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Automorphism groups of compact complex supermanifolds

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Abstract Let $\mathcal{M}$ be a compact complex supermanifold. We prove that the set $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms of $\mathcal{M}$ can be endowed with the structure of a complex Lie group acting holomorphically on $\mathcal{M}$, so that its Lie algebra is isomorphic to the Lie algebra of even holomorphic super vector fields on $\mathcal{M}$. Moreover, we prove the existence of a complex Lie supergroup $\mathrm{Aut}(\mathcal{M})$ acting holomorphically on $\mathcal{M}$ and satisfying a universal property. Its underlying Lie group is $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ and its Lie superalgebra is the Lie superalgebra of holomorphic super vector fields on $\mathcal{M}$. This generalizes the classical theorem by Bochner and Montgomery that the automorphism group of a compact complex manifold is a complex Lie group. Some examples of automorphism groups of complex supermanifolds over $\mathbb{P}_1(\mathbb{C})$ are provided.

Keywords Compact complex supermanifold · Automorphism group

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1 Introduction

The automorphism group of a compact complex manifold M carries the structure of a complex Lie group which acts holomorphically on M and whose Lie algebra consists of the

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holomorphic vector fields on M (see [6]). In this article, we investigate how this result can be extended to the category of compact complex supermanifolds.

Let $\mathcal{M}$ be a compact complex supermanifold, i.e. a complex supermanifold whose underlying manifold is compact. An automorphism of $\mathcal{M}$ is a biholomorphic morphism $\mathcal{M} \to \mathcal{M}$. A first candidate for the automorphism group of such a supermanifold is the set of automorphisms, which we denote by $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. However, every automorphism φ of a supermanifold $\mathcal{M}$ (with structure sheaf $\mathcal{O}_{\mathcal{M}}$) is "even" in the sense that its pullback $\varphi^* : \mathcal{O}_{\mathcal{M}} \to \tilde{\varphi}_*(\mathcal{O}_{\mathcal{M}})$ is a parity-preserving morphism. Therefore, we can (at most) expect this set of automorphisms of $\mathcal{M}$ to carry the structure of a classical Lie group if we require its action on $\mathcal{M}$ to be smooth or holomorphic. We cannot obtain a Lie supergroup of positive odd dimension.

We prove that the group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, endowed with an analogue of the compact-open topology, carries the structure of a complex Lie group such that the action on $\mathcal{M}$ is holomorphic and its Lie algebra is the Lie algebra of even holomorphic super vector fields on $\mathcal{M}$. It should be noted that the group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is in general different from the group $\mathrm{Aut}(M)$ of automorphisms of the underlying manifold M. There is a group homomorphism $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(M)$ given by assigning the underlying map to an automorphism of the supermanifold; this group homomorphism is in general neither injective nor surjective.

We define the automorphism group of a compact complex supermanifold $\mathcal{M}$ to be a complex Lie supergroup which acts holomorphically on $\mathcal{M}$ and satisfies a universal property. In analogy to the classical case, its Lie superalgebra is the Lie superalgebra of holomorphic super vector fields on $\mathcal{M}$, and the underlying Lie group is $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, the group of automorphisms of $\mathcal{M}$. Using the equivalence of complex Harish-Chandra pairs and complex Lie supergroups (see [24]), we construct the appropriate automorphism Lie supergroup of $\mathcal{M}$.

More precisely, the outline of this article is the following: First, we introduce a topology on the set $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms on a compact complex supermanifold $\mathcal{M}$ (cf. Sect. 3). This topology is an analogue of the compact-open topology in the classical case, which coincides in the case of a compact complex manifold with the topology of uniform convergence. We prove that the topological space $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ with composition and inversion of automorphisms as group operations is a locally compact topological group which satisfies the second axiom of countability.

In Sect. 4, the non-existence of small subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is proven, which means that there exists a neighbourhood of the identity in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ with the property that this neighbourhood does not contain any non-trivial subgroup. A result on the existence of Lie group structures on locally compact topological groups without small subgroups (see [25]) then implies that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a real Lie group.

In the case of a split compact complex supermanifold $\mathcal{M}$, the fact that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a Lie group follows more easily as described in Remark 8. In this case it can be proven that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is the semi-direct product of a finite-dimensional vector space and the automorphism group of the vector bundle corresponding to $\mathcal{M}$, which is by [17] a complex Lie group.

Then, continuous one-parameter subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ and their action on the supermanifold $\mathcal{M}$ are studied (see Sect. 5). This is done in order to obtain results on the regularity of the $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$-action on $\mathcal{M}$ and characterize the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. We prove that the action of each continuous one-parameter subgroup of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is analytic. As a corollary we get that the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is isomorphic to the Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even holomorphic super vector fields on $\mathcal{M}$, and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a complex Lie group so that its natural action on $\mathcal{M}$ is holomorphic.

Next, we show that the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of holomorphic super vector fields on a compact complex supermanifold $\mathcal{M}$ is finite-dimensional (see Sect. 6). Since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$

carries the structure of a complex Lie group, we already know that $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$, the even part of $\mathrm{Vec}(\mathcal{M})$, is finite-dimensional. The key point in the proof in the case of a split supermanifold $\mathcal{M}$ is that the tangent sheaf of $\mathcal{M}$ is a coherent sheaf of $\mathcal{O}_M$-modules on the compact complex manifold M, where $\mathcal{O}_M$ is the sheaf of holomorphic functions on M.

Let α denote the action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ by conjugation: $\alpha(\varphi)(X) = \varphi_*(X) = (\varphi^{-1})^* \circ X \circ \varphi^*$ for $\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $X \in \mathrm{Vec}(\mathcal{M})$. The restriction of this representation α to $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$, the even part of the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$, coincides with the adjoint action of the Lie group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on its Lie algebra, which is isomorphic to $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$. Hence α defines a Harish-Chandra pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$. The equivalence between Harish-Chandra pairs and complex Lie supergroups allows us to define the automorphism Lie supergroup of a compact complex supermanifold as follows (see Definition 2):

Definition *(Automorphism Lie supergroup)* Define the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a compact complex supermanifold to be the unique complex Lie supergroup associated with the Harish-Chandra pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$ with representation α.

The natural action of the automorphism Lie supergroup $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$ is holomorphic, i.e. we have a morphism $\Psi : \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ of complex supermanifolds. The automorphism Lie supergroup $\mathrm{Aut}(\mathcal{M})$ satisfies the following universal property (see Theorem 22):

Theorem *If $\mathcal{G}$ is a complex Lie supergroup with a holomorphic action $\Psi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$ on $\mathcal{M}$, then there is a unique morphism $\sigma : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups such that the diagram*

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{M} & \xrightarrow{\quad \Psi_{\mathcal{G}} \quad} & \mathcal{M} \\ & {\scriptstyle \sigma \times \mathrm{id}_{\mathcal{M}}} \searrow \qquad \nearrow {\scriptstyle \Psi} & \\ & \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} & \end{array}$$

is commutative.

The automorphism Lie supergroup of a compact complex supermanifold is the unique complex Lie supergroup satisfying the preceding universal property.

Using the "functor of points" approach to supermanifolds, an alternative definition of the automorphism group as a functor in analogy to [20,22] is possible, which is studied in Sect. 8. If $\mathcal{M}$ is a compact complex supermanifold, this functor from the category of supermanifolds to the category of sets can be defined by the assignment

$$\mathcal{N} \mapsto \{\varphi : \mathcal{N} \times \mathcal{M} \to \mathcal{N} \times \mathcal{M} \mid \varphi \text{ is invertible, and } \mathrm{pr}_{\mathcal{N}} \circ \varphi = \mathrm{pr}_{\mathcal{N}}\},$$

where $\mathrm{pr}_{\mathcal{N}} : \mathcal{N} \times \mathcal{M} \to \mathcal{N}$ denotes the projection onto the first component. The two approaches to the automorphism group are equivalent and the constructed automorphism group $\mathrm{Aut}(\mathcal{M})$ represents the just defined functor.

In the classical case, another class of complex manifolds where the automorphism group carries the structure of a Lie group is given by the bounded domains in $\mathbb{C}^m$ (see [8]). An analogue statement is false in the case of supermanifolds. In Sect. 9, we give an example showing that in the case of a complex supermanifold $\mathcal{M}$ whose underlying manifold is a bounded domain in $\mathbb{C}^m$ there does in general not exist a Lie supergroup acting on $\mathcal{M}$ and satisfying the universal property of the preceding theorem.

In Sect. 10, the automorphism group $\mathrm{Aut}(\mathcal{M})$ or its underlying Lie group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ are determined for some supermanifolds $\mathcal{M}$ with underlying manifold $M = \mathbb{P}_1\mathbb{C}$.

2 Preliminaries and notation

Throughout, we work with the "Berezin-Leĭtes-Kostant-approach" to supermanifolds (cf. [1,15,16]). If a supermanifold is denoted by a calligraphic letter $\mathcal{M}$, then we denote the underlying manifold by the corresponding uppercase standard letter M, and the structure sheaf by $\mathcal{O}_{\mathcal{M}}$. We call a supermanifold $\mathcal{M}$ compact if its underlying manifold M is compact. By a complex supermanifold we mean a supermanifold $\mathcal{M}$ with structure sheaf $\mathcal{O}_{\mathcal{M}}$ which is locally, on small enough open subsets $U \subset M$, isomorphic to $\mathcal{O}_U \otimes \bigwedge \mathbb{C}^n$, where $\mathcal{O}_U$ denotes the sheaf of holomorphic functions on U. For a morphism $\varphi : \mathcal{M} \to \mathcal{N}$ between supermanifolds $\mathcal{M}$ and $\mathcal{N}$, the underlying map $M \to N$ is denoted by $\tilde{\varphi}$ and its pullback by $\varphi^* : \mathcal{O}_{\mathcal{N}} \to \tilde{\varphi}_* \mathcal{O}_{\mathcal{M}}$. An automorphism of a complex supermanifold $\mathcal{M}$ is a biholomorphic morphism $\mathcal{M} \to \mathcal{M}$, i.e. an invertible morphism in the category of complex supermanifolds.

Let E be a vector bundle on a complex manifold M and $\mathcal{E}$ its sheaf of sections. Then we can associate a supermanifold $\mathcal{M} = (M, \mathcal{O}_{\mathcal{M}})$ by setting $\mathcal{O}_{\mathcal{M}} = \bigwedge \mathcal{E}$, which has a natural $\mathbb{Z}$-grading (and hence a $\mathbb{Z}/2\mathbb{Z}$-grading). Split supermanifolds are supermanifolds $\mathcal{M}$ such that there is a vector bundle on M with sheaf of sections $\mathcal{E}$ such that $\mathcal{M} \cong (M, \bigwedge \mathcal{E})$. If E is e.g. the trivial bundle of rank n on $M = \mathbb{C}^m$, then we get the supermanifold $\mathbb{C}^{m|n} = (\mathbb{C}^m, \bigwedge \mathcal{E}) = (\mathbb{C}^m, \mathcal{O}_{\mathbb{C}^m} \otimes \bigwedge \mathbb{C}^n)$.

For a complex supermanifold $\mathcal{M}$, let $\mathcal{T}_{\mathcal{M}}$ denote the tangent sheaf of $\mathcal{M}$. The Lie superalgebra of holomorphic vector fields on $\mathcal{M}$ is $\mathrm{Vec}(\mathcal{M}) = \mathcal{T}_{\mathcal{M}}(M)$, it consists of the subspace $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even and the subspace $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$ of odd super vector fields on $\mathcal{M}$.

Let $\mathcal{M}$ be a complex supermanifold of dimension $(m|n)$, and let $\mathcal{I}_{\mathcal{M}}$ be the subsheaf of ideals generated by the odd elements in the structure sheaf $\mathcal{O}_{\mathcal{M}}$ of a supermanifold $\mathcal{M}$. As described in [19], we have the filtration

$$\mathcal{O}_{\mathcal{M}} = (\mathcal{I}_{\mathcal{M}})^0 \supset (\mathcal{I}_{\mathcal{M}})^1 \supset (\mathcal{I}_{\mathcal{M}})^2 \supset \cdots \supset (\mathcal{I}_{\mathcal{M}})^{n+1} = 0$$

of the structure sheaf $\mathcal{O}_{\mathcal{M}}$ by the powers of $\mathcal{I}_{\mathcal{M}}$. Define the quotient sheaves $\mathrm{gr}_k(\mathcal{O}_{\mathcal{M}}) = (\mathcal{I}_{\mathcal{M}})^k/(\mathcal{I}_{\mathcal{M}})^{k+1}$. This gives rise to the $\mathbb{Z}$-graded sheaf $\mathrm{gr}\,\mathcal{O}_{\mathcal{M}} = \bigoplus_k \mathrm{gr}_k(\mathcal{O}_{\mathcal{M}})$. Furthermore, $\mathrm{gr}\,\mathcal{M} = (M, \mathrm{gr}\,\mathcal{O}_{\mathcal{M}})$ is a split complex supermanifold of the same dimension as $\mathcal{M}$.

Note that $\mathcal{E} := \mathrm{gr}_1(\mathcal{O}_{\mathcal{M}})$ defines a vector bundle E on M. An automorphism φ of $\mathcal{M}$ yields a pullback φ^* on $\mathcal{O}_{\mathcal{M}}$. Following [10], its reduction to the $\mathcal{O}_M$-module E yields a morphism of vector bundles $\varphi_0 \in \mathrm{Aut}(E)$ over the reduction $\tilde{\varphi} \in \mathrm{Aut}(M)$. By [17] the automorphism group of a principal fibre bundle over a compact complex manifold carries the structure of a complex Lie group. Since every automorphism of a vector bundle canonically induces an automorphism of the associated principal fibre bundle and vice versa, the automorphism group of the associated principal fibre bundle and $\mathrm{Aut}(E)$ may be identified. Moreover, this identification also respects the topology of compact convergence on both groups. Hence, the group $\mathrm{Aut}(E)$ also carries the structure of a complex Lie group. On local coordinate domains U, V with $\tilde{\varphi}(U) \subset V$ we can identify $\mathcal{O}_{\mathcal{M}}|_V \cong \Gamma_{\Lambda E}|_V$ and $\mathcal{O}_{\mathcal{M}}|_U \cong \Gamma_{\Lambda E}|_U$ and following [21] decompose $\varphi^* = \varphi_0^* \exp(Y)$ with $\mathbb{Z}$-degree preserving automorphism $\varphi_0^* : \Gamma_{\Lambda E}|_V \to \Gamma_{\Lambda E}|_U$ induced by φ_0 and where Y is an even super derivation on $\Gamma_{\Lambda E}|_V$ increasing the $\mathbb{Z}$-degree by 2 or more. Note that the exponential series $\exp(Y)$ is finite since Y is nilpotent.

More generally, there is a relation between nilpotent even super vector fields on a supermanifold and morphisms of this supermanifold satisfying a certain nilpotency condition. This is a direct consequence of a technical result on the relation of algebra homomorphisms and derivations (cf. [23], Proposition 2.1.3 and Lemma 2.1.4). If $\varphi : \mathcal{M} \to \mathcal{M}$ is a morphism of supermanifolds with underlying map $\tilde{\varphi} = \mathrm{id}_M$ and such that $\varphi^* - \mathrm{id}^*_{\mathcal{M}} : \mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ is nilpotent, i.e. there is $N \in \mathbb{N}$ with $(\varphi^* - \mathrm{id}^*_{\mathcal{M}})^N = 0$, then

$$X = \log(\varphi^*) = \sum_{n=1}^{N} \frac{(-1)^{n+1}}{n} (\varphi^* - \mathrm{id}^*_{\mathcal{M}})^n$$

is a nilpotent even super vector field on $\mathcal{M}$ and we have

$$\varphi^* = \exp(X) = \sum_{n \geq 0} \frac{1}{n!} X^n.$$

Furthermore, for any nilpotent even super vector fifeld X on $\mathcal{M}$, the (finite) sum $\exp(X)$ defines a map $\mathcal{O}_{\mathcal{M}} \to \mathcal{O}_{\mathcal{M}}$ which is the pullback of an invertible morphism $\mathcal{M} \to \mathcal{M}$ with the identity as underlying map, and the pullback of the inverse is $\exp(-X)$.

