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Vanishing polyhedron and collapsing map

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Abstract In this paper we give a detailed proof of the fact that the Milnor fiber X_t of an analytic complex isolated singularity function defined on a reduced n-equidimensional analytic complex space X is a regular neighborhood of a polyhedron $P_t \subset X_t$ of real dimension $n-1$. Moreover, we describe the degeneration of X_t onto the special fiber X_0, by giving a continuous collapsing map $\psi_t : X_t \to X_0$ which sends P_t to $\{0\}$ and which restricts to a homeomorphism $X_t \setminus P_t \to X_0 \setminus \{0\}$.

1 Introduction

Let $f\colon (X, x) \to (\mathbb{C}, 0)$ be a germ of complex analytic function f at a point x of a reduced equidimensional complex analytic space $X \subset \mathbb{C}^N$ (with arbitrary singularity). In [11] the first author proved that there exist sufficiently small positive real numbers ϵ and η with $0 < \eta \ll \epsilon \ll 1$ such that the restriction:

$$f_| : \mathbb{B}_\epsilon(x) \cap X \cap f^{-1}(\mathbb{D}^*_\eta) \to \mathbb{D}^*_\eta$$

is a locally trivial topological fibration, where $\mathbb{B}_\epsilon(x)$ is the closed ball of radius ϵ around $x \in \mathbb{C}^N$, $\mathbb{D}_\eta$ is the closed disk of radius η around $0 \in \mathbb{C}$ and $\mathbb{D}^*_\eta := \mathbb{D}_\eta \setminus \{0\}$.

The topology of the fiber $X_t := \mathbb{B}_\epsilon(x) \cap X \cap f^{-1}(t)$ does not depend on ϵ small enough (see Theorem 2.3.1 of [14]). We call X_t the Milnor fiber of f, with boundary $\partial X_t := X_t \cap \mathbb{S}_\epsilon(x)$. We also set $X_0 := \mathbb{B}_\epsilon(x) \cap X \cap f^{-1}(0)$.

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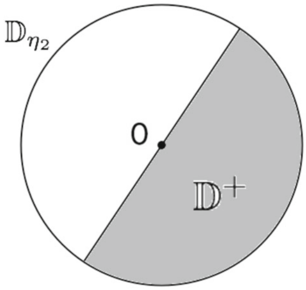

Fig. 1 The semi-disk $\mathbb{D}^+$

The first author sketched a proof of the following theorem in [13]:

Theorem 1 *Let $X \subset \mathbb{C}^N$ be a reduced equidimensional complex analytic space and let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of X. Let $f : (X, x) \to (\mathbb{C}, 0)$ be a germ of complex analytic function at a point $x \in X$. If f has an isolated singularity at x relatively to $\mathcal{S}$ and if ϵ and η are sufficiently small positive real numbers as above, then for each $t \in \mathbb{D}^*_\eta$ there exist:*

(i) *a polyhedron P_t of real dimension* $\dim_{\mathbb{C}} X_t$ *in the Milnor fiber X_t, compatible with the Whitney stratification $\mathcal{S}$, and a continuous simplicial map:*

$$\tilde{\xi}_t : \partial X_t \to P_t$$

compatible with $\mathcal{S}$, such that X_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$;

(ii) *a continuous map $\psi_t : X_t \to X_0$ that sends P_t to $\{0\}$ and that restricts to a homeomorphism $X_t \backslash P_t \to X_0 \backslash \{0\}$.*

The purpose of this paper is to give a complete and detailed proof of Theorem 1, following the strategy proposed in [13].

That theorem was conjectured by Thom in a seminar, in the early 70's, when X is smooth. He noticed that Pham gave an explicit construction of such a vanishing polyhedron in [19] when $f : \mathbb{C}^n \to \mathbb{C}$ is a polynomial of the form:

$$f(z_1, \ldots, z_n) = z_1^{\nu_1} + \cdots + z_n^{\nu_n},$$

with $\nu_i \geq 2$ integer, for $i = 1, \ldots, n$.

In this paper, we are going to prove the following stronger version of Theorem 1. Let $\mathbb{D}^+$ be a closed semi-disk in $\mathbb{D}_{\eta_2}$ as in Fig. 1 (with $0 \in \partial \mathbb{D}^+$).

So our main theorem is:

Theorem 2 *Let $X \subset \mathbb{C}^N$ be a n-dimensional reduced equidimensional complex analytic space and let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of X. Let $f : (X, x) \to (\mathbb{C}, 0)$ be a germ of complex analytic function with an isolated singularity at x, relatively to $\mathcal{S}$. Let ϵ and η be sufficiently small positive real numbers as above, and let $\mathbb{D}^+$ be a closed semi-disk in $\mathbb{D}_\eta \subset \mathbb{C}$ such that 0 is in its boundary. Then there exist:*

(i) *A polyhedron P^+ in $X^+ := X \cap f^{-1}(\mathbb{D}^+) \cap \mathbb{B}_\epsilon(x)$ of real dimension $n + 1$, compatible with the Whitney stratification $\mathcal{S}$, such that for each $t \in \mathbb{D}^+ \backslash \{0\}$ the intersection $P^+ \cap X_t$ is a polyhedron P_t of real dimension $n - 1$, compatible with the Whitney stratification $\mathcal{S}$.*

(ii) *A continuous simplicial map:*

$$\tilde{\xi}_+ : \partial X^+ \to P^+$$

compatible with $\mathcal{S}$*, such that* X^+ *is homeomorphic to the mapping cylinder of* $\tilde{\xi}_+$*, and such that for each* $t \in \mathbb{D}^+\backslash\{0\}$ *the map* $\tilde{\xi}_+$ *restricts to a continuous simplicial map:*

$$\tilde{\xi}_t : \partial X_t \to P_t$$

compatible with $\mathcal{S}$*, such that* X_t *is homeomorphic to the mapping cylinder of* $\tilde{\xi}_t$*, where* $\partial X^+ := X^+ \cap \mathbb{S}_\epsilon(x)$*,* $\partial X_t := X_t \cap \mathbb{S}_\epsilon(x)$ *and* $\mathbb{S}_\epsilon(x)$ *is the boundary of* $\mathbb{B}_\epsilon(x)$*.*

(iii) *A continuous map* $\psi_t : X_t \to X_0$ *that sends* P_t *to* $\{0\}$ *and that restricts to a homeomorphism* $X_t\backslash P_t \to X_0\backslash\{0\}$*, for any* $t \in \mathbb{D}^+\backslash\{0\}$*.*

In Sect. 2 we recall some classical definitions and results. In Sect. 3 we construct the *relative polar curve* of f, which is the main tool to prove Theorem 2. In Sect. 4 we prove Theorem 2 when X is two-dimensional. Then in Sect. 5 we present two propositions (Propositions 29 and 30) and we use them to prove Theorem 2 in the general case. In Sect. 6 we prove those Propositions by finite induction on the dimension of X. Finally, in Sect. 7 we make the detailed construction of a vector field (Lemma 35) that is used in Sect. 6.

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2 Background

In this section we recall some definitions, references and theorems that will be used in this paper.

2.1 Whitney stratification

Following [14] (Section 1, p. 67), we have:

Definition 3 Let X be a subanalytic set (resp. a reduced complex analytic space). We say that a locally finite family of non-singular subanalytic connected subsets $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ of X is a subanalytic stratification (resp. complex analytic stratification) of X if:

(i) the family $\mathcal{S}$ is a partition of X; and
(ii) the closure $\bar{S}_\alpha$ of S_α in X and $\bar{S}_\alpha \backslash S_\alpha$ are subanalytic (resp. complex analytic) subspaces of X, for any $\alpha \in A$.

The subsets S_α are called *strata* of the stratification $\mathcal{S}$ of X.

In this paper we will use both subanalytic and complex analytic stratifications. Although we work with a complex analytic space $X \subset \mathbb{C}^N$, when we intersect it with a closed ball $\mathbb{B}_\epsilon$ in $\mathbb{C}^N$ we obtain a subanalytic set.

We say that a (subanalytic or complex analytic) stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ as above satisfies the frontier condition if for any $(\alpha, \beta) \in A \times A$ such that $S_\alpha \cap \bar{S}_\beta \neq \emptyset$ ones has that $S_\alpha \subset \bar{S}_\beta$. In this case, $\bar{S}_\alpha$ and $\bar{S}_\alpha \backslash S_\alpha$ are union of strata of the stratification $\mathcal{S}$, for any $\alpha \in A$.

We have:

Definition 4 Let X be a subanalytic set (resp. reduced complex analytic space). We say that a subanalytic (resp. complex analytic) stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ of X is a subanalytic (resp. complex analytic) Whitney stratification if:

(i) the stratification $\mathcal{S}$ satisfies the frontier condition; and
(ii) for any $(\alpha, \beta) \in A \times A$ such that $S_\alpha \cap \bar{S}_\beta \neq \emptyset$ the pair of strata (S_α, S_β) satisfies the Whitney condition, which is the following: for any $y \in S_\alpha$ there exists a local embedding of (X, y) in $(\mathbb{R}^N, 0)$ such that for any sequence $(x_n, y_n)_{n\in\mathbb{N}}$ in $S_\beta \times S_\alpha$ that converges to (y, y) and such that the limit T of the tangent spaces $T_{x_n} S_\beta$ and the limit λ of the real secants $\overline{x_n y_n}$ in $\mathbb{R}^N$ exist, one has the inclusion $\lambda \subset T$.

One can verify that, for any $y \in S_\alpha$ fixed, if the condition above is satisfied for some local embedding, then it is satisfied for any local embedding.

In [23] and in [7] (for the complex case and for the subanalytic case respectively) it is proved the following:

Theorem 5 *Let X be a subanalytic set (resp. reduced complex analytic space) and let $(\Phi_i)_{i\in I}$ be a locally finite family of subanalytic (resp. complex analytic) closed subsets of X. There exists a subanalytic (resp. complex analytic) Whitney stratification of X such that each Φ_i is a union of strata, for $i \in I$.*

Next we give a lemma due to Cheniot [4] (see also Lemma 4.2.2. of Chapter III of [21]) that will be implicitly used many times in the paper:

Lemma 6 *Let X and Y be two subanalytic sets in $\mathbb{R}^N$ (or reduced complex analytic spaces in $\mathbb{C}^N$, in the complex case) such that X has a subanalytic (resp. complex analytic) Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha\in A}$ and such that Y is non-singular. If Y intersects each stratum S_α transversally in $\mathbb{R}^N$ (resp. in $\mathbb{C}^N$), then the Whitney stratification of X induces a subanalytic (resp. complex analytic) Whitney stratification $\mathcal{P} = (P_\alpha)_{\alpha\in A}$ of $X \cap Y$, where $P_\alpha := S_\alpha \cap Y$, for each $\alpha \in A$.*

Next we will present a stronger version of the Whitney stratification. But first we need the following definition. Given two vector subspaces A and B of $\mathbb{R}^N$ set:

$$\delta(A, B) := sup_{\substack{x\in A \\ \|x\|=1}} d(x, B),$$

where $d(x, B)$ is the distance between x and B. Then we have:

Definition 7 Let $X \subset \mathbb{R}^N$ be a subanalytic set (resp. reduced complex analytic space) with a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha\in A}$. We say that $\mathcal{S}$ has the property (w) if for any $(\alpha, \beta) \in A \times A$ such that $S_\alpha \cap \bar{S}_\beta \neq \emptyset$ the pair of strata (S_α, S_β) satisfies the Kuo–Verdier condition below:

For any $y' \in S_\alpha$ there exists a neighborhood U of y' in $\mathbb{R}^N$ and a real constant $C > 0$ such that for any $(x, y) \in (S_\beta \cap U, S_\alpha \cap U)$ one has that:

$$\delta\left(T_x(S_\beta), T_y(S_\alpha)\right) \leq C\|x - y\|.$$

In Theorem 1.2 of chapter V of [21], Teissier proved the following:

Lemma 8 *Let $X \subset \mathbb{C}^N$ be a reduced complex analytic space with a complex analytic Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha\in A}$. Then $\mathcal{S}$ has the property (w).*

The analogous of this result in the subanalytic case is not true (see Example 1 of [3] for instance).

Definition 9 Let X and Y be subanalytic sets and let $A \subset X$ be endowed with a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha\in A}$. We say that a morphism $f\colon X \to Y$ is transversal to $\mathcal{S}$ if, for any $\alpha \in A$, f induce a smooth morphism $f_\alpha : S_\alpha \to Y$.

According to Remark (3.7) of [22], we have:

Lemma 10 *Let X be a subanalytic set (resp. complex analytic set) and let $A \subset X$ be endowed with a subanalytic (resp. complex analytic) Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ with the property (w). If Y is smooth and if $f : X \to Y$ is transversal to $\mathcal{S}$, then for any smooth and locally closed subset $Z \subset Y$ one has that $f^{-1}(Z) \cap \mathcal{S}$ is a subanalytic (resp. complex analytic) Whitney stratification of $A \cap f^{-1}(Z)$ with the property (w).*

Considering $f : X \to \mathbb{R}$ the square of the distance function to a point $x \in X$ and Z a closed interval $[0, \epsilon^2]$, we have:

Corollary 11 *If $X \subset \mathbb{C}^N$ is a reduced complex analytic space with a complex analytic Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ and if $\mathbb{B}_\epsilon(x)$ is a ball around $x \in X$ in $\mathbb{C}^N$ of small enough radius $\epsilon > 0$, then $\mathcal{S}$ induces a subanalytic Whitney stratification on $X \cap \mathbb{B}_\epsilon(x)$ with the property (w).*

Let X be a subanalytic set endowed with a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ that has the property (w). We say that a Whitney stratification $\mathcal{S}' = (S'_\beta)_{\beta \in A'}$ with the property (w) is finer then $\mathcal{S}$ (or that $\mathcal{S}'$ is a refinement for $\mathcal{S}$) if for any $\beta \in A'$ there exists $\alpha \in A$ such that $S'_\beta \subset S_\alpha$.

As in Remark (3.6) of [22], we have:

Remark 12 Let X and Y be subanalytic sets and let $A \subset X$ be a closed subset. Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of A and let $\mathcal{Z} = (Z_\beta)_{\beta \in B}$ be a Whitney stratification of Y, both of them with the property (w). If $f : X \to Y$ is a morphism such that the restriction $f_{|A} : A \to Y$ is proper, then we can consider a refinement $\mathcal{S} = (S'_\alpha)_{\alpha \in A'}$ of $\mathcal{S}$ and a refinement $\mathcal{Z}' = (Z'_\beta)_{\beta \in B'}$ of $\mathcal{Z}$ such that, for any $\beta \in B'$, one has that:

(i) $f^{-1}(Z'_\beta) \cap A$ is a union of strata of $\mathcal{S}'$;
(ii) the restriction $f_| : f^{-1}(Z'_\beta) \to Z'_\beta$ is transversal to $\mathcal{S}' \cap f^{-1}(Z'_\beta)$.

Now let $f : X \to \mathbb{C}$ be a complex analytic function defined on a complex analytic space X with a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$. Following [12], we have:

Definition 13 We say that f has an isolated singularity at $x \in X$ relatively to the stratification $\mathcal{S}$ if:

(i) the restriction of f to S_α is a submersion, for any $\alpha \in A$ such that S_α does not contain x;
(ii) the restriction of f to $S_{\alpha(x)}$ has an isolated critical point at x, where $S_{\alpha(x)}$ is the stratum that contains x.

2.2 Rugose vector fields

Following [5], we will briefly define a *rugose vector field* on a subanalytic set $X \subset \mathbb{R}^N$ endowed with a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ with the property (w). See [22] for the detailed definitions.

We say that a real-valued function $f : X \to \mathbb{R}$ is a *rugose function* if for any $\alpha \in A$ one has that:

(i) the restriction of f to S_α is smooth;

(ii) for any $x \in S_\alpha$ there exists a neighborhood U of x in $\mathbb{R}^N$ and a real constant $C > 0$ such that for any $x' \in U \cap S_\alpha$ and for any $y \in U \cap X$ one has that:

$$\|f(x') - f(y)\| \leq C\|x - y\|.$$

We say that a map $f\colon X \to \mathbb{R}^M$ is a *rugose map* if each of its coordinate functions is a rugose function. We say that a vector bundle F over X is rugose if its chart change maps are rugose.

A *rugose vector bundle F on X tangent to the stratification $\mathcal{S}$* is a vector bundle over X such that, for every stratum S_α there is an injection $i_\alpha : F|_{S_\alpha} \to TS_\alpha$ such that the vector bundle morphism $F \to i^*T\mathbb{R}^N|_X$ induced by i_α is rugose.

A *stratified vector field $\vec{v}$ on X* is a section of the tangent bundle $T\mathbb{R}^N|_X$ such that at each $x \in X$, the vector $\vec{v}(x)$ is tangent to the stratum that contains x.

A stratified vector field $\vec{v}$ on X is called *rugose* near $y \in S_\alpha$ if there exists a neighborhood U of y in $\mathbb{R}^N$ and a real constant $C > 0$ such that:

$$\|\vec{v}(y') - \vec{v}(x)\| \leq C\|y' - x\|,$$

for every $(x, y') \in (U \cap S_\beta, U \cap S_\alpha)$ with $S_\alpha \subset \bar{S}_\beta$.

On the other hand, we say that a rugose stratified vector field $\vec{v}$ on X is *integrable* if there exists an open neighborhood U of $X \times \{0\}$ in $X \times \mathbb{R}$ and a rugose map $\theta : U \to X$ such that for any $\alpha \in A$ one has that:

(i) $\theta\left((S_\alpha \times \mathbb{R}) \cap U\right) \subset S_\alpha$;
(ii) for any $x \in S_\alpha$ such that $(x, t) \in (S_\alpha \cap \mathbb{R}) \cap U$ one has that $\frac{\partial}{\partial t}\theta(x, t) = \vec{v}\,(\theta(x, t))$.

We say that the map θ is the *flow* associated to the vector field $\vec{v}$.

For each $x_0 \in X$ we say that the restriction $\theta|_{(\{x_0\}\times\mathbb{R})\cap U} : (\{x_0\} \times \mathbb{R}) \cap U \to X$ is an *integral curve* for the vector field $\vec{v}$ with initial condition x_0. Condition (i) above assures that if x_0 is in a stratum S_α then the image of the integral curve for $\vec{v}$ with initial condition x_0 is contained in S_α.

