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Some remarks on the convergence of the Dirichlet series of L-functions and related questions

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Abstract First we show that the abscissae of uniform and absolute convergence of Dirichlet series coincide in the case of L-functions from the Selberg class $\mathcal{S}$. We also study the latter abscissa inside the extended Selberg class, indicating a different behavior in the two classes. Next we address two questions about majorants of functions in $\mathcal{S}$, showing links with the distribution of the zeros and with independence results.

Keywords Dirichlet series · Selberg class · Almost periodic functions

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1 Introduction

Let

$$F(s)=\sum_{n=1}^{\infty}\frac{a(n)}{n^s}$$

be a Dirichlet series which converges somewhere in the complex plane. It is well known that there are four classical abscissae associated with $F(s)$: the abscissa of *convergence* $\sigma_c(F)$, of *uniform convergence* $\sigma_u(F)$, of *absolute convergence* $\sigma_a(F)$ and of *boundedness* $\sigma_b(F)$. It may well be, in general, that $\sigma_c(F)=-\infty$, in which case the other three abscissae equal

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$-\infty$ as well. From the theory of Dirichlet series we know that

$$\sigma_c(F) \leq \sigma_b(F) = \sigma_u(F) \leq \sigma_a(F),$$

and in general this is best possible, i.e. inequalities cannot be replaced by equalities. We refer to Maurizi and Queffélec [15] for a modern reference for this sort of problems.

Our first result is that $\sigma_b(F) = \sigma_a(F)$ for an important class of Dirichlet series, namely those defining the L-functions of the *Selberg class* $\mathcal{S}$. We recall that the axiomatic class $\mathcal{S}$ contains, at least conjecturally, most L-functions from number theory and automorphic forms theory, and that $\sigma_b(F) = \sigma_a(F)$ is known in some special cases like the Riemann or the Dedekind zeta functions. The Selberg class $\mathcal{S}$ is defined, roughly, as the class of Dirichlet series absolutely convergent for $\sigma > 1$, having analytic continuation to $\mathbb{C}$ with at most a pole at $s = 1$, satisfying a functional equation of Riemann type and having an Euler product representation. Moreover, their coefficients satisfy the Ramanujan condition $a(n) \ll n^{\varepsilon}$ for any $\varepsilon > 0$. We also recall that the *extended Selberg class* $\mathcal{S}^\sharp$ is the larger class obtained by dropping the Euler product and Ramanujan condition requirements in the definition of $\mathcal{S}$. We refer to our survey papers [7,9,17–19] and to the forthcoming book [12] for definitions, examples and the basic theory of the Selberg classes $\mathcal{S}$ and $\mathcal{S}^\sharp$. In particular, we refer to these surveys for the definition of *degree* d_F, *conductor* q_F and *standard twist* of $F(s)$.

Theorem 1 *Suppose that $F(s)$ belongs to the Selberg class. Then*

$$\sigma_b(F) = \sigma_u(F) = \sigma_a(F). \tag{1}$$

Several months after submitting this result, the note by Brevig and Heap [3] appeared, where the authors prove the same theorem in the much more general framework of Dirichlet series with multiplicative coefficients. Trying to understand Brevig-Heap's proof, based on Bohr's theory, we noticed that their result was already known to Bohr himself in 1913 (see [1], Satz XI, p. 480); incidentally, Bohr's paper [1] appears as item [5] of the reference list in Brevig-Heap [3]. We wish to thank Dr. Mattia Righetti for bringing [1,3] to our attention and for his advice concerning these papers. We decided to keep Theorem 1 since our proof is different, easier and more direct; moreover, some points in the proof will be useful for the other results in the paper.

We expect that actually $\sigma_a(F) = 1$ *for all* $F \in \mathcal{S}$. This is known for most classical L-functions and, in the general case of the class $\mathcal{S}$, under the assumption of the *Selberg orthonormality conjecture*; however, an unconditional proof is missing at present. See again the above quoted references for definitions and results about such a conjecture.

Note that the abscissa of convergence $\sigma_c(F)$ can be smaller than 1 for functions in $\mathcal{S}$. For example, the Dirichlet L-functions $L(s, \chi)$ with a primitive non-principal character χ are convergent in the half-plane $\sigma > 0$. Actually, several general results are known about the abscissa $\sigma_c(F)$ for functions $F(s)$ in the extended Selberg class $\mathcal{S}^\sharp$. First of all

$$\textit{if } F \in \mathcal{S}^\sharp \textit{ is entire with degree } d \geq 1, \textit{ then } \frac{1}{2} - \frac{1}{2d} \leq \sigma_c(F) \leq 1 - \frac{2}{d+1} \tag{2}$$

