Math. Z. (2017) 286:1479–1493
DOI 10.1007/s00209-016-1810-y

Mathematische Zeitschrift

Radial multipliers and restriction to surfaces of the Fourier transform in mixed-norm spaces

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Received: 22 February 2016 / Accepted: 21 October 2016 / Published online: 5 November 2016

Abstract In this article we revisit some classical conjectures in harmonic analysis in the setting of mixed norm spaces $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$. We produce sharp bounds for the restriction of the Fourier transform to compact hypersurfaces of revolution in the mixed norm setting and study an extension of the disc multiplier. We also present some results for the discrete restriction conjecture and state an intriguing open problem.

1 Introduction

The well-known restriction conjecture, first proposed by E. M. Stein, asserts that the restriction of the Fourier transform of a given integrable function f to the unit sphere, $\hat{f}|_{S^{n-1}}$, yields a bounded operator from $L^p(\mathbb{R}^n)$, $n \geq 2$, to $L^q\left(S^{n-1}\right)$ so long as

$$1 \leq p < \frac{2n}{n+1}, \quad \frac{1}{q} \geq \frac{n+1}{n-1}\left(1 - \frac{1}{p}\right).$$

This conjecture has been fully proved only in dimension $n = 2$ by Fefferman [10] (see also [6] for an alternative geometrical proof). In higher dimensions, the best known result is the particular case $q = 2$ and $1 \leq p \leq \frac{2(n+1)}{n+3}$, which proof was obtained independently by Tomas and Stein [16].

Antonio Córdoba and Eric Latorre Crespo are partially supported by the Grant MTM2014-56350-P from the Ministerio de Ciencia e Innovación (Spain).

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The periodic analogue, i.e. for Fourier series, was observed by Zygmund [19], but also in two dimensions. It asserts that for any trigonometric polynomial

$$P(x) = \sum_{|\nu|=R} a_\nu e^{2\pi i \nu \cdot x},\ \nu \in \mathbb{Z}^2,$$

the following inequality holds:

$$\|P\|_{L^4(Q)} \lesssim \|P\|_{L^2(Q)},$$

uniformly on $R > 0$ and where Q is any unit square in the plane.

The alternative proof given in [6] allows us to connect both the periodic and the nonperiodic restriction theorems, explaining the reason for the apparently different numerologies of the corresponding (p, q) exponent ranges. It also raises an interesting question about the location of lattice points in small arcs of circles [5].

The first result in this paper goes further in that direction: given $\{\xi_j\}$ a finite set of points in the circle $\{\|\xi\| = R\}$ of the plane, let us consider

$$M := \sup_j \# \left\{\xi_k,\ \|\xi_k - \xi_j\| \le R^{\frac{1}{2}}\right\}.$$

We have:

Theorem 1 *The following inequality holds*

$$\sup_{\mu(Q)=1} \left[\int_Q \left|\sum a_k e^{2\pi i \xi_k \cdot x}\right|^4 d\mu(x)\right]^{\frac{1}{4}} \lesssim M^{\frac{1}{2}} \left(\sum |a_k|^2\right)^{\frac{1}{2}}, \tag{1.1}$$

where the supremum is taken over all unit squares of $\mathbb{R}^2$ *and* μ *corresponds to the Lebesgue measure.*

The corresponding result in higher dimensions ($n \ge 3$) is an interesting open problem:

Conjecture 2 *Let* $\{\xi_j\} \subset S_R^{n-1}$ *and* $M := \sup_j \# \left\{\xi_k,\ \|\xi_k - \xi_j\| \le R^{\frac{1}{2}}\right\}$, *is it true that*

$$\sup_{\mu(Q)=1} \left[\int_Q \left|\sum a_k e^{2\pi i \xi_k \cdot x}\right|^{\frac{2n}{n-1}} d\mu(x)\right]^{\frac{n-1}{2n}} \lesssim M^{\frac{1}{2}} \left(\sum |a_k|^2\right)^{\frac{1}{2}}. \tag{1.2}$$

Although there are many interesting publications by several authors throwing some light on the restriction conjecture, its proof remains open in dimension $n \ge 3$. One of the more remarkable improvements was B. Barcelo's thesis [15]. He proved that Fefferman's result also holds for the cone in $\mathbb{R}^3$. Another interesting result was given by L. Vega in his Ph.D. thesis [17], where he obtained the best result in the Stein–Tomas restriction inequality when the space $L^p(\mathbb{R}^n)$ is replaced by $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$.

Here we shall consider the restriction of the Fourier transform to other surfaces of revolution in these mixed norm spaces. Several special cases have already been treated [11,12] but we present a more general and unified proof for "all" compact surfaces of revolution:

$$\Gamma = \left\{(g(z), \theta, z) \in \mathbb{R}^{n+1},\ \theta \in S^{n-1},\ a \le z \le b,\ 0 \le g \in C^1(a, b)\right\}.$$

That is, in $\mathbb{R}^{n+1}$, $n \ge 2$, we consider cylindrical coordinates (r, θ, z) where the first components (r, θ) correspond to the standard polar coordinates in $\mathbb{R}^n$; $0 < r < \infty,\ \theta \in S^{n-1}$, and

$z \in \mathbb{R}$ denotes the zenithal coordinate. In this coordinate system, the $L^p_{rad}L^2_{zen}L^2_{ang}\left(\mathbb{R}^{n+1}\right)$ norm is given by

$$\left(\int_0^\infty r^{n-1}\left(\int_{-\infty}^\infty \int_{S^{n-1}} |f(r,\theta,z)|^2\, d\theta dz\right)^{\frac{p}{2}} dr\right)^{\frac{1}{p}}.$$

We can state our result.