We have (Proposition 4.8 of [22]):

Proposition 14 *Any rugose stratified vector field on a closed subanalytic set X of $\mathbb{R}^N$ is integrable. Moreover, given a rugose vector field, if θ and θ' are rugose maps defined in open neighborhoods U and U' of $X \times \{0\}$ in $X \times \mathbb{R}$, satisfying the properties (i) and (ii) above, then θ and θ' coincide in $U \cap U'$.*

2.3 Stratified maps

We have:

Definition 15 Let X and Y be subanalytic sets (or reduced complex analytic spaces) with Whitney stratifications $(X_\alpha)_{\alpha\in A}$ and $(Y_\beta)_{\beta\in B}$ respectively. A real analytic (resp. complex analytic) morphism $h : X \to Y$ is a stratified map if:

(i) h sends each stratum X_α to a unique stratum $Y_{\beta(\alpha)}$, for some $\beta(\alpha) \in B$;
(ii) the restriction of h to each stratum X_α induces a smooth map $h_\alpha : X_\alpha \to Y_{\beta(\alpha)}$.

We say that a stratified map h as above is a stratified submersion if each h_α is a (surjective) submersion.

We say that a stratified map h as above is a stratified homeomorphism if h is a homeomorphism and each h_α is a smooth diffeomorphism.

Let $h : X \to Y$ be a stratified map as above and let $\vec{\nu}$ be a stratified vector field on Y. We say that a stratified vector field $\vec{\mu}$ on X *lifts* $\vec{\nu}$ if for each $x \in X$ one has that $dh\,(\vec{\mu}(x)) = \vec{\nu}\,(h(x))$.

We have (see [22], Proposition 4.6):

Proposition 16 *Let X be a real analytic space endowed with a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha\in A}$ with the property (w), and let Z be a locally closed subset of X which is union of strata S_α. Let Y be a non-singular real analytic space and let $h : X \to Y$ be a stratified submersion. If $\vec{\nu}$ is a smooth vector field in Y, then there exists a rugose stratified vector field $\vec{\mu}$ on Z that lifts $\vec{\nu}$.*

There is a more general version of the theorem above, which includes the case when Y is not smooth or the case when the vector field $\vec{\nu}$ is non-everywhere smooth, as we present below (see [22], Remark 4.7):

Proposition 17 *Let X and Y be real analytic spaces with Whitney stratifications $\mathcal{S} = (S_\alpha)_{\alpha\in A}$ and $\mathcal{W} = (W_\beta)_{\beta\in B}$, respectively, both of them with the property (w). Let $h : X \to Y$ be a stratified submersion. Also, let Z be a locally closed subset of X which is a union of strata S_α. Suppose that each restriction:*

$$f_{|W_\beta} : Z \cap f^{-1}(W_\beta) \to W_\beta,$$

with $\beta \in B$, is transversal to $\mathcal{S} \cap Z \cap f^{-1}(W_\beta)$ (that is, $f_{|W_\beta}$ induces a smooth map on each stratum of $f^{-1}(W_\beta) \cap Z$). If $\vec{\nu}$ is a rugose stratified vector field on $f(Z)$, then there exists a rugose stratified vector field $\vec{\mu}$ on Z that lifts $\vec{\nu}$.

2.4 Simplicial maps

Let $\mathcal{X}$ be a topological space. A triangulation for $\mathcal{X}$ is a pair $(\mathcal{K}, h)$, where $\mathcal{K}$ is a simplicial complex and h is a homeomorphism $h : \mathcal{K} \to \mathcal{X}$. We say that $\mathcal{X}$ is triangulable if there exists a triangulation $(\mathcal{K}, h)$ for $\mathcal{X}$.

Hironaka proved in [7] that any subanalytic set is triangulable.

In this paper, a polyhedron is a compact topological space that is triangulable. We are only interested in the existence of a simplicial structure; a particular decomposition into faces is not important in this work.

We say that a map $f\colon P \to P'$ between two polyhedra is a *simplicial map* if there exist triangulations $(\mathcal{K}, h)$ and $(\mathcal{K}', h')$ for P and P', respectively, such that the induced map $\tilde{f} : \mathcal{K} \to \mathcal{K}'$ is a simplicial map in the usual sense (that is, $\tilde{f}$ has the property that whenever the vertices $v_0, \ldots, v_n$ of $\mathcal{K}$ span a simplex of $\mathcal{K}$, the points $f(v_0), \ldots, f(v_n)$ are vertices of a simplex of $\mathcal{K}'$).

Notice that our definition is slightly different from the usual definition of a simplicial map, since it relates spaces which do not have fixed simplicial structures (compare with Lemma 2.7 of [17]).

Finally, we have:

Definition 18 Let P be a polyhedron contained in a subanalytic space X endowed with a Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha\in A}$. We say that P is adapted to the stratification $\mathcal{S}$ if the interior of each simplex of P is contained in S_α for some $\alpha \in A$.

2.5 Some useful results

Now we will state two results that will be used later. The first one is Thom–Mather's first isotopy lemma (Proposition 11.1 of [15]):

Lemma 19 (Thom–Mather's first isotopy lemma) *Let M and P be smooth manifolds and let X be a closed subset of M with a Whitney stratification (real or complex). If $f : X \to P$ is proper stratified map and if it is a submersion on each stratum, then f is a locally trivial fibration.*

The second one is Remmert's theorem (see Corollary 1.68 of [6] for instance):

Theorem 20 (Finite mapping theorem) *Let $f : X \to Y$ be a finite morphism of complex analytic spaces and $Z \subset X$ a closed analytic complex subspace of X. Then $f(Z) \subset Y$ is an analytic subset of Y.*

3 Polar curves

In the rest of the paper, X will be a fixed reduced equidimensional complex analytic space in $\mathbb{C}^N$ such that $0 \in X$, and $f : (X, 0) \to (\mathbb{C}, 0)$ will be the germ of a complex analytic function. Notice that if we prove Theorem 2 with this assumption, then we clearly prove it in the general case $f : (X, x) \to (\mathbb{C}, 0)$, and we do so just to simplify the notation.

Moreover, we will endow the germ $(X, 0)$ with a fixed complex analytic Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ such that $f^{-1}(0)$ is a union of strata (see Lemma 5).

The notion of polar curve for a complex analytic function defined on an open neighborhood of $\mathbb{C}^N$ relatively to a linear form ℓ was introduced by Teissier and by the first author in [20] and [9], respectively. Later, in [10] the first author extended this concept to a germ of complex analytic function $f : (X, 0) \to (\mathbb{C}, 0)$ relatively to the Whitney stratification $\mathcal{S} = (S_\alpha)_{\alpha \in A}$. We are going to recall that.

Notice that by now we are not supposing that f has an isolated singularity (in the stratified sense). This hypothesis will be asked after the lemma below.

Let $f : X \to \mathbb{C}$ be a representative of the germ of function f such that X is closed in an open neighborhood U of 0 in $\mathbb{C}^N$. For any linear form:

$$\ell : \mathbb{C}^N \to \mathbb{C}$$

the function f and the restriction of ℓ to X induce the analytic morphism:

$$\phi_\ell : X \to \mathbb{C}^2$$

defined by $\phi_\ell(z) = \big(\ell(z), f(z)\big)$, for any $z \in X$.

We have the following lemma:

Lemma 21 *There is a representative X of $(X, 0)$ in an open neighborhood U of $0 \in \mathbb{C}^N$ and a non-empty Zariski open set Ω in the space of non-zero linear forms of $\mathbb{C}^N$ to $\mathbb{C}$ such that, for any $\ell \in \Omega$ and for any stratum S_α which is disjoint from $f^{-1}(0)$, the analytic morphism $\phi_\ell : X \to \mathbb{C}^2$ satisfies:*

(i) *The critical locus of the restriction of ϕ_ℓ to S_α is either empty or a smooth reduced complex curve, whose closure in X is denoted by Γ_α.*
(ii) *The image $(\Delta_\alpha, 0)$ of $(\Gamma_\alpha, 0)$ by ϕ_ℓ is the germ of a complex curve.*

Proof Let us choose an open neighborhood U of $0 \in \mathbb{C}^N$ such that the intersection $U \cap S_\alpha$ is not empty for finitely many indices α. Furthermore, we may assume that the closure $\overline{S}_\alpha$ in U is defined by an ideal $I(\overline{S}_\alpha)$ generated by complex analytic functions $g_1, \ldots, g_m$ defined on U, that is, $I(\overline{S}_\alpha) = (g_1, \ldots, g_m)$.

Now consider a linear form $\ell = a_1x_1 + \cdots + a_Nx_N$ and a stratum S_α that is not contained in $f^{-1}(0)$ and such that $0 \in \overline{S}_\alpha$. Let $C_{\ell,\alpha}$ be the critical set of the restriction of ϕ_ℓ to S_α. Consider the matrix:

$$J_\alpha = \begin{pmatrix} \partial g_1/\partial x_1 & \dots & \partial g_1/\partial x_N \\ \vdots & \ddots & \vdots \\ \partial g_m/\partial x_1 & \dots & \partial g_m/\partial x_N \end{pmatrix}.$$

A point z of S_α is a point where the rank of J_α at z is $\rho := \max_{z\in\overline{S}_\alpha} \operatorname{rank}\big(J_\alpha(z)\big)$, since it is a non-singular point of $\overline{S}_\alpha$. A point of $C_{\ell,\alpha}$ is a point of S_α where the matrix:

$$J_{\phi,\alpha} = \begin{pmatrix} \partial g_1/\partial x_1 & \dots & \partial g_1/\partial x_N \\ \vdots & \ddots & \vdots \\ \partial g_m/\partial x_1 & \dots & \partial g_m/\partial x_N \\ \partial f/\partial x_1 & \dots & \partial f/\partial x_N \\ a_1 & \dots & a_N \end{pmatrix}$$

has rank at most $\rho + 1$. So the determinants of the $(\rho + 2)$-minors:

$$\begin{pmatrix} \partial g_{i_1}/\partial x_{j_1} & \dots & \partial g_{i_1}/\partial x_{j_{\rho+2}} \\ \vdots & \ddots & \vdots \\ \partial g_{i_\rho}/\partial x_{j_1} & \dots & \partial g_{i_\rho}/\partial x_{j_{\rho+2}} \\ \partial f/\partial x_{j_1} & \dots & \partial f/\partial x_{i_{\rho+2}} \\ a_{i_1} & \dots & a_{i_{\rho+2}} \end{pmatrix}$$

are zero, that is:

$$\sum_{k=1}^{\rho+2} (-1)^{k+1} \cdot a_{i_k} \cdot \det \begin{pmatrix} \partial g_{i_1}/\partial x_{j_1} & \dots & \partial g_{i_1}/\partial x_{j_{k-1}} & \partial g_{i_1}/\partial x_{j_{k+1}} & \dots & \partial g_{i_1}/\partial x_{j_{\rho+2}} \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \partial g_{i_\rho}/\partial x_{j_1} & \dots & \partial g_{i_\rho}/\partial x_{j_{k-1}} & \partial g_{i_\rho}/\partial x_{j_{k+1}} & \dots & \partial g_{i_\rho}/\partial x_{j_{\rho+2}} \\ \partial f/\partial x_{j_1} & \dots & \partial f/\partial x_{j_{k-1}} & \partial f/\partial x_{j_{k+1}} & \dots & \partial f/\partial x_{j_{\rho+2}} \end{pmatrix} = 0.$$

An analytic version of a classical theorem of Bertini (see [1] and [2]) states that if $h_1, \dots, h_r$ are holomorphic functions defined on a complex analytic space Y and if the complex numbers λ_i are sufficiently generic, for $i = 1, \dots, r$, then the singular locus of the subvariety $\left\{\sum_{i=1}^r \lambda_i h_i = 0\right\}$ is contained in the union of the singular set of Y and the set:

$$\{h_1 = \cdots = h_r = 0\}.$$

So it follows from the analytic theorem of Bertini that there exists a non-empty Zariski open set Ω_α in the space of non-zero linear forms from $\mathbb{C}^N$ to $\mathbb{C}$ such that for any $\ell \in \Omega_\alpha$ one has that the singular points $\Sigma_{C_{\ell,\alpha}}$ of $C_{\ell,\alpha}$ are contained in the union of the set of the points where the determinants above are zero and of the singular locus of S_α. That is:

$$\Sigma_{C_{\ell,\alpha}} \subset \left(\text{Crit}\ \left(f_{|\overline{S}_\alpha}\right) \cup \Sigma_{\overline{S}_\alpha}\right) \cap S_\alpha.$$

Since this intersection is contained in $f^{-1}(0) \cap S_\alpha$, which is empty, it follows that $C_{\ell,\alpha}$ is either smooth or empty.

Now, since $\phi_\ell^{-1}(0,0) \cap (\Gamma_\alpha, 0) \subset \{0\}$, it follows from the geometric version of the Weierstrass preparation theorem given in [8] that the restriction of ϕ_ℓ to the germ $(\Gamma_\alpha, 0)$ is finite. So the finite mapping theorem (Theorem 20) implies that the image Δ_α of the analytic set Γ_α by ϕ_ℓ is a complex curve.

Finally, notice that there is a finite number of indices $\alpha \in A$ such that S_α is not contained in $f^{-1}(0)$ and such that 0 is contained in $\overline{S}_\alpha$. Let A_0 be the finite subset of A formed by such indices. So the set:

$$\Omega := \bigcap_{\alpha \in A_0} \Omega_\alpha$$

is the desired non-empty Zariski open set in the space of non-zero linear forms from $\mathbb{C}^N$ to $\mathbb{C}$. □

For any $\ell \in \Omega$ we say that the germ of curve at 0 given by:

$$\Gamma_\ell := \bigcup_{\alpha \in A} \Gamma_\alpha$$

is the *polar curve of f relatively to ℓ at* 0 and that the germ of curve at 0 given by:

$$\Delta_\ell := \bigcup_{\alpha \in A} \Delta_\alpha$$

is the *polar discriminant of f relatively to ℓ at* 0.

From now on, we fix a linear form $\ell \in \Omega$ and we set $\phi := \phi_\ell$, $\Gamma := \Gamma_\ell$ and $\Delta := \Delta_\ell$.

Notice that there exists an open neighborhood U of 0 in $\mathbb{C}^2$ and a representative X of $(X, 0)$ such that the map $\phi = (\ell, f) : X \to U$ is stratified and such that it induces a stratified submersion $X \backslash \phi^{-1}(\Delta) \to \phi(X) \backslash \Delta$.

From now on we will assume that f has an isolated singularity at 0 relatively to the stratification $\mathcal{S}$. So by an analytic version of Corollary 2.8 of [16], there exist ϵ and η_2 small enough positive real numbers with $0 < \eta_2 \ll \epsilon \ll 1$ such that, for any $t \in \mathbb{D}_{\eta_2}$, the sphere $\mathbb{S}_\epsilon$ of radius ϵ around 0 intersects $f^{-1}(t) \cap S_\alpha$ transversally, for any $\alpha \in A$.

We can also choose the linear form ℓ in such a way that there exists η_1 sufficiently small, with $0 < \eta_2 \ll \eta_1 \ll \epsilon \ll 1$, such that $\phi^{-1}(s, t) \cap S_\alpha = \ell^{-1}(s) \cap f^{-1}(t) \cap S_\alpha$ intersects $\mathbb{S}_\epsilon$ transversally, for any $(s, t) \in \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}$, where $\mathbb{D}_{\eta_1}$ and $\mathbb{D}_{\eta_2}$ are the closed disks in $\mathbb{C}$ centered at 0 and with radii η_1 and η_2, respectively.

So we have:

Proposition 22 *The map $\phi = (\ell, f)$ induces a stratified submersion:*

$$\phi_| : \mathbb{B}_\epsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta.$$

In particular, the first isotopy lemma of Thom–Mather (Lemma 19) gives:

Corollary 23 *The restriction $\phi_|$ above is a topological locally trivial fibration.*

Therefore the curve Δ plays the role of a local topological discriminant for the stratified map ϕ.

For any t in the disk $\mathbb{D}_{\eta_2}$ set:

$$D_t := \mathbb{D}_{\eta_1} \times \{t\}.$$

If $t \neq 0$, the Milnor fiber $f^{-1}(t) \cap \mathbb{B}_\epsilon$ of f is homeomorphic to $\phi^{-1}(D_t) \cap \mathbb{B}_\epsilon$ (see Theorem 2.3.1 of [14]). So, in order to simplify notation, we reset:

$$X_t := \phi^{-1}(D_t) \cap \mathbb{B}_\epsilon.$$

Notice that with this notation, the boundary ∂X_t of X_t is given by the union of $\phi^{-1}(\mathring{D}_t) \cap \mathbb{S}_\epsilon$ and $\phi^{-1}(\partial D_t) \cap \mathbb{B}_\epsilon$.

By Lemma 10 together with Corollary 11, the (complex analytic) Whitney stratification $\mathcal{S}$ of X induces a (subanalytic) Whitney stratification $\mathcal{S}(t)$ of X_t. Precisely, the strata of such stratification are the following intersections, for $\alpha \in A$:

(i) $\mathcal{S}_\alpha \cap (X_t \backslash \partial X_t)$
(ii) $\mathcal{S}_\alpha \cap \phi^{-1}(\mathring{D}_t) \cap \mathbb{S}_\epsilon$
(iii) $\mathcal{S}_\alpha \cap \phi^{-1}(\partial D_t) \cap \mathring{\mathbb{B}}_\epsilon$
(iv) $\mathcal{S}_\alpha \cap \phi^{-1}(\partial D_t) \cap \mathbb{S}_\epsilon$

Now, for any $t \in \mathbb{D}^*_{\eta_2}$ one has that ϕ induces a stratified map:

$$\ell_t : X_t \to D_t.$$

By construction, the restriction of ℓ_t to each stratum of X_t is a submersion at any point that is not in Γ. Therefore it induces a locally trivial fibration over $D_t \backslash (\Delta \cap D_t)$. That is, if we set:

$$\Delta \cap D_t = \{y_1(t), \ldots, y_k(t)\}$$

then the restriction of ℓ_t given by:

$$\varphi_t : X_t \backslash \ell_t^{-1}\big(\{y_1(t), \ldots, y_k(t)\}\big) \to D_t \backslash \{y_1(t), \ldots, y_k(t)\}$$

is a stratified submersion (see Definition 15) and a locally trivial fibration, by Thom–Mather first isotopy lemma (Lemma 19).

