Math. Z. (2017) 285:493–517
DOI 10.1007/s00209-016-1718-6

Mathematische Zeitschrift

Vanishing of Rabinowitz Floer homology on negative line bundles

Peter Albers[1] · **Jungsoo Kang**[1]

Received: 30 September 2015 / Accepted: 19 April 2016 / Published online: 20 June 2016

Abstract Following Frauenfelder (Rabinowitz action functional on very negative line bundles, Habilitationsschrift, Munich/München, 2008), Albers and Frauenfelder (Bubbles and onis, 2014. arXiv:1412.4360) we construct Rabinowitz Floer homology for negative line bundles over symplectic manifolds and prove a vanishing result. Ritter (Adv Math 262:1035–1106, 2014) showed that symplectic homology of these spaces does not vanish, in general. Thus, the theorem SH $= 0 \Leftrightarrow$ RFH $= 0$ (Ritter in J Topol 6(2):391–489, 2013), does *not* extend beyond the symplectically aspherical situation. We give a conjectural explanation in terms of the Cieliebak–Frauenfelder–Oancea long exact sequence Cieliebak et al. (Ann Sci Éc Norm Supér (4) 43(6):957–1015, 2010).

1 Introduction

Negative line bundles give rise to a rather special class of contact manifolds which nevertheless contains many interesting examples. They arise at many places in modern contact and symplectic geometry such as Givental's nonlinear Maslov index [17] and more generally contact rigidity [7,8,12,25] etc.

Let us be more specific. We choose a closed connected symplectic manifold (M, ω) with integral symplectic form $[\omega] \in \mathrm{H}^2(M, \mathbb{Z})$. We denote by $\wp : \Sigma \to M$ the principal S^1-bundle and by $\wp : E \to M$ the associated complex line bundle with first Chern class $c_1^E = -[\omega]$. We refer to these bundles as *negative line bundles*. There exists an S^1-invariant 1-form α on Σ, and hence $E \backslash M$, with the property

$$d\alpha = \wp^* \omega \tag{1.1}$$

✉ Peter Albers
peter.albers@wwu.de

Jungsoo Kang
jungsoo.kang@me.com

1 Mathematisches Institut, Westfälische Wilhelms-Universität Münster, Münster, Germany

which is a contact form on Σ. For more details we refer to [16, Section 7.2]. If we denote by r the radial coordinate on E then the 2-form

$$\Omega := d\big(\pi r^2\alpha\big) + \wp^*\omega = 2\pi r dr \wedge \alpha + \big(\pi r^2 + 1\big)\wp^*\omega \tag{1.2}$$

is a symplectic form on E. Throughout this article we make the assumption that (E, Ω) is semi-positive, see [20, Definition 6.4.1] and page 5 for an equivalent formulation.

Theorem 1.1 *The Rabinowitz Floer homology* $\mathrm{RFH}(\Sigma, E)$ *is well-defined.*

In many situations we are able to prove the following vanishing result.

Theorem 1.2 *We assume that one of the following is satisfied.*

(1) (M, ω) *is symplectically aspherical:* $\omega\big(\pi_2(M)\big) = 0$.
(2) There exists a constant $c \in \mathbb{Z}$ *such that*

$$c_1^{TM} = c\omega : \pi_2(M) \to \mathbb{Z} \tag{1.3}$$

and either $c \geq 1$ *or* $2c\nu \leq -\dim M$ *where* $\nu \in \mathbb{Z}_{>0}$ *is defined by* $\omega\big(\pi_2(M)\big) = \nu\mathbb{Z}$.

Then (E, Ω) *is semi-positive and Rabinowitz Floer homology vanishes*

$$\mathrm{RFH}(\Sigma, E) = 0\ . \tag{1.4}$$

Contact manifolds such as spheres, projective spaces etc. come from negative line bundles. In fact, the Boothby–Wang theorem [16, Theorem 7.2.5] characterizes these contact manifolds as those whose Reeb flow is periodic with all Reeb orbits having the same minimal period. On the sphere this corresponds to the Hopf fibration.

Remark 1.3 (1) If $c = 0$, i.e. $c_1^{TM}\big(\pi_2(M)\big) = 0$, then $c_1^{TE} = -\wp^*\omega : \pi_2(E) \to \mathbb{Z}$. Therefore, (E, Ω) is semi-positive if and only if $\nu \geq \frac{1}{2}\dim M - 1$ or $\omega\big(\pi_2(M)\big) = 0$, see Lemma 2.6. If (E, Ω) is semi-positive then $\mathrm{RFH}(\Sigma, E) = 0$ still holds.