Theorem 3 *Let Γ be a compact surface of revolution, then the restriction of the Fourier transform to Γ is a bounded operator from $L^p_{rad}L^2_{zen}L^2_{ang}\left(\mathbb{R}^{n+1}\right)$ to $L^2(\Gamma)$, i.e. there exists a finite constant C_p such that*

$$\left(\int_{-\infty}^\infty \int_{S^{n-1}} g(z)^{n-1}\sqrt{1+g'(z)^2}\left|\hat{f}(g(z),\theta,z)\right|^2 d\theta dz\right)^{\frac{1}{2}} \lesssim C_p \|f\|_{L^p_{rad}L^2_{zen}L^2_{ang}(\mathbb{R}^{n+1})}, \tag{1.3}$$

so long as $1 \leq p < \frac{2n}{n+1}$.

A central point in this area is C. Fefferman's observation that the disc multiplier in $\mathbb{R}^n$ for $n \geq 2$, given by the formula

$$\widehat{T_0 f}(\xi) = \chi_{B(0,1)}(\xi)\hat{f}(\xi),$$

is bounded on $L^p(\mathbb{R}^n)$ only in the trivial case $p = 2$. However, it was later proved (see Refs. [8,13]) that T_0 is bounded on the mixed norm spaces $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ if and only if $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. Here we extend that result to a more general class of radial multipliers.

Theorem 4 *Let T_m be a Fourier multiplier defined by*

$$(T_m f)\hat{}(\xi) := m(|\xi|)\hat{f}(\xi), \tag{1.4}$$

for all rapidly decreasing smooth functions f, where m satisfies the following hypothesis:

1. *$Supp(m) \subset [a,b] \subset \mathbb{R}^+$, and m is differentiable in the interior (a,b).*
2. *$\int_a^b |m'(x)|\, dx < \infty$.*

T_m is then bounded in $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ so long as $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

Finally, let us observe that this result was already proved by Duoandikoetxea et al. through the study of radial weights in [9]. We however give a direct proof that relies only on the decay of Bessel functions. We finally highlight that this theorem admits different extensions taking into account Littlewood–Paley theory.

2 Restriction in the discrete setting

Proof of Theorem 1 First let us observe that, by an easy argument, we can assume $M = 1$ without loss of generality. Next we take a smooth cut-off φ sot that

$$\varphi \equiv 1 \text{ on } B\left(0,\frac{1}{2}\right),$$
$$\varphi \equiv 0 \text{ when } \|x\| \geq 1,$$
$$\varphi \in C_0^\infty\left(\mathbb{R}^2\right).$$

We can then write

$$f(\xi) = \sum_k a_k \varphi(\xi + \xi_k) e^{2\pi i \xi \cdot q} = \sum_k a_k \varphi_k(\xi) e^{2\pi i \xi \cdot q},$$

where q is a point in $\mathbb{R}^2$. We have

$$\hat{f}(x) = \sum_k a_k \hat{\varphi}(x - q) e^{2\pi i \xi_k \cdot (x-q)}.$$

Note that the L^4 norm of $\hat{f}$ majorizes the left hand side of (1.1),

$$\int \left|\hat{f}(x)\right|^4 dx \geq \int_{x-q \in Q_0} \left|\sum a_k e^{2\pi i \xi_k \cdot (x-q)} \hat{\varphi}(x-q)\right|^4 dx \gtrsim \int_Q \left|\sum a_k e^{2\pi i \xi_k \cdot x}\right|^4 dx,$$

where $Q_0 = \left[-\frac{1}{2}, \frac{1}{2}\right]^2$ and $Q = q + Q_0$.

On the other hand, we have

$$\int \left|\hat{f}(x)\right|^4 dx = \int |f * f(\xi)|^2 d\xi = \int \left|\sum_{k,j} a_k a_j \varphi_k * \varphi_j(\xi) e^{i\xi \cdot q}\right|^2 d\xi.$$

Furthermore, because the supports of φ_k and φ_j have a finite overlapping, uniformly on the radius R.

$$\int \left|\hat{f}(x)\right|^4 dx \lesssim \left(\sum |a_k|^2\right)^2,$$

□

Using similar arguments we can obtain the following analogous result: In $\mathbb{R}^2$ let us consider the parabola $\gamma(t) = (t, t^2)$ and a set of real numbers $\{\xi_j\}$ so that $|t_{j+1} - t_j| \geq 1$, then

$$\sup_{\mu(Q)=1} \left\|\sum_j a_j e^{2\pi i \gamma(t_j) \cdot x}\right\|_{L^4(Q)} \lesssim \left(\sum |a_j|^2\right)^{\frac{1}{2}}.$$

An interesting open question is to decide if the L^4 norm could be replaced by an L^p norm ($p > 4$) in the inequality above. It is known that $p = 6$ fails, but for $4 < p < 6$ it is, as far as we know, an interesting open problem [2].