We notice that at this moment the parameter t is fixed, and the points $y_j(t)$ are numbered in an arbitrary way, as there is no natural way to do it. But in Sect. 4.2 and 6.2 those points will be defined in a continuous manner for t varying on a closed semi-disk $\mathbb{D}^+ \subset \mathbb{D}_{\eta_2}$ as in Fig. 1 above.

Remark 24 In the case that Γ is empty, one has that:

$$\phi_| : \mathbb{B}_\epsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}$$

is a locally trivial topological fibration, which implies a locally trivial topological fibration $\ell_t : X_t \to D_t$. Hence in this case the Milnor fiber X_t is homeomorphic to the product of D_t and the general fiber of ℓ_t.

So from now on we shall assume that the polar curve Γ is not empty.

4 The two-dimensional case

We shall prove Theorem 2 by induction on the dimension n of the analytic space X. We could start by proving the theorem for $n = 1$ and then proceed by induction for $n \geq 2$, but we choose to start with the 2-dimensional case, in order to provide the reader a better intuition of the constructions.

So in this section we prove Theorem 2 when $(X, 0)$ is a 2-dimensional reduced equidimensional germ of complex analytic space and $f: (X, 0) \to (\mathbb{C}, 0)$ has an isolated singularity at 0 in the stratified sense.

One particularity of this 2-dimensional case is that the singular set Σ of X has dimension at most one. If Σ has dimension one, we can put it inside the polar curve Γ. More precisely, only in this section we denote by Γ the union of the polar curve of f with Σ. We also denote by Δ the union of the polar discriminant of f with $\phi(\Sigma)$. Notice that if Δ is not empty, then it is a complex curve.

In order to make the constructions easier to understand, we will proceed in three steps. The first step will be to construct a polyhedron P_t in X_t, which we call a *vanishing polyhedron*, and a simplicial map $\tilde{\xi}_t$ in X_t, for any $t \in \mathbb{D}_{\eta_2}\backslash\{0\}$ fixed. In the second step, we do the construction of P_t and $\tilde{\xi}_t$ simultaneously, for t varying in a closed semi-disk $\mathbb{D}^+$ of $\mathbb{D}_{\eta_2}$ as in Fig. 1 above (with $0 \in \partial\mathbb{D}^+$). This will give a polyhedron P^+ in $X^+ := X \cap f^{-1}(\mathbb{D}^+) \cap \mathbb{B}_\epsilon$. In the third step, we will construct the map $\psi_t : X_t \to X_0$, for any $t \in \mathbb{D}^+\backslash\{0\}$. We call ψ_t a *collapsing map for* f.

4.1 First step: constructing the vanishing polyhedron P_t and the map $\tilde{\xi}_t$

Let us fix $t \in \mathbb{D}^*_{\eta_2}$ fixed. We recall that:

$$\Delta \cap D_t = \{y_1(t), \ldots, y_k(t)\}.$$

Let λ_t be a point in $D_t\backslash\{y_1(t), \ldots, y_k(t)\}$ and for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be the line segment in D_t starting at λ_t and ending at $y_j(t)$. We can choose λ_t in such a way that any two of these line segments intersect only at λ_t.

Set:

$$Q_t := \bigcup_{j=1}^{k} \delta(y_j(t))$$

and

$$P_t := \ell_t^{-1}(Q_t).$$

Since ℓ_t is finite, one can see that P_t is a one-dimensional polyhedron in X_t (see Sect. 2.4). And since the map $\varphi_t : X_t\backslash\ell_t^{-1}\big(\{y_1(t), \ldots, y_k(t)\}\big) \to D_t\backslash\{y_1(t), \ldots, y_k(t)\}$ is a stratified submersion, the interior of each 1-simplex of P_t is contained in some stratum $X_t \cap \mathcal{S}_\alpha$ of X_t, so P_t is adapted to the stratification $\mathcal{S}$ (see Definition 18).

We shall call P_t a *vanishing polyhedron for* f.

Recall that in this 2-dimensional case, by definition, the curve Γ contains the singular set Σ of X, so P_t contains the intersection $\Sigma \cap X_t$. Hence $X_t\backslash P_t$ is a smooth manifold.

Lemma 25 *There exists a subanalytic Whitney stratification $\mathcal{Z} = (Z_\beta)_{\beta\in B}$ of D_t with the property (w), and a continuous vector field $\vec{v}_t$ on D_t such that:*

1. *It is non-zero on $D_t\backslash Q_t$;*
2. *It vanishes on Q_t;*
3. *It is transversal to ∂D_t and points inwards;*
4. *It restricts to a rugose stratified vector field on the interior $\mathring{D}_t$ of D_t (relatively to the stratification $\mathcal{Z}$);*
5. *The associated flow $q_t : [0, \infty) \times (D_t\backslash Q_t) \to D_t\backslash Q_t$ defines a map:*

$$\begin{aligned} \xi_t \ : \ \partial D_t &\longrightarrow Q_t \\ u &\longmapsto \lim_{\tau\to\infty} q_t(\tau, u), \end{aligned}$$

such that ξ_t is continuous, simplicial (as defined in Sect. 2.4*) and surjective.*

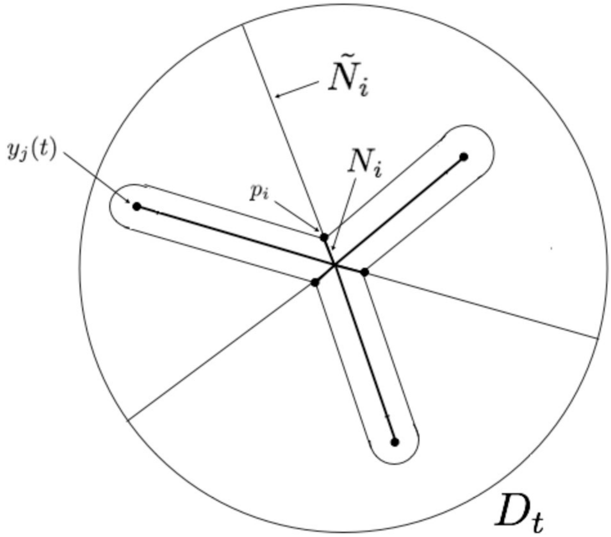


Fig. 2 The line-segments $\tilde{N}_i$

Proof Let $d_t : D_t \to \mathbb{R}$ be the function given by the distance to the set Q_t, that is $d_t(x) := d(x, Q_t)$. Consider the small closed neighborhood of Q_t in D_t given by:

$$\mathcal{R}_t := (d_t)^{-1}\big([0, r]\big)$$

for some small $r > 0$. Since both Q_t and d_t are subanalytic, it follows that $\mathcal{R}_t$ is subanalytic.

Since we have k-many points $y_1(t), \ldots, y_k(t)$, it follows that there exist exactly k-many points in $\partial\mathcal{R}_t$ whose distance to the point λ_t is r. Let us call them $p_1, \ldots, p_k$. For each p_i, let N_i be the closed line-segment in $\mathcal{R}_t$ that joins the points p_i and λ_t, and let $\tilde{N}_i$ be the closed line-segment in D_t that contains both the points λ_t and p_i, and also some point of the boundary of D_t. See Fig. 2.

We consider the Whitney stratification $\mathcal{Z}$ of D_t with the property (w) that has the smallest number of strata, such that Q_t and the line-segments $\tilde{N}_i$ are union of strata.

We will prove the lemma in two steps.

First step: We will endow $\mathcal{R}_t$ with a vector field $\vec{v}_1$, as follows:

For each point $y_j(t)$, let L_j be the line-segment in $\mathcal{R}_t$ that is ortogonal to the line-segment $\delta(y_j(t))$ at $y_j(t)$. Notice that the line-segments $L_1, \ldots, L_k$ together with the line-segments $N_1, \ldots, N_k$ give a decomposition of $\mathcal{R}_t$ in $2k$-many polygons $R_1, \ldots, R_{2k}$ that contain the point λ_t and k-many semi-disks $M_1, \ldots, M_k$ such that $y_j(t) \in M_j$, for $j = 1, \ldots, k$. See Fig. 3.

Now we endow each polygon R_m, for $m = 1, \ldots, 2k$, with a vector field ω_m as follows:

Let h_1 be the combination of the rotation and the translation in $\mathbb{R}^2$ that takes λ_t to the origin and that takes the line-segment $\delta(y_j(t))$ contained in R_m to a line-segment $[x_0, 0]$ in the first coordinate-axis. Notice that h_1 takes the line-segment N_i contained in R_m to a line-segment contained in the real line of equation $y = \alpha x$, for some $\alpha \in \mathbb{R}$. See Fig. 4.

Also, let h_2 be the diffeomorphism from the rectangle $[x_0, 0] \times [0, r]$ onto $h_1(R_m)$ given by:

$$h_2(x, y) := \left(x + \alpha\frac{(x_0 - x)}{x_0}y, y\right).$$

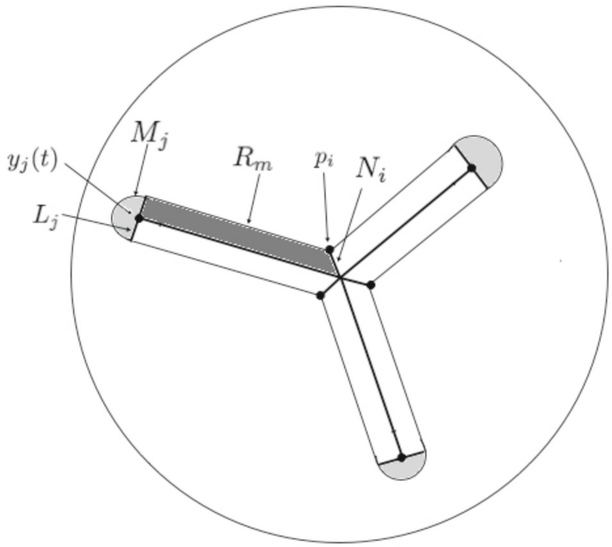


Fig. 3 Decomposition of $\mathcal{R}_t$

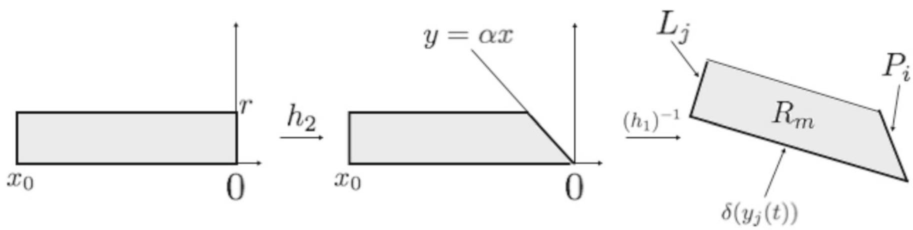


Fig. 4 The diffeomorphism h

Setting $h := h_1^{-1} \circ h_2$, we have that $h : [x_0, 0] \times [0, r] \to R_m$ is a diffeomorphism such that:

- h takes $[x_0, 0] \times \{0\}$ onto $\delta(y_j(t))$;
- h takes $\{x_0\} \times [0, r]$ onto L_j;
- h takes $\{0\} \times [0, r]$ onto N_i.

Now let $\rho : [x_0, 0] \times [0, r] \to \mathbb{R}$ be the function given by the square of the distance to $[x_0, 0] \times \{0\}$, that is, $\rho(x, y) := y^2$, and let $\vec{\omega}$ be the vector field in $[x_0, 0] \times [0, r]$ given by the opposite of the gradient vector field associated to ρ. Notice that $\vec{\omega}$ is continuous, smooth outside $[x_0, 0] \times \{0\}$, non-zero outside $[x_0, 0] \times \{0\}$ and zero on $[x_0, 0] \times \{0\}$.

The differential of h takes $\vec{\omega}$ to a vector field $\vec{\omega}_m$ on R_m that is smooth and non-zero outside $\delta(y_j(t))$ and zero on $\delta(y_j(t))$.

Moreover, the restriction of $\vec{\omega}_m$ to the line-segment L_j coincides with the vector field on L_j given by the opposite of the gradient of the square of the function given by the distance to the point $y_j(t)$. Also, the restriction of $\vec{\omega}_m$ to the line-segment N_i coincides with the vector field on N_i given by the opposite of the gradient of the square of the function given by the distance to the point λ_t. See Fig. 5.

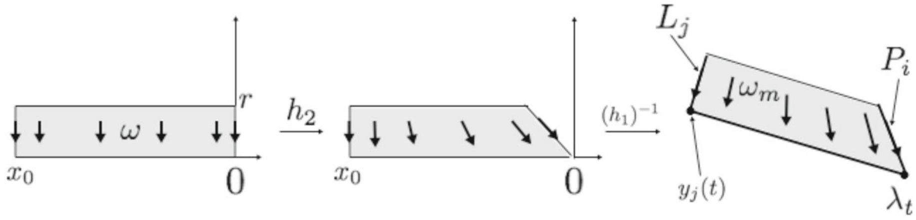


Fig. 5 The vector field $\vec{\omega}_m$

On the other hand, we endow each semi-disk M_j, for $j = 1, \dots, k$, with the vector field $\vec{w}_j$ given by the opposite of the gradient of the square of the function given by the distance to the point $y_j(t)$.

So putting all the polygons R_m and all the semi-disks M_j together, each of them endowed with the corresponding vector field $\vec{\omega}_m$ or $\vec{w}_j$, we get a continuous vector field $\vec{v}_1$ on $\mathcal{R}_t$ such that:

- $\vec{v}_1$ is smooth on each stratum of the stratification of $\mathcal{R}_t$ induced by $\mathcal{Z}$;
- $\vec{v}_1$ is non-zero outside Q_t;
- $\vec{v}_1$ is zero on Q_t;
- $\vec{v}_1$ never points in the direction of the gradient of the function given by the square of the distance to the point λ_t.

Let us show that:

- $\vec{v}_1$ is a rugose vector field.

Set $W_1 := D_t \backslash (Q_t \cup \tilde{N}_1 \cup \cdots \cup \tilde{N}_k)$ and set $W_2 := Q_t \cup \tilde{N}_1 \cup \cdots \cup \tilde{N}_k$.

If $z \in W_2 \cap \mathcal{R}_t$ is not λ_t, let U be a neighborhood of z in D_t that is contained in the union of two polygons R_m and $R_{m'}$. Then for any $z_1 \in W_1 \cap U$ and $z_2 \in W_2 \cap U$, we have that the line-segment $\overline{z_1 z_2}$ that joints z_1 and z_2 is contained either in R_m or in $R_{m'}$. Let us suppose that it is contained in R_m. Then we have:

$$\|\vec{\omega}_m(z_1) - \vec{\omega}_m(z_2)\| \leq K_m \|z_1 - z_2\|,$$

where K_m is the Lipschitz constant of the Lipshitz vector field $\vec{\omega}_m$.

If $z = \lambda_t$, let U be the open ball around λ_t of radius r. Given $z_1 \in W_1 \cap U$ and $z_2 \in W_2 \cap U$, consider the line-segment $\overline{z_1 z_2}$, starting from z_1 and going to z_2. It intersects the set Q at points $q_1, \dots, q_s$, and it intersects the line-segments N_i at the points $t_1, \dots, t_{s'}$, in this order. That is:

$$\overline{z_1 z_2} = \overline{z_1 q_1 t_1 q_2 t_2 \dots q_{s-1} t_{s-1} q_s z_2} \quad \text{or} \quad \overline{z_1 z_2} = \overline{z_1 t_1 q_1 t_2 q_2 \dots t_s q_s z_2}.$$

Let us consider the first case ($\overline{z_1 z_2} = \overline{z_1 q_1 t_1 q_2 t_2 \dots q_{s-1} t_{s-1} q_s z_2}$); the second case is analogous.

Let re-order the polygons R_m in such a way that, for each $i = 1, \dots, s$, we have:

- R_0 is the polygon that contains z_1;
- R_{2i-1} is the polygon that contains the line-segment $\overline{q_i t_i}$;
- R_{2i} is the polygon that contains the line-segment $\overline{t_i q_{i+1}}$.

Setting $K := \max_{m=1,\dots,2k}\{K_m\}$, we have:

$$\|\vec{\omega}_0(z_1) - \vec{\omega}_s(z_2)\|$$

$$\leq \|\vec{\omega}_0(z_1) - \vec{\omega}_0(q_1)\| + \|\vec{\omega}_1(q_1) - \vec{\omega}_1(t_1)\| + \|\vec{\omega}_2(t_1) - \vec{\omega}_2(q_2)\| + \|\vec{\omega}_3(q_2) - \vec{\omega}_3(t_2)\| + \cdots$$
$$\cdots + \|\vec{\omega}_{2s-2}(t_{s-1}) - \vec{\omega}_{2s-2}(q_s)\| + \|\vec{\omega}_{2s-1}(q_s) - \vec{\omega}_{2s-1}(z_2)\|$$
$$\leq K_0\|z_1 - q_1\| + K_1\|q_1 - t_1\| + K_2\|t_1 - q_2\| + \cdots + K_{2s-2}\|t_{s-1} - q_s\| + K_{2s-1}\|q_s - z_2\|$$
$$\leq 2sK\|z_1 - z_2\| \leq 2kK\|z_1 - z_2\|.$$

Second step: Constructing the vector field $\vec{v}_t$ on D_t.

Let r' be a small real number with $0 < r' < r$ and with $r - r' \ll 1$, and set $\mathcal{R}'_t := (d_t)^{-1}\big([0, r']\big)$, so $\mathcal{R}'_t \subset \mathcal{R}_t$. We endow $D_t \backslash int(\mathcal{R}'_t)$ with the vector field $\vec{v}_2$ given by the opposite of the gradient vector field of the function on $D_t \backslash int(\mathcal{R}'_t)$ given by the square of the distance to the point λ_t.