3 The topology on the group of automorphisms

Let $\mathcal{M}$ be a compact complex supermanifold. An automorphism of $\mathcal{M}$ is a biholomorphic morphism $\varphi : \mathcal{M} \to \mathcal{M}$. Denote by $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ the set of automorphisms of $\mathcal{M}$.

In this section, a topology on $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is introduced, which generalizes the compact-open topology and topology of compact convergence of the classical case. Then we show that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a locally compact topological group with respect to this topology.

Let $K \subseteq M$ be a compact subset such that there are local odd coordinates $\theta_1, \dots, \theta_n$ for $\mathcal{M}$ on an open neighbourhood of K. Moreover, let $U \subseteq M$ be open and $f \in \mathcal{O}_{\mathcal{M}}(U)$, and let U_ν be open subsets of $\mathbb{C}$ for $\nu \in (\mathbb{Z}_2)^n$. Let $\varphi : \mathcal{M} \to \mathcal{M}$ be an automorphism with $\tilde{\varphi}(K) \subseteq U$. Then there are holomorphic functions $\varphi_{f,\nu}$ on a neighbourhood of K such that

$$\varphi^*(f) = \sum_{\nu \in (\mathbb{Z}_2)^n} \varphi_{f,\nu} \theta^\nu.$$

Let

$$\Delta(K, U, f, \theta_j, U_\nu) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \,|\, \tilde{\varphi}(K) \subseteq U,\ \varphi_{f,\nu}(K) \subseteq U_\nu\},$$

and endow $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ with the topology generated by sets of this form, i.e. the sets of the form $\Delta(K, U, f, \theta_j, U_\nu)$ form a subbase of the topology.

For any open subset $U \subseteq M$ such that there exist coordinates for $\mathcal{M}$ on U, fix a set of coordinates functions $f_1^U, \dots, f_{m+n}^U \in \mathcal{O}_{\mathcal{M}}(U)$. Using Taylor expansion one can show that the sets of the form $\Delta(K, U, f_l^U, \theta_j, U_\nu)$ then also form a subbase of the topology.

Remark 1 In particular, the subsets of the form

$$\Delta(K, U) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \,|\, \tilde{\varphi}(K) \subseteq U\}$$

are open for $K \subseteq M$ compact and $U \subseteq M$ open. Hence the map $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(M)$, associating with an automorphism φ of $\mathcal{M}$ the underlying automorphism $\tilde{\varphi}$ of M, is continuous.

Remark 2 The group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ endowed with the above topology is a second-countable Hausdorff space since M is second-countable.

Let $U \subseteq M$ be open. Then we can define a topology on $\mathcal{O}_{\mathcal{M}}(U)$ as follows: If $K \subseteq U$ is compact such that there exist odd coordinates $\theta_1, \dots, \theta_n$ on a neighbourhood of K, write $f \in \mathcal{O}_{\mathcal{M}}(U)$ on K as $f = \sum_\nu f_\nu \theta^\nu$. Let $U_\nu \subseteq \mathbb{C}$ be open subsets. Then define a topology on $\mathcal{O}_{\mathcal{M}}(U)$ by requiring that the sets of the form $\{f \in \mathcal{O}_{\mathcal{M}}(U) | \, f_\nu(K) \subseteq U_\nu\}$ are a subbase of the topology. A sequence of functions f_k converges to f if and only if in all local coordinate domains with odd coordinates $\theta_1, \dots, \theta_n$ and $f_k = \sum_\nu f_{k,\nu}\theta^\nu$, $f = \sum_\nu f_\nu \theta^\nu$, the coefficient functions $f_{k,\nu}$ converge uniformly to f_ν on compact subsets. Note that for any open subsets $U_1, U_2 \subseteq M$ with $U_1 \subset U_2$ the restriction map $\mathcal{O}_{\mathcal{M}}(U_2) \to \mathcal{O}_{\mathcal{M}}(U_1)$, $f \mapsto f|_{U_1}$, is continuous.

Using Taylor expansion (in local coordinates) of automorphisms of $\mathcal{M}$ we can deduce the following lemma:

Lemma 3 *A sequence of automorphisms $\varphi_k : \mathcal{M} \to \mathcal{M}$ converges to an automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ with respect to the topology of* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *if and only if the following condition is satisfied: For all $U, V \subseteq M$ open subsets of M such that V contains the closure of $\tilde{\varphi}(U)$, there is an $N \in \mathbb{N}$ such that $\tilde{\varphi}_k(U) \subseteq V$ for all $k \geq N$. Furthermore, for any $f \in \mathcal{O}_{\mathcal{M}}(V)$ the sequence $(\varphi_k)^*(f)$ converges to $\varphi^*(f)$ on U in the topology of $\mathcal{O}_{\mathcal{M}}(U)$.*

Lemma 4 *If $U, V \subseteq M$ are open subsets, $K \subseteq M$ is compact with $V \subseteq K$, then the map*

$$\Delta(K, U) \times \mathcal{O}_{\mathcal{M}}(U) \to \mathcal{O}_{\mathcal{M}}(V), \ (\varphi, f) \mapsto \varphi^*(f)$$

is continuous.

Proof Let $\varphi_k \in \Delta(K, U)$ be a sequence of automorphisms of $\mathcal{M}$ converging to $\varphi \in \Delta(K, U)$, and $f_l \in \mathcal{O}_{\mathcal{M}}(U)$ a sequence converging to $f \in \mathcal{O}_{\mathcal{M}}(U)$. Choosing appropriate local coordinates and using Taylor expansion of the pullbacks $(\varphi_k)^*(f_l)$, it can be shown that $(\varphi_k)^*(f_l)$ converges to $\varphi^*(f)$ as $k, l \to \infty$. This uses that the derivatives of a sequence of uniformly converging holomorphic functions also uniformly converge. □

Lemma 5 *The topological space* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *is locally compact.*

The following remark about invertible morphisms is useful for the proof of this lemma.

Remark 6 (See e.g. Proposition 2.15.1 in [15] *or Corollary 2.3.3 in* [16]*)* Let $\mathcal{M}$ be a complex supermanifold and $\varphi : \mathcal{M} \to \mathcal{M}$ any morphism. Let $\xi_1, \dots, \xi_n$ and $\theta_1, \dots, \theta_n$ be local odd coordinates for $\mathcal{M}$, and superfunctions $\varphi_{j,k}$, $\varphi_{j,\nu}$ such that $\varphi^*(\xi_j) = \sum_{k=1}^n \varphi_{j,k}\theta_k + \sum_{||\nu|| \geq 3} \varphi_{j,\nu}\theta^\nu$, where $||\nu|| = ||(\nu_1, \dots, \nu_n)|| = \nu_1 + \dots + \nu_n \geq 3$. Then φ is locally biholomorphic if and only if the underlying map $\tilde{\varphi}$ is locally biholomorphic and $\det\big((\varphi_{j,k}(y))_{1 \leq j,k \leq n}\big) \neq 0$. The morphism φ is hence invertible if it is everywhere locally biholomorphic and $\tilde{\varphi}$ is biholomorphic.

Proof (of Lemma 5) Let $\psi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$. For each fixed $x \in M$ there are open neighbourhoods V_x and U_x of x and $\tilde{\psi}(x)$ respectively such that $\tilde{\psi}(K_x) \subseteq U_x$ for $K_x := \overline{V}_x$. We may additionally assume that there are local odd coordinates $\xi_1, \dots, \xi_n$ for $\mathcal{M}$ on U_x, and $\theta_1, \dots, \theta_n$ local odd coordinates on an open neighbourhood of K_x. For any automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ with $\tilde{\varphi}(K_x) \subseteq U_x$, let $\varphi_{j,k}, \varphi_{j,\nu}$ (for $||\nu|| = ||(\nu_1, \dots, \nu_n)|| = \nu_1 + \dots + \nu_n \geq 3$) be local holomorphic functions such that

$$\varphi^*(\xi_j) = \sum_{k=1}^n \varphi_{j,k}\theta_k + \sum_{||\nu|| \geq 3} \varphi_{j,\nu}\theta^\nu.$$

Choose bounded open subsets $U_{j,k}, U_{j,\nu} \subset \mathbb{C}$, such that $\psi_{j,k}(x) \in U_{j,k}$ and $\psi_{j,\nu}(x) \in U_{j,\nu}$. Since ψ is an automorphism, we have

$$\det\left((\psi_{j,k}(y))_{1\leq j,k\leq n}\right) \neq 0$$

for all $y \in K_x$ by Remark 6. For later considerations shrink $U_{j,k}$ such that $\det(C) \neq 0$ for all $C = (c_{j,k})_{1\leq j,k\leq n}$ with $c_{j,k} \in U_{j,k}$. After shrinking V_x we may assume $\psi_{j,k}(K_x) \subseteq U_{j,k}$ and $\psi_{j,\nu}(K_x) \subseteq U_{j,\nu}$. Hence ψ is contained in the set $\Theta(x) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \,|\, \tilde{\varphi}(K_x) \subseteq \overline{U}_x,\ \varphi_{j,k}(K_x) \subseteq \overline{U}_{j,k}, \varphi_{j,\nu}(K_x) \subseteq \overline{U}_{j,\nu}\}$, which contains an open neighbourhood of ψ. Since M is compact, M is covered by finitely many of the sets V_x, say $V_{x_1}, \ldots, V_{x_l}$. Then ψ is contained in $\Theta = \Theta(x_1) \cap \cdots \cap \Theta(x_l)$. We will now prove that Θ is sequentially compact: Let φ_k be any sequence of automorphisms contained in Θ. Then, using Montel's theorem and passing to a subsequence, the sequence φ_k converges to a morphism $\varphi : \mathcal{M} \to \mathcal{M}$. It remains to show that φ is an automorphism of $\mathcal{M}$.

The underlying map $\tilde{\varphi} : M \to M$ is surjective since if $p \notin \tilde{\varphi}(M)$, then $\varphi \in \Delta(M, M\setminus\{p\})$ and therefore $\varphi_k \in \Delta(M, M \setminus \{p\})$ for k large enough which contradicts the assumption that φ_k is an automorphism. This also implies that there is an $x \in M$ such that the differential $D\tilde{\varphi}(x)$ is invertible. Using Hurwitz's theorem (see e.g. [18], p. 80) it follows $\det(D\tilde{\varphi}(x)) \neq 0$ for all $x \in M$. Thus $\tilde{\varphi}$ is locally biholomorphic. Moreover, φ is locally invertible due to the special form of the sets $\Theta(x_i)$.

In order check that $\tilde{\varphi}$ is injective, let $p_1, p_2 \in M$, $p_1 \neq p_2$, such that $q = \tilde{\varphi}(p_1) = \tilde{\varphi}(p_2)$. Let Ω_j, $j = 1, 2$, be open neighbourhoods of p_j with $\Omega_1 \cap \Omega_2 = \emptyset$. By [18], p. 79, Proposition 5, there exists k_0 with the property that $q \in \tilde{\varphi}_k(\Omega_1)$ and $q \in \tilde{\varphi}_k(\Omega_2)$ for all $k \geq k_0$. The bijectivity of the φ_k's now yields a contradiction to $\Omega_1 \cap \Omega_2 = \emptyset$. □

Proposition 7 *The set* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *is a topological group with respect to composition and inversion of automorphisms.*

Proof Let φ_k and ψ_l be two sequences of automorphisms of $\mathcal{M}$ converging to φ and ψ respectively. By the classical theory, $\tilde{\varphi}_k \circ \tilde{\psi}_l$ converges to $\tilde{\varphi} \circ \tilde{\psi}$, and $\tilde{\varphi}_k{}^{-1}$ to $\tilde{\varphi}^{-1}$.

Let $U, V, W \subseteq M$ be open subsets with $\tilde{\varphi}(V) \subseteq W$, $\tilde{\varphi}_k(V) \subseteq W$, $\tilde{\psi}(U) \subseteq V$, $\tilde{\psi}_l(U) \subseteq V$, for k and l sufficiently large and let $f \in \mathcal{O}_{\mathcal{M}}(W)$. Then the sequence $(\varphi_k)^*(f) \in \mathcal{O}_{\mathcal{M}}(V)$ converges to $\varphi^*(f)$ on V, and by Lemma 4 $(\varphi_k \circ \psi_l)^*(f) = (\psi_l)^*\left((\varphi_k)^*(f)\right)$ converges to $\psi^*(\varphi^*(f)) = (\varphi \circ \psi)^*(f)$ on U as $k, l \to \infty$, which shows that the multiplication is continuous.

Consider now the inversion map $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $\varphi \mapsto \varphi^{-1}$. Let φ_k be a sequence in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ converging to $\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$. Note that since the automorphism group $\mathrm{Aut}(M)$ of the underlying manifold M is a topological group, the inversion map $\mathrm{Aut}(M) \to \mathrm{Aut}(M)$ is continuous. For any choice of local coordinate charts on $U, V \subseteq M$ such that the closure of $\tilde{\varphi}^{-1}(U)$ is contained in V we can conclude: Since $\tilde{\varphi}_k^{-1}$ converges to $\tilde{\varphi}^{-1}$, we have $\tilde{\varphi}_k{}^{-1}(U) \subseteq V$ for k sufficiently large. Identify $\mathcal{O}_{\mathcal{M}}(U) \cong \Gamma_{\Lambda E}(U)$, resp. $\mathcal{O}_{\mathcal{M}}(V) \cong \Gamma_{\Lambda E}(V)$ and decompose $\varphi^* = \varphi_0^* \exp(Y)$, $\varphi_k^* = \varphi_{k,0}^* \exp(Y_k)$ as in Section 2. Note that φ_0^* is induced by an automorphism φ_0 of the vector bundle E. We can verify by an observation in local coordinates that the map $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(E)$, $\varphi \mapsto \varphi_0$, is continuous. Hence, the sequence $\varphi_{k,0}$ converges to φ_0 and $\varphi_{k,0}^*$ converges to φ_0^*. By [17] the inversion on $\mathrm{Aut}(E)$ is continuous. Therefore, $(\varphi_{k,0}^{-1})^*$ converges to $(\varphi_0^{-1})^*$. Due to the finiteness of the logarithm and exponential series on nilpotent elements, Y_k converges to Y. Hence, $(\varphi_k^{-1})^* = \exp(-Y_k)(\varphi_{k,0}^*)^{-1}$ converges to $\exp(-Y)(\varphi_0^*)^{-1} = (\varphi^*)^{-1}$. □