(2) It is worth pointing out that Σ is *not* displaceable inside E since the zero-section $M \subset E$ is not even topologically displaceable. To our knowledge this is the first vanishing result for RFH result which is *not* due to a displaceability phenomenon, see also Ritter [23, Remark on p. 1044].

(3) There are very few *direct* computations of RFH. To our knowledge, the vanishing result in the displaceable case [9], the computation for cotangent bundles [5], and the computation for Brieskorn spheres [13] are the only ones. The long exact sequence, [11] leads often to computational results if the symplectic homology and the connecting maps are known. The latter rarely happens, though.

(4) Rabinowitz Floer homology, first constructed by Cieliebak and Frauenfelder in [9], is an invariant of contact type hypersurfaces in symplectic manifolds. It turned out to be an efficient tool for studying questions in symplectic topology and dynamics, see [2].
In [15] Frauenfelder studied the Rabinowitz Floer homology of negative line bundles under the additional assumption of the line bundle being very negative. The implication of the latter is the generic absence of holomorphic spheres. Amongst many other things he established C^∞_{loc}-compactness results, cf. [15, Theorem B]. Even though Rabinowitz Floer homology is not fully constructed in [15] all ingredients are basically contained therein, see also [3].
The purpose of this article is to complete and extend the construction of Rabinowitz Floer homology to negative line bundles in the presence of holomorphic spheres under a

semi-positivity assumption. In particular, we prove a transversality result made necessary due to the use of a rather restricted class of almost complex structures.
It is worth pointing out that this is the first instance where Rabinowitz Floer homology is constructed in the presence of holomorphic spheres. Holomorphic spheres are a source for interesting symplectic topology and big technical problems at the same time. The latter is the reason we require semi-positivity.

(5) The main new contribution is Theorem 1.2: Rabinowitz Floer homology vanishes in many cases. This should be contrasted with Ritter's result that symplectic homology does not necessarily vanishes, see [23]. Thus, the theorem SH $=0 \Leftrightarrow$ RFH $=0$, [22], does *not* extend beyond the symplectically aspherical situation. We give a conjectural explanation of this in Sect. 4 below.
For very negative line bundles Ritter in [23, Theorem 8] proved vanishing of symplectic homology. If we assume in addition that $c_1^{TM} = c\omega : \pi_2(M) \to \mathbb{Z}$ the Rabinowitz Floer homology vanishes according to Theorem 1.2 as well. The conjectural picture from Sect. 4 nicely relates these results.

2 Rabinowitz Floer homology and Hamiltonian Floer homology

2.1 Preliminaries

Let $\wp : E \to M$ be as described above. We denote by $\mathscr{L}(E)$ the component of contractible loops of the free loop space of E. Moreover, we denote by $\widetilde{\mathscr{L}}(E)$ the covering space of $\mathscr{L}(E)$ with deck transformations given by

$$\Gamma_E := \frac{\pi_2(E)}{\ker \Omega \cap \ker c_1^{TE}} \,. \tag{2.1}$$

We write elements in $\widetilde{\mathscr{L}}(E)$ as $[u, \bar{u}]$, where $u : S^1 \to E$ and $\bar{u} : D^2 \to E$ is a capping disk for u, i.e. $\bar{u}|_{S^1} = u$. Moreover, pairs $(u, \bar{u})$ and $(v, \bar{v})$ are equivalent if $u = v$ and $\Omega(-\bar{u}\#\bar{v}) = c_1^{TE}(-\bar{u}\#\bar{v}) = 0$, where $-\bar{u}\#\bar{v}$ is the sphere formed by $\bar{u}$ with orientation reversed and $\bar{v}$. The expression $[u, \bar{u}]$ denotes the corresponding equivalence class. Analogously we define

$$\Gamma_M := \frac{\pi_2(M)}{\ker \omega \cap \ker c_1^{TM}} \,, \tag{2.2}$$

$\mathscr{L}(M)$ and $\widetilde{\mathscr{L}}(M)$. By definition of E we have

$$c_1^{TE} = \wp^*\big(c_1^{TM} + c_1^E\big) \,. \tag{2.3}$$

Remark 2.1 We point out that under assumption (1.3), i.e. $c_1^{TM} = c\omega : \pi_2(M) \to \mathbb{Z}$, we have $\ker \omega \cap \ker c_1^{TM} = \ker \omega$ and $\ker \Omega \cap \ker c_1^{TE} = \ker \Omega$. If we instead assume $\omega\big(\pi_2(M)\big) = 0$ then $c_1^{TE} = \wp^* c_1^{TM} : \pi_2(E) \to \mathbb{Z}$. In particular, since $\pi_2(E) \cong \pi_2(M)$ via $\wp_*$, we can identify in both cases $\Gamma_E \cong \Gamma_M$.