3 The restriction conjecture in mixed norm spaces

Recall that in $\mathbb{R}^{n+1}$ we establish cylindrical coordinates (r, θ, z), where (r, θ) corresponds to the usual spherical coordinates in $\mathbb{R}^n$ and $z \in \mathbb{R}$ denotes the zenithal component. We will also use the notation (ρ, ϕ, ζ) to refer to the same coordinate system.

The $L^p_{rad}L^2_{zen}L^2_{ang}\left(\mathbb{R}^{n+1}\right)$ norm is therefore given by

$$\|f\|_{L^{p,2,2}}=\left(\int_0^\infty r^{n-1}\left(\int_{-\infty}^{\infty}\int_{S^{n-1}}|f(r,\theta,z)|^2\,d\theta dz\right)^{\frac{p}{2}}dr\right)^{\frac{1}{p}}. \tag{3.1}$$

Let g be a continuous positive function supported on a compact interval I of the real line that is almost everywhere differentiable, and consider the surface of revolution in $\mathbb{R}^{n+1}$ given by

$$\Gamma:=\left\{(g(z),\theta,z)\in\mathbb{R}^{n+1},\ \theta\in S^{n-1},-\infty<z<\infty\right\}. \tag{3.2}$$

We are interested in the restriction to Γ of the Fourier transform of functions in the Schwartz class $\mathcal{S}\left(\mathbb{R}^{n+1}\right)$. The restriction inequality

$$\left\|\hat{f}\right\|_{L^2(\Gamma)}\le C_p\,\|f\|_{L^{p,2,2}\left(\mathbb{R}^{n+1}\right)}$$

for $1\le p<\frac{2n}{n+1}$ is, by duality, equivalent to the extension estimate:

$$\left\|\widehat{fd\Gamma}\right\|_{L^{q,2,2}\left(\mathbb{R}^{n+1}\right)}\le C_q\,\|f\|_{L^2(\Gamma)}$$

for $q>\frac{2n}{n-1}$.

To compute $\widehat{fd\Gamma}$ let us recall

$$\begin{aligned} d\Gamma &= g(z)^{n-1}\sqrt{1+(g'(z))^2}\,dzd\theta\\ &= G_1(z)\,dzd\theta, \end{aligned}$$

so that

$$\widehat{fd\Gamma}(\rho,\phi,\zeta)=\int_{-\infty}^{\infty}\int_{S^{n-1}}G_1(z)\,f(z,\theta)\,e^{-iz\zeta}e^{-i(\rho g(z))\theta\cdot\phi}d\theta dz. \tag{3.3}$$

Next we use the spherical harmonic expansion

$$f(z,\theta)=\sum_{k,j}a_{k,j}(z)\,Y_k^j(\theta),$$

where for each k, $\left\{Y_k^j\right\}_{j=1,\dots,d(k)}$ is an orthonormal basis of the spherical harmonics degree k. We then obtain:

$$\begin{aligned}\widehat{fd\Gamma}(\rho,\phi,\zeta)=\sum_{k,j}2\pi i^k Y_k^j(\phi)\,\rho^{-\frac{n-2}{2}}\int_{-\infty}^{\infty}g(z)^{\frac{n}{2}}\left(1+\left(g'(z)\right)^2\right)^{\frac{1}{2}}\\ \cdot a_{k,j}(z)\,J_{k+\frac{n-2}{2}}(\rho g(z))\,e^{-iz\zeta}dz,\end{aligned}$$

where J_ν denotes Bessel's function of order ν (see Ref. [18]). Denoting by $G_2(z):=g(z)^{\frac{n}{2}}\left(1+\left(g'(z)\right)^2\right)^{\frac{1}{2}}$, the Fourier transform $\widehat{fd\Gamma}$ becomes

$$\sum_{k,j}2\pi i^k Y_k^j(\phi)\,\rho^{-\frac{n-2}{2}}\int_{-\infty}^{\infty}G_2(z)\,a_{k,j}(z)\,J_{k+\frac{n-2}{2}}(\rho g(z))\,e^{-iz\zeta}dz. \tag{3.4}$$

Taking into account the orthogonality of the elements of the basis $\{Y_k^j\}$ together with Plancherel's Theorem in the z-variable, we obtain that the mixed norm $\|\widehat{fd\Gamma}\|^q_{L^{q,2,2}}$ is up to a constant equal to

$$\int_0^\infty \rho^{-q\frac{n-2}{2}+n-1}\left(\sum_{k,j}\int_{-\infty}^{\infty}|g(\zeta)|^n\left|1+\left(g'(\zeta)\right)^2\right|\left|a_{k,j}(\zeta)\right|^2\left|J_{\nu_k}(\rho g(\zeta))\right|^2 d\zeta\right)^{\frac{q}{2}} d\rho, \tag{3.5}$$

where $\nu_k = k + \frac{n-2}{2}$. On the other hand we have

$$\begin{aligned}\int_\Gamma |f|^2 &= \int_{-\infty}^{\infty}\int_{S^{n-1}}\left|\sum_{j,k} a_{k,j}(z)\, Y_k^j(\theta)\right|^2 g(z)^{n-1}\sqrt{1+g'(z)^2}\,d\theta dz\\ &= \sum_{j,k}\int_{-\infty}^{\infty}\left|a_{k,j}(z)\right|^2 g(z)^{n-1}\sqrt{1+g'(z)^2}\,dz.\end{aligned} \tag{3.6}$$

Therefore our theorem will be a consequence of the following fact:

Lemma 5 *Given any sequence of positive indices $\{\nu_j\}$ with $\nu_j \geq \frac{n-2}{2}$ for all j and Schwartz functions a_j, the following inequality holds:*

$$\begin{aligned}&\int_0^\infty \rho^{-q\frac{n-2}{2}+n-1}\left(\sum_{j}\int_{-\infty}^{\infty}|g(z)|^n\left|1+\left(g'(z)\right)^2\right|\left|a_j(z)\right|^2\left|J_{\nu_j}(\rho g(z))\right|^2 dz\right)^{\frac{q}{2}} d\rho\\ &\lesssim \left(\sum_j\int_{-\infty}^{\infty}|g(z)|^{n-1}\left(1+\left(g'(z)\right)^2\right)^{\frac{1}{2}}\left|a_j(z)\right|^2 dz\right)^{\frac{q}{2}},\end{aligned} \tag{3.7}$$

for $q > \frac{2n}{n-1}$.

Remark 6 Taking into account the hypothesis about g we will look for estimates depending upon $A = \sup_{x\in I}|g(x)|$ and $B = \sup_{x\in I}\left|g'(x)\right|$, where I is the compact support of g. It is also easy to see that we can reduce ourselves to consider the sums over the family of indices $\{\nu_j\}_{j=1}^{\infty}$ such that $\nu_j \geq \frac{n-2}{2}$. Therefore it is enough to show

$$\begin{aligned}&\int_0^\infty \rho^{-q\frac{n-2}{2}+n-1}\left(\sum_{j}\int_{-\infty}^{\infty}\left|b_j(z)\right|^2\left|J_{\nu_j}(\rho g(z))\right|^2 dz\right)^{\frac{q}{2}} d\rho\\ &\lesssim \left(\sum_j\int_{-\infty}^{\infty}\left|b_j(z)\right|^2 dz\right)^{\frac{q}{2}}\end{aligned} \tag{3.8}$$

for a family of smooth functions $\{b_j\}_j$ and indexes $\nu_j \geq \frac{n-2}{2}$.

In order to show (3.8) we will need a sharp control of the decay of Bessel functions; namely the following estimates:

Lemma 7 *The following estimates hold for $\nu \geq 1$.*

1. $J_\nu(r) \leq \frac{1}{r^{1/2}}$, *when $r \geq 2\nu$.*
2. $J_\nu(r) \leq \frac{1}{\nu}$, *when $r \leq \frac{1}{2}\nu$.*

3. $J_\nu\left(\nu+\rho\nu^{1/3}\right)\le\frac{1}{\rho^{1/4}\nu^{1/3}}$, *when* $0\le\rho\le\frac{3}{2}\nu^{2/3}$.
4. $J_\nu\left(\nu-\rho\nu^{1/3}\right)\le\frac{1}{\rho\nu^{1/3}}$, *when* $1\le\rho\le\frac{3}{2}\nu^{2/3}$.
5. $J_\nu(r)\le r^\nu$, *as* $r\to 0$.

These asymptotics follow by the stationary phase method as it is shown in [1,7,18].

Proof of Lemma 5 To prove 3.8 we shall first decompose the ρ-integration in dyadic parts: $[0,\infty)=[0,1)\bigcup\cup_{n=0}^{\infty}[2^n,2^{n+1})$.

$$\int_0^1\rho^{-q\frac{n-2}{2}+n-1}\left(\sum_j\int_{-\infty}^{\infty}\left|b_j(z)\right|^2\left|J_{\nu_j}(\rho g(z))\right|^2dz\right)^{\frac{q}{2}}d\rho$$
$$+\sum_M\int_M^{2M}\rho^{-q\frac{n-2}{2}+n-1}\left(\sum_j\int_{-\infty}^{\infty}\left|b_j(z)\right|^2\left|J_{\nu_j}(\rho g(z))\right|^2dz\right)^{\frac{q}{2}}d\rho, \qquad (3.9)$$

where $M=2^m$, $m=0,1,\ldots$ □

For the lower integrand, we have the following splitting:

$$\int_0^1\rho^{-q\frac{n-2}{2}+n-1}[\ldots]^{\frac{q}{2}}d\rho=\int_0^{\frac{1}{A}}\rho^{-q\frac{n-2}{2}+n-1}[\ldots]^{\frac{q}{2}}d\rho+\int_{\frac{1}{A}}^1\rho^{-q\frac{n-2}{2}+n-1}[\ldots]^{\frac{q}{2}}d\rho$$
$$=I+II.$$

In order to bound I we invoke Minkowski's inequality and property 5. of Lemma 7.

$$I\lesssim\left[\int_{-\infty}^{\infty}\sum_j\left(\int_0^{\frac{1}{A}}\left\{\rho^{-(n-2)+\frac{2}{q}(n-1)}\left|b_j(z)\right|^2\left|J_{\nu_j}(\rho z)\right|^2\right\}^{\frac{q}{2}}d\rho\right)^{\frac{2}{q}}dz\right]^{\frac{q}{2}}.$$
$$\le\left[\int_{-\infty}^{\infty}\sum_j\left|b_j(z)\right|^2A^{2\nu_j}\left(\int_0^{\frac{1}{A}}\rho^{-q\frac{n-2}{2}+(n-1)+q\nu_j}d\rho\right)^{\frac{2}{q}}dz\right]^{\frac{q}{2}},$$

where $A=\|g\|_\infty$. Since the sum is taken over all $\nu_j\ge\frac{n-2}{2}$, the inner integrand is well defined and we can bound

$$I\lesssim A^{q\frac{n-1}{2}-n}\left[\sum_j\int_{-\infty}^{\infty}\left|b_j(z)\right|^2dz\right]^{\frac{q}{2}}. \qquad (3.10)$$