(recall that *there exist no functions* $F \in \mathcal{S}^\sharp$ *with degree* $0 < d < 1$, see [8] and Conrey and Ghosh [5]). Indeed, the first inequality in (2) is Corollary 3 in [11] and is based on the properties of the standard twist, while the second inequality follows from a well known theorem of Landau [13]. Moreover, in accordance with classical degree 2 conjectures and with the general Ω-theorem in Corollary 2 of [11], we expect that equality holds in the left inequality in (2). Further

$$\sigma_c(F) = -\infty \quad \textit{if and only if} \quad d_F = 0,$$

since the degree 0 functions of $\mathcal{S}^\sharp$ are Dirichlet polynomials (see [8]). From (2) we also deduce that

$$\sigma_c(F) = 1 \text{ if and only if } F(s) \text{ has a pole at } s = 1.$$

We also remark that *if the Lindelöf Hypothesis holds for* $F \in \mathcal{S}^\sharp$, *then* $\sigma_c(F) \leq 1/2$.

The behavior of $\sigma_a(F)$ in the extended class $\mathcal{S}^\sharp$ is different from the expected behavior in $\mathcal{S}$. Indeed, in the next section, which is also of independent interest, we show that

$$\text{there exist functions } F \in \mathcal{S}^\sharp \quad \text{with } \sigma_a(F) \text{ arbitrarily close to } 1/2.$$

We conclude this section with the following

Question. Does (1) hold for the functions in the extended Selberg class ? □

A variant of the question is: does (1) hold for linear combinations

$$F(s) = \sum_{j=1}^{N} c_j F_j(s)$$

with $F_j \in \mathcal{S}$ and $c_j \in \mathbb{C}$? If needed, one may assume that $F(s)$ belongs to $\mathcal{S}^\sharp$.

Since $\sigma_a(F) = 1$ for most classical L-functions $F(s)$, Theorem 1 prevents the possibility of getting information on the non-trivial zeros exploiting the properties of the abscissa of uniform convergence. On the other hand, if $F \in \mathcal{S}$ is bounded for $\sigma > 1 - \delta$ for some $\delta > 0$, then its Dirichlet series is absolutely convergent for $\sigma > 1 - \delta$ and hence $F(s) \neq 0$ by Euler's identity. In the next theorems we replace boundedness by more general majorants and deduce some consequences.

Let $F \in \mathcal{S}$ be of degree d, $N_F(\sigma, T)$ be the number of zeros $\rho = \beta + i\gamma$ with $\beta > \sigma$ and $|\gamma| \leq T$, and denote the *density abscissa* $\sigma_D(F)$ by

$$\sigma_D(F) = \inf\{\sigma : N_F(\sigma, T) = o(T)\}.$$

An inspection of the proof of Lemma 3 in [10], obtained by a rudimentary version of Montgomery's zero-detecting method, shows that

$$N_F(\sigma, T) \ll T^{4(d+3)(1-\sigma)+\varepsilon}.$$

Hence in general

$$\frac{1}{2} \leq \sigma_D(F) \leq 1 - \frac{1}{4(d+3)},$$

although it is well known that the classical L-functions $F(s)$ of degree 1 and 2 have $\sigma_D(F) = 1/2$, see e.g. Luo [14]. Actually, one can prove that $\sigma_D(F) = 1/2$ for all $F \in \mathcal{S}$ with degree $0 < d \leq 2$. Further, let $f(s)$ be holomorphic in $\sigma > 1-\delta$ for some $\delta > 0$ and almost periodic on the line $\sigma = A$ for some $A > 1$. We say that $f(s)$ is a δ-*almost periodic majorant* of $F(s)$ if

$$|F(s)| \leq c(\sigma)|f(s)| \tag{3}$$

in the half-plane $\sigma > 1 - \delta$, where $c(\sigma) > 0$ is a continuous function for $\sigma > 1 - \delta$.

Theorem 2 *Let* $F \in \mathcal{S}$ *and* $f(s)$ *be a* δ*-almost periodic majorant of* $F(s)$*. Then* $F(s)$ *and* $f(s)$ *have the same zeros, with the same multiplicity, in the half-plane* $\sigma > \max(1 - \delta, \sigma_D(F))$.

Remark Clearly, in view of (3) each zero of $f(s)$ is also a zero of $F(s)$; the non-trivial part of Theorem 2 says that the opposite assertion holds true as well. Note that we do not require that $f(s)$ is almost periodic for $\sigma > 1-\delta$, but only on some vertical line far on the right. We already noticed that, as a consequence of Theorem 1, $F(s) \neq 0$ in every right half-plane where it is bounded. An immediate consequence of Theorem 2 is that $F(s) \neq 0$ for $\sigma > \max(1-\delta, \sigma_D(F))$ if $f(s)$ is a non-vanishing δ-almost periodic majorant. In particular, from the density estimates reported above when $d \leq 2$, if $\delta = 1/2$ then the Riemann Hypothesis holds for such $F(s)$. □

Our final result is a kind of new independence statement for L-functions from the Selberg class. Several forms of independence are known in $\mathcal{S}$, such as the linear independence, the multiplicity one property and the orthogonality conjecture and some of its consequences; see our above quoted surveys on the Selberg class. The new independence result is expressed in terms of majorants as follows.

