- accordingly.

Replacing self-dual by dual-conjugate, the generalizations of the notions defined in **1.6** is straightforward and the theorem at the end remains valid.

The definition of the category $Rep_F^{\varphi_0}(\underline{H})$, for φ_0 a normed Langlands parameter for ${}^LH$, and its subcategories $Rep_F^{\varphi_0,\pm}(\underline{H})$ in the statement of the local Langlands correspondence **1.7** is then clear. (The cases with index "-" do not appear, i.e. these notions can remain undefined.) The L-functions and local factors which have to be used here are those coming from the Asai representation.

C.5 The theorem **1.8** is based on [15] and [14], which do not explicitly include unitary groups. However, [15] generalizes with only minor changes to pure inner forms of quasi-split unitary groups: as the Levi subgroups of H are of the form $GL_{k_1}(E) \times \cdots \times GL_{k_r}(E) \times H_l$, the assumptions made in [15, 1.13 - 17] and the results therein remain valid when taken the absolute value and a uniformizer for E at appropriate places. One remarks that the relative reduced roots for H form a root system of type B in the odd case and of type C in the even case. From this, the generalization of [15, 1.13] is straightforward. The same applies to section **6.** and **7.** of [15].

The Plancherel measure of a representation of type $\sigma_{S,\epsilon}$ of a Levi subgroup M_S can be computed as in [14] according to the results in [24,26] (especially [24, 8.4.4] already mentioned above in **C.3**), the relation with reducibility points remaining the same as in the orthogonal or symplectic case, using $|\cdot|_E$ instead of $|\cdot|_F$. Replacing self-dual by conjugate-dual, the generalization of **1.8** is straightforward. The corollary **1.9** is then a direct consequence.

C.6 Proposition [10, 3.4] *The trivial character of $E^\times$ is always a conjugate-orthogonal representation. The nontrivial unramified quadratic character of $E^\times$ is conjugate-symplectic, if and only if E/F is unramified. Otherwise, it is conjugate-orthogonal.*

C.7 The unitary group H is called unramified if E/F is an unramified extension. Denote by 1 the Langlands parameter for H such that 1_E is n times the trivial representation of W_E. We will write -1 for the Langlands parameter 1_{-1} for H in the above notations. From **C.6** and the definitions, it is immediate that the normed representation in the inertial class of 1 is $(-1)^{n-1}$, if H is unramified.

Proposition *Assume that H is unramified. Denote by $\widehat{T}$ the Langlands dual of the maximal torus of H. Let s be in $\widehat{T}$ such that $(-1)^{n-1}_{s,E}$ is a conjugate-dual representation of type ${}^LH$.*

Write $s = diag(x_1, \ldots, x_{[\frac{n}{2}]}, \widehat{1}, \overline{x}^{-1}_{[\frac{n}{2}]}, \ldots, \overline{x}^{-1}_1) \in GL_n(\mathbb{C})$ (with 1 appearing only when n is odd and $[\frac{n}{2}]$ denoting the integer part of $\frac{n}{2}$). For $x \in \{x_1, \ldots, x_{[\frac{n}{2}]}\}$, denote by $m(x, s)$ the multiplicity of x as an entry of s and put

$$C_x = \begin{cases} GL_{m(x;s)}, & \text{if } x \notin \{\pm 1\} \\ O_{m(1,s)}, & \text{if } x = 1, \\ Sp_{m(-1,s)}, & \text{if } x = -1. \end{cases}$$

Then, $C_{\widehat{H}}(Im((-1)^n_s))$ is isomorphic to $\prod_x C_x$, the product going over equivalence classes of elements in the set $\{x_1, \ldots, x_{[\frac{n}{2}]}\}$ with respect to the relation $x \sim x^{-1}$.

Proof This follows from [10, 8.1(iii) and section 4]. □

C.8 Proposition **2.1** remains valid, after replacing the Langlands parameter φ by φ_E and self-dual by conjugate-dual in appropriate places [10, sections 4 and 8]. In the same spirit, one gets the generalization of **2.2–2.6** with $C'^+ := C'$.

C.9 With all these changes, the corollary **3.5** is valid for the quasi-split unitary group.