Since the vector fields $\vec{v}_1$ and $\vec{v}_2$ never have opposite directions, the vector field $\vec{v}_t$ is obtained by gluing the vector fields $\vec{v}_1$ and $\vec{v}_2$, using a partition of unity. That is, we consider a pair (ρ_1, ρ_2) of continuous functions from the compact disk D_t to the closed unit interval [0, 1] such that:

- for every point $p \in D_t$ one has that $\rho_1(p) + \rho_2(p) = 1$,
- the support of ρ_1 is contained in $int(\mathcal{R}_t)$,
- the support of ρ_2 is contained in $D_t \backslash \mathcal{R}'_t$.

Hence for any $p \in \mathcal{R}'_t$ we have that $\big(\rho_1(p), \rho_2(p)\big) = (1, 0)$ and for any $p \in D_t \backslash int(\mathcal{R}_t)$ we have that $\big(\rho_1(p), \rho_2(p)\big) = (0, 1)$. So we set $\vec{v}_t := \rho_1\vec{v}_1 + \rho_2\vec{v}_2$.

Clearly, $\vec{v}_t$ is a continuous vector field on D_t that is non-zero on $D_t \backslash Q_t$, zero on Q_t and transversal to ∂D_t, pointing inwards, and that restricts to a rugose stratified vector field on the interior of D_t (relatively to the stratification $\mathcal{Z}$ of D_t). Moreover, each orbit associated to $\vec{v}_t$ has a limit point in Q_t.

So the flow $q_t : [0, \infty) \times (D_t \backslash Q_t) \to D_t \backslash Q_t$ associated to $\vec{v}_t$ defines a continuous, simplicial and surjective map:

$$\begin{aligned} \xi_t \ : \partial D_t &\longrightarrow Q_t \\ u &\longmapsto \lim_{\tau \to \infty} q_t(\tau, u). \end{aligned}$$

□

Remark 26 Lemma 25 is still true if the set Q_t is taken as the union of simple paths that intersect only at a point λ_t, instead of considering line-segments. It is enough to consider a suitable homeomorphism of the disk D_t onto itself which is a diffeomorphism outside the point λ_t and which sends the union of those paths to a union of line-segments.

Now recall the proper map $\ell_t : X_t \to D_t$ and recall that X_t has a subanalytic Whitney stratification $\mathcal{S}(t)$ with the property (w) induced by the Whitney stratification $\mathcal{S}$ of X. By Remark 12, we can consider a refinement $\mathcal{S}'(t)$ of $\mathcal{S}(t)$ and a refinement $\mathcal{Z}' = (Z'_\beta)_{\beta \in B'}$ of $\mathcal{Z}$ such that:

(i) $\ell_t^{-1}(Z'_\beta)$ is a union of strata of $\mathcal{S}'(t)$;
(ii) the restriction $\ell_{t|} : \ell_t^{-1}(Z'_\beta) \to Z'_\beta$ is transversal to $\mathcal{S}'(t) \cap f^{-1}(Z'_\beta)$.

One has:

Proposition 27 *We can choose a lifting of the vector field $\vec{v}_t$ of Lemma 25 to a continuous vector field $\vec{\vartheta}_t$ on X_t so that:*

1. *It is non-zero outside P_t and zero on P_t;*

2. *It is transversal to ∂X_t and points inwards;*
3. *It restricts to a stratified rugose vector field on the interior of X_t (relatively to the stratification $\mathcal{S}'(t)$);*
4. *The flow $\tilde{q}_t : [0, \infty) \times (X_t \backslash P_t) \to X_t$ associated to $\vec{\vartheta}_t$ defines a map:*

$$\begin{aligned} \tilde{\xi}_t \ : \ \partial X_t &\longrightarrow P_t \\ z &\longmapsto \lim_{\tau \to \infty} \tilde{q}_t(\tau, z) \end{aligned}$$

such that $\tilde{\xi}_t$ is continuous, simplicial and surjective;

5. *The fiber X_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$.*

Proof Recall from Proposition 22 that the restriction ℓ_t of the linear form ℓ to the Milnor fiber X_t induces a stratified submersion:

$$\varphi_t : X_t \backslash \ell_t^{-1}\big(\{y_1(t), \dots, y_k(t)\}\big) \to D_t \backslash \{y_1(t), \dots, y_k(t)\}.$$

So, by Proposition 17, we can lift the vector field $\vec{v}_t$ to a continuous vector field $\vec{\vartheta}_t$ in X_t that satisfies properties (1), (2) and (3).

Let us show that we can choose $\vec{\vartheta}_t$ satisfying also condition (4). Fix $z \in \partial X_t$. We want to show that $\lim_{\tau \to \infty} \tilde{q}_t(\tau, z)$ exists, that is, that there exists a point $\tilde{p} \in P_t$ such that for any open neighborhood $\tilde{U}$ of $\tilde{p}$ in X_t there exists $\theta > 0$ such that $\tau > \theta$ implies that $\tilde{q}_t(\tau, z) \in \tilde{U}$.

From Lemma 25 we know that there exists $p \in Q_t$ such that $\lim_{\tau \to \infty} q_t(\tau, \ell_t(z)) = p$, where $q_t : [0, \infty) \times (D_t \backslash Q_t) \to D_t$ is the flow associated to the vector field $\vec{v}_t$. So for any small open neighborhood U of p in D_t there exists $\theta > 0$ such that $\tau > \theta$ implies that $q_t(\tau, \ell_t(z)) \in U$. Setting $\{\tilde{p}_1, \dots, \tilde{p}_r\} := \ell_t^{-1}(p)$, we can consider U sufficiently small such that there are disjoint connected components $\tilde{U}_1, \dots, \tilde{U}_r$ of $\ell_t^{-1}(U)$ such that each $\tilde{U}_j$ contains $\tilde{p}_j$.

Since $\vec{\vartheta}_t$ is a lifting of $\vec{v}_t$, we have that $q_t(\tau, \ell_t(z)) = \ell_t(\tilde{q}_t(\tau, z))$ for any $\tau \geq 0$. So $\tau > \theta$ implies that $\ell_t^{-1}\big(\ell_t(\tilde{q}_t(\tau, z))\big) \subset \ell_t^{-1}(U)$. Hence for some $j \in \{1, \dots, r\}$ we have that $\tilde{q}_t(\tau, z) \in \tilde{U}_j$. Therefore $\lim_{\tau \to \infty} \tilde{q}_t(\tau, z) = \tilde{p}_j$. This proves (4).

Now we show that X_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$. In fact, the integration of the vector field $\vec{\vartheta}_t$ on X_t gives a surjective continuous map:

$$\alpha : [0, \infty] \times \partial X_t \to X_t$$

that restricts to a homeomorphism:

$$\alpha_| : [0, \infty) \times \partial X_t \to X_t \backslash P_t.$$

Since the restriction $\alpha_\infty : \{\infty\} \times \partial X_t \to P_t$ is equal to $\tilde{\xi}_t$, which is surjective, it follows that the induced map:

$$[\alpha_\infty] : \big((\{\infty\} \times \partial X_t)\big/ \sim\big) \to P_t$$

is a homeomorphism, where $\sim$ is the equivalence relation given by identifying $(\infty, z) \sim (\infty, z')$ if $\alpha_\infty(z) = \alpha_\infty(z')$. Hence the map:

$$[\alpha] : \big(([0, \infty] \times \partial X_t)\big/ \sim\big) \to X_t$$

induced by α defines a homeomorphism between X_t and the mapping cylinder of $\tilde{\xi}_t$. This proves (5). □

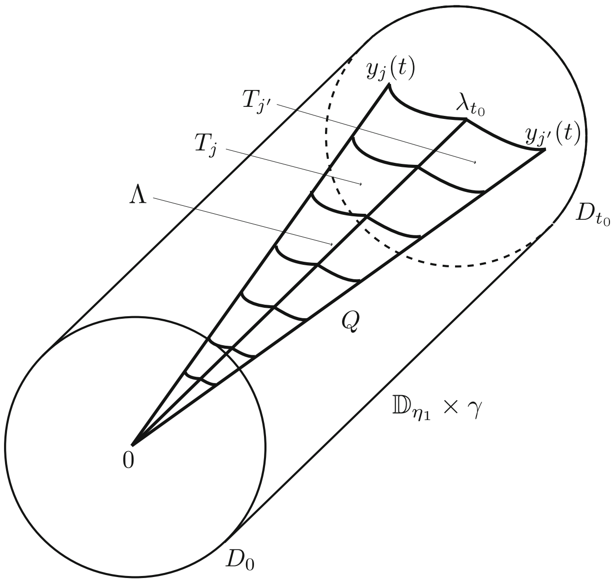


Fig. 6 The triangles T_j

4.2 Second step: constructing P^+ and $\tilde{\xi}_+$

Recall from Lemma 21 that the polar curve Γ is not contained in $f^{-1}(0)$. Hence Δ is not contained in $\mathbb{D}_{\eta_1} \times \{0\}$, and so the natural projection $\pi : \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \to \mathbb{D}_{\eta_2}$ restricted to Δ induces a ramified covering:

$$\pi_| : \Delta \to \mathbb{D}_{\eta_2}$$

of degree k, whose ramification locus is $\{0\} \subset \Delta$.

So the intersection of the polar discriminant Δ with the product $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ give semi-disks $Y_1, \ldots, Y_k$ in $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ such that Y_j projects differentially onto $\mathbb{D}^+$ outside $0 \in Y_j$, for each $j = 1, \ldots, k$. The set $\Lambda := \{0\} \times \mathbb{D}^+$ is also a semi-disk in $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$, which can be supposed to intersect Y_j only at $0 \in \mathbb{C}^2$, for any $j = 1, \ldots, k$.

We can choose the simple paths $\delta(y_1(t)), \ldots, \delta(y_k(t))$ for each $t \in \mathbb{D}^+$ in such a way that $\delta(y_j(t))$ depends continuously on the parameter $t \in \mathbb{D}^+$, for each $j = 1, \ldots, k$; and it forms a 3-dimensional triangle $\mathcal{T}_j$ in $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ bounded by the semi-disks Y_j and Λ and by the union of paths $\bigcup_{t \in \mathbb{D}^+ \cap \partial \mathbb{D}_{\eta_2}} \delta_j(y_j(t))$.

The Fig. 6 below represents the 2-dimensional triangles $T_j := \mathcal{T}_j \cap (\mathbb{D}_{\eta_1} \times \gamma)$, for $j = 1, \ldots, k$, where γ is a simple path in $\mathbb{D}^+$ going joining some $t_0 \in \mathbb{D}^+ \cap \partial \mathbb{D}_{\eta_2}$ to $0 \in \mathbb{C}^2$. It helps the reader to understand the construction of each $\mathcal{T}_j$.

Setting $Q^+ := \bigcup_{j=1}^k \mathcal{T}_j$, we define:

$$P^+ := \phi^{-1}(Q^+),$$

which is contained in $X^+ := X \cap f^{-1}(\mathbb{D}^+) \cap \mathbb{B}_\epsilon$. It is a polyhedron adapted to the stratification $\mathcal{S}^+$ of X^+ induced by $\mathcal{S}$, and the intersection $P^+ \cap X_t$ is a vanishing polyhedron P_t as in Sect. 4.1, for any $t \in \mathbb{D}^+ \setminus \{0\}$. Moreover, $P^+ \cap X_0 = \{0\}$.

Notice that the complex Whitney stratification $\mathcal{S}$ of X in fact induces a subanalytic Whitney stratification $\mathcal{S}^+$ of X^+ with the property (w). This is because $\mathcal{S}$ has the property (w), by Lemma 8, and hence one can use Lemma 10.

Now, analogously to Lemma 25, we can construct a Whitney stratification $\mathcal{Z}^+$ of $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ with the property (w), and an integrable vector field $\vec{v}_+$ in $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ that deformation retracts $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ onto Q^+. That is, we can consider a continuous vector field $\vec{v}_+$ in $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ such that:

- It is non-zero outside Q^+ and it is zero on Q^+;
- It is transversal to $\partial\mathbb{D}_{\eta_1} \times \mathbb{D}^+$;
- It restricts to a rugose stratified vector field on the interior of $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ (with respect to $\mathcal{Z}^+$);
- The projection of $\vec{v}_+$ onto $\mathbb{D}^+$ is zero;
- The flow $q_+ : [0, \infty) \times \big((\mathbb{D}_{\eta_1} \times \mathbb{D}^+)\backslash Q^+\big) \to \mathbb{D}_{\eta_1} \times \mathbb{D}^+$ associated to $\vec{v}_+$ defines a map:

$$\begin{aligned} \xi_+ \ : \partial\mathbb{D}_{\eta_1} \times \mathbb{D}^+ &\longrightarrow Q^+ \\ z &\longmapsto \lim_{\tau\to\infty} q_+(\tau, z) \end{aligned}$$

  that is continuous, simplicial and surjective.

As we did above, we can use Remark 12 to obtain a refinement $(\mathcal{S}^+)'$ of $\mathcal{S}^+$ and a refinement $(\mathcal{Z}^+)' = (Z_\beta^{+'})_{\beta\in B'}$ of $\mathcal{Z}^+$ such that:

(i) $\phi^{-1}(Z_\beta^{+'})$ is a union of strata of $(\mathcal{S}^+)'$;
(ii) the restriction $\phi_| : \phi^{-1}(Z_\beta^{+'}) \to (Z_\beta^{+'})$ is transversal to $(\mathcal{S}^+)' \cap f^{-1}(Z_\beta^{+'})$.

So by Proposition 16 we have:

Proposition 28 *The vector field $\vec{v}_+$ can be lifted to an integrable vector field $\vec{\vartheta}_+$ in X^+ such that:*

(i) *For any $t \in \mathbb{D}^+\backslash\{0\}$, the restriction of $\vec{\vartheta}_+$ to X_t gives a vector field $\vec{\vartheta}_t$ as in Proposition 27, relatively to the polyhedron $P_t = P^+ \cap X_t$.*
(ii) *The vector field $\vec{\vartheta}_+$ is non-zero on $X^+\backslash P^+$, zero on P^+, transversal to ∂X^+, pointing inwards, and it restricts to a rugose stratified vector field on the interior of X^+ (relatively to the refinement $(\mathcal{S}^+)'$).*

Analogously to the proof of (6) of Proposition 27, one can show that the flow associated to the vector field $\vec{\vartheta}_+$ defines a map $\tilde{\xi}_+$ with the desired properties.

This proves (i) and (ii) of Theorem 2 in the 2-dimensional case.

4.3 Third step: constructing the collapsing map ψ_t

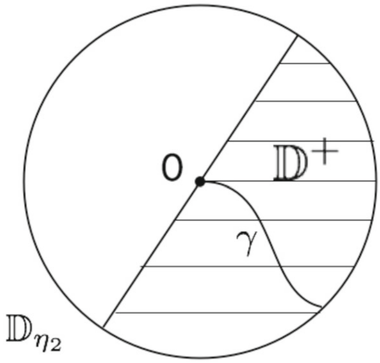
First, let us recall that $X_t := \mathbb{B}_\epsilon \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \{t\})$ and that the map:

$$\phi = (\ell, f) : X \to \mathbb{C}^2$$

induces a stratified submersion $\phi_| : \mathbb{B}_\epsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}\backslash\Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}\backslash\Delta$ (see Proposition 22).

Let γ be a simple path in $\mathbb{D}_{\eta_2}$ joining 0 and some $t_0 \in \partial\mathbb{D}_{\eta_2}$, such that γ is transverse to $\partial\mathbb{D}_{\eta_2}$. See Fig. 7. We want to describe the collapsing of f along γ, that is, how X_t degenerates to X_0 as $t \in \gamma$ goes to 0.

Fig. 7 The path γ



Recall the sets Q^+ and P^+ defined above, and set $Q_\gamma := Q^+ \cap (\mathbb{D}_{\eta_1} \times \gamma)$. It is a union of triangles $T_1, \ldots, T_k$, as in Fig. 6. Also set $P_\gamma := P^+ \cap f^{-1}(\gamma)$.

The vector field $\vec{v}_+$ constructed above restricts to a continuous stratified vector field $\vec{v}_\gamma$ in $\mathbb{D}_{\eta_1} \times \gamma$ such that:

- It is non-zero outside Q_γ and it is zero on Q_γ;
- It is transversal to $\partial\mathbb{D}_{\eta_1} \times \gamma$;
- It restricts to a rugose stratified vector field on the interior of $\mathbb{D}_{\eta_1} \times \gamma$ (relatively to the Whitney stratification $\mathcal{Z}^+ \cap (\mathbb{D}_{\eta_1} \times \gamma)$ (with the property (w)) induced by $\mathcal{Z}^+$);
- The projection of $\vec{v}_\gamma$ onto γ is zero;
- The flow $q_\gamma : [0, \infty) \times \big((\mathbb{D}_{\eta_1} \times \gamma)\backslash Q_\gamma\big) \to \mathbb{D}_{\eta_1} \times \gamma$ associated to $\vec{v}_\gamma$ defines a map:

$$\begin{aligned} \xi_\gamma \ : \partial\mathbb{D}_{\eta_1} \times \gamma &\longrightarrow Q_\gamma \\ z &\longmapsto \lim_{\tau \to \infty} q_\gamma(\tau, z) \end{aligned}$$

  that is continuous, simplicial and surjective.

Now, for any real number $A > 0$ set:

$$V_A(Q_\gamma) := (\mathbb{D}_{\eta_1} \times \gamma)\backslash q_\gamma\big([0, A) \times \partial\mathbb{D}_{\eta_1} \times \gamma\big),$$

which is a closed neighborhood of Q_γ in $\mathbb{D}_{\eta_1} \times \gamma$. This gives a system of closed neighborhoods of Q_γ in $\mathbb{D}_{\eta_1} \times \gamma$, such that:

(i) The boundary $\partial V_A(Q_\gamma)$ of $V_A(Q_\gamma)$ is a stratified topological manifold, for any $A \geq 0$, since it is the image of $\partial\mathbb{D}_{\eta_1} \times \gamma$ by ξ_γ;
(ii) $V_0(Q_\gamma) = \mathbb{D}_{\eta_1} \times \gamma$;
(iii) For any $A_1 > A_2$ one has $V_{A_1}(Q_\gamma) \subset V_{A_2}(Q_\gamma)$;
(iv) For any open neighborhood U of Q_γ in $\mathbb{D}_{\eta_1} \times \gamma$, there exists $A_U \geq 0$ sufficiently big such that $V_{A_U}(Q_\gamma)$ is contained in U.