Theorem 3 *Let $F, G \in \mathcal{S}$ be such that $F(s) \ll |G(s)|$ for $\sigma > 1/2$. Then $F(s) = G(s)$.*

The special nature of the majorant is very important here. Indeed, suppose that $G(s)$ is entire; then Theorem 2 gives only that $F(s)$ and $G(s)$ have the same zeros for $\sigma > \sigma_D(F)$. Instead, exploiting the information that $G \in \mathcal{S}$, Theorem 3 shows that actually $F(s) = G(s)$. In other words, no function from $\mathcal{S}$ can dominate in $\sigma > 1/2$ another function from $\mathcal{S}$. We may regard this as a weak form of a well known result obtained, under stronger assumptions, by Selberg [20] and Bombieri and Hejhal [2] about the statistical independence of the values of L-functions.

2 The lift operator

Let $Q > 0$, $\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_r)$ with $\lambda_j > 0$, $\boldsymbol{\mu} = (\mu_1, \ldots, \mu_r)$ with $\mu_j \in \mathbb{C}$ and $|\omega| = 1$. We denote by $W(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ the $\mathbb{R}$-linear space of the Dirichlet series $F(s)$ solutions of the functional equation

$$Q^s \prod_{j=1}^{r} \Gamma(\lambda_j s + \mu_j) F(s) = \omega Q^{1-s} \prod_{j=1}^{r} \Gamma(\lambda_j (1-s) + \overline{\mu_j}) \overline{F(1-\overline{s})}. \tag{4}$$

Given an integer $k \geq 1$, we define the *k-lift operator* by

$$F(s) \longmapsto F_k(s) = F\left(ks + \frac{1-k}{2}\right);$$

clearly, the operator is trivial for $k = 1$. A simple computation shows that

$$if\, F \in W(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega) \quad then \quad F_k \in W\left(Q^k, k\boldsymbol{\lambda}, \boldsymbol{\mu} + \frac{1-k}{2}\boldsymbol{\lambda}, \omega\right). \tag{5}$$

In particular, from (5) we have that degree d_{F_k} and conductor q_{F_k} of $F_k(s)$ satisfy

$$d_{F_k} = k d_F \quad q_{F_k} = q_F^k k^{k d_F}. \tag{6}$$

We recall (see the above references) that the class $\mathcal{S}^\sharp$ consists of the Dirichlet series satisfying a functional equation of type (4), where now $\Re \mu_j \geq 0$, with the following properties:

$F(s)$ is absolutely convergent for $\sigma > 1$ and $(s-1)^m F(s)$ is entire of finite order for some integer $m \geq 0$. Therefore we consider

$$B_F = 2 \min_{1 \leq j \leq r} \frac{\Re \mu_j}{\lambda_j} + 1,$$

which *is an invariant of* $\mathcal{S}^\sharp$ (see again the above references) since a simple computation shows that

$$B_F = -2 \max_\rho \Re \rho + 1,$$

where ρ runs over the trivial zeros of $F(s)$. From the definition of the k-lift operator and (5) we see that, given $F \in \mathcal{S}^\sharp$, *the lifted function* $F_k(s)$ *also belongs to* $\mathcal{S}^\sharp$ *provided* $1 \leq k \leq B_F$ *and, if* $B_F \geq 2$, $F(s)$ *is entire*. Indeed, if $k \geq 2$, $F(s)$ has to be holomorphic at $s = 1$ otherwise the pole of $F_k(s)$ is not at $s = 1$, and the bound $k \leq B_F$ is needed to have non-negative real part of the μ's data of $F_k(s)$. Therefore, defining $V(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$ to be the $\mathbb{R}$-linear space of the entire functions $F \in \mathcal{S}^\sharp$ satisfying (4) (again with $\Re \mu_j \geq 0$), we have that

$$\begin{gathered} for\ 1 \leq k \leq B_F, the\ k-lift\ operator\ maps\ V(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)\ into \\ V\left(Q^k, k\boldsymbol{\lambda}, \boldsymbol{\mu} + \frac{1-k}{2}\boldsymbol{\lambda}, \omega\right). \end{gathered}$$

Note that B_F depends only on $\boldsymbol{\lambda}$ and $\boldsymbol{\mu}$, so it is the same for all functions in $V(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$. Note also that *the Selberg class* $\mathcal{S}$ *is not preserved under the above mappings* since the Ramanujan condition is not (necessarily) satisfied by $F_k(s)$ even if $F(s)$ does; see the examples below. Further, a simple computation shows that *the* k*-lift operator commutes with the map sending* $F(s)$ *to its standard twist*. We also remark that the requirement $\Re \mu_j \geq 0$ in the definition of $\mathcal{S}^\sharp$, which is responsible for the limitation $k \leq B_F$ in (6), is apparently not of primary importance in the theory of the Selberg class. Hence, although formally not belonging to $\mathcal{S}^\sharp$, the lifts $F_k(s)$ of entire $F \in \mathcal{S}^\sharp$ with $k > B_F$ are further examples of Dirichlet series with continuation over $\mathbb{C}$ and functional equation. A similar remark applies to the other condition in the definition of $V(Q, \boldsymbol{\lambda}, \boldsymbol{\mu}, \omega)$, namely the holomorphy at $s = 1$.