References

1. Arthur, J.: The endoscopic classification of representations—orthogonal and symplectic groups AMS (2013)
2. Bernstein, J.N.: (rédigé par P. Deligne) Le "centre" de Bernstein dans Représentations des groupes réductifs sur un corps local (J.N. Bernstein, P. Deligne, D. Kazhdan, M.-F. Vignéras eds.) Travaux en cours, Hermann Paris (1984)
3. Choiy, K., Goldberg, D.: Invariance of R-groups between p-adic inner forms of quasi-split classical groups. Trans. Am. Math. Soc. **368**, 1387–1410 (2016)
4. Choiy, K., Goldberg, D.: Behavior of R-groups for p-adic inner forms of quasi-split special unitary groups. arXiv:1605.05299 368 (2016)
5. Cogdell, J.W., Shahidi, F., Tsai, T.-L.: Local Langlands correspondence for GL_n and the exterior and symmetric square ϵ-factors. arXiv:1412.1448 (2014)
6. Cohen, J.: A matrix Paley–Wiener theorem for non-connected p-adic reductive groups. arXiv:1406.4897 (2014)
7. Dat, J.-F.: A functoriality principle for blocks of p-adic linear groups, (à para î tre dans Contemp. Math. (2016), 30pp
8. Deligne, P.: Catégories tannakiennes, The Grothendieck Festschrift, vol. II, pp. 111–195. Birkhäuser Classics (1990)
9. Gan, W.-T., Takeda, S.: The local Langlands conjecture for Sp(4). Int. Math. Res. Not. **2010**, 2987–3038 (2010)
10. Gan, W.-T., Gross, B., Prasad, D.: Symplectic local root numbers, central critical L-values and restriction problems in the representation theory of classical groups. Astérisque **346**, 1–109 (2012)
11. Goldberg, D., Herb, R.: Some results on the admissible representations of non connected-reductive p-adic groups. Ann. Sci. ENS **30**, 97–146 (1997)
12. Heiermann, V.: Orbites unipotentes et pôles d'ordre maximal de la fonction μ de Harish-Chandra. Can. J. Math. **58**, 1203–1228 (2006)
13. Heiermann, V.: Unipotent orbits and local L-functions. J. Reine Angew. Math. **596**, 103–114 (2006)
14. Heiermann, V.: Paramètres de Langlands et algèbres d'entrelacement. Int. Math. Res. Not. **2010**(9), 1607–1623 (2010)
15. Heiermann, V.: Opérateurs d'entrelacement et algèbres de Hecke avec paramètres d'un groupe réductif p-adique: le cas des groupes classiques. Sel. Math. New Ser. **17**, 713–756 (2011)
16. Heiermann, V.: Algèbres de Hecke avec paramètres et représentations d'un groupe p-adique classique: préservation du spectre tempéré. J. Algebra **371**, 596–608 (2012)
17. Henniart, G.: La conjecture de Langlands locale numérique pour GL(n). Ann. Sci. Éc. Norm. Sup. t. **21**, 497–544 (1988)
18. Henniart, G.: Correspondance de Langlands et fonctions L des carrés extérieur et symé-trique. Int. Math. Res. Not. **2010**, 633–673 (2010)
19. Kazhdan et, D., Lusztig, D.G.: Proof of the Deligne–Langlands conjecture for Hecke algebras. Invent. Math. **87**, 153–215 (1987)
20. Lusztig, G.: Affine Hecke algebras and their graded version. J. AMS **2**, 599–635 (1989)
21. Lusztig, G.: Classification of unipotent representations of simple p-adic groups. Int. Math. Res. Not. (IMRN) **1995**, 517–589 (1995)
22. Lusztig, G.: Classification of unipotent representations of simple p-adic groups II. Represent. Theory **6**, 243–289 (2002)
23. Lusztig, G.: Twelve bridges from a reductive group to its Langlands dual. Contemp. Math. **478**, 125–143 (2009)
24. Moeglin, C.: Classification et changement de base pour les séries discrètes des groupes unitaires p-adiques. Pac. J. Math. **233**, 159–204 (2007)
25. Moeglin, C.: Multiplicité 1 dans les paquets d'Arthur, dans 'On Certain L-Functions: Proceedings of a conference in honor of Freydoon Shahidi's 60th Birthday (J. Arthur, J. Cogdell, S. Gelbart, D. Goldberg, S. Kudla, D. Ramakrishnan and J.-K- Yu eds). AMS/Clay Math Institute - Proceedings (2011)
26. Moeglin, C.: Paquets stables des séries discrètes accessibles par endoscopie tordue; leur paramètre de Langlands. Contemp. Math. **614**, 295–336 (2014)
27. Moy, A.: The irreducible orthogonal and symplectic Galois representations of a p-adic field (the tame case). J. Number Theory **19**, 341–344 (1984)
28. Opdam, E.M.: Spectral transfer morphisms for unipotent affine Hecke algebras. arXiv:1310.7790 [math.RT] (2013)
29. Opdam, E.M.: Spectral correspondences for affine Hecke algebras. Adv. Math. **286**, 912–957 (2016)

30. Shahidi, F.: A proof of Langlands' conjecture on Plancherel measures; complementary series of p-adic groups. Ann. Math. **132**, 273–330 (1990)
31. Vogan, D.: The local Langlands conjecture. Contemp. Math. AMS **145**, 305–379 (1993)
32. Waldspurger, J.-L.: Représentations de réduction unipotente pour SO(2n+1): quelques conséquences d'un article de Lusztig, Contributions to Automorphic Forms, Geometry and Number Theory: A Volume in Honor of Joseph Shalika (H. Hida, D. Ramakrishnan, F. Shahidi eds). The John Hopkins University Press, pp. 803–910 (2004)