Notice that $\partial V_A(Q_\gamma)$ fibers over γ with fiber a circle, by the restriction of the projection $\pi : \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \to \mathbb{D}_{\eta_2}$.

Now, setting:

$$X_\gamma := X \cap f^{-1}(\gamma) \cap \mathbb{B}_\epsilon,$$

we can finally construct a stratified rugose vector field $\vec{\zeta}_\gamma$ in $X_\gamma \backslash P_\gamma$ (relatively to the stratification $\mathcal{S}'(\gamma)$ of X_γ induced by the stratification $(\mathcal{S}^+)'$ of X^+) whose flow gives the degeneration of X_{t_0} to X_0. We will say that $\vec{\zeta}_\gamma$ is a *collapsing vector field* that gives the degeneration of f along the path γ.

Recall that the restriction:

$$\phi_| : \phi^{-1}\big((\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2})\backslash Q\big) \to (\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2})\backslash Q$$

is a stratified submersion (relatively to the stratification $\mathcal{S}$), and that the restriction of $\pi \circ \phi$ to $\phi^{-1}\big(\partial V_A(Q_\gamma)\big) \cap \mathbb{B}_\epsilon$ is a proper locally trivial fibration over γ.

Let $\vec{\theta}$ be a vector field on γ that goes from t_0 to 0 in time $a > 0$ and fix $A > 0$. We are going to construct a smooth and integrable vector field $\vec{\zeta}_\gamma$ in $X_\gamma \backslash P_\gamma$ that lifts $\vec{\theta}$ outside $\{0\}$, and such that $\vec{\zeta}_\gamma$ is tangent to $\phi^{-1}\big(\partial V'_A(Q_\gamma)\big)$, for any $A' \geq A$. We will construct it locally, that is, for each point $p \in X_\gamma \backslash P_\gamma$ we will construct a vector field $\vec{\zeta}_p$ in some neighborhood U_p of p, and then we will glue all of them using a partition of unity associated to the covering given by the neighborhoods U_p (see Lemma 41.6 of [18] for the proof of the existence of a partition of unity associated to an infinite covering, since $X_\gamma \backslash P_\gamma$ is not compact).

Each $\vec{\zeta}_p$ is constructed in the following way:

(a) If $p \notin \phi^{-1}\big(V_A(Q_\gamma)\big) \cap \mathbb{B}_\epsilon$, there is an open neighborhood U_p of p in X_γ that does not intersect the closed set $\phi^{-1}\big(V_A(Q_\gamma)\big) \cap \mathbb{B}_\epsilon$. Then we define a smooth vector field $\vec{\zeta}_p$ on U_p that lifts $\vec{\theta}$.

(b) If $p \in \big[\phi^{-1}\big(V_A(Q_\gamma)\big) \cap \mathbb{B}_\epsilon\big] \backslash P_\gamma$, there is an open neighborhood U_p of p in X_γ that does not intersect P_γ. We define a smooth vector field $\vec{\zeta}_p$ on U_p that lifts $\vec{\theta}$ and that is tangent to each stratum of $\phi^{-1}\big(\partial V_{A'}(Q_\gamma)\big) \cap \mathbb{B}_\epsilon$, for any $A' \geq A$. This is possible because the restriction of $\pi \circ \phi$ to $\phi^{-1}\big(V_A(Q_\gamma)\backslash Q_\gamma\big) \cap \mathbb{B}_\epsilon$ is a stratified submersion, which restricts to a locally trivial fibration $\phi^{-1}\big(\partial V_{A'}(Q_\gamma)\big) \cap \mathbb{B}_\epsilon \to \gamma$, for each $A' \geq A$.

Then the collapsing vector field $\vec{\zeta}_\gamma$ is obtained by gluing the vector fields $\vec{\zeta}_p$ using a partition of unity. Notice that $\vec{\zeta}_\gamma$ lifts $\vec{\theta}$ outside $\{0\}$.

Hence the flow $h : [0, a] \times X_\gamma \backslash P_\gamma \to X_\gamma \backslash P_\gamma$ associated to $\vec{\zeta}_\gamma$ defines a stratified homeomorphism $\tilde{\psi}_{t_0}$ from $X_{t_0} \backslash P_{t_0}$ to $X_0 \backslash \{0\}$.

Moreover, we can show that the extension $\psi_{t_0} : X_{t_0} \to X_0$ given by:

$$\psi_{t_0}(x) := \begin{cases} \tilde{\psi}_{t_0}(x) & \text{if} \quad x \in X_{t_0} \backslash P_{t_0} \\ 0 & \text{if} \quad x \in P_{t_0} \end{cases}$$

is continuous. It is enough to show that if $(x_r)_{r \in \mathbb{R}}$ is a sequence of points in $X_{t_0} \backslash P_{t_0}$ such that $x_r \in \phi^{-1}\left(\partial V_r(Q_\gamma)\right)$, for r sufficiently large (so x_r converges to a point $x \in P_{t_0}$), then the sequence of points $\left(\tilde{\psi}_{t_0}(x_r)\right)_{r \in \mathbb{R}}$ in X_0 converges to 0. Since the collapsing vector field $\vec{\zeta}_\gamma$ is tangent to $\phi^{-1}\left(\partial V_r(Q_\gamma)\right)$, for each r sufficiently large, it follows that:

$$\tilde{\psi}_{t_0}(x_r) \in \phi^{-1}\left(\partial V_r(Q_\gamma)\right) \cap X_0.$$

On the other hand, by the condition (iv) above, we have that the system of neighborhoods $V_r(Q_\gamma)$ is such that $\cap_{r \in \mathbb{N}} V_r(Q_\gamma) = Q_\gamma$, so $\phi^{-1}\left(\partial V_r(Q_\gamma)\right) \cap X_0$ goes to $\{0\}$ when r goes to infinity, in the sense of condition (iv) above. Precisely, for any open neighborhood U of 0 in X_0 there exists $R > 0$ sufficiently large such that $\phi^{-1}\left(\partial V_R(Q_\gamma)\right) \cap X_0$ is contained in U. Therefore, the sequence of points $(\tilde{\psi}_{t_0}(x_r))_{r \in \mathbb{R}}$ in X_0 converges to 0.

This finishes the proof of Theorem 2 in the 2-dimensional case.

5 Elements of the proof of the main theorem

Now we go back to the general case of a germ of complex analytic function

$$f : (X, x) \to (\mathbb{C}, 0)$$

at a point x of a reduced equidimensional complex analytic space $X \subset \mathbb{C}^N$ of any dimension. Let $\mathcal{S} = (S_\alpha)_{\alpha \in A}$ be a Whitney stratification of X and suppose that f has an isolated singularity at x in the stratified sense.

In order to simplify the notations, suppose further that x is the origin in $\mathbb{C}^N$.

Recall the polar curve Γ of f relatively to a generic linear form ℓ, as well as the polar discriminant $\Delta := \phi(\Gamma)$, where ϕ is the stratified map:

$$\phi := (\ell, f) : (X, 0) \to (\mathbb{C}^2, 0).$$

As before, we assume that the polar curve Γ is non-empty.

In the next section, we will prove the following proposition:

Proposition 29 *For any $t \in \mathbb{D}^*_{\eta_2}$, there exists a refinement $\mathcal{S}'(t)$ of the Whitney stratification $\mathcal{S}(t)$ of X_t (with the property (w)) induced by $\mathcal{S}$, such that there are:*

- (i) *A polyhedron P_t of dimension $\dim_{\mathbb{C}} X - 1$ that is contained in the Milnor fiber $X_t := \mathbb{B}_\epsilon \cap f^{-1}(t)$ and that is adapted to the stratification $\mathcal{S}'(t)$;*
- (ii) *A continuous vector field $\vec{\vartheta}_t$ in X_t so that:*
  1. *It is non-zero outside P_t and it is zero on P_t;*
  2. *It is transversal to ∂X_t (in the stratified sense) and pointing inwards;*
  3. *It restricts to a rugose stratified vector field on the interior of X_t (relatively to the stratification $\mathcal{S}'(t)$)*
  4. *The flow $\tilde{q}_t : [0, \infty) \times (X_t \backslash P_t) \to X_t$ associated to $\vec{\vartheta}_t$ defines a map:*

$$\begin{aligned} \tilde{\xi}_t \ :\ \partial X_t &\longrightarrow P_t \\ z &\longmapsto \lim_{\tau \to \infty} \tilde{q}_t(\tau, z) \end{aligned}$$

  such that $\tilde{\xi}_t$ is continuous, stratified, simplicial and surjective;

  5. *The Milnor fiber X_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$.*

We say that the polyhedron P_t above is a *vanishing polyhedron for f*.

The idea of the construction of P_t is quite simple and we will briefly describe it here. First recall the stratified map $\ell_t : X_t \to D_t$ given by the restriction of ϕ to X_t.

By induction hypothesis, we have a vanishing polyhedron P'_t for the restriction of f to the hyperplane section $X \cap \{\ell = 0\}$.

For each point $y_j(t)$ in the intersection of the polar discriminant Δ with the disk $D_t := \mathbb{D}_{\eta_1} \times \{t\}$ as above, let $x_j(t)$ be a point in the intersection of the polar curve Γ with $\ell_t^{-1}\big(y_j(t)\big)$. To simplify, we can assume that $x_j(t)$ is the only point in such intersection (see Conjecture 34 below).

Also by the induction hypothesis, we have a collapsing cone P_j for the restriction of the map ℓ_t to a small neighborhood of $x_j(t)$. The "basis" of a such cone is the polyhedron $P_j(a_j) := P_j \cap \ell_t^{-1}(a_j)$, where a_j is a point in $\delta(y_j(t))\backslash y_j(t)$ close to $y_j(t)$.

Since ℓ_t is a locally trivial fiber bundle over $\delta(y_j(t))\backslash y_j(t)$, we can "extend" the cone P_j until it reaches the "central" polyhedron P'_t. This gives a polyhedron C_j. The union of all the polyhedra C_j together with P'_t gives our vanishing polyhedron P_t (see Fig. 8).

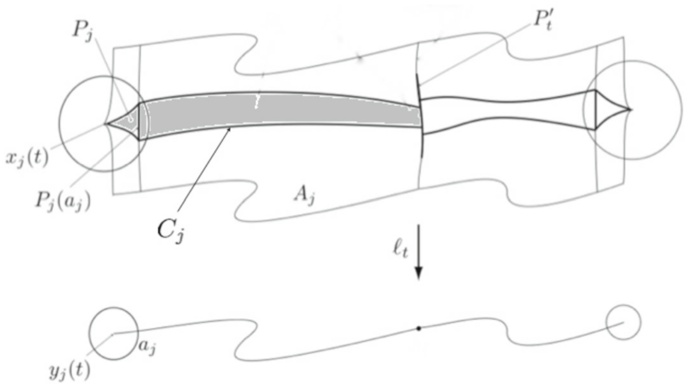


Fig. 8 The vanishing polyhedron P_t

The detailed construction of P_t will be given in the next section. We will also show in the next section that the construction of the polyhedron P_t and the vector field $\vec{\vartheta}_t$ can be done simultaneously, for any t in a closed semi-disk $\mathbb{D}^+$ of $\mathbb{D}_{\eta_2}$ such that 0 is in its boundary, as in Fig. 1.

Precisely, we will prove the following:

Proposition 30 *Let $\mathbb{D}^+$ be a closed semi-disk in $\mathbb{D}_{\eta_2}$ such that 0 is in its boundary, and set $X^+ := X \cap f^{-1}(\mathbb{D}^+) \cap \mathbb{B}_\epsilon$. There exists a refinement $(\mathcal{S}^+)'$ of the Whitney stratification $\mathcal{S}^+$ of X^+ (with the property (w)) induced by the stratification $\mathcal{S}$ of X such that there are a polyhedron P^+, adapted to the stratification $(\mathcal{S}^+)'$, and an integrable vector field $\vec{\vartheta}_+$ in X^+, so that:*

- (i) *the intersection $P^+ \cap X_t$ is a vanishing polyhedron P_t as in Proposition 29, for any $t \in \mathbb{D}^+ \backslash \{0\}$, and $P^+ \cap X_0 = \{0\}$;*
- (ii) *for any $t \in \mathbb{D}^+ \backslash \{0\}$, the restriction of $\vec{\vartheta}_+$ to X_t gives a vector field $\vec{\vartheta}_t$ as in Proposition 29, relatively to the polyhedron $P_t = P^+ \cap X_t$;*
- (iii) *the vector field $\vec{\vartheta}_+$ is non-zero outside P^+, zero on P^+, transversal to $\partial X^+ := X^+ \cap \mathbb{S}_\epsilon$ in the stratified sense, pointing inwards, and it restricts to a rugose stratified vector field on the interior of X^+ (relatively to the stratification $(\mathcal{S}^+)'$).*

We say that the polyhedron P^+ is a *collapsing cone for f along the semi-disk $\mathbb{D}^+$*.

As an immediate corollary, we have:

Corollary 31 *Let γ be a simple path in $\mathbb{D}_{\eta_2}$ joining 0 and some $t_0 \in \mathbb{D}_{\eta_2}$ (as in Fig. 7), and set $X_\gamma := X \cap f^{-1}(\gamma) \cap \mathbb{B}_\epsilon$. There exists a refinement $\mathcal{S}'(\gamma)$ of the Whitney stratification $\mathcal{S}(\gamma)$ of X_γ (with the property (w)) induced by the stratification $\mathcal{S}$ of X such that there are a polyhedron P_γ in X_γ, adapted to the stratification $\mathcal{S}'(\gamma)$, and an integrable vector field $\vec{\vartheta}_\gamma$ in X_γ, so that:*

- (i) *the intersection $P_\gamma \cap X_t$ is a vanishing polyhedron P_t as in Proposition 29, for any $t \in \gamma \backslash \{0\}$, and $P_\gamma \cap X_0 = \{0\}$;*
- (ii) *for any $t \in \gamma \backslash \{0\}$, the restriction of $\vec{\vartheta}_\gamma$ to X_t gives a vector field $\vec{\vartheta}_t$ as in the proposition above, relatively to the polyhedron $P_t = P_\gamma \cap X_t$;*
- (iii) *the vector field $\vec{\vartheta}_\gamma$ is non-zero outside P_γ, zero on P_γ, transversal to $\partial X_\gamma := X_\gamma \cap \mathbb{S}_\epsilon$ in the stratified sense, pointing inwards, and it restricts to a rugose stratified vector field on the interior of X_γ.*

We say that the polyhedron P_γ above is a *collapsing cone for f along the path γ*.

One can check that the flow $\tilde{q}_\gamma : [0, \infty) \times (X_\gamma \backslash P_\gamma) \to X_\gamma$ given by the integration of the vector field $\vec{\vartheta}_\gamma$ on $X_\gamma \backslash P_\gamma$ defines a continuous, simplicial and surjective map:

$$\begin{aligned} \tilde{\xi}_\gamma \ : \partial X_\gamma &\longrightarrow P_\gamma \\ z &\longmapsto \lim_{\tau \to \infty} \tilde{q}_\gamma(\tau, z) \end{aligned}$$

such that X_γ is homeomorphic to the mapping cylinder of $\tilde{\xi}_\gamma$ (see Proposition 27).

Remark 32 In order to prove Theorem 1, we just need Corollary 31. Nevertheless, we will need Proposition 29 to prove Proposition 30 and we will need Proposition 30 when $\dim_{\mathbb{C}} X = n - 1$ to prove Proposition 29 when $\dim_{\mathbb{C}} X = n$.

So let us assume now that Proposition 30 is true. Itens (i) and (ii) of Theorem 2 follow easily from Proposition 30 (the proof that X^+ is homeomorphic to the mapping cylinder of the map $\tilde{\xi}_+$ given by the vector field $\vec{\vartheta}_+$ is analogous to the proof of item (7) of Proposition 27). Then we can easily prove (iii) of Theorem 2 as follows:

Fix $t \in \mathbb{D}^*_{\eta_2}$ and let γ be a simple path in $\mathbb{D}_{\eta_2}$ connecting t and 0. Consider the polyhedron P_γ and the vector field $\vec{\vartheta}_\gamma$ in X_γ given by Corollary 31, as well as the flow $\tilde{q}_\gamma$ given by the integration of $\vec{\vartheta}_\gamma$.

For any positive real $A > 0$ set:

$$\tilde{V}_A(P_\gamma) := X_\gamma \backslash \tilde{q}_\gamma\big([0, A) \times \partial X_\gamma\big),$$

which is a closed neighborhood of P_γ in X_γ. Notice that using the first isotopy lemma of Thom–Mather (Lemma 19), the boundary $\partial \tilde{V}_A(P_\gamma)$ of $\tilde{V}_A(P_\gamma)$ is a locally trivial topological fibration over γ.

Following the steps (a) and (b) of the end of Sect. 4.3 and using Proposition 16, we can construct a collapsing vector field $\vec{\zeta}_\gamma$ on $X_\gamma \backslash P_\gamma$ such that:

- it is a rugose stratified vector field (relatively to the stratification $\mathcal{S}'(\gamma)$ of X_γ):
- it projects on a smooth vector field $\vec{\theta}$ on γ that goes from t_0 to 0 in a time $a > 0$;
- it is tangent to $\partial \tilde{V}_A(P_\gamma)$, for any $A > 0$.

So the flow $g : [0, a] \times X_\gamma \backslash P_\gamma \to X_\gamma \backslash P_\gamma$ associated to the collapsing vector field $\vec{\zeta}_\gamma$ defines a homeomorphism ψ_t from $X_t \backslash P_t$ to $X_0 \backslash \{0\}$ that extends to a continuous map from X_t to X_0 and that sends P_t to $\{0\}$, for any $t \in \gamma \backslash \{0\}$. This proves (iii) of Theorem 2.

Remark 33 Notice that the collapsing vector field $\vec{\zeta}_\gamma$ on $X_\gamma \backslash P_\gamma$ that gives the collapsing of f along the path γ can be extended to a collapsing vector field $\vec{\zeta}_+$ on $X^+ \backslash P^+$ that gives the collapsing of f along the semi-disk $\mathbb{D}^+$.