The second part is similarly bounded

$$II\lesssim\left(1+A^{q\frac{n-1}{2}-n}\right)\left[\sum_j\int_{-\infty}^{\infty}\left|b_j(z)\right|^2dz\right]^{\frac{q}{2}}. \qquad (3.11)$$

Then Lemma 5 will be a consequence of the following claim:

Claim 8 *For all $q > 4$, the following inequality holds true*

$$\int_M^{2M} \rho \left(\sum_j \int_{-\infty}^{\infty} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim M^{\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_j \left| b_j(z) \right|^2 dz \right)^{\frac{q}{2}}. \tag{3.12}$$

Indeed, if $q > 4$ we need only to note that

$$\int_M^{2M} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim M^{(n-2)(-\frac{q}{2}+1)} \int_M^{2M} \rho \left(\sum_j \int_{-\infty}^{\infty} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho,$$

invoke our claim and sum over all dyadic intervals in (3.9):

$$\sum_m \int_{2^m}^{2^{m+1}} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim \sum_m 2^{m(n-2)(-\frac{q}{2}+1)+m\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_j \left| b_j(z) \right|^2 dz \right)^{\frac{q}{2}}. \tag{3.13}$$

It is then a simple matter to check that the exponent is negative for $q > \frac{2n}{n-1}$.

If the exponent q is however smaller, $\frac{2n}{n-1} < q \leq 4$, we need to use an extra trick. Note that Eq. (3.12) implies

$$\int_M^{2M} \left(\sum_j \int_{-\infty}^{\infty} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q_1}{2}} d\rho \lesssim M^{1-\frac{q_1}{2}} \left(\int_{-\infty}^{\infty} \sum_j \left| b_j(z) \right|^2 dz \right)^{\frac{q_1}{2}},$$

for all $q_1 > 4$. Then using Hölder's inequality and the previous inequality,

$$\int_M^{2M} \left(\sum_j \int_{-\infty}^{\infty} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim M^{1-\frac{q}{q_1}} \left(\int_M^{2M} \left(\sum_j \int_{-\infty}^{\infty} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 dz \right)^{\frac{q_1}{2}} d\rho \right)^{\frac{q}{q_1}}.$$

Therefore, summing over all intervals, we obtain

$$\sum_m \int_{2^m}^{2^{m+1}} \rho^{-q\frac{n-2}{2}+n-1} \left(\sum_j \int_{-\infty}^{\infty} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\lesssim \sum_m 2^{m\left\{-q\frac{n-2}{2}+n-1+1-\frac{q}{2}\right\}} \left(\int_{-\infty}^{\infty} \sum_j \left|b_j(z)\right|^2 dz \right)^{\frac{q}{2}},$$

where the exponent $-q\frac{n-1}{2}+n$ is negative for all $q > \frac{2n}{n-1}$.

To prove Claim 8 let us split each dyadic integrand in (3.9) in three parts corresponding to the different ranges of control of Bessel functions.

$$\int_M^{2M} \rho \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^0} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$+ \int_M^{2M} \rho \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^c} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$+ \int_M^{2M} \rho \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^\infty} \left|b_j(z)\right|^2 \left|J_{\nu_j}(\rho g(z))\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$= \sum_M \left(I_M^0 + I_M^c + I_M^\infty\right),$$

where $I_M^0 = [0, Mg(z)/2)$, $I_M^c = [Mg(z)/2, 4Mg(z))$, and $I_M^\infty = [4Mg(z), \infty)$.

Recall that if $2k < r$, $|J_k(r)| \le r^{-1/2}$; in I_M^0 we have $2\nu_j < Mg(z) < \rho g(z)$, hence

$$I_M^0 \le A^{-\frac{q}{2}} \int_M^{2M} \rho^{1-\frac{q}{2}} \left(\int_{-\infty}^{\infty} \sum_{\nu_j \in I^0} \left|b_j(z)\right|^2 dz \right)^{\frac{q}{2}} d\rho$$
$$\le A^{-\frac{q}{2}} M^{\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_{\nu_j} \left|b_j(z)\right|^2 dz \right)^{\frac{q}{2}}. \tag{3.14}$$

Similarly, I_M^∞ is also easily bounded as if $k > 2r$, $|J_k(r)| \le k^{-1}$, and in I_M^∞, $k > 4Mg(z) > 2\rho g(z)$. Furthermore, since $\rho g(z) > 1$, $(\rho g(z))^{-2} < (\rho g(z))^{-1}$ and, in I_M^∞, we have $|J_k(\rho g(z))|^2 \le (\rho g(z))^{-1}$. This shows that again

$$I_M^\infty \le A^{-\frac{q}{2}} M^{\frac{4-q}{2}} \left(\int_{-\infty}^{\infty} \sum_{\nu_j} \left|b_j(z)\right|^2 dz \right)^{\frac{q}{2}}. \tag{3.15}$$