Example The Riemann zeta function $\zeta(s)$ cannot be lifted inside $\mathcal{S}^\sharp$ since it has $B_\zeta = 1$. The same holds for the Dirichlet L-functions with even primitive characters, while those associated with odd primitive characters may be lifted inside $\mathcal{S}^\sharp$ for $k = 2$ and $k = 3$. However, after lifting their Dirichlet coefficients do not satisfy the Ramanujan condition, hence the lifted Dirichlet L-functions do not belong to $\mathcal{S}$. Note that, once suitably normalized, the lifts with $k = 2$ become the L-functions associated with half-integral weight modular forms; see the books by Hecke [6] and Ogg [16]. Concerning degree 2, we consider the L-functions associated with holomorphic eigenforms of level N and integral weight K; see Ogg [16]. Denoting by $F(s)$ their normalization satisfying a functional equation reflecting $s \mapsto 1 - s$ (instead of the original $s \mapsto K - s$), we have that

$$B_F = K.$$

Hence the normalized L-functions of eigenforms of weight K may be lifted inside $\mathcal{S}^\sharp$ with k up to their weight. Here we consider only eigenforms since in general the L-functions of modular forms of level N satisfy a slightly different functional equation, not of $\mathcal{S}^\sharp$ type.

□

We finally turn to the problem of the absolute convergence abscissa in $\mathcal{S}^\sharp$. Let $F \in \mathcal{S}^\sharp$ be of degree $d \geq 1$. Then, thanks again to the properties of the standard twist, we know that

$$\sigma_a(F) \geq \frac{1}{2} + \frac{1}{2d}; \tag{7}$$

this folows from Theorem 1 of [11]. On the other hand, if $F \in \mathcal{S}^\sharp$ we have that the series

$$\sum_{n=1}^{\infty} \frac{|a(n)|}{n^{k\sigma+(1-k)/2}}$$

converges for $\sigma > 1/2 + 1/(2k)$. Hence from (6) and (7) we obtain that if both $F(s)$ of degree $d \geq 1$ and $F_k(s)$ belong to $\mathcal{S}^\sharp$, then

$$\frac{1}{2} + \frac{1}{2kd} \leq \sigma_a(F_k) \leq \frac{1}{2} + \frac{1}{2k}. \tag{8}$$

Since the above examples show that there exist functions $F \in \mathcal{S}^\sharp$ with arbitrarily large B_F (e.g. the holomorphic eigenforms with arbitrarily large weight K), (8) shows that $\sigma_a(F)$ can be arbitrarily close to $1/2$ inside $\mathcal{S}^\sharp$. Hence the behavior of $\sigma_a(F)$ in the extended class $\mathcal{S}^\sharp$ is definitely different from its expected behavior in the class $\mathcal{S}$.

3 Proof of Theorem 1

Observe that the case $d = 0$ is trivial, since $F(s)$ is identically 1; see Conrey and Ghosh [5]. For d positive we have $\sigma_b(F) \geq 1/2$, since $F(s)$ is unbounded for $\sigma < 1/2$ by the functional equation and the properties of the Γ function. Therefore, to prove the assertion it suffices to show the following fact: if for a certain $1/2 < \sigma_0 \leq 1$ the function $F(s)$ is bounded for $\sigma > \sigma_0$, then $\sigma_a(F) \leq \sigma_0$.

Let us fix an $\varepsilon \in (0, \sigma_0 - 1/2)$, and let $c_0 = c_0(\varepsilon)$ be such that $|a(n)| \leq c_0 n^{\varepsilon/2}$ for all $n \geq 1$. Without loss of generality we may assume that $c_0 \geq 3$. Consider the finite set of primes

$$S_\varepsilon = \{p : |a(p)| > p^{\varepsilon/2} \ \text{ or } \ p < c_0^{2/\varepsilon}\}.$$

Let

$$F_p(s) = \sum_{m=0}^{\infty} \frac{a(p^m)}{p^{ms}} \tag{9}$$

denote the pth Euler factor of $F(s)$. We split the Euler product as

$$\begin{aligned} F(s) &= \prod_{p \notin S_\varepsilon} \left(1 + \frac{a_F(p)}{p^s}\right) \prod_{p \in S_\varepsilon} F_p(s) \prod_{p \notin S_\varepsilon} \left(F_p(s) \left(1 + \frac{a_F(p)}{p^s}\right)^{-1}\right) \\ &= P_1(s) P_2(s) P_3(s), \end{aligned} \tag{10}$$

say. Both $P_2(s)$ and its inverse $1/P_2(s)$ have Dirichlet series representations which converge absolutely for $\sigma > \theta$ for some $\theta < 1/2$. This is a simple consequence of the definition of the Selberg class; see the above quoted references. Therefore, $P_2(s)$ and $1/P_2(s)$ are bounded for $\sigma > \sigma_0$.