6 Proof of Propositions 29 and 30

We will prove Propositions 29 and 30 by induction on the dimension of X, in the following way: we will prove that if Proposition 30 is true whenever $\dim_{\mathbb{C}} X = n-1$, then Proposition 29 is true whenever $\dim_{\mathbb{C}} X = n$, and this implies that Proposition 30 is true whenever $\dim_{\mathbb{C}} X = n$.

Notice that in Sects. 4.1 and 4.2 we have proved Propositions 29 and 30, respectively, when $\dim_{\mathbb{C}} X = 2$.

6.1 Proof of Proposition 29: constructing the vanishing polyhedron

As we said above, the polyhedron P_t will consist of a "central" polyhedron P_t' on which we will attach the polyhedra C_j. The first step will be to construct the central polyhedron P_t', and then we will construct the polyhedra C_j.

Recall that we have fixed a linear form $\ell : \mathbb{C}^N \to \mathbb{C}$ that satisfies the conditions of Lemma 21. Then Γ is the polar curve of f relatively to ℓ at 0 and Δ is the polar discriminant of f relatively to ℓ at 0.

Also recall from Proposition 22 that the map $\phi = (\ell, f)$ induces a stratified submersion (relatively to the stratification induced by $\mathcal{S}$):

$$\phi_| : \mathbb{B}_\epsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta$$

and that for each $t \in \mathbb{D}^*_{\eta_2}$ fixed, the restriction ℓ_t of ℓ to the Milnor fiber X_t induces a topological locally trivial fibration:

$$\varphi_t : X_t \backslash \ell_t^{-1}\big(\{y_1(t), \ldots, y_k(t)\}\big) \to D_t \backslash \{y_1(t), \ldots, y_k(t)\},$$

where $D_t = \mathbb{D}_{\eta_1} \times \{t\}$ and $\{y_1(t), \ldots, y_k(t)\} = \Delta \cap D_t$.

For any $t \in \mathbb{D}_{\eta_2}$ set $\lambda_t := (0, t)$. Since the complex line $\{0\} \times \mathbb{C}$ is not a component of Δ, we can suppose that $\lambda_t \notin \{y_1(t), \ldots, y_k(t)\}$.

For each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a smooth simple path in D_t starting at λ_t and ending at $y_j(t)$, such that two of them intersect only at λ_t.

First step: constructing the central polyhedron P_t':

Consider the restriction f' of f to the intersection $X \cap \{\ell = 0\}$, which has complex dimension $n - 1$. Then we can apply the induction hypothesis to f' to obtain a vanishing polyhedron P_t' in the fiber $X_t \cap \{\ell = 0\}$ and an integrable vector field $\vec{\vartheta}_t'$ in $X_t \cap \{\ell = 0\}$ that deformation retracts $X_t \cap \{\ell = 0\}$ onto P_t'.

Second step: constructing the polyhedra C_j:

First of all, in order to make it easier for the reader to understand the constructions, we will suppose that Γ intersects $\ell_t^{-1}\big(y_j(t)\big)$ in only one point, which we call $x_j(t)$. The proof of the general case follows the same steps. In fact, we make the following conjecture:

Conjecture 34 *For ℓ general enough, the map-germ $\phi_\ell = (\ell, f) : (X, x) \to (\mathbb{C}^2, 0)$ induces a bijective morphism from Γ onto Δ.*

Now recall that ℓ_t induces a locally trivial fibration over $\delta(y_j(t)) \backslash \{y_j(t)\}$. If we look at the local situation at $x_j(t)$, we can apply the induction hypothesis to the germ $\ell_{t|} : (X_t, x_j(t)) \to (D_t, y_j(t))$, which has an isolated singularity at $x_j(t)$ in the stratified sense, in lower dimension. That is, considering a small ball B_j in $\mathbb{C}^N$ centered at $x_j(t)$; a small disk D_s in D_t centered at $y_j(t)$ and a semi-disk D_s^+ of D_s which contains $\delta(y_j(t)) \cap \mathring{D}_s$ in its interior, we obtain:

- a collapsing cone P_j^+ for ℓ_t along the semi-disk D_s^+;
- a collapsing cone P_j for ℓ_t along the path $D_s \cap \delta(y_j(t))$, of real dimension $n - 1$;

which give the collapsing of the map $\ell_{t|} : B_j \cap \ell_t^{-1}(D_s) \to D_s$ along the path $D_s \cap \delta(y_j(t))$. See Fig. 9.

Now we are going to extend the cone P_j until it hits P_t', as follows.

First we need to construct the following vector fields on $A_j := \ell_t^{-1}\big(\delta(y_j(t)) \backslash \{y_j(t)\}\big)$ that will be used to extend the cone P_j and glue it on the central polyhedron P_t':

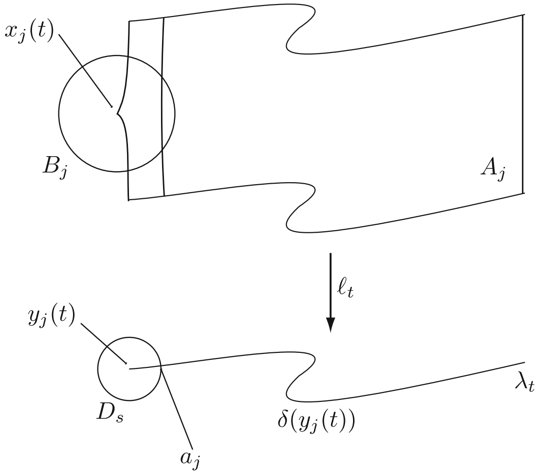


Fig. 9 The restriction ℓ_t

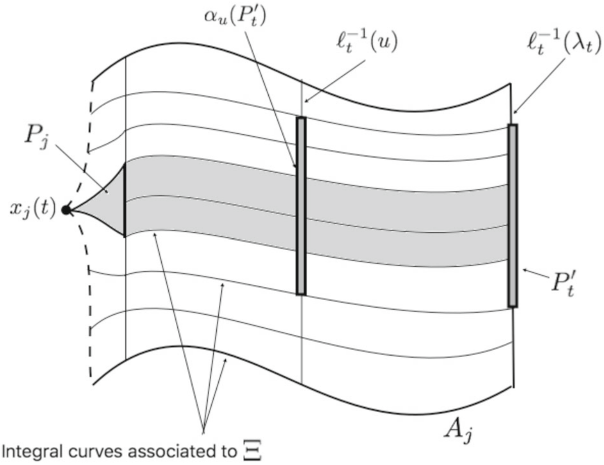


Fig. 10 The vector field $\vec{\Xi}$

- *Vector field* $\vec{\Xi}$: let $\vec{\xi}$ be a smooth non-singular vector field on $\delta(y_j(t))\backslash\{y_j(t)\}$ that goes from $y_j(t)$ to $\lambda_t = (0, t)$. Since the restriction of ℓ_t to each Whitney stratum of $\mathcal{S}$ has maximum rank over $\delta(y_j(t))\backslash\{y_j(t)\}$, we can lift $\vec{\xi}$ to an rugose (and hence integrable) stratified vector field $\vec{\Xi}$ on A_j (see Proposition 16). In particular, for any $u \in \delta(y_j(t))\backslash\{y_j(t)\}$ we can use the vector field $\vec{\Xi}$ to obtain a stratified homeomorphism $\alpha_u : \ell_t^{-1}(\lambda_t) \to \ell_t^{-1}(u)$, which takes P_t' to a polyhedron $\alpha_u(P_t')$ in $\ell_t^{-1}(u)$. See Fig. 10.

- *Vector field* $\vec{\kappa}$: we can transport the vector field $\vec{\vartheta}_t'$ of $\ell_t^{-1}(\lambda_t) = X_t \cap \{\ell = 0\}$ given by the induction hypothesis to all the fibers $\ell_t^{-1}(u)$, for any $u \in \delta(y_j(t))\backslash\{y_j(t)\}$. The transportation of $\vec{\vartheta}_t'$ to $\ell_t^{-1}(u)$ is the vector field on $\ell_t^{-1}(u)$ given by the flow obtained as image by α_u of the flow given by $\vec{\vartheta}_t'$. So we obtain a vector field $\vec{\kappa}$ on A_j whose restriction to $\ell_t^{-1}(\lambda_t)$ is $\vec{\vartheta}_t'$. The flow associated to $\vec{\kappa}$ takes a point $z \in \ell_t^{-1}(u)$ to the polyhedron $\alpha_u(P_t')$. See Fig. 11.

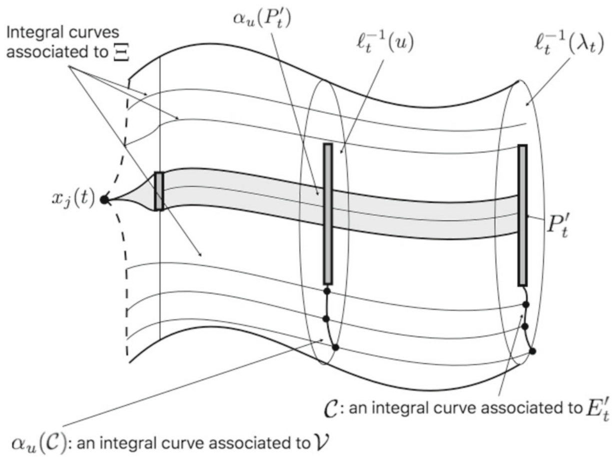


Fig. 11 The vector field $\vec{\kappa}$

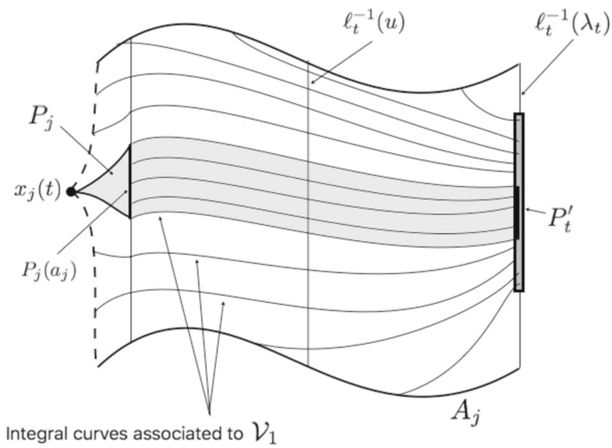


Fig. 12 The vector field $\vec{\kappa}_1$

- *Vector field* $\vec{\kappa}_1$: let θ be a smooth function on $\delta(y_j(t))$ such that $\theta(\lambda_t) = 0$ and such that θ is non-singular and positive on $\delta(y_j(t))\setminus\{\lambda_t\}$. It induces a function $\tilde{\theta} := \theta \circ \ell_t$ defined on A_j. Set:

$$\vec{\kappa}_1 := \vec{\kappa} + \tilde{\theta} \cdot \vec{\Xi},$$

which is an integrable vector field, tangent to the strata of the interior of A_j induced by $\mathcal{S}$. Furthermore, this vector field $\vec{\kappa}_1$ is pointing inwards on the boundary ∂A_j, i.e. transversal in A_j to the strata of ∂A_j induced by $\mathcal{S}$. See Fig. 12.

Since the vectors $\vec{\kappa}(z)$ and $\vec{\Xi}(z)$ are not parallel for any $z \in A_j \setminus P'_t$, the vector field $\vec{\kappa}_1$ is zero only on the vanishing polyhedron P'_t of $\ell_t^{-1}(\lambda_t)$. Then if z is a point in $A_j \setminus \ell_t^{-1}(\lambda_t)$, the orbit of $\vec{\kappa}_1$ that passes through z has its limit point z'_1 in P'_t.

Moreover, since the integral curve associated to $\vec{\kappa}$ that contains $z \in A_j$ has its limit point z' in the polyhedron $\alpha_{\ell_t(z)}(P'_t)$ (which is the transportation of P'_t to $\ell_t^{-1}\big(\ell_t(z)\big)$ by the flow

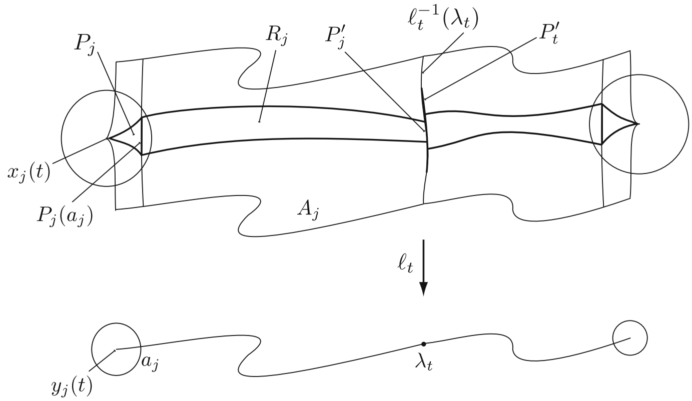


Fig. 13 Decomposition of the vanishing polyhedron P_t

associated to $\vec{\Xi}$), it follows that z'_1 is the point corresponding to z' by $\vec{\Xi}$, that is, $z'_1 = \alpha_{\ell_t(z)}(z')$. In fact, if $u := \ell_t(z)$ and if $w := (\alpha_u)^{-1}(z)$ is the corresponding point in $\ell_t^{-1}(\lambda_t)$, then by construction the integral curve $\mathcal{C}_\kappa(z)$ associated to $\vec{\kappa}$ that contains z is given by $\alpha_u(\mathcal{C}(w))$, where $\mathcal{C}(w)$ is the integral curve associated to $\vec{\vartheta}'_t$ that contains w.

Set $a_j := \partial D_s \cap \delta(y_j(t))$ and $P_j(a_j) := P_j \cap \ell_t^{-1}(a_j)$, where P_j is the collapsing cone for ℓ_t at $x_j(t)$ along the path $D_s \cap \delta(y_j(t))$, as defined above. By the previous paragraph, $\vec{\kappa}_1$ takes $P_j(a_j)$ to P'_t.

Since the action of the flow given by $\vec{\kappa}$ is simplicial, we can assume that the action of the flow given by $\vec{\kappa}_1$ is simplicial. Then the image of $P_j(a_j)$ by the flow of $\vec{\kappa}_1$ is a subpolyhedron P'_j of P'_t. Moreover, the orbits of the points in $P_j(a_j)$ give a polyhedron R_j. See Fig. 13.

Set:

$$C_j := P_j \cup R_j \cup P'_j.$$

It is a polyhedron in P_t of real dimension $n-1$. We call C_j a *wing* of the polyhedron P_t. In the case when X is smooth, it corresponds to a Lefschetz thimble.

Then the polyhedron we are going to consider is:

$$P_t := P'_t \bigcup_{j=1}^{k} C_j.$$

It is adapted to the stratification $\mathcal{S}$, since P_j is adapted to $\mathcal{S}$ and the vector field $\vec{\kappa}_1$ is tangent to the strata of $\mathcal{S}$.

Now we have:

Lemma 35 *There exists a refinement $\mathcal{S}'(t)$ of the Whitney stratification $\mathcal{S}(t)$ of X_t (with the property (w)) induced by $\mathcal{S}$, such that there is a continuous vector field $\vec{\vartheta}_t$ on X_t such that:*

(i) *It is non-zero on $X_t \backslash P_t$ and it is zero on P_t;*
(ii) *It is transversal to the strata of ∂X_t, pointing inwards;*
(iii) *It restricts to a rugose stratified vector field on the interior of X_t (relatively to the stratification $\mathcal{S}'(t)$);*
(iv) *The orbits associated to $\vec{\vartheta}_t$ have a limit point at P_t when the parameter goes to infinity.*

The vector field $\vec{\vartheta}_t$ is obtained by gluing several vector fields on X_t given by the lifting by φ_t of suitable vector fields on D_t. The detailed proof of Lemma 35 is quite involved since it contains too many technical steps and constructions, so we present it separately in Sect. 7.

The flow defined by the vector field $\vec{\vartheta}_t$ of Lemma 35 gives a continuous, surjective and simplicial map $\tilde{\xi}_t$ from ∂X_t to P_t such that X_t is homeomorphic to the mapping cylinder of $\tilde{\xi}_t$ (see the proof of Proposition 27). This proves Proposition 29.

We remark that although we have used just Corollary 31 (in lower dimension) in the construction of the polyhedron P_t, we will need the stronger Proposition 30 (in lower dimension) for the construction of the vector field $\vec{\vartheta}_t$ of Lemma 35, in Sect. 7.

6.2 Proof of Proposition 30: constructing the polyhedron P^+

Given a closed semi-disk $\mathbb{D}^+$ in $\mathbb{D}_{\eta_2}$ as in Fig. 1 above (with 0 in its boundary), we want to construct a polyhedron P^+ in $X^+ := X \cap f^{-1}(\mathbb{D}^+) \cap \mathbb{B}_\epsilon$, adapted to the stratification induced by $\mathcal{S}$, and a continuous vector field $\vec{\vartheta}_+$ in X^+, tangent to each stratum of X^+, satisfying the conditions (i), (ii) and (iii) of Proposition 30.

Recall that in Sect. 4.2 we already did that when X has complex dimension 2. Also recall the 3-dimensional triangles $\mathcal{T}_1, \ldots, \mathcal{T}_k$ in $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$, bounded by the semi-disks Y_j and Λ, where $Y_1, \ldots, Y_k$ are the semi-disks given by the intersection of the polar discriminant Δ with $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$ and $\Lambda := \{0\} \times \mathbb{D}^+$. For each $j = 1, \ldots, k$ and for each $t \in \mathbb{D}^+$, the intersection $\mathcal{T}_j \cap D_t$ gives a simple path $\delta(y_j(t))$ used to construct a vanishing polyhedron P_t as in Sect. 6.1.

Finally, also recall from Proposition 22 that the map $\phi = (\ell, f)$ induces a stratified submersion:

$$\phi_| : \mathbb{B}_\epsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta,$$

where $\Delta \subset \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2}$ is the polar discriminant of f relatively to the linear form ℓ.

We are going to construct P^+. The construction of the vector field $\vec{\vartheta}_+$ is analogous to the construction of the vector field $\vec{\vartheta}_t$ of Lemma 35, which is described with details in the next section, so we leave it to the reader.