Remark 8 Let $\mathcal{M}$ be a split supermanifold and let $E \to M$ be a vector bundle with associated sheaf of sections $\mathcal{E}$ such that the structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$. By [17] the group of automorphisms $\mathrm{Aut}(E)$ of the vector bundle E is a complex Lie group. Each automorphism φ of the supermanifold $\mathcal{M}$ induces an automorphism φ_0 of the vector bundle E over the underlying map $\tilde{\varphi}$ of φ, and the map $\pi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(E)$, $\varphi \mapsto \varphi_0$, is continuous. An automorphism of the bundle E lifts to an automorphism of the supermanifold $\mathcal{M}$ if we fix a splitting $\mathcal{O}_{\mathcal{M}} \cong \bigwedge \mathcal{E}$. If $\chi : E \to E$ is an automorphism with pullback χ^* we define an automorphism of $\mathcal{M}$ by the pullback $f_1 \wedge \ldots \wedge f_k \mapsto \chi^*(f_1) \wedge \ldots \wedge \chi^*(f_k)$ for $f_1 \wedge \ldots \wedge f_k \in \bigwedge^k \mathcal{E}$. This assignment defines a section of π. In particular, π is surjective and we have an exact sequence

$$0 \to \ker \pi \to \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(E) \to 0,$$

which splits. Consequently, the topological group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a semidirect product

$$\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \ker \pi \rtimes \mathrm{Aut}(E).$$

The kernel of π consists of those automorphisms φ of $\mathcal{M}$ whose underlying map $\tilde{\varphi}$ is the identity on M and whose pullback φ^* satisfies

$$(\varphi^* - \mathrm{id}^*)(\mathcal{E}) \subseteq \bigoplus_{k\geq 2} \left(\bigwedge^k \mathcal{E}\right).$$

In this case $(\varphi^* - \mathrm{id}^*)$ is nilpotent and there is an even super vector field X on $\mathcal{M}$ with $\exp(X) = \varphi^*$ as mentioned in Sect. 2. The super vector field X is nilpotent and fulfills

$$X\left(\bigwedge^k \mathcal{E}\right) \subseteq \bigoplus_{l\geq k+2} \left(\bigwedge^l \mathcal{E}\right)$$

for all k. More generally, the map

$$\left\{ X \in \mathrm{Vec}_{\bar{0}}(\mathcal{M}) \,\middle|\, X\left(\bigwedge^k \mathcal{E}\right) \subseteq \bigoplus_{l\geq k+2} \left(\bigwedge^l \mathcal{E}\right) \text{ for all } k \right\} \longrightarrow \ker \pi, \quad X \mapsto \exp(X),$$

which assigns to a super vector field X the automorphism of $\mathcal{M}$ with pullback $\exp(X)$, is bijective. In Sect. 6, we will prove that the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of super vector fields on $\mathcal{M}$ and thus subspaces of $\mathrm{Vec}(\mathcal{M})$ are finite-dimensional. Therefore, the topological group $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \ker \pi \rtimes \mathrm{Aut}(E)$ carries the structure of a complex Lie group.

In the general case of a not necessarily split supermanifold $\mathcal{M}$, the proof that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ can be endowed with the structure of a complex Lie group is more difficult. In order to prove the corresponding result also for non-split supermanifolds, the structure of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is further studied in the next two sections.

4 Non-existence of small subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$

In this section, we prove that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ does not contain small subgroups, i.e. that there exists an open neighbourhood of the identity in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ such that each subgroup contained in this neighbourhood consists only of the identity. As a consequence, the topological group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a real Lie group by a result of Yamabe (cf. [25]).

Before proving the non-existence of small subgroups, a few technical preparations are needed: Consider $\mathbb{C}^{m|n}$ and let $z_1, \dots, z_m, \xi_1, \dots, \xi_n$ denote coordinates on $\mathbb{C}^{m|n}$. Let $U \subseteq \mathbb{C}^m$ be an open subset. For $f = \sum_\nu f_\nu \xi^\nu \in \mathcal{O}_{\mathbb{C}^{m|n}}(U)$ define

$$||f||_U = \left|\left|\sum_\nu f_\nu \xi^\nu\right|\right|_U := \sum_\nu ||f_\nu||_U ,$$

where $||f_\nu||_U$ denotes the supremum norm of the holomorphic function f_ν on U. For any morphism $\varphi : \mathcal{U} = (U, \mathcal{O}_{\mathbb{C}^{m|n}}|_U) \to \mathbb{C}^{m|n}$ define

$$||\varphi||_U := \sum_{i=1}^{m} ||\varphi^*(z_i)||_U + \sum_{j=1}^{n} ||\varphi^*(\xi_j)||_U.$$

Lemma 9 *Let $\mathcal{U} = (U, \mathcal{O}_{\mathbb{C}^{m|n}}|_U)$ be a superdomain in $\mathbb{C}^{m|n}$. For any relatively compact open subset U' of U there exists $\varepsilon > 0$ such that any morphism $\psi : \mathcal{U} \to \mathbb{C}^{m|n}$ with the property $||\psi - \mathrm{id}||_U < \varepsilon$ is biholomorphic as a morphism from $\mathcal{U}' = (U', \mathcal{O}_{\mathbb{C}^{m|n}}|_{U'})$ onto its image.*

Proof Let $r > 0$ such that the closure of the polydisc

$$\Delta_r^n(z) = \{(w_1, \dots, w_m) |\, |w_j - z_j| < r\}$$

is contained in U for any $z = (z_1, \dots, z_m) \in U'$. Let $v \in \mathbb{C}^m$ be any non-zero vector. Then we have $z + \zeta v \in U$ for any $z \in U'$ and ζ in the closure of $\Delta_{\frac{r}{||v||}}(0) = \{t \in \mathbb{C} |\, |t| < \frac{r}{||v||}\}$. If for given $\varepsilon > 0$ it is $||\psi - \mathrm{id}||_U < \varepsilon$ then we have in particular $||\tilde{\psi} - \mathrm{id}||_U < \varepsilon$ for the supremum norm of the underlying maps $\tilde{\psi}, \mathrm{id} : U \to \mathbb{C}^m$. Then, for the differential $D\tilde{\psi}$ of $\tilde{\psi}$ and any non-zero vector $v \in \mathbb{C}^m$ and any $z \in U'$ we have

$$\begin{aligned}
\left|\left|D\tilde{\psi}(z)(v) - v\right|\right| &= \left|\left|\frac{d}{dt}\left(\tilde{\psi}(z + tv) - (z + tv)\right)\right|\right| \\
&= \frac{1}{2\pi}\left|\left|\int_{\partial\Delta_{\frac{r}{||v||}}(0)} \frac{\tilde{\psi}(z + \zeta v) - (z + \zeta v)}{\zeta^2} d\zeta\right|\right| \\
&\leq \frac{1}{2\pi}\int_{\partial\Delta_{\frac{r}{||v||}}(0)} \left|\left|\frac{\tilde{\psi}(z + \zeta v) - (z + \zeta v)}{\zeta^2}\right|\right| d\zeta \\
&< \frac{\varepsilon ||v||}{r}.
\end{aligned}$$

This implies $||D\tilde{\psi}(z) - \mathrm{id}|| < \frac{\varepsilon}{r}$ with respect to the operator norm, for any $z \in U'$. Thus $\tilde{\psi}$ is locally biholomorphic on U' if ε is small enough. Moreover, ε might now be chosen such that $\tilde{\psi}$ is injective (see e.g. [13], Chapter 2, Lemma 1.3).
Let $\psi_{j,k}, \psi_{j,\nu}$ be holomorphic functions on U such that $\psi^*(\xi_j) = \sum_{k=1}^n \psi_{j,k}\xi_k + \sum_{||\nu|| \geq 3} \psi_{j,\nu}\xi^\nu$. By Remark 6 it is now enough to show

$$\det((\psi_{j,k})_{1 \leq j,k \leq n}(z)) \neq 0$$

for all $z \in U'$ and ε small enough in order to prove that ψ is a biholomorphism form $\mathcal{U}'$ onto its image. This follows from the fact that we assumed, via $||\psi - \mathrm{id}||_U < \varepsilon$, that $||\psi_{j,k}||_U < \varepsilon$ if $j \neq k$ and $||\psi_{j,j} - 1||_U < \varepsilon$. □

This lemma now allows us to prove that $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ contains no small subgroups; for a similar result in the classical case see [5], Theorem 1.

Proposition 10 *The topological group* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *has no small subgroups, i.e. there is a neighbourhood of the identity which contains no non-trivial subgroup.*

Proof Let $U \subset V \subset W$ be open subsets of M such that U is relatively compact in V and V is relatively compact in W. Suppose that $\mathcal{W} = (W, \mathcal{O}_{\mathcal{M}}|_W)$ is isomorphic to a superdomain in $\mathbb{C}^{m|n}$ and let $z_1, \ldots, z_m, \xi_1, \ldots, \xi_n$ be local coordinates on $\mathcal{W}$. By definition $\Delta(\overline{V}, W) = \{\varphi \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}) | \, \tilde{\varphi}(\overline{V}) \subseteq W\}$ and $\Delta(\overline{U}, V)$ are open neighbourhoods of the identity in $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. Choose $\varepsilon > 0$ as in the preceding lemma such that any morphism $\chi : \mathcal{V} \to \mathbb{C}^{m|n}$ with $||\chi - \mathrm{id}||_V < \varepsilon$ is biholomorphic as a morphism from $\mathcal{U}$ onto its image. Let $\Omega \subseteq \Delta(\overline{V}, W) \cap \Delta(\overline{U}, V)$ be the subset whose elements φ satisfy $||\varphi - \mathrm{id}||_V < \varepsilon$. The set Ω is open and contains the identity. Since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is locally compact by Lemma 5, it is enough to show that each compact subgroup $Q \subseteq \Omega$ is trivial. Otherwise for non-compact Q, let Ω' be an open neighbourhood of the identity with compact closure $\overline{\Omega}'$ which is contained in Ω, and suppose $Q \subseteq \Omega'$. Then $\overline{Q} \subseteq \overline{\Omega}' \subset \Omega$ is a compact subgroup, and Q is trivial if $\overline{Q}$ is trivial.

Define a morphism $\psi : \mathcal{V} \to \mathbb{C}^{m|n}$ by setting

$$\psi^*(z_i) = \int_Q q^*(z_i)\, dq \quad \text{and} \quad \psi^*(\xi_j) = \int_Q q^*(\xi_j)\, dq,$$

where the integral is taken with respect to the normalized Haar measure on Q. This yields a holomorphic morphism $\psi : \mathcal{V} \to \mathbb{C}^{m|n}$ since each $q \in Q$ defines a holomorphic morphism $\mathcal{V} \to \mathcal{W} \subseteq \mathbb{C}^{m|n}$. Its underlying map is $\tilde{\psi}(z) = \int_Q \tilde{q}(z)\, dq$. The morphism ψ satisfies

$$||\psi^*(z_i) - z_i||_V = \left|\left| \int_Q (q^*(z_i) - z_i)\, dq \right|\right|_V \leq \int_Q ||q^*(z_i) - z_i||_V\, dq$$

and similarly

$$||\psi^*(\xi_j) - \xi_j||_V \leq \int_Q ||q^*(\xi_j) - \xi_j||_V\, dq.$$

Consequently, we have

$$\begin{aligned} ||\psi - \mathrm{id}||_V &= \sum_{i=1}^m ||\psi^*(z_i) - z_i||_V + \sum_{j=1}^n ||\psi^*(\xi_j) - \xi_j||_V \\ &\leq \int_Q \left(\sum_{i=1}^m ||q^*(z_i) - z_i||_V + \sum_{j=1}^n ||q^*(\xi_j) - \xi_j||_V \right) dq \\ &= \int_Q ||q - \mathrm{id}||_V\, dq < \varepsilon. \end{aligned}$$

Thus by the preceding lemma, $\psi|_U$ is a biholomorphic morphism onto its image. Furthermore, on U we have $\psi \circ q' = \psi$ for any $q' \in Q$ since

$$\begin{aligned} (\psi \circ q')^*(z_i) &= (q')^*(\psi^*(z_i)) = (q')^* \left(\int_Q q^*(z_i)\, dq \right) = \int_Q (q')^*(q^*(z_i))\, dq \\ &= \int_Q (q \circ q')^*(z_i)\, dq = \int_Q q^*(z_i)\, dq = \psi^*(z_i) \end{aligned}$$

due to the invariance of the Haar measure, and also

$$(\psi \circ q')^*(\xi_j) = \psi^*(\xi_j).$$

The equality $\psi \circ q' = \psi$ on U implies $q'|_U = \mathrm{id}_{\mathcal{U}}$ because of the invertibility of ψ. By the identity principle it follows that $q' = \mathrm{id}_{\mathcal{M}}$ if M is connected, and hence $Q = \{\mathrm{id}_{\mathcal{M}}\}$.

In general, M has only finitely many connected components since M is compact. Therefore, a repetition of the preceding argument yields the existence of a neighbourhood of the identity of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ without any non-trivial subgroups. □

By Theorem 3 in [25], the preceding proposition implies the following:

Corollary 11 *The topological group $Aut_{\bar{0}}(\mathcal{M})$ can be endowed with the structure of a real Lie group.*

5 One-parameter subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$

In order to obtain results on the regularity of the action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on the compact complex supermanifold $\mathcal{M}$ and to characterize the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, we study continuous one-parameter subgroups of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$. Each continuous one-parameter subgroup $\mathbb{R} \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is an analytic map between the Lie groups $\mathbb{R}$ and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$.

We prove that the action of each continuous one-parameter subgroup of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is analytic and induces an even holomorphic super vector field on $\mathcal{M}$. Consequently, the Lie algebra of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ may be identified with the Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even holomorphic super vector fields on $\mathcal{M}$, and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a complex Lie group whose action on the supermanifold $\mathcal{M}$ is holomorphic.

Definition 1 A continuous one-parameter subgroup φ of automorphisms of $\mathcal{M}$ is a family of automorphisms $\varphi_t : \mathcal{M} \to \mathcal{M}$, $t \in \mathbb{R}$, such that the map $\varphi : \mathbb{R} \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $t \mapsto \varphi_t$, is a continuous group homomorphism.

Remark 12 Let $\varphi_t : \mathcal{M} \to \mathcal{M}, t \in \mathbb{R}$, be a family of automorphisms satisfying $\varphi_{s+t} = \varphi_s \circ \varphi_t$ for all $s, t \in \mathbb{R}$, and such that $\tilde{\varphi} : \mathbb{R} \times M \to M$, $\tilde{\varphi}(t, p) = \tilde{\varphi}_t(p)$ is continuous. Then φ_t is a continuous one-parameter subgroup if and only if the following condition is satisfied: Let $U, V \subset M$ be open subsets, and $[a, b] \subset \mathbb{R}$ such that $\tilde{\varphi}([a, b] \times U) \subseteq V$. Assume moreover that there are local coordinates $z_1, \ldots, z_m, \xi_1, \ldots, \xi_n$ for $\mathcal{M}$ on U. Then for any $f \in \mathcal{O}_{\mathcal{M}}(V)$ there are continuous functions $f_\nu : [a, b] \times U \to \mathbb{C}$ with $(f_\nu)_t = f_\nu(t, \cdot) \in \mathcal{O}_{\mathcal{M}}(U)$ for fixed $t \in [a, b]$ such that

$$(\varphi_t)^*(f) = \sum_\nu f_\nu(t, z)\xi^\nu.$$

We say that the action of the one-parameter subgroup φ on $\mathcal{M}$ is analytic if each $f_\nu(t, z)$ is analytic in both components.

This equivalent characterization of continuous one-parameter subgroups of automorphisms also allows us to define this notion for non-compact complex supermanifolds.

Proposition 13 *Let φ be a continuous one-parameter subgroup of automorphisms on $\mathcal{M}$. Then the action of φ on $\mathcal{M}$ is analytic.*

Remark 14 The statement of Proposition 13 also holds true for complex supermanifolds $\mathcal{M}$ with non-compact underlying manifold M as compactness of M is not needed for the proof.