In view of (9) we have

$$P_3(s) = \prod_{p \notin S_\varepsilon} \left(1 + \sum_{m=2}^{\infty} \frac{b(p^m)}{p^{ms}}\right)$$

with

$$b(p^m) = \sum_{l=0}^{m} (-1)^l a(p)^l a(p^{m-l}).$$

Hence, recalling that $p \notin S_\varepsilon$, $m \geq 2$ and $c_0 \geq 3$, we have

$$|b(p^m)| \leq \sum_{l=0}^{m} |a(p)|^l |a(p^{m-l})| \leq c_0 m p^{m\varepsilon/2} \leq p^{m\varepsilon}.$$

Thus for $\sigma > 1/2 + \varepsilon$ and $p \notin S_\varepsilon$ we have

$$\sum_{m=2}^{\infty} \frac{|b(p^m)|}{p^{m\sigma}} < 1 \quad \text{and} \quad \sum_{p \notin S_\varepsilon} \sum_{m=2}^{\infty} \frac{|b(p^m)|}{p^{m\sigma}} \ll 1.$$

Hence both $P_3(s)$ and $1/P_3(s)$ are bounded and have Dirichlet series representations which converge absolutely for $\sigma > \sigma_0$ (recall that $\sigma_0 > 1/2 + \varepsilon$).

We therefore see that $P_1(s) = F(s)/(P_2(s)P_3(s))$ is bounded for $\sigma > \sigma_0$. Let us write

$$P_1(s) = \sum_{n=1}^{\infty} \frac{c(n)}{n^s}.$$

The coefficients $c(n)$ are completely multiplicative, and the series converges for $\sigma > \sigma_0$. Fix such a σ, and a positive $\delta < \sigma - \sigma_0$. Consider the following familiar Mellin's transform

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^{\sigma+it}} e^{-n/Y} = \frac{1}{2\pi i} \int_{1-i\infty}^{1+i\infty} \frac{F(w+\sigma+it)}{P_2(w+\sigma+it)P_3(w+\sigma+it)} \Gamma(w) Y^w \mathrm{d}w.$$

We shift the line of integration to $\Re(w) = -\delta$ and obtain

$$\sum_{n=1}^{\infty} \frac{c(n)}{n^{\sigma+it}} e^{-n/Y} = \frac{F(\sigma+it)}{P_2(\sigma+it)P_3(\sigma+it)} + O(Y^{-\delta}) \ll 1$$

uniformly in $t \in \mathbb{R}$ and $Y \geq 1$. Since $|c(n)| \leq n^{\varepsilon/2}$, due to the decay of the exponential we may cut the sum on the left hand side to $n \leq 3Y \log Y$, say, producing an extra error term of size $O(1/Y)$. Thus

$$\sum_{n \leq 3Y \log Y} \frac{c(n)}{n^{\sigma+it}} e^{-n/Y} \ll 1 \tag{11}$$

uniformly in $t \in \mathbb{R}$ and $Y \geq 1$.

Now we apply Kronecker's theorem in the following form, see Theorem 8 of Ch. VIII of Chandrasekharan [4]. If $\theta_1, \ldots, \theta_k \in \mathbb{R}$ are linearly independent over $\mathbb{Z}$, $\beta_1, \ldots, \beta_k \in \mathbb{R}$ and $T, \eta > 0$, then there exist $t > T$ and $n_1, \ldots, n_k \in \mathbb{Z}$ such that

$$|t\theta_\ell - n_\ell - \beta_\ell| < \eta \qquad \ell = 1, \ldots, k. \tag{12}$$