For $\tau > 0$ we denote by $\mu_\tau : E \to \mathbb{R}$ the function $\mu_\tau = \pi r^2 - \tau$ where as above r denotes the radial coordinate on E. We point out that along $\Sigma_\tau := \{\mu_\tau = 0\}$ the Hamiltonian vector field X_{μ_τ} of μ_τ agrees with the Reeb vector field R associated to the contact form α. In particular, we use the convention $\Omega(X_{\mu_\tau}, \cdot) = -d\mu_\tau$. The Rabinowitz action functional $\mathscr{A}^\tau$ is defined as

$$\begin{aligned}\mathscr{A}^\tau &: \widetilde{\mathscr{L}}(E) \times \mathbb{R} \to \mathbb{R}\\ \big([u, \bar{u}], \eta\big) &\mapsto \int_{D^2} \bar{u}^*\Omega - \eta \int_0^1 \mu_\tau\big(u(t)\big)dt.\end{aligned} \tag{2.4}$$

In the article [9] Cieliebak and Frauenfelder developed a Floer theory for this functional in a slightly simpler set-up and for non-degenerate contact forms. The current set-up has been developed and studied Frauenfelder in [15] for very negative line bundles, see also [3].

In our setting the contact form is Morse–Bott non-degenerate. A general Morse–Bott approach to Rabinowitz Floer homology is currently not available in the literature. Instead of perturbing the contact form we choose the following perturbation. We fix $f : M \to \mathbb{R}$ and set

$$F := \big(\pi r^2 + 1\big) f \circ \wp : E \to \mathbb{R}. \tag{2.5}$$

The perturbed Rabinowitz action functional is

$$\begin{aligned}\mathscr{A}^\tau_f &: \widetilde{\mathscr{L}}(E) \times \mathbb{R} \to \mathbb{R}\\ \big([u, \bar{u}], \eta\big) &\mapsto \int_{D^2} \bar{u}^*\Omega - \eta \int_0^1 \mu_\tau\big(u(t)\big)dt - \int_0^1 F\big(u(t)\big)dt\ .\end{aligned} \tag{2.6}$$

Critical points of $\mathscr{A}^\tau_{f=0}$ correspond to capped Reeb orbits traversed in forward and backward direction and, in addition, to constant loops contained in Σ_τ together with cappings. In Lemma 2.3 we show that for C^2-small Morse functions f the critical points of $\mathscr{A}^\tau_f$ correspond to capped Reeb orbits which lie via $\wp : E \to M$ over $\mathrm{Crit}(f)$. The functional $\mathscr{A}^\tau_f$ is still Morse–Bott due to the remaining S^1-symmetry. This can be dealt with as in the article by Bourgeois-Oancea [6].

Remark 2.2 We split the tangent bundle $TE \cong V \oplus H$ in vertical resp. horizontal subspaces V resp. H according to α. In particular, $\wp_* : (H, d\alpha) \xrightarrow{\cong} (TM, \omega)$ is an isomorphism of symplectic vector bundles and V is spanned by the Reeb vector field R and the radial vector field $r\partial_r$.

Lemma 2.3 *If $f : M \to \mathbb{R}$ is C^2-small then $\big([u, \bar{u}], \eta\big)$ is a critical point of $\mathscr{A}^\tau_f$ if and only if the following equations are satisfied.*

$$\begin{cases}\pi r(u)^2 = \tau\\ q := \wp(u) \in \mathrm{Crit}(f)\\ \dot{u} = \big(\eta + f(q)\big)R(u)\end{cases} \tag{2.7}$$

In particular, necessarily $\eta + f(q) \in \mathbb{Z}$ and $u \subset \Sigma_\tau$ is a $(\eta + f(q))$-fold cover of the underlying simple periodic orbit.