For the proof of the proposition the following technical lemma is needed:

Lemma 15 *Let $U \subseteq V \subseteq \mathbb{C}^m$ be open subsets, $p \in U$, $\Omega \subseteq \mathbb{R}$ an open connected neighbourhood of* 0*, and let $\alpha : \Omega \times U \to V$ be a continuous map satisfying*

$$\alpha(t, z) = \alpha(t + s, z) - f(t, s, z)$$

for (t, s, z) in a neighbourhood of $(0, 0, p)$ and for some continuous function f which is analytic in (t, z). If α is holomorphic in the second component, then it is analytic on a neighbourhood of $(0, p)$.

Proof For small $t, h > 0$, z near p, we have

$$\begin{aligned} h \cdot \alpha(t, z) &= \int_0^h \alpha(t + s, z)ds - \int_0^h f(t, s, z)ds \\ &= \int_t^{h+t} \alpha(s, z)ds - \int_0^h \alpha(s, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds \\ &= \int_h^{h+t} \alpha(s, z)ds - \int_0^t \alpha(s, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds \\ &= \int_0^t (\alpha(s + h, z) - \alpha(s, z))ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds \\ &= \int_0^t f(s, h, z)ds - \int_0^h (f(t, s, z) - \alpha(s, z))ds. \end{aligned}$$

The assumption that f is a continuous function which is analytic in the first and third component therefore implies that α is analytic. □

Proof (of Proposition 13) Due to the action property $\varphi_{s+t} = \varphi_s \circ \varphi_t$ it is enough to show the statement for the restriction of φ to $(-\varepsilon, \varepsilon) \times \mathcal{M}$ for some $\varepsilon > 0$. Let $U, V \subseteq M$ be open subsets such that U is relatively compact in V, and such that there are local coordinates $z_1, \ldots, z_m, \xi_1, \ldots, \xi_n$ on V for $\mathcal{M}$. Choose $\varepsilon > 0$ such that $\tilde{\varphi}_t(U) \subseteq V$ for any $t \in (-\varepsilon, \varepsilon)$. Let $\alpha_{i,\nu}, \beta_{j,\nu}$ be continuous functions on $(-\varepsilon, \varepsilon) \times U$ with

$$(\varphi_t)^*(z_i) = \sum_{|\nu|=0} \alpha_{i,\nu}(t, z)\xi^\nu$$

and

$$(\varphi_t)^*(\xi_j) = \sum_{|\nu|=1} \beta_{j,\nu}(t, z)\xi^\nu,$$

where $|\nu| = |(\nu_1, \ldots, \nu_n)| = (\nu_1 + \ldots + \nu_n) \bmod 2 \in \mathbb{Z}_2$. We have to show that α and β are analytic in (t, z). The induced map $\psi' : (-\varepsilon, \varepsilon) \times U \times \mathbb{C}^n \to V \times \mathbb{C}^n$ on the underlying vector bundle is given by

$$\left(t, \begin{pmatrix} z_1 \\ \vdots \\ z_m \\ v_1 \\ \vdots \\ v_n \end{pmatrix} \right) \mapsto \begin{pmatrix} \alpha_{1,0}(t, z) \\ \vdots \\ \alpha_{m,0}(t, z) \\ \sum_{k=1}^n \beta_{1,k}(t, z)v_k \\ \vdots \\ \sum_{k=1}^n \beta_{n,k}(t, z)v_k \end{pmatrix},$$

where $\beta_{j,k} = \beta_{j,e_k}$ if $e_k = (0, \dots, 0, 1, 0, \dots, 0)$ denotes the k-th unit vector. The map ψ' is a local continuous one-parameter subgroup on $U \times \mathbb{C}^n$ because φ is a continuous one-parameter subgroup. By a result of Bochner and Montgomery the map ψ' is analytic in (t, z, v) (see [4], Theorem 4). Hence, the map $\psi : (-\varepsilon, \varepsilon) \times \mathcal{U} \to \mathcal{V}$ given by $(\psi_t)^*(z_i) = \alpha_i(t, z)$, $(\psi_t)^*(\xi_j) = \sum_{k=1}^n \beta_{j,k}(t, z)\xi_k$ is analytic. Let X be the local vector field on $\mathcal{U}$ induced by ψ, i.e.

$$X(f) = \left.\frac{\partial}{\partial t}\right|_0 (\psi_t)^*(f).$$

We may assume that X is non-degenerate, i.e. the evaluation of X in p, $X(p)$, does not vanish for all $p \in U$. Otherwise, consider, instead of φ, the diagonal action on $\mathbb{C} \times \mathcal{M}$ acting by addition of t in the first component and φ_t in the second, and note that this action is analytic precisely if φ is analytic. For the differential $d\psi$ of ψ in $(0, p)$ we have

$$d\psi\left(\left.\frac{\partial}{\partial t}\right|_{(0,p)}\right) = \left.\frac{\partial}{\partial t}\right|_{(0,p)} \circ \psi^* = X(p) \neq 0.$$

Therefore, the restricted map $\psi|_{(-\varepsilon,\varepsilon)\times\{p\}}$ is an immersion and its image $\psi((-\varepsilon, \varepsilon) \times \{p\})$ is a subsupermanifold of $\mathcal{V}$. Let $\mathcal{S}$ be a subsupermanifold of $\mathcal{U}$ transversal to $\psi((-\varepsilon, \varepsilon) \times \{p\})$ in p. The map $\psi|_{(-\varepsilon,\varepsilon)\times\mathcal{S}}$ is a submersion in $(0, p)$ since $d\psi(T_{(0,p)}(-\varepsilon, \varepsilon) \times \{p\})) = T_p\psi((-\varepsilon, \varepsilon) \times \{p\})$ and $d\psi(T_{(0,p)}\{0\} \times \mathcal{S}) = T_p\mathcal{S}$ because $\psi|_{\{0\}\times\mathcal{U}} = \mathrm{id}$. Hence $\chi := \psi|_{(-\varepsilon,\varepsilon)\times\mathcal{S}}$ is locally invertible around $(0, p)$, and thus invertible as a map onto its image after possibly shrinking U and ε, and

$$\chi_*\left(\frac{\partial}{\partial t}\right) = (\chi^{-1})^* \circ \frac{\partial}{\partial t} \circ \chi^* = (\chi^{-1})^* \circ \chi^* \circ X = X.$$

Therefore, after defining new coordinates $w_1, \dots, w_m, \theta_1, \dots, \theta_n$ for $\mathcal{M}$ on U via χ, we have $X = \frac{\partial}{\partial w_1}$ and $(\varphi_t)^*$ is of the form

$$(\varphi_t)^*(w_1) = w_1 + t + \sum_{|\nu|=0,\nu\neq 0} \alpha_{1,\nu}(t, w)\theta^\nu,$$
$$(\varphi_t)^*(w_i) = w_i + \sum_{|\nu|=0,\nu\neq 0} \alpha_{i,\nu}(t, w)\theta^\nu \quad \text{for } i \neq 1,$$
$$(\varphi_t)^*(\theta_j) = \theta_j + \sum_{|\nu|=1,||\nu||\neq 1} \beta_{j,\nu}(t, w)\theta^\nu,$$

for appropriate $\alpha_{i,\nu}$, $\beta_{j,\nu}$, where $||\nu|| = ||(\nu_1, \dots, \nu_n)|| = \nu_1 + \dots + \nu_n$.

For small s and t we have

$$\begin{aligned}\varphi_t^*\left(\varphi_s^*(w_i)\right) &= \varphi_t^*\left(w_i + \delta_{1,i}s + \sum_{|\nu|=0,||\nu||\neq 0} \alpha_{i,\nu}(s, w)\theta^\nu\right)\\ &= w_i + \delta_{i,1}(t + s) + \sum_{|\nu|=0,||\nu||\neq 0} \alpha_{i,\nu}(t, w)\theta^\nu + \sum_{|\nu|=0,||\nu||\neq 0} \varphi_t^*(\alpha_{i,\nu}(s, w)\theta^\nu).\end{aligned} \tag{1}$$

Let $f_{i,\nu}(t, s, w)$ be such that

$$\sum_{|\nu|=0,||\nu||\neq 0} \varphi_t^*(\alpha_{i,\nu}(s, w)\theta^\nu) = \sum_{|\nu|=0,||\nu||\neq 0} f_{i,\nu}(t, s, w)\theta^\nu. \tag{2}$$

For fixed ν_0 the coefficient $f_{i,\nu_0}(t, s, w)$ of θ^{ν_0} depends only on $\alpha_{i,\nu_0}(s, w+te_1)$, $\beta_{j,\mu}(t, w)$ for μ with $||\mu|| \le ||\nu_0||-1$, and $\alpha_{j,\nu}(t, w)$ and its partial derivatives in the second component for ν with $||\nu|| \le ||\nu_0|| - 2$. This can be shown by a calculation using the special form of $\varphi_t^*(w_j)$ and $\varphi_t^*(\theta_j)$ and general properties of the pullback of a morphism of supermanifolds. Assume now that the analyticity near $(0, p)$ of $\alpha_{i,\nu}$, $\beta_{j,\mu}$ is shown for $||\nu||, ||\mu|| < 2k$ and all i, j. Let ν_0 be such that $||\nu_0|| = 2k$. Then $f_{i,\nu_0}(t, s, w)$ is a continuous function which is analytic in (t, w) near $(0, p)$ for fixed s. Since $\varphi_t^*(\varphi_s^*(w_i)) = \varphi_{t+s}^*(w_i)$, using (1) and (2) we get

$$\alpha_{i,\nu_0}(t, w) + f_{i,\nu_0}(t, s, w) = \alpha_{i,\nu_0}(t + s, w),$$

and thus $\alpha_{i,\nu_0}(t, w)$ is analytic near $(0, p)$ by Lemma 15. Similarly, it can be shown that β_{j,μ_0} is analytic for $||\mu_0|| = 2k + 1$ if $\alpha_{i,\nu}$, $\beta_{j,\mu}$ for $||\nu||, ||\mu|| < 2k + 1$. □

Corollary 16 *The Lie algebra of* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *is isomorphic to the Lie algebra* $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ *of even super vector fields on* $\mathcal{M}$*, and* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *is a complex Lie group.*

Proof If $\gamma : \mathbb{R} \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $t \mapsto \gamma_t$ is a continuous one-parameter subgroup, then by Proposition 13 the action of φ on $\mathcal{M}$ is analytic. Therefore, γ induces an even holomorphic super vector field $X(\gamma)$ on $\mathcal{M}$ by setting

$$X(\gamma) = \left.\frac{\partial}{\partial t}\right|_0 (\gamma_t)^*,$$

and γ is the flow map of $X(\gamma)$. On the other hand, since M is compact, the underlying vector field of each $X \in \mathrm{Vec}_{\bar{0}}(\mathcal{M})$ is globally integrable and the proof of Theorem 5.4 in [12] then shows that X is also globally integrable. Its flow defines a one-parameter subgroup γ^X of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, which is continuous. This yields an isomorphism of Lie algebras

$$\mathrm{Lie}(\mathrm{Aut}_{\bar{0}}(\mathcal{M})) \to \mathrm{Vec}_{\bar{0}}(\mathcal{M}).$$

Consequently, we have $\mathrm{Lie}(\mathrm{Aut}_{\bar{0}}(\mathcal{M})) \cong \mathrm{Vec}_{\bar{0}}(\mathcal{M})$ and since $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ is a complex Lie algebra, $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ carries the structure of a complex Lie group. □

The Lie group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ naturally acts on $\mathcal{M}$; this action $\psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ is given by $\mathrm{ev}_g \circ \psi^* = g^*$ where ev_g denotes the evaluation in $g \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ in the first component.

Corollary 17 *The natural action of* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *on* $\mathcal{M}$ *defines a holomorphic morphism of supermanifolds* $\psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$.

Proof Since the action of each continuous one-parameter subgroup of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is holomorphic by the preceding considerations, and each $g \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a biholomorphic morphism $g : \mathcal{M} \to \mathcal{M}$, the action ψ is a holomorphic. □

If a Lie supergroup $\mathcal{G}$ (with Lie superalgebra $\mathfrak{g}$ of right-invariant super vector fields) acts on a supermanifold $\mathcal{M}$ via $\psi : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$, this action ψ induces an infinitesimal action $d\psi : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ defined by $d\psi(X) = (X(e) \otimes \mathrm{id}_{\mathcal{M}}^*) \circ \psi^*$ for any $X \in \mathfrak{g}$, where $X \otimes \mathrm{id}_{\mathcal{M}}^*$ denotes the canonical extension of the vector field X on $\mathcal{G}$ to a vector field on $\mathcal{G} \times \mathcal{M}$, and $(X(e) \otimes \mathrm{id}_{\mathcal{M}}^*)$ is its evaluation in the neutral element e of $\mathcal{G}$.

Corollary 18 *Identifying the Lie algebra of* $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ *with* $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ *as in Corollary 16, the induced infinitesimal action of the action* $\psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ *in Corollary 17 is the inclusion* $\mathrm{Vec}_{\bar{0}}(\mathcal{M}) \hookrightarrow \mathrm{Vec}(\mathcal{M})$.

6 The Lie superalgebra of vector fields

In this section, we prove that the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of holomorphic super vector fields on a compact complex supermanifold $\mathcal{M}$ is finite-dimensional.

First, we prove that $\mathrm{Vec}(\mathcal{M})$ is finite-dimensional if $\mathcal{M}$ is a split supermanifold using that its tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is a coherent sheaf of $\mathcal{O}_M$-modules, where $\mathcal{O}_M$ denotes again the sheaf of holomorphic functions on the underlying manifold M. Then the statement in the general case is deduced using a filtration of the tangent sheaf.

Remark that since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a complex Lie group with Lie algebra isomorphic to the Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even holomorphic super vector fields on $\mathcal{M}$ (see Corollary 16), we already know that the even part of $\mathrm{Vec}(\mathcal{M}) = \mathrm{Vec}_{\bar{0}}(\mathcal{M}) \oplus \mathrm{Vec}_{\bar{1}}(\mathcal{M})$ is finite-dimensional.

Lemma 19 *Let $\mathcal{M}$ be a split complex supermanifold. Then its tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is a coherent sheaf of $\mathcal{O}_M$-modules.*

Proof Since $\mathcal{M}$ is split, its structure sheaf $\mathcal{O}_{\mathcal{M}}$ is isomorphic to $\bigwedge \mathcal{E}$ as an $\mathcal{O}_M$-module, where $\mathcal{E}$ is the sheaf of sections of a holomorphic vector bundle on the underlying manifold M. Thus, the structure sheaf $\mathcal{O}_{\mathcal{M}}$, and hence also the tangent sheaf $\mathcal{T}_{\mathcal{M}}$, carry the structure of a sheaf of $\mathcal{O}_M$-modules. Let $U \subset M$ be an open subset such that there exist even coordinates $z_1, \ldots, z_m$ and odd coordinates $\xi_1, \ldots, \xi_n$. Any derivation $D \in \mathcal{T}_{\mathcal{M}}(U)$ on U can uniquely be written as

$$D = \sum_{\nu \in (\mathbb{Z}_2)^n} \left(\sum_{i=1}^{m} f_{i,\nu}(z) \xi^{\nu} \frac{\partial}{\partial z_i} + \sum_{j=1}^{n} g_{j,\nu}(z) \xi^{\nu} \frac{\partial}{\partial \xi_j} \right)$$

where $f_{i,\nu}, g_{j,\nu}$ are holomorphic functions on U. Therefore, the restricted sheaf $\mathcal{T}_{\mathcal{M}}|_U$ is isomorphic to $(\mathcal{O}_M|_U)^{2^n(m+n)}$ and $\mathcal{T}_{\mathcal{M}}$ is coherent over $\mathcal{O}_M$. □

Proposition 20 *The Lie superalgebra* $\mathrm{Vec}(\mathcal{M})$ *of holomorphic super vector fields on a compact complex supermanifold $\mathcal{M}$ is finite-dimensional.*

Proof First, assume that $\mathcal{M}$ is split. Then the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ is a coherent sheaf of $\mathcal{O}_M$-modules. Thus, the space of global sections of $\mathcal{T}_{\mathcal{M}}$, $\mathrm{Vec}(\mathcal{M}) = \mathcal{T}_{\mathcal{M}}(M)$, is finite-dimensional since M is compact (cf. [9]).