The construction of the polyhedron P^+ will be made in three steps:

First step: fixing an initial polyhedron P_{t_0}:

Fix some $t_0 \in \mathbb{D}^+ \cap \partial\mathbb{D}_{\eta_2}$. By Proposition 29, we can choose a vanishing polyhedron P_{t_0} in the Milnor fiber:

$$X_{t_0} = X \cap f^{-1}(t_0) \cap \mathbb{B}_\epsilon,$$

which has the form:

$$P_{t_0} = P'_{t_0} \bigcup_{j=1}^{k} C_j,$$

where each C_j is a wing glued to the central polyhedron P'_{t_0} along a subpolyhedron $(P_j)'_{t_0}$ of P'_{t_0} (recall that P'_{t_0} is a vanishing polyhedron for the restriction f' of f to $X \cap \{\ell = 0\}$).

Second step: extending P_{t_0} over $\mathbb{D}^+ \cap \partial\mathbb{D}_{\eta_2}$.

By the induction hypothesis, the vanishing polyhedron $P'_{t_0} \subset X_{t_0} \cap \{\ell = 0\}$ can be extended to a collapsing cone $(P^+)' \subset X^+ \cap \{\ell = 0\}$ for the restriction of f to $\{\ell = 0\}$ along the semi-disk $\mathbb{D}^+$.

Now set $\tilde{X} := \phi^{-1}\big(\mathbb{D}_{\eta_1} \times (\mathbb{D}^+ \cap \partial\mathbb{D}_{\eta_2})\big)$ and recall that ϕ induces a topological trivial fibration from $\tilde{X}$ onto $\mathbb{D}^+ \cap \partial\mathbb{D}_{\eta_2}$ (see Fig. 7).

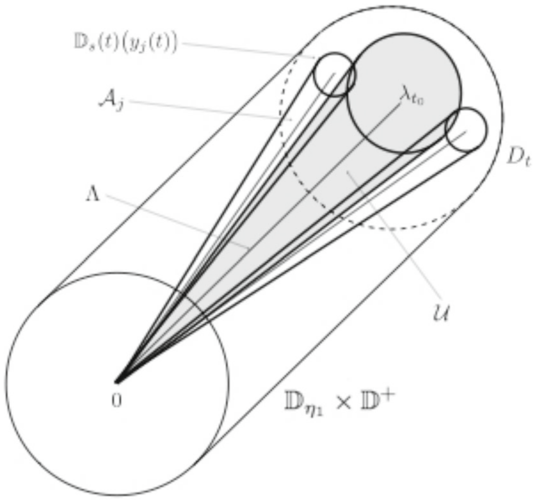


Fig. 14 Some suitable neighborhoods

So we can extend the polyhedron P_{t_0} constructed above for t varying in $\mathbb{D}^+ \cap \partial\mathbb{D}_{\eta_2}$. That is, we consider a polyhedron $\tilde{P}$ in $\tilde{X}$ such that $\tilde{P} \cap X_{t_0} = P_{t_0}$ and such that for any $t \in \mathbb{D}^+ \cap \partial\mathbb{D}_{\eta_2}$ one has that $\tilde{P} \cap X_t$ is a vanishing polyhedron P_t in X_t with central $P'_t := (P^+)' \cap X_t$ and k-many wings, each one of them conic from the corresponding point $x_j(t) \in \Gamma \cap X_t$.

Third step: constructing some suitable neighborhoods:

For each $x_j(t)$ over $y_j(t)$, with $t \in \mathbb{D}^+$, choose a small radius $r(t)$ such that the set:

$$\mathcal{B}_j := \bigcup_{t \in \mathbb{D}^+} \mathbb{B}_{r(t)}(x_j(t))$$

is a neighborhood of:

$$\bigcup_{t \in \mathbb{D}^+ \setminus \{0\}} \{x_j(t)\}$$

conic from 0, where $r(t)$ can be taken as a real analytic function of t with $r(0) = 0$, by Puiseux's theorem.

To each $\mathcal{B}_j$ one can associate a neighborhood:

$$\mathcal{A}_j := \bigcup_{t \in \mathbb{D}^+} \mathbb{D}_{s(t)}(y_j(t))$$

in $\mathbb{D}_{\eta_1} \times \mathbb{D}^+$, where $s(t)$ is an analytic function of $t \in \mathbb{D}^+$ with $0 < s(t) \ll r(t)$, if $t \neq 0$, and $s(0) = 0$.

Finally, let $\mathcal{U}$ be a neighborhood of $\Lambda \setminus \{0\}$, conic from 0, that meets all the $\mathcal{A}_j$'s, but not containing any $y_j(t)$. See Fig. 14.

Fourth step: constructing a suitable vector field:

By the induction hypothesis, we have a collapsing cone $(P^+)'$ in $X^+ \cap \{\ell = 0\}$ and a collapsing vector field $\vec{\zeta}'_+$ in $X^+ \cap \{\ell = 0\}$ that give the degeneration of f' along $\mathbb{D}^+$. That is, $\vec{\zeta}'_+$ is a rugose (and hence integrable) vector field, tangent to each stratum of the interior of $X^+ \cap \{\ell = 0\}$, whose associated flow defines a homeomorphism from $(X_{t_0} \cap \{\ell = 0\}) \setminus P'_{t_0}$ to $(X_{t_0} \cap \{\ell = 0\}) \setminus \{0\}$ that extends to a continuous map from $X_{t_0} \cap \{\ell = 0\}$ to $X_0 \cap \{\ell = 0\}$

and that sends P'_{t_0} to $\{0\}$ (see Remark 33). Notice that the vector field $\vec{\zeta}'_+$ lifts a radial vector field in $\{0\} \times \mathbb{D}^+$ that goes to 0.

Set:

$$\tilde{\mathcal{U}} := \phi^{-1}(\mathcal{U}) \cap X^+.$$

Since $\mathcal{U}$ is a cone over a contractible space, we can extend the collapsing vector field $\vec{\zeta}'_+$ to a integrable vector field $\vec{\zeta}_{\mathcal{U}}$ on $\tilde{\mathcal{U}}$, tangent to each stratum of its interior, and that lifts a radial vector field in $\{0\} \times \mathbb{D}^+$ that goes to 0. Notice that the flow given by the vector field $\vec{\zeta}_{\mathcal{U}}$ sends the intersection $P'_{t_0} \cap \tilde{\mathcal{U}}$ to $\{0\}$, where $P'_{t_0} = P_{t_0} \cap \{\ell = 0\}$.

Since each $\mathcal{B}_j \backslash \{0\}$ is a stratified topological locally trivial fibration over $\mathbb{D}^+ \backslash \{0\}$, one can also construct a rugose vector field $\vec{\sigma}_j$ on $\mathcal{B}_j \backslash \{0\}$ that trivializes it over $\mathbb{D}^+ \backslash \{0\}$ and that is tangent to the intersection of the polar curve Γ with $\mathcal{B}_j \backslash \{0\}$ (which is the set of the points $x_j(t) \in \mathcal{B}_j$ for $t \in \mathbb{D}^+ \backslash \{0\}$).

Then, using a partition of unity $(\rho_{\mathcal{U}}, \rho_1, \ldots, \rho_k)$ adapted to $\tilde{\mathcal{U}}, \mathcal{B}_1, \ldots, \mathcal{B}_k$, we glue all the vector fields $\vec{\sigma}_j$'s and $\vec{\zeta}_{\mathcal{U}}$ together. We obtain a continuous trivializing vector field:

$$\vec{\sigma} := \rho_{\mathcal{U}} \vec{\zeta}_{\mathcal{U}} + \sum_{j=1}^{k} \rho_j \vec{\sigma}_j$$

in $X^+ \cap (\tilde{\mathcal{U}} \cup_{j=1}^k \mathcal{B}_j)$ such that:

- it is tangent on each stratum of the interior of $X^+ \cap (\tilde{\mathcal{U}} \cup_{j=1}^k \mathcal{B}_j)$;
- it is rugose and hence integrable;
- it projects to a radial vector field in $\mathbb{D}^+$ that converges to 0.

So the flow associated to the vector field $\vec{\sigma}$ goes from $\tilde{X} \cap (\tilde{\mathcal{U}} \cup_{j=1}^k \mathcal{B}_j)$ to 0, and the action of this flow over $\tilde{P}$ give the polyhedron P^+.

7 Proof of Lemma 35: constructing the vector field $\vec{\vartheta}_t$

In this section we give the detailed construction of the vector field $\vec{\vartheta}_t$ on $X_t := \phi^{-1}(\mathbb{D}_{\eta_1} \times \{t\}) \cap \mathbb{B}_\epsilon$ of Lemma 35, whose flow gives a continuous, surjective and simplicial map $\tilde{\xi}_t$ from the boundary of the Milnor fiber $\partial X_t := \phi^{-1}(\mathbb{D}_{\eta_1} \times \{t\}) \cap \mathbb{S}_\epsilon$ to the polyhedron P_t constructed in the previous section, such that X_t is homeomorphic to the simplicial map cylinder of $\tilde{\xi}_t$.

Recall that we have fixed a linear form $\ell : \mathbb{C}^N \to \mathbb{C}$ that satisfies the conditions of Lemma 21. Then Γ is the polar curve of f relatively to ℓ at 0 and Δ is the polar discriminant of f relatively to ℓ at 0.

Also recall from Proposition 22 that the map $\phi = (\ell, f)$ induces a stratified submersion (relatively to the stratification induced by the Whitney stratification $\mathcal{S}$ of X):

$$\phi_| : \mathbb{B}_\epsilon \cap X \cap \phi^{-1}(\mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta) \to \mathbb{D}_{\eta_1} \times \mathbb{D}_{\eta_2} \backslash \Delta.$$

As before, fix $t \in \mathbb{D}_{\eta_2} \backslash \{0\}$ and take a point λ_t in $D_t := \mathbb{D}_{\eta_1} \times \{t\}$ such that $\lambda_t \notin \{y_1(t), \ldots, y_k(t)\}$, where $\{y_1(t), \ldots, y_k(t)\} := \Delta \cap D_t$. Also, for each $j = 1, \ldots, k$, let $\delta(y_j(t))$ be a simple path in D_t starting at λ_t and ending at $y_j(t)$, such that two of them intersect only at λ_t. We defined the set $Q_t := \cup_{j=1}^k \delta(y_j(t))$.

Recall that in Lemma 25 we constructed a Whitney stratification $\mathcal{Z} = (Z_\beta)_{\beta \in B}$ of D_t (with the property (w)) and a continuous vector field $\vec{v}_t$ on D_t such that:

1. It is non-zero on $D_t \setminus Q_t$;
2. It vanishes on Q_t;
3. It is transversal to ∂D_t and points inwards;
4. It restricts to a rugose stratified vector field on the interior $\mathring{D}_t$ of D_t (relatively to the stratification $\mathcal{Z}$);
5. The associated flow $q_t : [0, \infty) \times (D_t \setminus Q_t) \to D_t \setminus Q_t$ defines a map:

$$\xi_t : \partial D_t \longrightarrow Q_t$$
$$u \longmapsto \lim_{\tau \to \infty} q_t(\tau, u),$$

such that ξ_t is continuous, simplicial (as defined in Sect. 2.4) and surjective.

After that, we considered a refinement $\mathcal{S}'(t)$ of the Whitney stratification $\mathcal{S}(t)$ (with the property (w)) induced by $\mathcal{S}$, and we considered a refinement $\mathcal{Z}' = (Z'_\beta)_{\beta \in B'}$ of $\mathcal{Z}$ such that the restriction $\ell_t : X_t \to D_t$ of ℓ to the Milnor fiber X_t is a stratified map. So ℓ_t induces a stratified submersion:

$$\varphi_t : X_t \setminus \ell_t^{-1}\big(\{y_1(t), \ldots, y_k(t)\}\big) \to D_t \setminus \{y_1(t), \ldots, y_k(t)\}.$$

Finally, recall that we can apply the induction hypothesis to the restriction f' of f to the intersection $X \cap \{\ell = 0\}$, which has complex dimension $n - 1$. We obtain a vanishing polyhedron P'_t in the intersection $X_t \cap \{\ell = 0\}$ and a vector field $\vec{\vartheta}'_t$ that deformation retracts it onto P'_t.

The vector field $\vec{\vartheta}_t$ is obtained by gluing several vector fields on X_t given by the lift of suitable vector fields on the disk D_t by φ_t. By Proposition 16, the resulting vector fields are rugose, and hence integrable.

Recall that the polyhedron P_t is the union of the wings C_j and that the polyhedron P'_t is given by the induction hypothesis (as in Sect. 6.1). Moreover, each wing C_j consists of a collapsing cone P_j, a product R_j and the gluing polyhedron P'_j on P'_t, that is:

$$C_j = P_j \cup R_j \cup P'_j.$$

See Fig. 13.

Then it is natural that the construction of the vector field $\vec{\vartheta}_t$ concerns at least three subsets of the Milnor fiber X_t: the points that are taken to $P'_t \setminus P'_j$ by the flow associated to $\vec{\vartheta}_t$; the points that are taken to P'_j and the points that are taken to $C_j \setminus P'_j$. This justifies the complexity of the construction given below.

7.1 First step: decomposing D_t

Let $q_t : [0, \infty) \times \partial D_t \to D_t$ be the flow associated to the vector field $\vec{\nu}_t$ defined in Lemma 25. Set:

$$V := D_t - q_t([0, A) \times \partial D_t),$$

for some $A \gg 0$, which is a closed neighborhood of Q_t whose boundary:

$$\partial V = q_t(\{A\} \times \partial D_t)$$

is transversal to each $\partial D_s(y_j(t))$, that is, the vector field $\vec{\nu}_t$ is transversal to the boundary $\partial D_s(y_j(t))$ of each disk $D_s(y_j(t))$. See Fig. 15.

Then we will construct the vector field $\vec{\vartheta}_t$ on X_t by gluing a vector field $\vec{\tau}$ in $\ell_t^{-1}(D_t \setminus V')$, where $V' := D_t - q_t([0, A'[\times \partial D_t)$, with $A' > A$, $A' - A \ll 1$; and a vector field $\vec{v}$ in $\ell_t^{-1}(V)$, using a partition of unity.

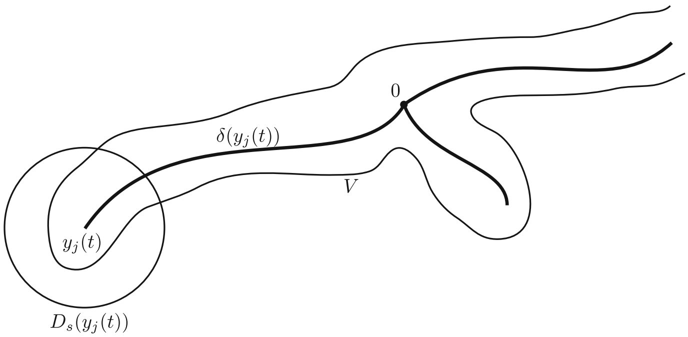


Fig. 15 The neighborhood V

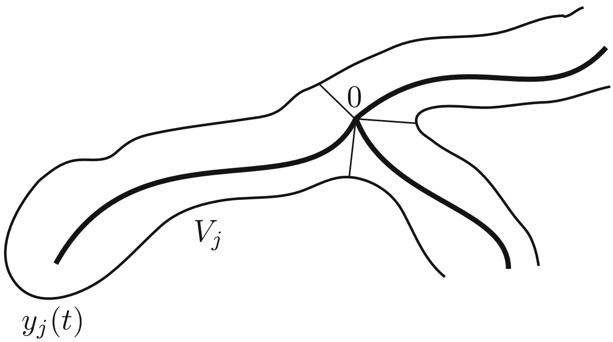


Fig. 16 The branch V_j

The vector field $\vec{\tau}$ in $\ell_t^{-1}(D_t \backslash V')$ is a lifting of the vector field $\vec{\nu}_t$. It is transversal to the boundary of X_t, pointing inwards, and it restricts to a rugose stratified vector field on $\phi^{-1}(D_t \backslash V') \cap \mathring{\mathbb{B}}_\epsilon$.

The construction of the vector field $\vec{\upsilon}$ in $\ell_t^{-1}(V)$ is much more complicated. We are going to do it in the rest of this subsection.

7.2 Second step: decomposing *V*

We first decompose V into "branches" V_j as follows: each "branch" V_j is a closed neighborhood of $\delta(y_j(t))\backslash\{0\}$ whose boundary is composed by $\partial V \cap V_j$ and two simple paths that one can suppose to be orbits of the vector field $\vec{\nu}_t$ constructed above. See Fig. 16.

We will construct the vector field $\vec{\upsilon}$ by gluing the vector fields $\vec{\upsilon}_j$ that we are going to construct on each $\ell_t^{-1}(V_j)$. In other words, we will construct a vector field $\vec{\upsilon}_j$ on $\ell_t^{-1}(V_j)$, for each j fixed, which is continuous, integrable, tangent to the strata of $\mathcal{S}$, non-zero and smooth on $\ell_t^{-1}(V_j)\backslash C_j$, and zero on C_j, where C_j is the polyhedron defined in Sect. 6.1.

7.3 Third step: covering $\ell_t^{-1}(V_j)$ by open sets $W_{j,i}$

Fix $j \in \{1, \dots, k\}$. The approach of the construction of each vector field $\vec{\upsilon}_j$ will be the following: we will cover $\ell_t^{-1}(V_j)$ by open sets $W_{j,1}$, $W_{j,2}$, $W_{j,3}$ and $W_{j,4}$. Then we will

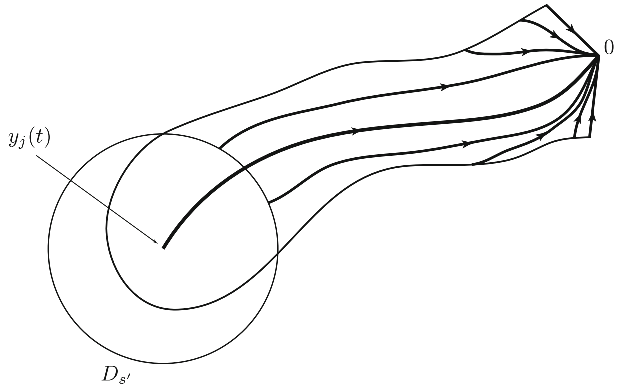


Fig. 17 The vector field $\vec{\omega}_j$

construct the vector fields $\vec{\upsilon}_{j,i}$ on $W_{j,i}$, for $i = 1, \dots, 4$, in such a way that each orbit of the vector field $\vec{\upsilon}_j$ obtained by gluing them with a partition of unity has a limit point in P_t.