Proof The critical point equation for $\mathscr{A}^\tau_f$ is

$$\begin{cases}\dot{u} = \eta X_{\mu_\tau}(u) + X_F(u)\\ \displaystyle\int_0^1 \mu_\tau(u)dt = 0\ .\end{cases} \tag{2.8}$$

The Hamiltonian vector field of F is $X_F = (f \circ \wp)X_{\mu_\tau} + X_f^h$ where X_f^h is the horizontal lift of X_f, i.e. $X_f^h \in H$ and $\wp_*(X_f^h) = X_f$. Indeed,

$$\begin{aligned}\Omega(X_F, \cdot) &= \Big(2\pi r dr \wedge \alpha + \big(\pi r^2 + 1\big)\wp^*\omega\Big)\Big((f \circ \wp)X_{\mu_\tau} + X_f^h, \cdot\Big)\\ &= -2\pi r(f \circ \wp) + \big(\pi r^2 + 1\big)\wp^*\big(\omega(X_f, \cdot)\big)\\ &= -2\pi r(f \circ \wp) - \big(\pi r^2 + 1\big)\wp^* df\\ &= -d\big[\big(\pi r^2 + 1\big) f \circ \wp\big]\\ &= -dF\ .\end{aligned} \tag{2.9}$$

We point out that $\Omega(X_{\mu_\tau}, X_F) = dF(X_{\mu_\tau}) = 0$ since $\wp_*(X_{\mu_\tau}) = 0 = dr(X_{\mu_\tau})$. Therefore the critical point Eq. (2.8) simplifies to

$$\begin{cases} \dot{u} = \big(\eta + f(\wp(u))\big)X_{\mu_\tau}(u) + X_f^h(u) \\ \mu_\tau\big(u(t)\big) = 0 \quad \forall t \in S^1\ . \end{cases} \tag{2.10}$$

The last equation translates into $r(u(t))$ being constant and $\pi r(u)^2 = \tau$. The critical point equation together with $\wp_*(X_{\mu_\tau}) = 0$ implies that $\wp(u)$ is a 1-periodic solution of X_f in M. Now, if the C^2-norm of f is sufficiently small the only 1-periodic solutions of X_f are the critical points of f, see [19, p. 185]. Thus, $q := \wp(u) \in \mathrm{Crit}(f)$ and u corresponds to a $(\eta + f(q))$-periodic orbit of R on Σ_τ. This implies that $\eta + f(q) \in \mathbb{Z}$ due to our convention $S^1 = \mathbb{R}/\mathbb{Z}$. □

Remark 2.4 To summarize critical points of $\mathscr{A}_f^\tau$ correspond to all Reeb orbits over $\mathrm{Crit}(f)$ together with cappings. More precisely, all forward (i.e. $\eta + f(q) > 0$) and backward (i.e. $\eta + f(q) < 0$) iterations and also the "constants" (i.e. $\eta + f(q) = 0$) together with cappings.

Convention 2.5
- From now on we assume that the Morse function $f : M \to \mathbb{R}$ is chosen C^2-small so that Lemma 2.3 applies.
- Every simple periodic Reeb orbit $v \subset \Sigma_\tau$ has a capping by its fiber disk $d_v \subset E$ and correspondingly the n-fold cover v^n has d_v^n as capping disk for $n \in \mathbb{Z}\backslash\{0\}$. Every non-constant critical point $\big([u, \bar{u}], \eta\big)$ can be expressed in the form $u = v^n$ and $\bar{u} = d_v^n \# A$ for some $A \in \Gamma_E$. If u is a constant critical point, the capping disk $\bar{u}$ can be thought of as a sphere $A \in \Gamma_E$. We are going to adopt the notation $[u, \bar{u}] = [u, A]$.

Using Lemma 2.3 we compute the action value for a critical point $\big([u, \bar{u}], \eta\big) = \big([v^n, A], \eta\big)$ with $q = \wp(u)$.