Now, let $\mathcal{M}$ be an arbitrary compact complex supermanifold. We associate the split complex supermanifold $\mathrm{gr}\,\mathcal{M} = (M, \mathrm{gr}\,\mathcal{O}_{\mathcal{M}})$ as described in Section 2. Let $\mathcal{I}_{\mathcal{M}}$ denote as before the subsheaf of ideal in $\mathcal{O}_{\mathcal{M}}$ generated by the odd elements. Define the filtration of sheaves of Lie superalgebras

$$\mathcal{T}_{\mathcal{M}} =: (\mathcal{T}_{\mathcal{M}})_{(-1)} \supset (\mathcal{T}_{\mathcal{M}})_{(0)} \supset (\mathcal{T}_{\mathcal{M}})_{(1)} \supset \cdots \supset (\mathcal{T}_{\mathcal{M}})_{(n+1)} = 0$$

of the tangent sheaf $\mathcal{T}_{\mathcal{M}}$ by setting

$$(\mathcal{T}_{\mathcal{M}})_{(k)} = \{D \in \mathcal{T}_{\mathcal{M}} \,|\, D(\mathcal{O}_{\mathcal{M}}) \subset (\mathcal{I}_{\mathcal{M}})^k,\ D(\mathcal{I}_{\mathcal{M}}) \subset (\mathcal{I}_{\mathcal{M}})^{k+1}\}$$

for $k \geq 0$. Moreover, define $\mathrm{gr}_k(\mathcal{T}_{\mathcal{M}}) = (\mathcal{T}_{\mathcal{M}})_{(k)}/(\mathcal{T}_{\mathcal{M}})_{(k+1)}$ and set

$$\mathrm{gr}(\mathcal{T}_{\mathcal{M}}) = \bigoplus_{k \geq -1} \mathrm{gr}_k(\mathcal{T}_{\mathcal{M}}).$$

By [19], Proposition 1, the sheaf $\mathrm{gr}(\mathcal{T}_{\mathcal{M}})$ is isomorphic to the tangent sheaf of the associated split supermanifold $\mathrm{gr}\,\mathcal{M}$. By the preceding considerations, the space of holomorphic super vector fields on $\mathrm{gr}\,\mathcal{M}$,

$$\mathrm{Vec}(\mathrm{gr}\,\mathcal{M}) = \mathrm{gr}(\mathcal{T}_\mathcal{M})(M) = \bigoplus_{k\geq -1} \mathrm{gr}_k(\mathcal{T}_\mathcal{M})(M),$$

is of finite dimension. The projection onto the quotient yields

$$\dim(\mathcal{T}_\mathcal{M})_{(k)}(M) - \dim(\mathcal{T}_\mathcal{M})_{(k+1)}(M) \leq \dim(\mathrm{gr}_k(\mathcal{T}_\mathcal{M})(M))$$

and $\dim(\mathcal{T}_\mathcal{M})_{(n)}(M) = \dim(\mathrm{gr}_n(\mathcal{T}_\mathcal{M})(M))$ and hence by induction

$$\dim(\mathcal{T}_\mathcal{M})_{(k)}(M) \leq \sum_{j\geq k} \dim(\mathrm{gr}_j(\mathcal{T}_\mathcal{M})(M)),$$

which gives

$$\dim(\mathcal{T}_\mathcal{M}(M)) = \dim\left((\mathcal{T}_\mathcal{M})_{(-1)}(M)\right) \leq \dim\left(\mathrm{gr}(\mathcal{T}_\mathcal{M})(M)\right).$$

In particular, $\dim(\mathcal{T}_\mathcal{M}(M))$ is finite. □

Remark 21 The proof of the preceding proposition also shows the following inequality:

$$\dim(\mathrm{Vec}(\mathcal{M})) \leq \dim(\mathrm{Vec}(\mathrm{gr}\,\mathcal{M}))$$

7 The automorphism group

In this section, the automorphism group of a compact complex supermanifold is defined. This is done via the formalism of Harish-Chandra pairs for complex Lie supergroups (cf. [24]). The underlying classical Lie group is $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ and the Lie superalgebra is $\mathrm{Vec}(\mathcal{M})$, the Lie superalgebra of super vector fields on $\mathcal{M}$. Moreover, we prove that the automorphism group satisfies a universal property.

Consider the representation α of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$ given by

$$\alpha(g)(X) = g_*(X) = (g^{-1})^* \circ X \circ g^* \quad \text{for} \quad g \in \mathrm{Aut}_{\bar{0}}(\mathcal{M}),\ X \in \mathrm{Vec}(\mathcal{M}).$$

This representation α preserves the parity on $\mathrm{Vec}(\mathcal{M})$, and its restriction to $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ coincides with the adjoint action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on its Lie algebra $\mathrm{Lie}(\mathrm{Aut}_{\bar{0}}(\mathcal{M})) \cong \mathrm{Vec}_{\bar{0}}(\mathcal{M})$. Moreover, the differential $(d\alpha)_{\mathrm{id}}$ at the identity $\mathrm{id} \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is the adjoint representation of $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$:

Let X and Y be super vector fields on $\mathcal{M}$. Assume that X is even and let φ^X denote the corresponding one-parameter subgroup. Then we have

$$(d\alpha)_{\mathrm{id}}(X)(Y) = \left.\frac{\partial}{\partial t}\right|_0 (\varphi_t^X)_*(Y) = [X, Y];$$

see e.g. [2], Corollary 3.8. Therefore, the pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$ together with the representation α is a complex Harish-Chandra pair, and using the equivalence between the category of complex Harish-Chandra pairs and complex Lie supergroups (cf. [24], § 2), we can define the automorphism group of a compact complex supermanifold $\mathcal{M}$ as follows:

Definition 2 Define the automorphism group $\mathrm{Aut}(\mathcal{M})$ of a compact complex supermanifold to be the unique complex Lie supergroup associated with the Harish-Chandra pair $(\mathrm{Aut}_{\bar{0}}(\mathcal{M}), \mathrm{Vec}(\mathcal{M}))$ with adjoint representation α.

Since the action $\psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ induces the inclusion $\mathrm{Vec}_{\bar{0}}(\mathcal{M}) \hookrightarrow \mathrm{Vec}(\mathcal{M})$ as infinitesimal action (see Corollary 18), there exists a Lie supergroup action $\Psi : \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ with the identity $\mathrm{Vec}(\mathcal{M}) \to \mathrm{Vec}(\mathcal{M})$ as induced infinitesimal action and $\Psi|_{\mathrm{Aut}_{\bar{0}}(\mathcal{M})\times\mathcal{M}} = \psi$ (cf. Theorem 5.35 in [2]).

The automorphism group together with Ψ satisfies a universal property:

Theorem 22 *Let $\mathcal{G}$ be a complex Lie supergroup with a holomorphic action $\Psi_{\mathcal{G}} : \mathcal{G} \times \mathcal{M} \to \mathcal{M}$. Then there is a unique morphism $\sigma : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups such that the diagram*

$$\begin{array}{ccc} \mathcal{G} \times \mathcal{M} & \xrightarrow{\quad \Psi_{\mathcal{G}} \quad} & \mathcal{M} \\ & {\scriptstyle \sigma \times \mathrm{id}_{\mathcal{M}}} \searrow \qquad \nearrow {\scriptstyle \Psi} & \\ & \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} & \end{array}$$

is commutative.

Proof Let G be the underlying Lie group of $\mathcal{G}$. For each $g \in G$, we have a morphism $\Psi_{\mathcal{G}}(g) : \mathcal{M} \to \mathcal{M}$ by setting $(\Psi_{\mathcal{G}}(g))^* = \mathrm{ev}_g \circ (\Psi_{\mathcal{G}})^*$. This morphism $\Psi_{\mathcal{G}}(g)$ is an automorphism of $\mathcal{M}$ with inverse $\Psi_{\mathcal{G}}(g^{-1})$ and gives rise to a group homomorphism $\tilde{\sigma} : G \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $g \mapsto \Psi_{\mathcal{G}}(g)$.

Let $\mathfrak{g}$ denote the Lie superalgebra (of right-invariant super vector fields) of $\mathcal{G}$, and $d\Psi_{\mathcal{G}} : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ the infinitesimal action induced by $\Psi_{\mathcal{G}}$. The restriction of $d\Psi_{\mathcal{G}}$ to the even part $\mathfrak{g}_{\bar{0}} = \mathrm{Lie}(G)$ of $\mathfrak{g}$ coincides with the differential $(d\tilde{\sigma})_e$ of $\tilde{\sigma}$ at the identity $e \in G$.

Moreover, if $\alpha_{\mathcal{G}}$ denotes the adjoint action of G on $\mathfrak{g}$, and α denotes, as before, the adjoint action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$, we have

$$\begin{aligned} d\Psi_{\mathcal{G}}(\alpha_{\mathcal{G}}(g)(X)) &= (\Psi_{\mathcal{G}}(g^{-1}))^* \circ d\Psi_{\mathcal{G}}(X) \circ (\Psi_{\mathcal{G}}(g))^* \\ &= (\tilde{\sigma}(g^{-1}))^* \circ d\Psi_{\mathcal{G}}(X) \circ (\tilde{\sigma}(g))^* \\ &= \alpha(\tilde{\sigma}(g))(d\Psi_{\mathcal{G}}(X)) \end{aligned}$$

for any $g \in G$, $X \in \mathfrak{g}$. Using the correspondence between Lie supergroups and Harish-Chandra pairs, it follows that there is a unique morphism $\sigma : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups with underlying map $\tilde{\sigma}$ and derivative $d\Psi_{\mathcal{G}} : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ (see e.g. [24], § 2), and σ satisfies $\Psi \circ (\sigma \times \mathrm{id}_{\mathcal{M}}) = \Psi_{\mathcal{G}}$.

The uniqueness of σ follows from the fact that each morphism $\tau : \mathcal{G} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups fulfilling the same properties as σ necessarily induces the map $d\Psi_{\mathcal{G}} : \mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ on the level of Lie superalgebras and its underlying map $\tilde{\tau}$ has to satisfy $\tilde{\tau}(g) = \Psi_{\mathcal{G}}(g) = \tilde{\sigma}(g)$. □

Remark 23 Since the morphism σ in Theorem 22 is unique, the automorphism group of a compact complex supermanifold $\mathcal{M}$ is the unique Lie supergroup satisfying the universal property formulated in Theorem 22.

Remark 24 We say that a real Lie supergroup $\mathcal{G}$ acts on $\mathcal{M}$ by holomorphic transformations if the underlying Lie group G acts on the complex manifold M by holomorphic transformations and if there is a homomorphism of Lie superalgebras $\mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ which is compatible with the action of G on M. Using the theory of Harish-Chandra pairs, we also have the Lie supergroup $\mathcal{G}^{\mathbb{C}}$, the universal complexification of $\mathcal{G}$; see [14]. The underlying Lie group of $\mathcal{G}^{\mathbb{C}}$ is the universal complexification $G^{\mathbb{C}}$ of the Lie group G. Let $\mathfrak{g} = \mathfrak{g}_{\bar{0}} \oplus \mathfrak{g}_{\bar{1}}$ denote the Lie superalgebra of $\mathcal{G}$, $\mathfrak{g}_{\bar{0}}$ the Lie algebra of G. Then the Lie algebra $\mathfrak{g}_{\bar{0}}^{\mathbb{C}}$ of $G^{\mathbb{C}}$ is a quotient of $\mathfrak{g}_{\bar{0}} \otimes \mathbb{C}$, and the Lie superalgebra of $\mathcal{G}^{\mathbb{C}}$ can be realized as $\mathfrak{g}_{\bar{0}}^{\mathbb{C}} \oplus (\mathfrak{g}_{\bar{1}} \otimes \mathbb{C})$. The action of G on $\mathcal{M}$ extends to a holomorphic $G^{\mathbb{C}}$-action on $\mathcal{M}$, and the homomorphism $\mathfrak{g} \to \mathrm{Vec}(\mathcal{M})$ extends to a homomorphism $\mathfrak{g}_{\bar{0}}^{\mathbb{C}} \oplus (\mathfrak{g}_{\bar{1}} \otimes \mathbb{C}) \to \mathrm{Vec}(\mathcal{M})$ of complex Lie superalgebras, which is compatible with the $G^{\mathbb{C}}$-action on $\mathcal{M}$. Thus, we have a holomorphic $\mathcal{G}^{\mathbb{C}}$-action on $\mathcal{M}$ extending the $\mathcal{G}$-action. Moreover, there is a morphism $\sigma : \mathcal{G}^{\mathbb{C}} \to \mathrm{Aut}(\mathcal{M})$ of Lie supergroups as in Theorem 22.

Example 25 Let $\mathcal{M} = \mathbb{C}^{0|1}$. Denoting the odd coordinate on $\mathbb{C}^{0|1}$ by ξ, each super vector field on $\mathbb{C}^{0|1}$ is of the form $X = a\xi\frac{\partial}{\partial\xi} + b\frac{\partial}{\partial\xi}$ for $a, b \in \mathbb{C}$. The flow $\varphi : \mathbb{C} \times \mathcal{M} \to \mathcal{M}$ of $a\xi\frac{\partial}{\partial\xi}$ is given by $(\varphi_t)^*(\xi) = e^{at}\xi$, and the flow $\psi : \mathbb{C}^{0|1} \times \mathcal{M} \to \mathcal{M}$ of $b\frac{\partial}{\partial\xi}$ by $\psi^*(\xi) = b\tau + \xi$. Let $X_0 = \xi\frac{\partial}{\partial\xi}$ and $X_1 = \frac{\partial}{\partial\xi}$. Then $\mathrm{Vec}(\mathbb{C}^{0|1}) = \mathbb{C}X_0 \oplus \mathbb{C}X_1 = \mathbb{C}^{1|1}$, where the Lie algebra structure on $\mathbb{C}^{1|1}$ is given by $[X_0, X_1] = -X_1$ and $[X_1, X_1] = 0$. Note that this Lie superalgebra is isomorphic to the Lie superalgebra of right-invariant vector fields on the Lie supergroup $(\mathbb{C}^{1|1}, \mu_{0,1})$, where the multiplication $\mu = \mu_{0,1}$ is given by $\mu^*(t) = t_1 + t_2$ and $\mu^*(\tau) = \tau_1 + e^{t_1}\tau_2$; for the Lie supergroup structures on $\mathbb{C}^{1|1}$ see e.g. [12], Lemma 3.1. In particular, the Lie superalgebra $\mathrm{Vec}(\mathbb{C}^{0|1})$ is not abelian.

Since each automorphism φ of $\mathbb{C}^{0|1}$ is given by $\varphi^*(\xi) = c \cdot \xi$ for some $c \in \mathbb{C}$, $c \neq 0$, we have $\mathrm{Aut}_{\bar{0}}(\mathbb{C}^{0|1}) \cong \mathbb{C}^*$.