We choose the θ's as $-\frac{1}{2\pi} \log p$ with the primes $p \leq 3Y \log Y$ not in S_ε and, correspondingly, the β's such that $|c(p)| = c(p)e^{2\pi i \beta_p}$ for each such p. Hence by (12) there exists a sequence of real numbers $t_\nu \to +\infty$ such that

$$c(p)p^{-it_\nu} \to |c(p)| \qquad \nu \to \infty$$

uniformly for the primes $p \leq 3Y \log Y$ not in S_ε. By the complete multiplicativity of $c(n)$ we infer that

$$c(n)n^{-it_\nu} \to |c(n)| \qquad \nu \to \infty$$

uniformly for $n \leq 3Y \log Y$. Thus putting $t = t_\nu$ in (11) and making $\nu \to \infty$ we obtain

$$\sum_{n \leq Y} \frac{|c(n)|}{n^\sigma} \leq e \sum_{n \leq 3Y \log Y} \frac{|c(n)|}{n^\sigma} e^{-n/Y} = e \lim_{\nu \to \infty} \sum_{n \leq 3Y \log Y} \frac{c(n)}{n^{\sigma + it_\nu}} e^{-n/Y} \ll 1$$

uniformly for $Y \geq 1$. Letting $Y \to \infty$, we see that the Dirichlet series of $P_1(s)$ converges absolutely for $\sigma > \sigma_0$.

Summarizing, we have shown that the Dirichlet series of $P_1(s)$, $P_2(s)$ and $P_3(s)$ are absolutely convergent for $\sigma > \sigma_0$, hence the Dirichlet series of $F(s)$ is also absolutely convergent for $\sigma > \sigma_0$ thanks to (10), and the result follows.

4 Proof of Theorem 2

As in Theorem 1 the case $d = 0$ is trivial, hence we assume $d > 0$ and consider the function

$$h(s) = \frac{F(s)}{f(s)}$$

for $\sigma > 1 - \delta$. From (3) we have that $h(s)$ is holomorphic for $\sigma > 1 - \delta$, bounded on every closed vertical strip inside $\sigma > 1 - \delta$ and almost periodic on the line $\sigma = A$. For a given $\varepsilon > 0$, let τ be an ε-almost period of $h(A + it)$, namely for every $t \in \mathbb{R}$

$$|h(A + i(t + \tau)) - h(A + it)| < \varepsilon.$$

Then, by the convexity following from Phragmén-Lindelöf's theorem applied to $h(s + i\tau) - h(s)$, given $\eta > 1 - \delta$ and any $\eta < \sigma < A$ we have

$$\sup_{t \in \mathbb{R}} |h(\sigma + i(t + \tau)) - h(\sigma + it)| \leq \Big(\sup_{t \in \mathbb{R}} |h(\eta + i(t + \tau)) - h(\eta + it)| \Big)^{\frac{A - \sigma}{A - \eta}} \times \Big(\sup_{t \in \mathbb{R}} |h(A + i(t + \tau)) - h(A + it)| \Big)^{\frac{\sigma - \eta}{A - \eta}}.$$

Hence we obtain that

$$\sup_{t \in \mathbb{R}} |h(\sigma + i(t + \tau)) - h(\sigma + it)| \ll \varepsilon^c$$

uniformly in any closed strip contained in $\eta < \sigma < A$, where $c > 0$ depends on the strip. Since ε is arbitrarily small, $h(s)$ is uniformly almost periodic in such strips. Suppose now that $h(\rho) = 0$ for some ρ with $\Re\rho > 1 - \delta$. Then by a well known argument based on Rouché's theorem we have that for any $1 - \delta < \eta < \Re\rho$

$$T \ll N_h(\eta, T) \leq N_F(\eta, T) = o(T)$$

if $\eta > \sigma_D(F)$, a contradiction. Thus $h(s) \neq 0$ for $\sigma > \max(1 - \delta, \sigma_D(F))$, hence every zero of $F(s)$ in this half-plane is a zero of $f(s)$. Theorem 2 is therefore proved, since the opposite implication is a trivial consequence of (3).

5 Proof of Theorem 3

Again the case $d = 0$ is trivial, since in this case $F(s) \equiv 1$ and so $G(s)$ does not vanish inside the critical strip, thus its degree is 0 and hence $G(s) \equiv 1$ as well. Let $F, G \in \mathcal{S}$ be with positive degrees and coefficients $a_F(n)$ and $a_G(n)$, respectively, and consider the function

$$H(s) = \frac{F(s)}{G(s)} = \sum_{n=1}^{\infty} \frac{h(n)}{n^s},$$

say. By our hypothesis $H(s)$ is bounded, and hence holomorphic, for $\sigma > 1/2$. We modify the proof of Theorem 1 at several points. By Lemma 1 of [10] we have that for every $\varepsilon > 0$ there exists an integer $K = K(\varepsilon)$ such that the coefficients $a_G^{-1}(n)$ of $1/G(s)$ satisfy

$$a_G^{-1}(n) \ll n^{\varepsilon} \qquad (n, K) = 1,$$

and hence

$$h(n) \ll n^{2\varepsilon} \qquad (n, K) = 1.$$

Therefore the set

$$S = \{p : |h(p^m)| > p^{m/10} \quad \text{for some} \quad m \geq 1 \text{ or } p \leq 10^4\}$$