$$\begin{aligned}\mathscr{A}_f^\tau\big([v^n, A], \eta\big) &= \int_{D^2} \big(d_v^n\big)^*\Omega + \omega(A) - \eta \int_0^1 \underbrace{\mu_\tau(v^n)}_{=0} dt - \int_0^1 F(v^n)dt\\ &= \int_{D^2} \big(d_v^n\big)^*\big[d\big(\pi r^2\alpha\big)\big] - \int_0^1 \big(\pi r^2 + 1\big) f \circ \wp(v^n)dt + \omega(A)\\ &= \int_{S^1} (v^n)^*\big(\pi r^2\alpha\big) - (\tau + 1) f(q) + \omega(A)\\ &= \int_{S^1} \tau\alpha\big((\eta + f(q))R(v^n)\big) - (\tau + 1) f(q) + \omega(A)\\ &= \tau\big(\eta + f(q)\big) - (\tau + 1) f(q) + \omega(A)\end{aligned} \tag{2.11}$$

$$= \tau\eta + \omega(A) - f(q)$$
$$= \tau n + \omega(A) - (\tau+1) f(q),$$

where we use $n = \eta + f(q)$ and $\Omega = \omega : \Gamma_E \cong \Gamma_M \to \mathbb{Z}$.

Next we explain how to define Floer homology for $\mathscr{A}_f^\tau$. This mainly follows the lines of Frauenfelder [15] and Albers and Frauenfelder [3]. We assume throughout that (E, Ω) is semi-positive. According to McDuff and Salamon [20, Exercise 6.4.3] the symplectic manifold (E, Ω) is semi-positive if and only if

- (E, Ω) is symplectically aspherical,
- (E, Ω) is monotone,
- $c_1^{TE} : \pi_2(E) \to \mathbb{Z}$ vanishes,
- the minimal Chern number N_E of E satisfies $N_E \geq \frac{1}{2}\dim E - 2$.

Since $\pi_2(E) \cong \pi_2(M)$ via $\wp_*$ the first condition is equivalent to (M, ω) being symplectically aspherical which is condition (1) in Theorem 1.2.

If we assume that there exists a constant $c \in \mathbb{Z}$ such that $c_1^{TM} = c\omega : \pi_2(M) \to \mathbb{Z}$ then

$$c_1^{TE} = \wp^*\big(c_1^{TM} + c_1^E\big) = (c-1)\wp^*\omega. \tag{2.12}$$

Thus, if $c > 1$ the symplectic manifold (E, Ω) is monotone and for $c = 1$ we have $c_1^{TE} = 0$ on $\pi_2(E)$. Furthermore, if we denote by $\nu \in \mathbb{Z}_{\geq 0}$ the generator of $\omega\big(\pi_2(M)\big) = \nu\mathbb{Z}$ then the minimal Chern number N_E of E is

$$N_E = |c-1|\nu. \tag{2.13}$$

Thus, we proved the following Lemma.

Lemma 2.6 *The symplectic manifold (E, Ω) is semi-positive if*

- *(M, ω) is symplectically aspherical or*
- *$c_1^{TM} = c\omega : \pi_2(M) \to \mathbb{Z}$ with $c \geq 1$ or $N_E = |c-1|\nu \geq \frac{1}{2}\dim E - 2$.*

In the following Lemma we use the notation of Convention 2.5. For $\big([u, \bar{u}], \eta\big) = \big([u, A], \eta\big) \in \mathrm{Crit}(\mathscr{A}_f^\tau)$, we denote by $\mu_{\mathrm{CZ}}^E(u, \bar{u}) \equiv \mu_{\mathrm{CZ}}^E(u, A)$ the Conley–Zehnder index of u with respect to the capping disk $\bar{u}$. We refer to [24] for a thorough discussion of the Conley–Zehnder index.

Lemma 2.7 *The Conley–Zehnder index of a n-fold cover v^n with its fiber disk d_v^n is*

$$\mu_{\mathrm{CZ}}^E(v^n, d_v^n) = 2n. \tag{2.14}$$

More generally, for any capping $\bar{u} = d_v^n \# A$ of $u = v^n$,

$$\mu_{\mathrm{CZ}}^E(v^n, d_v^n \# A) = 2n + 2c_1^{TE}(A). \tag{2.15}$$

If (M, ω) is symplectically aspherical then all iterates v^n are non-contractible inside Σ. Otherwise, the first iterate of v, which is contractible in Σ, is the orbit v^ν.