8 The functor of points of the automorphism group

In [22], the diffeomorphism supergroup of a real compact supermanifold is proven to carry the structure of a Fréchet Lie supergroup. This diffeomorphism supergroup is defined using the "functor of points" approach to supermanifolds, i.e. a supermanifold is a representable contravariant functor from the category of supermanifolds to the category of sets. Starting with a supermanifold $\mathcal{M}$ we define the corresponding functor $\mathrm{Hom}(-, \mathcal{M})$ by the assignment $\mathcal{N} \mapsto \mathrm{Hom}(\mathcal{N}, \mathcal{M})$, where $\mathrm{Hom}(\mathcal{N}, \mathcal{M})$ denotes the set of morphisms of supermanifolds $\mathcal{N} \to \mathcal{M}$, and for morphisms $\alpha : \mathcal{N}_1 \to \mathcal{N}_2$ between supermanifolds $\mathcal{N}_1$ and $\mathcal{N}_2$ we define $\mathrm{Hom}(-, \mathcal{M})(\alpha) : \mathrm{Hom}(\mathcal{N}_2, \mathcal{M}) \to \mathrm{Hom}(\mathcal{N}_1, \mathcal{M})$ by $\varphi \mapsto \varphi \circ \alpha$.

In analogy to the definition in [22] for the diffeomorphism supergroup, a functor $\overline{\mathrm{Aut}}(\mathcal{M})$ associated with a complex supermanifold $\mathcal{M}$ can be defined. In the case of a compact complex supermanifold $\mathcal{M}$, the automorphism Lie supergroup as defined in Section 7 represents the functor $\overline{\mathrm{Aut}}(\mathcal{M})$, i.e. the functors $\overline{\mathrm{Aut}}(\mathcal{M})$ and $\mathrm{Hom}(-, \mathrm{Aut}(\mathcal{M}))$ are isomorphic. This is proven in [3], Section 5.4. Here we give an outline of the main steps in the proof.

Definition 3 Let $\mathcal{M}$ be a complex supermanifold. We define the functor $\overline{\mathrm{Aut}}(\mathcal{M})$ from the category of supermanifolds to the category of groups as follows:
On objects, we define $\overline{\mathrm{Aut}}(\mathcal{M})$ by the assignment

$$\mathcal{N} \mapsto \{\varphi : \mathcal{N} \times \mathcal{M} \to \mathcal{N} \times \mathcal{M} \mid \varphi \text{ is invertible, and } \mathrm{pr}_{\mathcal{N}} \circ \varphi = \mathrm{pr}_{\mathcal{N}}\},$$

where $\mathrm{pr}_{\mathcal{N}} : \mathcal{N} \times \mathcal{M} \to \mathcal{N}$ is the projection. For morphisms $\alpha : \mathcal{N}_1 \to \mathcal{N}_2$, we set $\overline{\mathrm{Aut}}(\mathcal{M})(\alpha) : \overline{\mathrm{Aut}}(\mathcal{M})(\mathcal{N}_2) \to \overline{\mathrm{Aut}}(\mathcal{M})(\mathcal{N}_1)$,

$$\varphi \mapsto (\mathrm{id}_{\mathcal{N}_1} \times (\mathrm{pr}_{\mathcal{M}} \circ \varphi \circ (\alpha \times \mathrm{id}_{\mathcal{M}}))) \circ (\mathrm{diag} \times \mathrm{id}_{\mathcal{M}}),$$

denoting by $\mathrm{diag} : \mathcal{N}_1 \to \mathcal{N}_1 \times \mathcal{N}_1$ the diagonal map and by $\mathrm{pr}_{\mathcal{M}}$ the projection onto $\mathcal{M}$. Thus $\overline{\mathrm{Aut}}(\mathcal{M})(\alpha)(\varphi)$ is the unique automorphism $\psi : \mathcal{N}_1 \times \mathcal{M} \to \mathcal{N}_1 \times \mathcal{M}$ with $\mathrm{pr}_{\mathcal{N}_1} \circ \psi = \mathrm{pr}_{\mathcal{N}_1}$ and $\mathrm{pr}_{\mathcal{M}} \circ \psi = \mathrm{pr}_{\mathcal{M}} \circ \varphi \circ (\alpha \times \mathrm{id}_{\mathcal{M}})$.

The group structure on $\overline{\mathrm{Aut}}(\mathcal{M})(\mathcal{N})$ is defined by the composition and inversion of automorphisms $\mathcal{N} \times \mathcal{M} \to \mathcal{N} \times \mathcal{M}$, and the neutral element is the identity map $\mathcal{N} \times \mathcal{M} \to \mathcal{N} \times \mathcal{M}$.

Let $\chi : \mathcal{N} \to \mathrm{Aut}(\mathcal{M})$ be an arbitrary morphism of complex supermanifolds and let $\Psi : \mathrm{Aut}(\mathcal{M}) \times \mathcal{M} \to \mathcal{M}$ denote the natural action of $\mathrm{Aut}(\mathcal{M})$ on $\mathcal{M}$. Then the composition

$$\varphi_\chi = (\mathrm{id}_{\mathcal{N}} \times (\Psi \circ (\chi \times \mathrm{id}_{\mathcal{M}}))) \circ (\mathrm{diag} \times \mathrm{id}_{\mathcal{M}})$$

is an invertible map $\mathcal{N} \times \mathcal{M} \to \mathcal{N} \times \mathcal{M}$ with $\mathrm{pr}_{\mathcal{N}} = \mathrm{pr}_{\mathcal{N}} \circ \varphi_\chi$. This defines a natural transformation:

Lemma 26 *The assignments* $\mathrm{Hom}(\mathcal{N}, \mathrm{Aut}(\mathcal{M})) \to \overline{\mathrm{Aut}}(\mathcal{M})(\mathcal{N})$, $\chi \mapsto \varphi_\chi$, *define a natural transformation* $\mathrm{Hom}(-, \mathrm{Aut}(\mathcal{M})) \to \overline{\mathrm{Aut}}(\mathcal{M})$.

This statement of the lemma can be verified by direct calculations; see also Lemma 5.4.2 in [3].

The natural transformation between $\mathrm{Hom}(-, \mathrm{Aut}(\mathcal{M}))$ and $\overline{\mathrm{Aut}}(\mathcal{M})$ is actually an isomorphism of functors. The injectivity of the assignment $\chi \mapsto \varphi_\chi$ follows from the fact that the $\mathrm{Aut}(\mathcal{M})$-action on $\mathcal{M}$ is effective. As a generalization of the classical definition of effectiveness, we call an action Ψ of a Lie supergroup $\mathcal{G}$ on a supermanifold $\mathcal{M}$ effective if for arbitrary morphisms $\chi_1, \chi_2 : \mathcal{N} \to \mathcal{G}$ of supermanifolds the equality

$$\Psi \circ (\chi_1 \times \mathrm{id}_{\mathcal{M}}) = \Psi \circ (\chi_2 \times \mathrm{id}_{\mathcal{M}})$$

implies $\chi_1 = \chi_2$; cf. Section 2.5 in [3].

In the proof of the surjectivity a "normal form" of the pullback of automorphisms $\varphi : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k} \times \mathcal{M}$ with $\mathrm{pr}_{\mathbb{C}^{0|k}} \circ \varphi = \mathrm{pr}_{\mathbb{C}^{0|k}}$ is used. Let $\mathcal{M}$ be a complex supermanifold and $\varphi : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k} \times \mathcal{M}$ be an invertible morphism with $\mathrm{pr}_{\mathbb{C}^{0|k}} \circ \varphi = \mathrm{pr}_{\mathbb{C}^{0|k}}$. Let $\iota : \mathcal{M} \hookrightarrow \{0\} \times \mathcal{M} \subset \mathbb{C}^{0|k} \times \mathcal{M}$ denote the canonical inclusion. The composition $\bar{\varphi} = \mathrm{pr}_{\mathcal{M}} \circ \varphi \circ \iota$ is an automorphism of $\mathcal{M}$. Then φ is uniquely determined by $\bar{\varphi}$ and a set of super vector fields on $\mathcal{M}$:

Lemma 27 *Let* $\varphi : \mathbb{C}^{0|k} \times \mathcal{M} \to \mathbb{C}^{0|k} \times \mathcal{M}$ *be an invertible morphism with* $\mathrm{pr}_{\mathbb{C}^{0|k}} \circ \varphi = \mathrm{pr}_{\mathbb{C}^{0|k}}$. *Let* $\tau_1, \ldots, \tau_k$ *denote coordinates on* $\mathbb{C}^{0|k} \subset \mathbb{C}^{0|k} \times \mathcal{M}$. *Then there are super vector fields* X_ν *on* $\mathcal{M}$, *of parity* $|\nu|$ *for* $\nu \in (\mathbb{Z}_2)^k$, $\nu \neq 0$, *such that*

$$\varphi^* = (\mathrm{id}_{\mathbb{C}^{0|k}} \times \bar{\varphi})^* \exp\left(\sum_{\nu \neq 0} \tau^\nu X_\nu\right),$$

By $\tau^\nu X_\nu$ *we mean the super vector field on* $\mathbb{C}^{0|k} \times \mathcal{M}$ *which is induced by the extension of the super vector field* X_ν *on* $\mathcal{M}$ *to a super vector field on the product* $\mathbb{C}^{0|k} \times \mathcal{M}$ *followed by the multiplication with* $\tau^\nu = \tau_1^{\nu_1} \ldots \tau_k^{\nu_k}$. *In other words for* $U \subseteq M$ *open we have* $\tau^\nu X_\nu(f) = 0$ *for* $f \in \mathcal{O}_{\mathbb{C}^{0|k}}(\{0\}) \subset \mathcal{O}_{\mathbb{C}^{0|k} \times \mathcal{M}}(\{0\} \times U)$ *and* $(\tau^\nu X_\nu)(g) = \tau^\nu X_\nu(g)$ *for* $g \in \mathcal{O}_{\mathcal{M}}(U) \subset \mathcal{O}_{\mathbb{C}^{0|k} \times \mathcal{M}}(\{0\} \times U)$ *considering* $X_\nu(g)$ *as a function on the product.*

Moreover,

$$\exp\left(\sum_{\nu\neq 0}\tau^{\nu}X_{\nu}\right)=\sum_{n\geq 0}\frac{1}{n!}\left(\sum_{\nu\neq 0}\tau^{\nu}X_{\nu}\right)^{n}$$

is a finite sum since $\left(\sum_{\nu\neq 0}\tau^{\nu}X_{\nu}\right)^{k+1}=0.$

A version of this lemma is also proven in [22], Theorem 5.1. A different proof using the relation between nilpotent even super vector fields on a supermanifold and morphisms of this supermanifold satisfying a certain nilpotency condition as formulated in Sect. 2 is also possible; for details see also [3], Lemma 5.4.3.

Using the normal form of the lemma, we can prove that the assignment $\chi\mapsto\varphi_{\chi}$ defines a surjective map by directly constructing a morphism χ with $\varphi_{\chi}=\varphi$ for any $\varphi:\mathcal{N}\times\mathcal{M}\to\mathcal{N}\times\mathcal{M}$ with $\mathrm{pr}_{\mathcal{N}}\circ\varphi=\mathrm{pr}_{\mathcal{N}}$. It is here enough to prove this statement locally (in $\mathcal{N}$) and thus to consider the case where $\mathcal{N}=N\times\mathbb{C}^{0|k}$ for a classical complex manifold N. In the following we indicate how such a morphism χ can be defined; for the proof that χ fulfills the desired property $\varphi_{\chi}=\varphi$ see Proposition 5.4.4 in [3].

Let $\varphi:N\times\mathbb{C}^{0|k}\times\mathcal{M}\to N\times\mathbb{C}^{0|k}\times\mathcal{M}$ be an invertible morphism with $\mathrm{pr}_{N\times\mathbb{C}^{0|k}}\circ\varphi=\mathrm{pr}_{N\times\mathbb{C}^{0|k}}$. Each $z\in N$ induces an invertible morphism $\varphi_{z}:\mathbb{C}^{0|k}\times\mathcal{M}\to\mathbb{C}^{0|k}\times\mathcal{M}$ with $\mathrm{pr}_{\mathbb{C}^{0|k}}\circ\varphi_{z}=\mathrm{pr}_{\mathbb{C}^{0|k}}$, and the family φ_{z}, $z\in N$, uniquely determines φ.

Let $X_{\nu,z}$ be super vector fields on $\mathcal{M}$ of parity $|\nu|$, $\nu\in(\mathbb{Z}_{2})^{k}$, $\nu\neq 0$, and $\bar{\varphi}_{z}:\mathcal{M}\to\mathcal{M}$ automorphisms such that $\varphi_{z}^{*}=(\mathrm{id}_{\mathbb{C}^{0|k}}\times\bar{\varphi}_{z})^{*}\exp\left(\sum_{\nu\neq 0}\tau^{\nu}X_{\nu,z}\right)$ as in Lemma 27. Since φ is holomorphic, the coefficients of the super vector fields $X_{\nu,z}$ and the pullbacks $\bar{\varphi}_{z}^{*}$ in local coordiantes depend holomorphically on $z\in N$. Each $\bar{\varphi}_{z}$ is the automorphism of $\mathcal{M}$ induced by the evalutation in $(z,0)\in N\times\mathbb{C}^{0|k}$ and an element of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ by definition. Let $\mathrm{ev}_{\bar{\varphi}_{z}}$ denote the evaluation in $\bar{\varphi}_{z}$, i.e. $\mathrm{ev}_{\bar{\varphi}_{z}}$ is the pullback of the canonical inclusion $\{\bar{\varphi}_{z}\}\hookrightarrow\mathrm{Aut}(\mathcal{M})$, and let $\mathrm{pr}_{\mathrm{Aut}(\mathcal{M})}:N\times\mathbb{C}^{0|k}\times\mathrm{Aut}(\mathcal{M})\to\mathrm{Aut}(\mathcal{M})$ be the projection. We define $\chi:N\times\mathbb{C}^{0|k}\to\mathrm{Aut}(\mathcal{M})$ as the morphism whose underlying map is $\{z\}\hookrightarrow\{\bar{\varphi}_{z}\}\subset\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ and whose pullback evaluated in $z\in N$ is

$$\chi_{z}^{*}=(\mathrm{id}_{\mathbb{C}^{0|k}}^{*}\otimes\mathrm{ev}_{\bar{\varphi}_{z}})\circ\exp\left(\sum_{\nu\neq 0}\tau^{\nu}(X_{\nu,z})_{R}\right)\circ\mathrm{pr}_{\mathrm{Aut}(\mathcal{M})}^{*},$$

where $(X_{\nu,z})_{R}$ denotes the right-invariant super vector field on $\mathrm{Aut}(\mathcal{M})$ corresponding to the super vector field $X_{\nu,z}$ on $\mathcal{M}$ which is an element of the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of $\mathrm{Aut}(\mathcal{M})$.

The next proposition is then a consequence of Lemma 26 and the surjectivity of the assignment $\chi\mapsto\varphi_{\chi}$.

Proposition 28 *(See [3], Corollary 5.4.5) The functors* $\overline{\mathrm{Aut}}(\mathcal{M})$ *and* $\mathrm{Hom}(-,\mathrm{Aut}(\mathcal{M}))$ *are isomorphic. This isomorphism is realized by the natural transformation introduced in Lemma 26.*

9 The case of a superdomain with bounded underlying domain

In the classical case, the automorphism group of a bounded domain $U\subset\mathbb{C}^{m}$ is a (real) Lie group (see Theorem 13 in “Sur les groupes de transformations analytiques” in [8]). If

$\mathcal{U} \subset \mathbb{C}^{m|n}$ is a superdomain whose underlying set U is a bounded domain in $\mathbb{C}^m$, it is in general not possible to endow its set of automorphisms with the structure of a Lie group such that the action on $\mathcal{U}$ is smooth, as will be illustrated in an example. In particular, there is no Lie supergroup satisfying the universal property as the automorphism group of a compact complex supermanifold $\mathcal{M}$ does as formulated in Theorem 22.