Finally, we need to work a little bit harder than in the previous cases to obtain a suitable estimate for I_M^c. First of all note that Minkowski's inequality yields

$$I_M^c \leq \left[\int_{-\infty}^{\infty} \left\{ \int_M^{2M} \rho \left(\sum_{\nu_j \in I^c} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 \right)^{\frac{q}{2}} d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}}. \tag{3.16}$$

In I_M^c we want to use estimate (3) of Lemma 7, we thus need to split the inner integral so that $\rho g(z) \sim \nu_j + \alpha \nu_j$ in the according range of α. Consider the family of sets

$$G_\alpha = \left[\frac{M}{2} + \alpha M^{\frac{1}{3}} g(z)^{-\frac{2}{3}}, \frac{M}{2} + (\alpha+1) M^{\frac{1}{3}} g(z)^{-\frac{2}{3}} \right],$$

for $\alpha = 0, 1, 2, \ldots, \left[(Mg(z))^{\frac{2}{3}} \right]$, so that $\bigcup G_\alpha \supseteq [M, 2M]$ and in each interval $\rho g(z) \sim \nu_j + \alpha \nu_j^{\frac{1}{3}}$, and split (3.16) in the following way

$$I_M^c \lesssim \left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \rho \left(\sum_{\nu_j \in I^c} \left| b_j(z) \right|^2 \left| J_{\nu_j}(\rho g(z)) \right|^2 \right)^{\frac{q}{2}} d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}},$$

Let us also define

$$A_\beta = \sum_{\nu_j \in G_\beta} \left| b_j(z) \right|^2.$$

We can then invoke Lemma 7 and rearragne the sums to bound I_M^c by

$$\left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \left(\sum_{\beta \leq \alpha} A_\beta \frac{1}{(|\alpha-\beta|+1)^{1/2} M^{\frac{2}{3}} g(z)^{-\frac{4}{3}}} \right)^{\frac{q}{2}} \rho d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}}$$
$$+ \left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \int_{G_\alpha} \left(\sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha-\beta|+1)^{2} M^{\frac{2}{3}} g(z)^{-\frac{4}{3}}} \right)^{\frac{q}{2}} \rho d\rho \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}}.$$

Note that the second sum is easier to control than the first. We shall, therefore, focus on the first term, $I_{M,}^{c,1}$. Since the intervals G_α have length $M^{\frac{1}{3}} g(z)^{-\frac{2}{3}}$,

$$I_M^{c,1} \lesssim M^{\frac{4-q}{3}} A^{2\frac{(q-1)}{3}} \left[\int_{-\infty}^{\infty} \left\{ \sum_\alpha \left(\sum_{\beta \geq \alpha} A_\beta \frac{1}{(|\alpha-\beta|+1)^2} \right)^{\frac{q}{2}} \right\}^{\frac{2}{q}} dz \right]^{\frac{q}{2}}.$$

Furthermore, using Young's inequality, since $q > 4$, taking $2/q = 1/s - 1/2$ we obtain

$$\sum_{\alpha}\left(\sum_{\beta\geq\alpha} A_{\beta}\frac{1}{(|\alpha-\beta|+1)^{2}}\right)^{\frac{q}{2}} \lesssim \left(\sum_{\gamma} A_{\gamma}^{s}\right)^{\frac{q}{2s}} \lesssim \left(\sum_{\gamma} A_{\gamma}\right)^{\frac{q}{2}}.$$

We have thus showed that the central integrand I_M^c can also be bounded in the desired way;

$$I_M^{c,1} \lesssim A^{2\left(\frac{q-1}{3}\right)} M^{\frac{4-q}{3}}\left[\int_{-\infty}^{\infty}\sum_{k\in I_M^c}|a_k|^2\,dz\right]^{\frac{q}{2}}. \tag{3.17}$$

□

4 Generalized disc multiplier

In the late 80's it was proved independently in [8,13] that the disc multiplier operator is bounded in the mixed norm spaces $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ for all $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. Let us here explore further the theory of radial fourier multipliers following the ideas presented in the aforementioned articles.

Let m be a radial function and consider the fourier multiplier

$$(T_m f)\hat{\ }(\xi) = m(|\xi|)\,\hat{f}(\xi).$$

Once again, recall the expansion of a given function f in terms of its spherical harmonics,

$$f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} f_{k,j}(|x|)\,Y_k^j\left(\frac{x}{|x|}\right).$$

Then, the classical formula relating the Fourier transform and the spherical harmonics expansion, [14], yields

$$\hat{f}(\xi) = \sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} Y_k^j\left(\frac{\xi}{|\xi|}\right) 2\pi i^k\,|\xi|^{-\left(k+\frac{n-2}{2}\right)}\int_0^{\infty} f_{k,j}(t)\,J_{k+\frac{n-2}{2}}(2\pi|\xi|t)\,t^{k+\frac{n}{2}}\,dt.$$

The expression of T_m in terms of its spherical harmonics expansion is then

$$T_m f(x) = \sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} 2\pi i^k\int_{\mathbb{R}^n} e^{2\pi i x\xi} m(|\xi|)\,Y_k^j\left(\frac{\xi}{|\xi|}\right)|\xi|^{-\left(k+\frac{n-2}{2}\right)} \int_0^{\infty} f_{k,j}(t)\,J_{k+\frac{n-2}{2}}(2\pi|\xi|t)\,t^{k+\frac{n-2}{2}}\,dt\,d\xi.$$