If we assume $c_1^{TM} = c\omega$ then the Conley–Zehnder index of v^n for $n \in \nu\mathbb{Z}\setminus\{0\}$ with respect to a capping disk $\bar{v}$ contained entirely in Σ is

$$\mu_{\mathrm{CZ}}^E(u, \bar{v}) = 2cn. \tag{2.16}$$

Proof Since the linearized map of the Reeb flow is the identity in horizontal directions, the first assertion follows from the corresponding computation for $S^1 \subset \mathbb{C}$. The relevant bit of the homotopy long exact sequence of the S^1-bundle $\Sigma \to M$ is

$$\cdots \longrightarrow \pi_2(M) \xrightarrow{\delta} \pi_1(S^1) \xrightarrow{i_*} \pi_1(\Sigma) \longrightarrow \cdots . \tag{2.17}$$

If we identify $\pi_1(S^1) \cong \mathbb{Z}$ then $i_*(k) = [v^k]$ and $\delta(s) = -\omega(s)$ with respect to the homomorphism $\omega : \pi_2(M) \to \mathbb{Z}$. Thus, if (M, ω) is symplectically aspherical then all iterates v^k are non-contractible in Σ. Otherwise $\omega\big(\pi_2(M)\big) = \nu\mathbb{Z}$ with $\nu > 0$ and therefore the first iterate of v which is contractible in Σ is the orbit v^ν.

Now we assume that $c_1^{TM} = c\omega$ and recall that $\bar{v}$ is a capping which is entirely contained inside Σ.

$$\begin{aligned}
\mu_{\mathrm{CZ}}^E(v^n, \bar{v}) &= \mu_{\mathrm{CZ}}^E\Big(v^n, d_v^n\#\big(\underbrace{\bar{v}\# - d_v^n}_{\in\Gamma_E}\big)\Big)\\
&= \mu_{\mathrm{CZ}}^E\big(v^n, d_v^n\big) + 2c_1^{TE}\big(\bar{v}\# - d_v^n\big)\\
&= 2n + 2(c-1)\wp^*\omega\big(\bar{v}\# - d_v^n\big)\\
&= 2n + 2(c-1)\Big[\int_{\bar{v}} \wp^*\omega - \underbrace{\int_{d_v^n} \wp^*\omega}_{=0}\Big]\\
&\overset{(*)}{=} 2n + 2(c-1)\int_{\bar{v}} d\alpha\\
&= 2n + 2(c-1)\int_{v^n} \alpha\\
&= 2n + 2(c-1)n\\
&= 2cn
\end{aligned} \tag{2.18}$$

where we used in $(*)$ that the disk $\bar{v}$ is contained inside Σ. □

Definition 2.8 We point out that critical points of $\mathscr{A}_f^\tau$ are S^1-families, cf. Lemma 2.3. We choose a perfect Morse function $h : \mathrm{Crit}(\mathscr{A}_f^\tau) \to \mathbb{R}$ such that every critical manifold $S^1 \cdot \big([u, A], \eta\big) \subset \mathrm{Crit}(\mathscr{A}_f^\tau)$ gives rise to two critical points of h which we denote by $\big([u, A]^\pm, \eta\big)$ according to the maximum resp. minimum of h on $S^1 \cdot \big([u, A], \eta\big)$. We define the index of a critical point by

$$\mu\big([u, A]^\pm, \eta\big) := \mu_{\mathrm{CZ}}^E(u, A) - \mu_{\mathrm{Morse}}(\wp(u), f) + \tfrac{1}{2}\dim M \pm \tfrac{1}{2} \in \tfrac{1}{2} + \mathbb{Z}\,, \tag{2.19}$$

where $\mu_{\mathrm{Morse}}(\wp(u), f)$ is the Morse index of $\wp(u) \in \mathrm{Crit}(f)$. In case that u is a constant critical point we define $\mu_{\mathrm{CZ}}^E(u, A = 0) := 0$. We set

$$\mathfrak{C} := \mathrm{Crit}(h) = \big\{\big([u, A]^\pm, \eta\big)\big\} \subset \mathrm{Crit}(\mathscr{A}_f^\tau) \tag{2.20}$$

and

$$\mathfrak{C}_k := \big\{\big([u, A]^\bullet, \eta\big) \in \mathfrak{C} \,\big|\, \mu\big([u, A]^\bullet, \eta)\big) = k\big\}\ . \tag{2.21}$$

Here, $\bullet$ indicates some choice of $\pm$.