Example 29 Consider a superdomain $\mathcal{U}$ of dimension $(1|2)$ whose underlying set is a bounded domain $U \subset \mathbb{C}$. Let z, θ_1, θ_2 denote coordinates for $\mathcal{M}$. For any holomorphic function f on U, define the even super vector field $X_f = f(z)\theta_1\theta_2 \frac{\partial}{\partial z}$. The reduced vector field $\tilde{X}_f = 0$ is completely integrable and thus the flow of X_f can be defined on $\mathbb{C} \times \mathcal{U}$ (cf. [12] Lemma 5.2). The flow is given by $(\varphi_t)^*(z) = z + t \cdot f(z)\theta_1\theta_2$ and $(\varphi_t)^*(\theta_j) = \theta_j$. For all holomorphic functions f and g we have $[X_f, X_g] = 0$, and thus their flows locally commute (cf. [2], Corollary 3.8). Therefore, $\{X_f \,|\, f \in \mathcal{O}(U)\} \cong \mathcal{O}(U)$ is an uncountably infinite-dimensional abelian Lie algebra. If the set of automorphisms of $\mathcal{U}$ carried the structure of a Lie group such that its action on $\mathcal{U}$ was smooth, its Lie algebra would necessarily contain $\{X_f \,|\, f \in \mathcal{O}(U)\} \cong \mathcal{O}(U)$ as a Lie subalgebra, which is not possible.

10 Examples

In this section, we determine the automorphism group $\mathrm{Aut}(\mathcal{M})$ for some complex supermanifolds $\mathcal{M}$ with underlying manifold $M = \mathbb{P}_1\mathbb{C}$.

Let L_1 denote the hyperplane bundle on $M = \mathbb{P}_1\mathbb{C}$ with sheaf of sections $\mathcal{O}(1)$, and $L_k = (L_1)^{\otimes k}$ the line bundle of degree k, $k \in \mathbb{Z}$, on $\mathbb{P}_1\mathbb{C}$, and sheaf of sections $\mathcal{O}(k)$. Each holomorphic vector bundle on $\mathbb{P}_1\mathbb{C}$ is isomorphic to a direct sum of line bundles $L_{k_1} \oplus \ldots \oplus L_{k_n}$ (see [11]). Therefore, if $\mathcal{M}$ is a split supermanifold with $M = \mathbb{P}_1\mathbb{C}$ and $\dim \mathcal{M} = (1|n)$, there exist $k_1, \ldots, k_n \in \mathbb{Z}$ such that the structure sheaf $\mathcal{O}_\mathcal{M}$ of $\mathcal{M}$ is isomorphic to

$$\bigwedge(\mathcal{O}(k_1) \oplus \ldots \oplus \mathcal{O}(k_n)).$$

Let $U_j = \{[z_0 : z_1] \in \mathbb{P}_1\mathbb{C} \,|\, z_j \neq 0\}$, $j = 1, 2$, and $\mathcal{U}_j = (U_j, \mathcal{O}_\mathcal{M}|_{U_j})$. Moreover, define $U_0{}^* = U_0 \setminus \{[1:0]\}$ and $U_1{}^* = U_1 \setminus \{[0:1]\}$, and let $\mathcal{U}_j{}^* = (U_j{}^*, \mathcal{O}_\mathcal{M}|_{U_j{}^*})$. We can now choose local coordinates $z, \theta_1, \ldots, \theta_n$ for $\mathcal{M}$ on U_0, and local coordinates $w, \eta_1, \ldots, \eta_n$ on U_1 so that the transition map $\chi : \mathcal{U}_0{}^* \to \mathcal{U}_1{}^*$, which determines the supermanifold structure of $\mathcal{M}$, is given by

$$\chi^*(w) = \frac{1}{z} \quad \text{and} \quad \chi^*(\eta_j) = z^{k_j}\theta_j.$$

Example 30 Let $\mathcal{M} = (\mathbb{P}_1\mathbb{C}, \mathcal{O}_\mathcal{M})$ be a complex supermanifold of dimension $(1|1)$. Since the odd dimension is 1, the supermanifold $\mathcal{M}$ has to be split. Let $-k \in \mathbb{Z}$ be the degree of the associated line bundle. Choose local coordinates z, θ for $\mathcal{M}$ on U_0 and w, η on U_1 as above so that the transition map $\chi : \mathcal{U}_0{}^* \to \mathcal{U}_1{}^*$ is given by $\chi^*(w) = \frac{1}{z}$ and $\chi^*(\eta) = \frac{1}{z^k}\theta$.

We first want to determine the Lie superalgebra $\mathrm{Vec}(\mathcal{M})$ of super vector fields on $\mathcal{M}$. A calculation in local coordinates verifying the compatibility condition with the transition map χ yields that the restriction to U_0 of any super vector field on $\mathcal{M}$ is of the form

$$\left((\alpha_0 + \alpha_1 z + \alpha_2 z^2)\frac{\partial}{\partial z} + (\beta + k\alpha_2 z)\theta\frac{\partial}{\partial \theta}\right) + \left(p(z)\frac{\partial}{\partial \theta} + q(z)\theta\frac{\partial}{\partial z}\right),$$

where $\alpha_0, \alpha_1, \alpha_2, \beta \in \mathbb{C}$, p is a polynomial of degree at most k, and q is a polynomial of degree at most $2-k$. If $k < 0$ (respectively $2-k < 0$), the polynomial p (respectively q)

is 0. The Lie algebra $\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ of even super vector fields is isomorphic to $\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}$, where an isomorphism $\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}\to\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ is given by

$$\left(\begin{pmatrix} a & b \\ c & -a \end{pmatrix}, d\right) \mapsto (-b-2az+cz^2)\frac{\partial}{\partial z} + ((d-ka)+kcz)\theta\frac{\partial}{\partial\theta}.$$

Note that since the odd dimension of $\mathcal{M}$ is 1 each automorphism $\varphi:\mathcal{M}\to\mathcal{M}$ gives rise to an automorphism of the line bundle L_{-k} and vice versa. Hence, the automorphism group $\mathrm{Aut}(L_{-k})$ of the line bundle L_{-k} and $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ coincide.

A calculation yields that the group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms $\mathcal{M}\to\mathcal{M}$ can be identified with $\mathrm{PSL}_2(\mathbb{C})\times\mathbb{C}^*$ if k is even and with $\mathrm{SL}_2(\mathbb{C})\times\mathbb{C}^*$ if k is odd. Consider the element $\left(\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right), s\right)$, where $s\in\mathbb{C}^*$ and $\left(\begin{smallmatrix} a & b \\ c & d\end{smallmatrix}\right)$ is either an element of $\mathrm{SL}_2(\mathbb{C})$ or the representative of the corresponding class in $\mathrm{PSL}_2(\mathbb{C})$. The action of the corresponding element $\varphi\in\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathcal{M}$ is then given by

$$\varphi^*(z) = \frac{c+dz}{a+bz} \quad\text{and}\quad \varphi^*(\theta) = \left(\frac{1}{(a+bz)^k}+s\right)\theta$$

as a morphism over appropriate subsets of U_0 and by

$$\varphi^*(w) = \frac{aw+b}{cw+d} \quad\text{and}\quad \varphi^*(\eta) = \left(\frac{1}{(cw+d)^k}+s\right)\eta$$

over appropriate subsets of U_1.

The Lie supergroup structure on $\mathrm{Aut}(\mathcal{M})$ is now uniquely determined by $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$, $\mathrm{Vec}(\mathcal{M})$, and the adjoint action of $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ on $\mathrm{Vec}(\mathcal{M})$. Since $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ is a connected Lie group, it is enough to calculate the adjoint action of $\mathrm{Vec}_{\bar{0}}(\mathcal{M})\cong\mathfrak{sl}_2\mathbb{C}\oplus\mathbb{C}$ on $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$.

Let P_l denote the space of polynomials of degree at most l, and set $P_l=\{0\}$ for $l<0$. The space of odd super vector fields $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$ is isomorphic to $P_k\oplus P_{2-k}$ via $\left(p(z)\frac{\partial}{\partial\theta}+q(z)\theta\frac{\partial}{\partial z}\right)\mapsto(p(z),q(z))$.

The element $H=\left(\begin{smallmatrix} 1 & 0 \\ 0 & -1\end{smallmatrix}\right)\in\mathfrak{sl}_2(\mathbb{C})\subset\mathfrak{sl}_2(\mathbb{C})\oplus\mathbb{C}\cong\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ corresponds to $-2z\frac{\partial}{\partial z}-k\theta\frac{\partial}{\partial\theta}$. The adjoint action of this super vector field on the first factor P_k of $\mathrm{Vec}_{\bar{1}}(\mathcal{M})$ is given by by $-2z\frac{\partial}{\partial z}+k\cdot\mathrm{Id}$, and on the second factor P_{2-k} by $-2z\frac{\partial}{\partial z}+(2-k)\cdot\mathrm{Id}$. Calculating the weights of the $\mathfrak{sl}_2(\mathbb{C})$-representation on P_k and P_{2-k}, we get that P_k is the unique irreducible $(k+1)$-dimensional representation and P_{2-k} the unique irreducible $(3-k)$-dimensional representation. Moreover, a calculation yields that $d\in\mathbb{C}$ corresponding to $d\cdot\theta\frac{\partial}{\partial\theta}\in\mathrm{Vec}_{\bar{0}}(\mathcal{M})$ acts on P_k by multiplication with $-d$ and on P_{2-k} by multiplication with d.

If $k<0$ or $k>2$, we have

$$\left[\mathrm{Vec}_{\bar{1}}(\mathcal{M}),\mathrm{Vec}_{\bar{1}}(\mathcal{M})\right]=0.$$

In the case $k=0$, we have $P_k\cong\mathbb{C}$. Since $[\frac{\partial}{\partial\theta}, q(z)\theta\frac{\partial}{\partial z}]=q(z)\frac{\partial}{\partial z}$ for any $q\in P_2$, we get

$$\left[\mathrm{Vec}_{\bar{1}}(\mathcal{M}),\mathrm{Vec}_{\bar{1}}(\mathcal{M})\right]=\left\{a(z)\frac{\partial}{\partial z}\,\middle|\, a\in P_2\right\}\cong\mathfrak{sl}_2(\mathbb{C}),$$

and the map $P_0\times P_2\to\mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(X,Y)\mapsto[X,Y]$, corresponds to $\mathbb{C}\times P_2\to\mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(p,q(z))\mapsto p\cdot q(z)\frac{\partial}{\partial z}$.

Similarly, if $k = 2$, we have $P_{2-k} \cong \mathbb{C}$, and

$$[\mathrm{Vec}_{\bar{1}}(\mathcal{M}), \mathrm{Vec}_{\bar{1}}(\mathcal{M})] = \left\{ (\alpha_0 + \alpha_1 z + \alpha_2 z^2)\frac{\partial}{\partial z} + (\alpha_1 + 2\alpha_2 z)\theta\frac{\partial}{\partial \theta} \,\middle|\, \alpha_j \in \mathbb{C} \right\}$$
$$\cong \mathfrak{sl}_2(\mathbb{C})$$

since $[p(z)\frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial z}] = p(z)\frac{\partial}{\partial z} + p'(z)\theta\frac{\partial}{\partial\theta}$, and the map $P_2 \times P_0 \to \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(X, Y) \mapsto [X, Y]$, corresponds to $P_2 \times \mathbb{C} \to \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, $(p(z), q) \mapsto q \cdot p(z)\frac{\partial}{\partial z} + q \cdot p'(z)\theta\frac{\partial}{\partial\theta}$.

If $k = 1$, then $P_k \oplus P_{2-k} \cong \mathbb{C}^2 \oplus \mathbb{C}^2$. We have

$$\left[\frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial z}\right] = \frac{\partial}{\partial z}, \left[z\frac{\partial}{\partial\theta}, \theta\frac{\partial}{\partial z}\right] = z\frac{\partial}{\partial z} + \theta\frac{\partial}{\partial\theta},$$
$$\left[\frac{\partial}{\partial\theta}, z\theta\frac{\partial}{\partial z}\right] = z\frac{\partial}{\partial z}, \left[z\frac{\partial}{\partial\theta}, z\theta\frac{\partial}{\partial z}\right] = z^2\frac{\partial}{\partial z} + z\theta\frac{\partial}{\partial\theta},$$

and consequently $[\mathrm{Vec}_{\bar{1}}(\mathcal{M}), \mathrm{Vec}_{\bar{1}}(\mathcal{M})] = \mathrm{Vec}_{\bar{0}}(\mathcal{M})$.

Remark that $\mathrm{Aut}(\mathcal{M})$ carries the structure of a split Lie supergroup if and only if $k < 0$ or $k > 2$ (cf. Proposition 4 in [24]).

Example 31 Let $\mathcal{M} = (\mathbb{P}_1\mathbb{C}, \mathcal{O}_\mathcal{M})$ be a split complex supermanifold of dimension $\dim \mathcal{M} = (1|2)$ associated with $\mathcal{O}(-k_1) \oplus \mathcal{O}(-k_2)$, $k_1, k_2 \in \mathbb{Z}$. We will determine the group $\mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of automorphisms $\mathcal{M} \to \mathcal{M}$.

We choose coordinates z, θ_1, θ_2 for $\mathcal{U}_0$ and w, η_1, η_2 for $\mathcal{U}_1$ as described above such that the transition map χ is given by $\chi^*(w) = z^{-1}$ and $\chi^*(\eta_j) = z^{-k_j}\theta_j$.

The action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{P}_1\mathbb{C}$ by Möbius transformations lifts to an action of $\mathrm{SL}_2(\mathbb{C})$ on $\mathcal{M}$ by letting $A = \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ act by the automorphism $\varphi_A : \mathcal{M} \to \mathcal{M}$ with pullback

$$\varphi_A^*(z) = \frac{c + dz}{a + bz} \quad \text{and} \quad \varphi_A^*(\theta_j) = (a + bz)^{-k_j}\theta_j$$

as a morphism over appropriate subsets of U_0, and

$$\varphi_A^*(w) = \frac{aw + b}{cw + d} \quad \text{and} \quad \varphi_A^*(\eta_j) = (cw + d)^{-k_j}\eta_j$$

over appropriate subsets of U_1. Using the transition map χ one might also calculate the representation of φ in coordinates as a morphism over subsets $U_0 \to U_1$ and $U_1 \to U_0$.

If k_1 and k_2 are both even, we have $\varphi_A = \mathrm{Id}_\mathcal{M}$ for $A = \left(\begin{smallmatrix} -1 & 0 \\ 0 & -1 \end{smallmatrix}\right)$ and thus we get an action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{M}$.