Exchanging the order of integration, the previous expression becomes

$$\sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} 2\pi i^k\int_0^{\infty} f_{k,j}(t)\,t^{k+\frac{n-2}{2}}\,\hat{g}_t(x)\,dx,$$

where

$$g_t(\xi) = m(|\xi|)\, J_{k+\frac{n-2}{2}}(2\pi\,|\xi|\,t)\,|\xi|^{-\left(k+\frac{n-2}{2}\right)}\, Y_k^j\left(\frac{\xi}{|\xi|}\right).$$

Therefore, computing once more the Fourier transform of a radial function,

$$T_m f(r\theta) = \sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} 4\pi^2 (-1)^k\, Y_k^j(\theta)\, T_m^{k,j} f(r),$$

with

$$T_m^{k,j} f(r) = \int_0^{\infty} f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} K_{k+\frac{n-2}{2}}(t,r)\, dt,$$

where

$$K_\nu(t,r) = \sqrt{rt}\int_a^b m(s)\, J_\nu(2\pi ts)\, J_\nu(2\pi rs)\, s ds.$$

In order to simplify the notation, note that

$$T_m f(r\theta) \approx \sum_{k=0}^{\infty}\sum_{j=1}^{d(k)} Y_k^j(\theta)\, T_m^{k,j} f(r) \tag{4.1}$$

with $T_m^{k,j}$ defined as before, but

$$K_\nu(t,r) = \sqrt{rt}\int_a^b m(s)\, J_\nu(ts)\, J_\nu(rs)\, s ds.$$

Let us take a closer look at the kernel of the operator K_α,

$$K_\alpha(t,r) = \sqrt{rt}\int_a^b m(s)\, J_\alpha(ts)\, J_\alpha(rs)\, s ds. \tag{4.2}$$

It is suitable to decode these kernels in terms of an auxiliary function $\mathcal{U}_r(s) = \sqrt{rs} J_\alpha(rs)$. The use of Bessel's equation yields

$$\frac{\partial}{\partial s}\left\{\mathcal{U}_r(s)\,\mathcal{U}_t'(s) - \mathcal{U}_t(s)\,\mathcal{U}_r'(s)\right\} = \left(t^2 - r^2\right)\sqrt{tr} J_\alpha(rs)\, J_\alpha(ts)\, s.$$

Therefore, after an integration by parts in (4.2), we obtain

$$\begin{aligned} K_\alpha(t,r) = &\left[m(s)\,\frac{1}{t^2-r^2}\left\{\mathcal{U}_r(s)\,\mathcal{U}_t'(s) - \mathcal{U}_t(s)\,\mathcal{U}_r'(s)\right\}\right]_a^b \\ &- \int_a^b m'(s)\,\frac{1}{t^2-r^2}\left\{\mathcal{U}_r(s)\,\mathcal{U}_t'(s) - \mathcal{U}_t(s)\,\mathcal{U}_r'(s)\right\} ds. \end{aligned}$$

Hence, we express the modified disc multiplier in the following way

$$\begin{aligned} T_m f(r\theta) = &\sum_{k,j} Y_k^j(\theta)\int_0^{\infty} f_{k,j}(t)\, t^{\frac{n+2k-1}{2}} r^{-\frac{n+2k-1}{2}} \\ &\cdot \left(m(b)\, k(r,t,b) - m(a)\, k(r,t,a) - \int_a^b m'(s)\, k(r,t,s)\, ds\right) dt, \end{aligned} \tag{4.3}$$

where $k_\alpha(t,r,s)$ denotes the kernel $\frac{1}{t^2-r^2}\left\{\mathcal{U}_r(s)\mathcal{U}_t'(s)-\mathcal{U}_t(s)\mathcal{U}_r'(s)\right\}$. A simple expansion of k_α reveals the underlying singularities of the operator K_α;

$$k_\alpha(t,r,s)=\left(s\frac{\sqrt{t}J_\alpha'(ts)J_\alpha(rs)\sqrt{r}}{2(t-r)}+s\frac{\sqrt{t}J_\alpha'(ts)J_\alpha(rs)\sqrt{r}}{2(t+r)}\right.$$
$$\left.+s\frac{\sqrt{t}J_\alpha(ts)J_\alpha'(rs)\sqrt{r}}{2(r-t)}+s\frac{\sqrt{t}J_\alpha'(ts)J_\alpha(rs)\sqrt{r}}{2(t+r)}\right). \tag{4.4}$$

A thorough study of the kernel $k_\alpha(r,t,1)$ was carried out in [8] using the decay properties of Bessel functions (Lemma 7) in order to show that the disc multiplier is bounded in the mixed norm spaces $L^p_{rad}L^2_{ang}(\mathbb{R}^n)$ in the optimal range $\frac{2n}{n+1}<p<\frac{2n}{n-1}$.

Although nothing really new has been done, we have brought to light a more general family of operators underlying the disc multiplier, that is the family of operators T^s defined as

$$T^s f(r\theta)=\sum_{k,j}Y_k^j(\theta)\int_0^\infty f_{k,j}(t)\,t^{\frac{n+2k-1}{2}}r^{-\frac{n+2k-1}{2}}k(r,t,s)\,dt. \tag{4.5}$$

Indeed, any bound on operators T^s that is uniform in s implies a bound on T_m for a suitable m.