As before, given positive real numbers r and s, let B_r denote the ball around $x_j(t)$ in $\mathbb{C}^N$ of radius r and let D_s denote the disk around $y_j(t)$ in D_t of radius s.

Let r and r' be small enough positive real numbers such that $r' < r$ and $r - r' \ll 1$. Let us cover $\ell_t^{-1}(V_j)$ by the open sets $W_{j,1}$, $W_{j,2}$, $W_{j,3}$ and $W_{j,4}$ defined as follows:

- $W_{j,1} := \ell_t^{-1}(\mathring{D}_s \cap \mathring{V}_j) \cap \mathring{B}_r$

and

- $W_{j,2} := \ell_t^{-1}(\mathring{D}_s \cap \mathring{V}_j) \backslash B_{r'}$.

To define $W_{j,3}$ and $W_{j,4}$ we have to do a construction first. Set:

$$W'_{j,3} := \ell_t^{-1}(\mathring{V}_j \backslash D_{s'}),$$

where $s' < s$ and $s - s' \ll 1$.

We can construct a vector field $\vec{\omega}_j$ in $V_j \backslash \mathring{D}_{s'}$ which is smooth, non zero outside $\{0\}$, zero on $\{0\}$, with trajectories transversal to $\partial V \cap (V_j \backslash D_{s'})$ and to $\partial D_{s'} \cap V_j$, as in Fig. 17.

Since the Whitney stratification $\mathcal{S}'(t)$ of X_t with the property (w) induces a Whitney stratification on $\ell_t^{-1}(V_j \backslash \mathring{D}_{s'})$ with the property (w), and since the restriction of ℓ_t to $\ell_t^{-1}(V_j \backslash \mathring{D}_{s'}) \cap S_\alpha$ is a submersion for each $\alpha \in A$, we can lift $\vec{\omega}_j$ to a continuous vector field $\vec{\Omega}_j$ in $\ell_t^{-1}(V_j \backslash \mathring{D}_{s'})$ that is rugose, smooth and tangent to each stratum, and that trivializes $\ell_t^{-1}(V_j \backslash \mathring{D}_{s'})$ over $V_j \backslash \mathring{D}_{s'}$.

Recall that the induction hypothesis applied to the restriction of f to $X \cap \{\ell = 0\}$ gives a vanishing polyhedron P'_t in $X_t \cap \{\ell = 0\}$ and a vector field $\vec{\vartheta}'_t$ that deformation retracts $X_t \cap \{\ell = 0\}$ onto P'_t.

Then one can transport the vector field $\vec{\vartheta}'_t$ on $\ell_t^{-1}(0)$ to all the fibers $\ell_t^{-1}(u)$ for $u \in V_j \backslash \mathring{D}_{s'}$. This way we obtain a vector field $\vec{\mathcal{V}}_j$ in $\ell_t^{-1}(V_j \backslash \mathring{D}_{s'})$ which is integrable, tangent to each stratum of $\ell_t^{-1}(u) \cap \mathring{\mathbb{B}}_\epsilon$ and transversal to each stratum of $\ell_t^{-1}(u) \cap \mathbb{S}_\epsilon$, for any $u \in V_j \backslash \mathring{D}_{s'}$.

Now consider the vector field $\vec{\Upsilon}_j$ in $\ell_t^{-1}(V_j \backslash \mathring{D}_{s'})$ given by:

$$\vec{\Upsilon}_j := \vec{\mathcal{V}}_j + \vec{\Omega}_j,$$

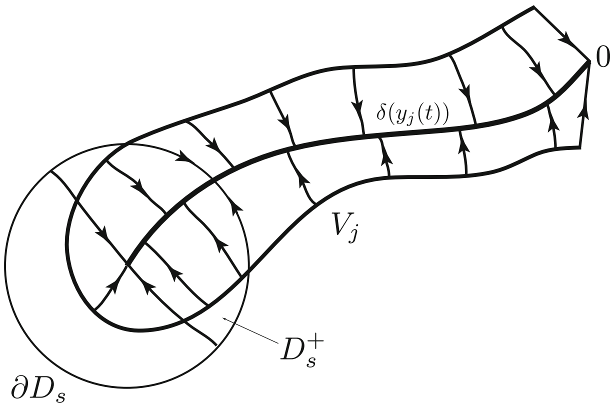


Fig. 18 The vector field $\vec{\upsilon}_j$

which is integrable, tangent to the strata of the stratification $\mathcal{S}'(t)$, transversal to the strata of $\mathbb{S}_\epsilon \cap \ell_t^{-1}(V_j \backslash \mathring{D}_{s'})$, non-zero outside P_t' and zero on P_t'.

One can see (as in the case of the vector field $\vec{\kappa}_1$ of Sect. 6.1) that each orbit of the vector field $\vec{\Upsilon}_j$ has a limit point in P_t'.

The orbits by the action of $\vec{\Upsilon}_j$ which intersect $\ell_t^{-1}(\mathring{V}_j \cap \partial D_s) \cap \mathring{B}_r$ define a set that we call $A(V_j, r)$. We set $W_{j,4}' := A(V_j, r)$ and:

- $W_{j,4} := W_{j,4}' \cup W_{j,1}$.

Finally, the set $W_{j,3}$ is given by:

- $W_{j,3} := W_{j,3}' \backslash A(V_j', r')$,

where $r' < r$, with $r - r' \ll 1$, and $V_j' := D_t \backslash q_t([0, A'[\times \partial D_t)$, with $A' < A$ and $A - A' \ll 1$. One can check that both $W_{j,3}$ and $W_{j,4}$ are open sets.

7.4 Fourth step: constructing the vector fields $\vec{\upsilon}_{j,i}$

1. Construction of $\vec{\upsilon}_{j,1}$: We can consider a continuous vector field $\vec{\upsilon}_j$ on V_j which is smooth and non-zero outside$\delta(y_j(t))$, zero on $\delta(y_j(t))$, transversal to ∂V_j and tangent to $\partial D_s \cap V_j$, like in Fig. 18 (see the construction of the vector field $\vec{\upsilon}_t$ of Lemma 25).

Let D_s^+ be a semi-disk of D_s which contains $\delta(y_j(t)) \cap \mathring{D}_s$ in its interior. We will lift $\vec{\upsilon}_j$ to a rugose vector field $\vec{\chi}_j$ in $\ell_t^{-1}(\mathring{D}_s \cap \mathring{V}_j) \cap B_r$, which is zero on $\ell_t^{-1}\big(\mathring{D}_s \cap \delta(y_j(t))\big)$, tangent to the strata of $\mathcal{S}'(t)$ and of $\ell_t^{-1}(\mathring{D}_s \cap \mathring{V}_j) \cap S_r$, where $S_r := \partial B_r$, in the following way:

- Recall that we can apply the induction hypothesis to the restriction:
$$(\ell_t)_| : \ell_t^{-1}(D_s^+ \cap \mathring{D}_s \cap \mathring{V}_j) \cap \mathring{B}_r \to D_s^+ \cap \mathring{D}_s \cap \mathring{V}_j,$$
which has an isolated singularity at $x_j(t)$ in the stratified sense, since $\ell_t^{-1}(D_s^+ \cap \mathring{D}_s \cap \mathring{V}_j)$ has complex dimension $n-1$, where n is the dimension of X.
- Then we obtain a collapsing vector field $\vec{\vartheta}_+(j)$ and a collapsing cone P_j^+. Let:
$$q_j : [0, +\infty[\times\big(\ell_t^{-1}(\mathring{D}_s^+ \cap \mathring{V}_j) \cap S_r\big) \to \ell_t^{-1}(\mathring{D}_s^+ \cap \mathring{V}_j) \cap B_r$$

be the flow associated to $\vec{\vartheta}_+(j)$ and set:

$$P_j(u) := q_j\big(\{u\} \times \ell_t^{-1}(\overset{\circ}{D}{}_s^+ \cap \mathring{V}_j) \cap S_r\big),$$

where $u \geq 0$.

- The Whitney stratification $\mathcal{S}'(t)$ induces a Whitney stratification of $P_j(u)$ (see Lemma 6 and notice that $P_j(0) = \ell_t^{-1}(\overset{\circ}{D}{}_s^+ \cap \mathring{V}_j) \cap S_r$ is the intersection of $\ell^{-1}(\overset{\circ}{D}{}_s^+ \cap \mathring{V}_j)$ with the stratified space $X_t \cap S_r$). Moreover, the restriction of ℓ_t to each stratum has maximum rank. So by Proposition 16 we can lift the vector field $\vec{v}_j$ over $\overset{\circ}{D}{}_s^+ \cap \mathring{V}_j$ to a rugose stratified vector field that is tangent to the strata of $P_j(u)$.
- On the other hand, for any point in $\ell_t^{-1}\big((\mathring{D}_s \backslash \overset{\circ}{D}{}_s^+) \cap \mathring{V}_j\big)$ we just ask the vector field $\vec{\chi}_j$ to be tangent to the strata of $\mathcal{S}'(t)$ and to lift $\vec{v}_j$. This can be done locally and then $\vec{\chi}_j$ is obtained by a partition of unity.

Notice that at any point of $\mathring{B}_r \cap \ell_t^{-1}\big(\mathring{D}_s^+ \cap \mathring{V}_j\big) - \{x_j(t)\}$ and at any point of $(\mathring{B}_r \backslash B_{r'}) \cap \ell_t^{-1}\big((\mathring{D}_s \backslash D_s^+) \cap \mathring{V}_j\big)$, for $r' < r$ with $r - r' \ll 1$, one can extend the vector field $\vec{\vartheta}_+(j)$ on a small open neighborhood. Now we construct $\vec{v}_{j,1}$ as follows:

- Over a small open neighborhood $U_{x_j(t)}$ of $x_j(t)$, consider the zero vector field.
- For any $z \in \mathring{B}_r \cap \ell_t^{-1}(D_s^+ \cap \mathring{D}_s \cap \mathring{V}_j) \backslash \{x_j(t)\}$, take an open neighborhood U_z of z small enough such that it does not contain $x_j(t)$, it is contained in $\mathring{B}_r \cap \ell_t^{-1}(\mathring{D}_s \cap \mathring{V}_j)$ and $\vec{\vartheta}_+(j)$ is well defined on it. Then in U_z we define the vector field:

$$\vec{\iota}_z := \vec{\vartheta}_+(j)_{|U_z} + \vec{\chi}_{j\,|U_z},$$

where $\vec{\vartheta}_+(j)_{|U_z}$ and $\vec{\chi}_{j\,|U_z}$ denote the restrictions of the vector fields $\vec{\vartheta}_+(j)$ and $\vec{\chi}_j$, respectively, to the neighborhood U_z. This vector field is rugose, tangent to the strata of $\mathcal{S}'(t)$, non-zero outside the intersection of U_z and P_j and zero on $P_j \cap U_z$, where:

$$P_j := P_j^+ \cap \ell_t^{-1}\big(\delta(y_j(t))\big).$$

- For any $z \in \mathring{B}_{r'} \cap \ell_t^{-1}\big((\mathring{D}_s \backslash D_s^+) \cap V_j\big)$, take a small open neighborhood U_z of z and set

$$\vec{\iota}_z := \vec{\chi}_{j\,|U_z}.$$

- For any $z \in (\mathring{B}_r \backslash B_{r'}) \cap \ell_t^{-1}\big((\mathring{D}_s \backslash D_s^+) \cap V_j\big)$, take a small open neighborhood U_z of z contained in $(\mathring{B}_r \backslash B_{r'}) \cap \ell_t^{-1}\big((\mathring{D}_s \backslash D_s^+) \cap V_j\big)$ and set:

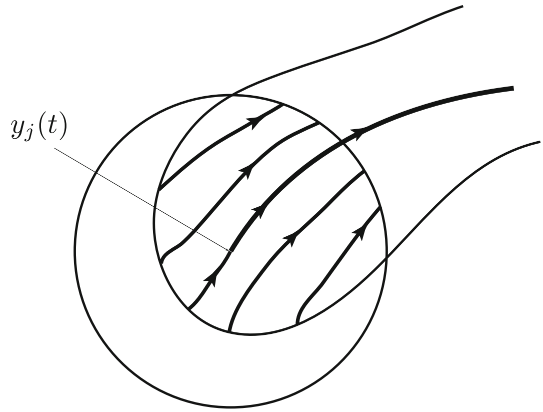
$$\vec{\iota}_z := \vec{\vartheta}_+(j)_{|U_z} + \vec{\chi}_{j\,|U_z}.$$

- Then considering a partition of unity (ρ_z) associated to the covering (U_z), we set the vector field:

$$\vec{v}_{j,1} := \sum \rho_z \vec{\iota}_z$$

in $\ell_t^{-1}(\mathring{D}_s \cap \mathring{V}_j) \cap \mathring{B}_r$, which is continuous, rugose outside the point $x_j(t)$ (and therefore in $W_{j,1} \backslash P_j$), tangent to the strata of $\mathcal{S}'(t)$, non-zero outside P_j and zero on $P_j \cap \big(\ell_t^{-1}(\mathring{D}_s \cap \mathring{V}_j)\big) \cap \mathring{B}_r$.
- Notice that if $z \in \ell_t^{-1}(\mathring{D}_s \backslash D_s^+) \cap \mathring{B}_r$, its orbit by $\vec{v}_{j,1}$ has $\{x_j(t)\}$ as limit point, and the orbit by $\vec{v}_{j,1}$ of a point $z \in \ell_t^{-1}(D_s^+) \cap \mathring{B}_r$ has its limit point in P_j.

Fig. 19 The vector field $\vec{\eta}_j$



2. Construction of $\vec{\upsilon}_{j,2}$: Consider a smooth non-zero vector field $\vec{\eta}_j$ in $\mathring{V}_j \cap \mathring{D}_s$ as Fig. 19 and such that, for any $u \in \mathring{V}_j \cap \mathring{D}_s$ one has the following implication:

$$\lambda\vec{\nu}_j(u) + \mu\vec{\eta}_j(u) = 0, \quad \lambda \geq 0, \quad \mu \geq 0 \implies \lambda = \mu = 0,$$

where $\vec{\nu}_j$ is the vector field defined above.

Then $\vec{\upsilon}_{j,2}$ is a lifting of $\vec{\eta}_j$ in $W_{j,2}$, which is rugose vector field, tangent to the strata of $\mathcal{S}'(t)$ and of $\ell_t^{-1}(\mathring{D}_s) \cap S_r$.

3. Construction of $\vec{\upsilon}_{j,3}$: We set $\vec{\upsilon}_{j,3}$ to be the restriction of the vector field $\vec{\Upsilon}_j$ constructed above to $W_{j,3}$.
4. Construction of $\vec{\upsilon}_{j,4}$: Recall the vector field $\vec{\vartheta}_+(j)$ in $\ell_t^{-1}(D_s^+) \cap \mathring{B}_r$, obtained by the induction hypothesis, and restrict it to $\ell_t^{-1}(\partial D_s \cap \mathring{V}_j)$. Then transport it by the action of the vector field $\vec{\Upsilon}_j$. We obtain a vector field $\vec{\sigma}$ on $W'_{j,4}$ that is rugose and tangent to the strata of $\mathcal{S}'(t)$.

Over $W_{j,4} = W'_{j,4} \cup W_{j,1}$, the vector fields $\vec{\sigma}$ and $\vec{\upsilon}_{j,1}$ glue in a vector field $\vec{\upsilon}_{j,4}$ that is continuous, rugose and non-zero on $W_{j,4} \backslash P_t$. The orbits of the points of $W_{j,4}$ by $\vec{\upsilon}_{j,4}$ has limit points in P_t.

7.5 Fifth step: gluing all the vector fields to obtain $\vec{\vartheta}_t$

Now, considering ρ_2, ρ_3 and ρ_4 a partition of unity associated to $W_{j,2}$, $W_{j,3}$ and $W_{j,4}$, we obtain the vector field:

$$\vec{\upsilon}_j := \rho_2\vec{\upsilon}_{j,2} + \rho_3\vec{\upsilon}_{j,3} + \rho_4\vec{\upsilon}_{j,4}$$

in $\ell_t^{-1}(\mathring{V}_j)$, which is continuous, rugose, non-zero on $\ell_t^{-1}(V_j) \backslash P_t$ and zero on P_t.

Gluing these vector fields $\vec{\upsilon}_j$, for $j = 1, \ldots, k$, we get the vector field $\vec{\upsilon}$.

Finally, gluing the vector field $\vec{\upsilon}$ in $\ell_t^{-1}(V)$ and the vector field $\vec{\tau}$ in $\ell_t^{-1}(D_t \backslash V')$ constructed in Section 6.1, we obtain a continuous vector field $\vec{\vartheta}_t$ in X_t with the properties (i) to (v) of Proposition 35. We just have to check that the orbits of this vector field have a limit point when the parameter goes to infinity:

(a) If $z \in \ell_t^{-1}(D_t \backslash V')$, the orbit of z arrives to $W_{j,2} \cup W_{j,3} \cup W_{j,4}$ after a finite time.
(b) If $z \in W_{j,2}$, the orbit of z arrives to $W_{j,3} \cup W_{j,4}$ after a finite time.

(c) If $z \in W_{j,3} \backslash W_{j,4}$, it has a limit point on $\cup_{j=1}^{k} C_j$.
(d) If $z \in W_{j,4} \backslash W_{j,3}$, it has a limit point on P_t'.
(e) If $z \in W_{j,3} \cap W_{j,4}$, we have that the orbit passing through z has a limit point that is the limit point by $\vec{\Upsilon}_j$ of the limit point of the orbit of z by $\vec{\sigma}$. Hence this limit point is on $P_j' = P_t' \cap \overline{C}_j$.

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