Consider the homomorphism of Lie groups $\Psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(\mathbb{P}_1\mathbb{C})$ assigning to each automorphism $\varphi : \mathcal{M} \to \mathcal{M}$ the underlying biholomorphic map $\tilde{\varphi} : \mathbb{P}_1\mathbb{C} \to \mathbb{P}_1\mathbb{C}$. This homomorphism Ψ is surjective since $\mathrm{Aut}(\mathbb{P}_1\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C})$ and since the $\mathrm{PSL}_2(\mathbb{C})$-action on $\mathbb{P}_1\mathbb{C}$ lifts to an action (of $\mathrm{SL}_2(\mathbb{C})$) on the supermanifold $\mathcal{M}$. The kernel $\ker \Psi$ of the homomorphism Ψ consists of those automorphisms $\varphi : \mathcal{M} \to \mathcal{M}$ whose underlying map $\tilde{\varphi}$ is the identity $\mathbb{P}_1\mathbb{C} \to \mathbb{P}_1\mathbb{C}$. This kernel $\ker \Psi$ is a normal subgroup, $\mathrm{SL}_2(\mathbb{C})$ acts on $\ker \Psi$, and we have

$$\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \ker \Psi \rtimes \mathrm{SL}_2(\mathbb{C})$$

if k_1 and k_2 are not both even, and $\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C})$ if k_1 and k_2 are even. Thus, it remains to determine $\ker \Psi$.

Let $\varphi : \mathcal{M} \to \mathcal{M}$ be an automorphism with $\tilde{\varphi} = \mathrm{Id}$. Let f and b_{jk}, $j, k = 1, 2$, be holomorphic functions on $U_0 \cong \mathbb{C}$ such that the pullback of φ over U_0 is given by

$$\varphi^*(z) = z + f(z)\theta_1\theta_2 \quad \text{and} \quad \varphi^*(\theta) = B(z)\theta,$$

where $B(z) = \left(\begin{smallmatrix} b_{11}(z) & b_{12}(z) \\ b_{21}(z) & b_{22}(z) \end{smallmatrix}\right)$ and $\varphi^*(\theta) = B(z)\theta$ is an abbreviation for

$$\varphi^*(\theta_j) = b_{j1}(z)\theta_1 + b_{j2}(z)\theta_2 \text{ for } j = 1, 2.$$

Similarly, let g and c_{jk} be holomorphic functions on $U_1 \cong \mathbb{C}$ such that the pullback of φ over U_1 is given by

$$\varphi^*(w) = w + g(w)\eta_1\eta_2 \quad \text{and} \quad \varphi^*(\eta) = C(z)\eta,$$

where $C(z) = \left(\begin{smallmatrix} c_{11}(z) & c_{12}(z) \\ c_{21}(z) & c_{22}(z) \end{smallmatrix}\right)$. The compatibility condition with the transition map χ gives now the relation

$$f(z) = -z^{2-(k_1+k_2)} g\left(\frac{1}{z}\right) \quad \text{for } z \in \mathbb{C}^*.$$

Therefore, f and g are both polynomials of degree at most $2 - (k_1 + k_2)$, and they are 0 in the case $k_1 + k_2 > 2$. For the matrices B and C we get the relation

$$B(z) = \begin{pmatrix} z^{k_1} & 0 \\ 0 & z^{k_2} \end{pmatrix} C\left(\frac{1}{z}\right) \begin{pmatrix} z^{-k_1} & 0 \\ 0 & z^{-k_2} \end{pmatrix} \quad \text{for } z \in \mathbb{C}^*.$$

If $k_1 = k_2$, this implies $B(z) = C\left(\frac{1}{z}\right)$ for all $z \in \mathbb{C}^*$. Thus, $B(z) = B$ and $C(w) = C$ are constant matrices, and $B = C \in \mathrm{GL}_2(\mathbb{C})$ since φ was assumed to be invertible. Consequently, we have

$$\ker \Psi \cong P_{2-(k_1+k_2)} \rtimes \mathrm{GL}_2(\mathbb{C})$$

in the case $k_1 = k_2$, where $P_{2-(k_1+k_2)}$ denotes the space of polynomials of degree at most $2 - (k_1 + k_2)$ if $k_1 + k_2 < 2$ and $P_{2-(k_1+k_2)} = \{0\}$ otherwise. The group structure on the semidirect product is given by $(f_1(z), B_1) \cdot (f_2(z), B_2) = (\det B_1 f_1(z) + f_2(z), B_1 B_2)$.

Let now $k_1 \neq k_2$. After possibly changing coordinates we may assume $k_1 > k_2$. Then we have

$$B(z) = \begin{pmatrix} z^{k_1} & 0 \\ 0 & z^{k_2} \end{pmatrix} C\left(\frac{1}{z}\right) \begin{pmatrix} z^{-k_1} & 0 \\ 0 & z^{-k_2} \end{pmatrix} = \begin{pmatrix} c_{11}\left(\frac{1}{z}\right) & z^{k_1-k_2} c_{12}\left(\frac{1}{z}\right) \\ z^{k_2-k_1} c_{21}\left(\frac{1}{z}\right) & c_{22}\left(\frac{1}{z}\right) \end{pmatrix}$$

for all $z \in \mathbb{C}^*$. This implies that $b_{11} = c_{11}$ and $b_{22} = c_{22}$ are constants. Since we assume $k_1 > k_2$, we also get $b_{21} = c_{21} = 0$ and b_{12} and c_{12} are polynomials of degree at most $k_1 - k_2$. Therefore,

$$\ker \Psi \cong P_{2-(k_1+k_2)} \rtimes \left\{ \begin{pmatrix} \lambda & p(z) \\ 0 & \mu \end{pmatrix} \,\middle|\, \lambda, \mu \in \mathbb{C}^*,\ p \in P_{k_1-k_2} \right\},$$

and the group structure is again given by

$$(f_1(z), B_1) \cdot (f_2(z), B_2) = (\det B_1 f_1(z) + f_2(z), B_1 B_2)$$

for $f_1, f_2 \in P_{2-(k_1+k_2)}$, $B_1, B_2 \in \left\{ \left(\begin{smallmatrix} \lambda & p(z) \\ 0 & \mu \end{smallmatrix}\right) \,\middle|\, \lambda, \mu \in \mathbb{C}^*,\ p \in P_{k_1-k_2} \right\}$.

The semidirect product $\ker \Psi \rtimes \mathrm{SL}_2(\mathbb{C})$ (or $\ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C})$) is a direct product if and only if $k_1 = k_2$ and $k_1 + k_2 \geq 2$.

Example 32 Let $\mathcal{M} = (\mathbb{P}_1\mathbb{C}, \mathcal{O}_{\mathcal{M}})$ be the complex supermanifold of dimension $\dim \mathcal{M} = (1|2)$ given by the transition map $\chi : \mathcal{U}_0{}^* \to \mathcal{U}_1{}^*$ with pullback

$$\chi^*(w) = \frac{1}{z} + \frac{1}{z^3}\theta_1\theta_2 \quad \text{and} \quad \chi^*(\eta_j) = \frac{1}{z^2}\theta_j.$$

The supermanifold $\mathcal{M}$ is not split and the associated split supermanifold corresponds to $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$; see e.g. [7].

As in the previous example, the action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathbb{P}_1\mathbb{C}$ by Möbius transformations lifts to an action of $\mathrm{PSL}_2(\mathbb{C})$ on $\mathcal{M}$. Let A denote the class of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{SL}_2(\mathbb{C})$ in $\mathrm{PSL}_2(\mathbb{C})$. Then A acts by the morphism $\varphi_A : \mathcal{M} \to \mathcal{M}$ whose pullback as a morphism over appropriate subsets of U_0 is given by

$$\varphi_A^*(z) = \frac{c + dz}{a + bz} - \frac{b}{(a + bz)^3}\theta_1\theta_2 \quad \text{and} \quad \varphi_A^*(\theta_j) = \frac{1}{(a + bz)^2}\theta_j.$$

Let $\Psi : \mathrm{Aut}_{\bar{0}}(\mathcal{M}) \to \mathrm{Aut}(\mathbb{P}_1\mathbb{C}) \cong \mathrm{PSL}_2(\mathbb{C})$ denote again the Lie group homomorphism which assigns to an automorphism of $\mathcal{M}$ the underlying automorphism of $\mathbb{P}_1\mathbb{C}$. The assignment $A \mapsto \varphi_A \in \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ defines a section $\mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ of Ψ, and we have

$$\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C}).$$

The section $\mathrm{PSL}_2(\mathbb{C}) \to \mathrm{Aut}_{\bar{0}}(\mathcal{M})$ induces on the level of Lie algebras the morphism $\sigma : \mathfrak{sl}_2(\mathbb{C}) \hookrightarrow \mathrm{Vec}_{\bar{0}}(\mathcal{M})$, which maps an element $\left(\begin{smallmatrix} a & b \\ c & -a \end{smallmatrix}\right) \in \mathfrak{sl}_2(\mathbb{C})$ to the super vector field on $\mathcal{M}$ whose restriction to $\mathcal{U}_0$ is

$$\left(c - 2az - bz^2 - b\theta_1\theta_2\right)\frac{\partial}{\partial z} - 2(a + bz)\left(\theta_1\frac{\partial}{\partial\theta_1} + \theta_2\frac{\partial}{\partial\theta_2}\right).$$

We now calculate the kernel $\ker \Psi$. Let $\varphi \in \ker \Psi$. Its underlying map $\tilde{\varphi}$ is the identity and we thus have

$$\varphi^*(z) = z + a_0(z)\theta_1\theta_2 \quad \text{and} \quad \varphi^*(\theta) = A_0(z)\theta$$

on U_0 and

$$\varphi^*(w) = w + a_1(w)\eta_1\eta_2 \quad \text{and} \quad \varphi^*(\eta) = A_1(w)\eta$$

on U_1 for holomorphic functions a_0 and a_1 and invertible matrices A_0 and A_1 whose entries are holomorphic functions. The notation $\varphi^*(\theta) = A_0(z)\theta$ (and similarly $\varphi^*(\eta) = A_1(w)\eta$) is again an abbreviation for $\varphi^*(\theta_j) = (A_0(z))_{j1}\theta_1 + (A_0(z))_{j2}\theta_2$, where $A_0(z) = \left((A_0(z))_{jk}\right)_{1 \le j,k \le 2}$. A calculation with the transition map χ then yields the relations

$$A_1(w) = A_0\left(\frac{1}{w}\right) \quad \text{and} \quad a_1(w) = \frac{1}{w}\left(\left(\det A_0\left(\frac{1}{w}\right) - 1\right) - \frac{1}{w}a_0\left(\frac{1}{w}\right)\right)$$

for any $w \in \mathbb{C}^*$. Since a_0, a_1, A_0, and A_1 are holomorphic on $\mathbb{C}$, we get that $A_0 = A_1$ are constant matrices, $\det A_0 = 1$, and $a_0 = a_1 = 0$. Therefore, $\ker \Psi \cong \mathrm{SL}_2(\mathbb{C})$, and its Lie algebra is

$$\left\{ (a_{11}\theta_1 + a_{12}\theta_2)\frac{\partial}{\partial\theta_1} + (a_{21}\theta_1 + a_{22}\theta_2)\frac{\partial}{\partial\theta_2} \;\middle|\; \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} \in \mathfrak{sl}_2(\mathbb{C}) \right\}.$$

Since $\mathrm{Lie}(\ker \Psi)$ and $\sigma(\mathrm{Lie}(\mathrm{PSL}_2(\mathbb{C}))$ commute, the semidirect product $\ker \Psi \rtimes \mathrm{PSL}_2(\mathbb{C})$ is direct and we have

$$\mathrm{Aut}_{\bar{0}}(\mathcal{M}) \cong \mathrm{SL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C}).$$

Remark in particular that this group is different from the automorphism group of the corresponding split supermanifold $\mathcal{N}$, which is associated with $\mathcal{O}(-2) \oplus \mathcal{O}(-2)$, with $\mathrm{Aut}_{\bar{0}}(\mathcal{N}) \cong \mathrm{GL}_2(\mathbb{C}) \times \mathrm{PSL}_2(\mathbb{C})$.

References

1. Berezin, F.A.: Introduction to Superanalysis, Mathematical Physics and Applied Mathematics, vol. 9, D. Reidel Publishing Co., Dordrecht, Translation edited by Dimitri Leĭtes (1987)
2. Bergner, H.: Globalizations of infinitesimal actions on supermanifolds. J. Lie Theory **24**(3), 809–847 (2014)
3. Bergner, H.: Symmetries of supermanifolds, PhD Thesis, Ruhr-Universität Bochum, November (2015)
4. Bochner, S., Montgomery, D.: Groups of differentiable and real or complex analytic transformations. Ann. Math. **46**, 685–694 (1945)
5. Bochner, S., Montgomery, D.: Locally compact groups of differentiable transformations. Ann. Math. **47**, 639–653 (1946)
6. Bochner, S., Montgomery, D.: Groups on analytic manifolds. Ann. Math. **48**, 659–669 (1947)
7. Bunegina, V.A., Onishchik, A.L.: Homogeneous supermanifolds associated with the complex projective line. J. Math. Sci. **82**(4), 3503–3527 (1996). Algebraic geometry, 1
8. Cartan, H.: Œuvres. Vol. I, II, III, Springer-Verlag, Berlin, New York, Edited by Reinhold Remmert and Jean-Pierre Serre (1979)
9. Cartan, H., Serre, J.-P.: Un théorème de finitude concernant les variétés analytiques compactes. C. R. Acad. Sci. Paris **237**, 128–130 (1953)
10. Green, P.: On holomorphic graded manifolds. Proc. Am. Math. Soc. **85**(4), 587–590 (1982)
11. Grothendieck, A.: Sur la classification des fibrés holomorphes sur la sphère de Riemann. Am. J. Math. **79**, 121–138 (1957)
12. Garnier, S., Wurzbacher, T.: Integration of vector fields on smooth and holomorphic supermanifolds. Doc. Math. **18**, 519–545 (2013)
13. Hirsch, M .W.: Differential Topology. Springer, New York (1976). Graduate Texts in Mathematics, No. 33
14. Kalus, M.: On the relation of real and complex Lie supergroups. Can. Math. Bull. **58**(2), 281–284 (2015)
15. Kostant, B.: Graded manifolds, graded Lie theory, and prequantization. In: Differential geometrical methods in mathematical physics. (Proceedings of symposium, University of Bonn, Bonn, 1975) Lecture Notes in Mathematics, vol. 570, pp. 177–306. Springer, Berlin (1977)
16. Leĭtes, D.A.: Introduction to the theory of supermanifolds, Uspekhi Mat. Nauk **35** , no. 1(211), 3–57, 255 (1980)
17. Morimoto, A.: Sur le groupe d'automorphismes d'un espace fibré principal analytique complexe. Nagoya Math. J. **13**, 157–168 (1958)
18. Narasimhan, R.: Several Complex Variables. Chicago Lectures in Mathematics. The University of Chicago Press, Chicago (1971)
19. Onishchik, A.L.: A spectral sequence for the tangent sheaf cohomology of a supermanifold. Lie groups and Lie algebras, Math. Appl., vol. 433, pp. 199–215. Kluwer Academic Publishers, Dordrecht (1998)
20. Ostermayr, D.: Automorphism supergroups of supermanifolds, arXiv:1504.02653v1 (2015)
21. Rothstein, M.J.: Deformations of complex supermanifolds. Proc. Am. Math. Soc. **95**(2), 255–260 (1985)
22. Sachse, C., Wockel, C.: The diffeomorphism supergroup of a finite-dimensional supermanifold. Adv. Theor. Math. Phys. **15**(2), 285–323 (2011)
23. van den Essen, A.: Polynomial Automorphisms and the Jacobian Conjecture, Progress in Mathematics, vol. 190. Birkhäuser Verlag, Basel (2000)
24. Vishnyakova, E.G.: On complex Lie supergroups and split homogeneous supermanifolds. Transform. Groups **16**(1), 265–285 (2011)
25. Yamabe, H.: A generalization of a theorem of Gleason. Ann. Math. **58**, 351–365 (1953)