Proposition 9 *Let f be a rapidly decreasing function then, for every $\frac{2n}{n+1}<p<\frac{2n}{n-1}$*

$$\left\|T^s f\right\|_{p,2}\leq C_{p,n}\|f\|_{p,2}, \tag{4.6}$$

where the constant $C_{p,n}$ is uniform in s.

Proof In order to simplify the expressions we will just write one of the four core kernels of k_α apparent in (4.4), that is

$$T^s f(r\theta)\sim\sum_{k,j}Y_k^j(\theta)\int_0^\infty f_{k,j}(t)\,t^{\frac{n+2k-1}{2}}r^{-\frac{n+2k-1}{2}}s\frac{\sqrt{t}J_\alpha'(ts)J_\alpha(rs)\sqrt{r}}{2(t-r)}dt, \tag{4.7}$$

for any fixed $s\in\mathbb{R}$. The orthonormality in $L^2\left(\mathbb{S}^{n-1}\right)$ of spherical harmonics can now be used in our advantage to complute the $L^p_{rad}L^2_{ang}$ norm of T^s. Indeed, $\|T^s f\|_{p,2,}$ is up to the notation reduction equal to

$$\left(\int_0^\infty r^{n-1}\left\{\sum_{k,j}\left|\int_0^\infty f_{k,j}(t)\,t^{\frac{n+2k-1}{2}}r^{-\frac{n+2k-1}{2}}s\frac{\sqrt{t}J_\alpha'(ts)J_\alpha(rs)\sqrt{r}}{2(t-r)}dt\right|^2\right\}^{\frac{p}{2}}dr\right)^{\frac{1}{p}}.$$

Two simple changes of variables, $t'=st$ and $r'=sr$, yield

$$s^{-\frac{n}{p}}\left(\int_0^\infty r^{n-1}\left\{\sum_{k,j}\left|\int_0^\infty f_{k,j}\left(\frac{t}{s}\right)t^{\frac{n+2k-1}{2}}r^{-\frac{n+2k-1}{2}}\frac{\sqrt{t}J_\alpha'(t)J_\alpha(r)\sqrt{r}}{2(t-r)}dt\right|^2\right\}^{\frac{p}{2}}dr\right)^{\frac{1}{p}}.$$

Note that this expression corresponds to that of the disc multiplier T_0 analyzed by in [8]. We can therefore bound it by

$$C_{p,n}s^{-\frac{n}{p}}\left(\int_0^\infty r^{n-1}\left\{\sum_{k,j}\left|f_{k,j}\left(\frac{r}{s}\right)\right|^2\right\}^{\frac{p}{2}}dr\right)^{\frac{1}{p}},$$

for every $\frac{2n}{n+1} < p < \frac{2n}{n-1}$. One last change of variables produces the estimate

$$\left\| T^s f \right\|_{p,2} \leq C \left\| f \right\|_{p,2},$$

where C is uniform on s.

It is now a simple matter to produce a bound for the operator T_m.

$$\|T_m f\|_{p,2} \lesssim |m(b)| \left\| T^b f \right\|_{p,2} + |m(a)| \left\| T^b f \right\|_{p,2} + \int_a^b \left| m'(s) \right| \left\| T^s f \right\|_{p,2} ds, \quad (4.8)$$

and Theorem 4 follows from the uniformity in the bound (4.6). That is

$$\|T_m f\|_{p,2} \leq C \left(\sup_{s \in [a,b]} |m(s)| + \int_a^b \left| m'(s) \right| ds \right) \|f\|_{p,2}.$$

Remark 10 Let us highlight that, once obtained the expression (4.5), it is possible to control the operator $T^s f(x)$ by the *universal Kakeya maximal function* using the techniques developed by Carbery et al. [3,4]. That is, for every $\alpha > 1$ and every radial weight g there exists a finite constant C_α so that for every rapidly decreasing function f,

$$\int_{\mathbb{R}^n} \left| T^s f(x) \right|^2 g(x)\, dx \leq C_\alpha \int_0^\infty |f(x)|^2 \mathcal{M}_\alpha g(x)\, dx, \quad (4.9)$$

where the constant C is uniform in s. Here $\mathcal{M}$ denotes the universal Kakeya maximal function

$$\mathcal{M}g(x) = \sup_{x \in R \in \mathcal{R}_n} \frac{1}{|R|} \int_R |g(y)|\, dy, \quad (4.10)$$

where the supremum is taken over all recantles in $\mathbb{R}^n$ containing the point x, and $\mathcal{M}_\alpha g = (\mathcal{M}(|g|^\alpha))^{\frac{1}{\alpha}}$.

Indeed, this approach has the advantage that one can easily derive Littlewood–Paley estimates using the boundedness of the universal Kakeya maximal function acting on radial functions. Such work was already carried out by Duoandikoetxea et al. [9] and produces the following result:

Corollary 11 *Let T be a Fourier multiplier operator with a radial multiplier m, satisfying $m \in L^\infty(\mathbb{R})$ and for each dyadic interval I,*

$$\int_I \left| m'(t) \right| dt \leq C, \quad (4.11)$$

uniformly in I. Then m is an $L^p_{rad} L^2_{ang}(\mathbb{R}^n)$ multiplier for all $\frac{2n}{n+1} < p < \frac{2n}{n-1}$.

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