is finite and we write

$$\begin{aligned} H(s) &= \prod_p \frac{F_p(s)}{G_p(s)} = \prod_p H_p(s) \\ &= \prod_{p \notin S} \left(1 + \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}}\right) \prod_{p \in S} H_p(s) \prod_{p \notin S} \left(H_p(s) \left(1 + \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}}\right)^{-1}\right) \\ &= Q_1(s) Q_2(s) Q_3(s), \end{aligned} \tag{13}$$

say. As in the prof of Theorem 1, $Q_2(s)$ and $1/Q_2(s)$ are holomorphic and bounded for $\sigma \geq 1/2$. Moreover we have

$$Q_3(s) = \prod_{p \notin S} \left(1 + \frac{\sum_{m=3}^{\infty} \frac{h(p^m)}{p^{ms}}}{1 + \frac{h(p)}{p^s} + \frac{h(p^2)}{p^{2s}}}\right) = \prod_{p \notin S} \left(1 + \sum_{m=3}^{\infty} \frac{k(p^m)}{p^{ms}}\right),$$

say, and a computation shows that for $\sigma \geq 1/2$

$$\sum_{m=3}^{\infty} \frac{|k(p^m)|}{p^{m\sigma}} \leq \frac{1}{3} \quad \text{for every} \quad p \notin S \quad \text{and} \quad \sum_{p \notin S} \sum_{m=3}^{\infty} \frac{|k(p^m)|}{p^{m\sigma}} \ll 1.$$

Therefore, no factor of the product vanishes, and $Q_3(s)$ and $1/Q_3(s)$ are holomorphic and bounded for $\sigma \geq 1/2$ as well.

In order the treat $Q_1(s)$ we need the following elementary lemma.

Lemma *For every $a, b \in \mathbb{C}$ there exists $\theta \in \mathbb{C}$ with $|\theta| = 1$ such that*

$$|1 + \theta a + \theta^2 b| \geq 1 + \frac{1}{24}(|a| + |b|).$$

Proof Suppose first that $|a| \le |b|/2$. Then

$$\max_{|\theta|=1} |1+\theta a+\theta^2 b| \ge 1+|b|-|a| \ge 1+\frac{1}{2}|b| \ge 1+\frac{1}{3}(|a|+|b|),$$

and the result follows in this case. In the opposite case $|a| > |b|/2$ we apply the maximum modulus principle to the function $f(z) = 1+az+bz^2$, thus obtaining

$$\begin{aligned}\max_{|\theta|=1} |1+\theta a+\theta^2 b| &\ge \max_{|\theta|=1} \left|1+\frac{1}{4}\theta a+\frac{1}{16}\theta^2 b\right| \\ &\ge 1+\frac{1}{4}|a|-\frac{1}{16}|b| \ge 1+\frac{1}{24}(|a|+|b|),\end{aligned}$$

and the Lemma follows. Note that the constant $1/24$ is neither optimal nor important in what follows; moreover, in general it cannot be made arbitrarily close to 1. □

From (13), our hypothesis and the above information on $Q_2(s)$ and $Q_3(s)$ we deduce that there exists $M > 0$ such that for $\sigma > 1/2$

$$|Q_1(s)| = \prod_{p\notin S} \left|1+p^{-it}\frac{h(p)}{p^\sigma}+p^{-2it}\frac{h(p^2)}{p^{2\sigma}}\right| \le M.$$

By the Lemma, for every σ and p there exists $|\theta_{p,\sigma}| = 1$ such that

$$\left|1+\theta_{p,\sigma}\frac{h(p)}{p^\sigma}+\theta_{p,\sigma}^2\frac{h(p^2)}{p^{2\sigma}}\right| \ge 1+\frac{1}{24}\left(\frac{|h(p)|}{p^\sigma}+\frac{|h(p^2)|}{p^{2\sigma}}\right).$$

Assuming that $\sigma > 1/2$ and $p \notin S$, applying Kronecker's theorem as in the last part of the proof of Theorem 1 we find that

$$\prod_{p\notin S}\left(1+\frac{1}{24}\left(\frac{|h(p)|}{p^\sigma}+\frac{|h(p^2)|}{p^{2\sigma}}\right)\right) \le M.$$

Then, letting $\sigma \to 1/2^+$, we deduce that the product

$$\prod_{p\notin S}\left(1+\frac{1}{24}\left(\frac{|h(p)|}{p^{1/2}}+\frac{|h(p^2)|}{p}\right)\right)$$

is convergent. Thus the series

$$\sum_{p\notin S}\left(\frac{|h(p)|}{p^{1/2}}+\frac{|h(p^2)|}{p}\right)$$

is convergent as well and, in turn, the product

$$\prod_{p\notin S}\left(1+\left(\frac{|h(p)|}{p^{1/2}}+\frac{|h(p^2)|}{p}\right)\right)$$

converges. Hence $Q_1(s)$ and $Q_1(s)^{-1}$ are non-vanishing for $\sigma \ge 1/2$.

From (13) and the above properties of $Q_j(s)$, $j = 1, 2, 3$, we immediately see that $H(s)$ is holomorphic and non-vanishing for $\sigma \ge 1/2$. Denoting by $\gamma_F(s)$ and $\gamma_G(s)$ the γ-factors of $F(s)$ and $G(s)$, thanks to the functional equation we deduce that

$$\frac{\gamma_F(s)}{\gamma_G(s)}H(s)$$

is a non-vanishing entire function of order ≤ 1, and hence by Hadamard's theory we have

$$H(s) = \frac{\gamma_G(s)}{\gamma_F(s)} e^{as+b} \tag{14}$$

with some $a, b \in \mathbb{C}$. Now we can conclude by means of the almost periodicity argument that we used in our proof of the multiplicity one property of $\mathcal{S}$. For this we refer to Lemma 2.1 of [9] and to Theorem 2.3.2 of [7]; in particular, (14) is exactly the last displayed formula of p. 167 of [7]. This way we get that $H(s) \equiv 1$, hence Theorem 3 is proved.

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References

1. Bohr, H.: Über die Bedeutung der Potenzreihen unendlich vieler Variabeln in der Theorie der Dirichletschen Reihen $\sum \frac{a_n}{n^s}$. Nachr. Akad. Wiss. Göttingen Math. Phys. Kl. 441–488 (1913); see also Collected Mathematical Works, vol. I, Dansk Matematisk Forening (1952)
2. Bombieri, E., Hejhal, D.A.: On the distribution of zeros of linear combinations of Euler products. Duke Math. J. **80**, 821–862 (1995)
3. Brevig, O.F., Heap, W.: Convergence abscissas for Dirichlet series with multiplicative coefficients. Expo. Math. (2015, to appear). doi:10.1016/j.exmath.2015.10.005
4. Chandrasekharan, K.: Introduction to Analytic Number Theory. Springer, New York (1968)
5. Conrey, J.B., Ghosh, A.: On the Selberg class of Dirichlet series: small degrees. Duke Math. J. **72**, 673–693 (1993)
6. Hecke, E.: Lectures on Dirichlet Series, Modular Functions and Quadratic Forms. Vanderhoeck & Ruprecht, Gottingen (1983)
7. Kaczorowski, J.: Axiomatic theory of L-functions: the Selberg class. In: Perelli, A., Viola, C. (eds.) Analytic Number Theory, C.I.M.E. Summer School, Cetraro (Italy). Lecture Notes in Mathematics, vol. 1891, pp. 133–209. Springer, Berlin, Heidelberg (2006)
8. Kaczorowski, J., Perelli, A.: On the structure of the Selberg class, I: $0 \leq d \leq 1$. Acta Math. **182**, 207–241 (1999)
9. Kaczorowski, J., Perelli, A.: The Selberg class: a survey. In: Győry, K. et al. (eds.) Proceedings of the Conference in Honor of A. Schinzel on Number Theory in Progress, pp. 953–992. de Gruyter (1999)
10. Kaczorowski, J., Perelli, A.: On the prime number theorem for the Selberg class. Arch Math. **80**, 255–263 (2003)
11. Kaczorowski, J., Perelli, A.: On the structure of the Selberg class, VI: non-linear twists. Acta Arith. **116**, 315–341 (2005)
12. Kaczorowski, J., Perelli, A.: Introduction to the Selberg Class of L-Functions (**in preparation**)
13. Landau, E.: Über die Anzahl der Gitterpunkte in gewissen Bereichen (Zweite Abhandlung). Nachr. Ges. Wiss. Göttingen Math.-Phys. Kl. 209–243 (1915)
14. Luo, W.: Zeros of Hecke L-functions associated with cusp forms. Acta Arith. **71**, 139–157 (1995)
15. Maurizi, B., Queffélec, H.: Some remarks on the algebra of bounded Dirichlet series. J. Fourier Anal. Appl. **16**, 676–692 (2010)
16. Ogg, A.: Modular Forms and Dirichlet Series. Benjamin, New York (1969)
17. Perelli, A.: A survey of the Selberg class of L-functions, part I. Milan J. Math. **73**, 19–52 (2005)
18. Perelli, A.: A survey of the Selberg class of L-functions, part II. Riv. Mat. Univ. Parma **3**(7), 83–118 (2004)
19. Perelli, A.: Non-linear twists of L-functions: a survey. Milan J. Math. **78**, 117–134 (2010)
20. Selberg, A.: Old and new conjectures and results about a class of Dirichlet series. In: Bombieri, E. et al. (eds.) Proceedings of the Amalfi Conference on Analytic Number Theory, pp. 367–385. Universitá di Salerno 1992; Collected Papers vol. II, pp. 47–63. Springer (1991)