In order to define Rabinowitz Floer homology and to prove the vanishing result we rely on a fairly special class of almost complex structures which we describe next. In the next subsection we prove that this class is big enough to prove the necessary transversality results. We recall that we split the tangent bundle $TE \cong V \oplus H$ in vertical resp. horizontal subspaces V resp. H, see Remark 2.2. Let us abbreviate by

$$\mathfrak{j} := \big\{ j \in \Gamma(S^1 \times M, \mathrm{Aut}(TM)) \mid j_t := j(t,\cdot) \text{ is an } \omega\text{-compatible almost complex structure} \big\}$$

the space of S^1-families of compatible almost complex structures on (M,ω). Next we fix disjoint open balls around each point in $\mathrm{Crit}(f)$. The union of these balls is denoted by $\mathcal{U}$. For a fixed $j \in \mathfrak{j}$ we denote by $\mathcal{B}(j)$ the set of $B \in \Gamma_0(S^1 \times E, \mathrm{L}(H,V))$ where $B_t := B(t,\cdot)$ satisfies

$$iB_t + B_t j_t = 0 \;\forall t \in S^1 \quad \text{and} \quad B_t(e) = 0 \;\forall e \in \wp^{-1}(\mathcal{U})\,. \tag{2.22}$$

Here the subscript 0 indicates compact support and $\mathrm{L}(H,V)$ is the space of linear maps. To describe the Floer equation we will choose a S^1-family J_t of almost complex structures on E of the form

$$J_t = \begin{pmatrix} i & B_t \\ 0 & j_t \end{pmatrix}\,. \tag{2.23}$$

The matrix representation refers to the splitting $TE \cong V \oplus H$. Moreover, $j \in \mathfrak{j}$ and $B \in \mathcal{B}(j)$ and i is the standard complex structure on $V_e \cong \mathbb{C}$, $e \in E$. We point out that J_t is *not* Ω-compatible. But, since $\begin{pmatrix} i & 0 \\ 0 & j_t \end{pmatrix}$ is tame (even compatible) and B has compact support, the almost complex structure J_t is Ω-tame for sufficiently small B_t. We denote by $\mathcal{B}^T(j) \subset \mathcal{B}(j)$ the non-empty open convex subset consisting of those $B \in \mathcal{B}(j)$ for which the corresponding J_t is tame.

We use J_t to introduce a bilinear form $\mathfrak{m}$ on $T\big(\widetilde{\mathscr{L}}_E \times \mathbb{R}\big)$ as follows. For $(\hat u_1, \hat\eta_1), (\hat u_2, \hat\eta_2) \in T_{([u,A],\eta)}\big(\widetilde{\mathscr{L}}_E \times \mathbb{R}\big) = \Gamma(S^1, u^*TE) \times \mathbb{R}$ we set

$$\mathfrak{m}\big((\hat u_1, \hat\eta_1), (\hat u_2, \hat\eta_2)\big) := -\int_0^1 \Omega\Big(J_t\big(u(t)\big)\hat u_1(t), \hat u_2(t)\Big)dt + \hat\eta_1\hat\eta_2\,. \tag{2.24}$$

The bilinear form $\mathfrak{m}$ is not symmetric but positive definite since J is tame. Therefore we can define the vector field $\nabla\mathscr{A}_f^\tau(w)$ at $w = \big([u,A],\eta\big)$ implicitly by

$$d\mathscr{A}_f^\tau(w)\hat w = \mathfrak{m}\big(\nabla\mathscr{A}_f^\tau(w), \hat w\big) \quad \forall \hat w \in T_w\big(\widetilde{\mathscr{L}}_E \times \mathbb{R}\big)\,. \tag{2.25}$$

An explicit expression is

$$\nabla\mathscr{A}_f^\tau(w) = \begin{pmatrix} -J_t(u)\big(\partial_t u - \eta X_{\mu_\tau}(u) - X_F(u)\big) \\ -\displaystyle\int_0^1 \mu_\tau(u)dt \end{pmatrix}\,. \tag{2.26}$$

$\nabla\mathscr{A}_f^\tau$ is a gradient-like vector field for $\mathscr{A}_f^\tau$ since J is tame and B vanishes near critical points of $\mathscr{A}_f^\tau$: $B_t(e) = 0$ for all $e \in \wp^{-1}(\mathcal{U})$. Indeed, $d\mathscr{A}_f^\tau(w)\nabla\mathscr{A}_f^\tau(w) = \mathfrak{m}\big(\nabla\mathscr{A}_f^\tau(w), \nabla\mathscr{A}_f^\tau(w)\big) \geq 0$ with equality if and only if $w \in \mathrm{Crit}(\mathscr{A}_f^\tau)$. Moreover, $\mathfrak{m}$ is an inner product near critical points.

To construct Floer homology for $\mathscr{A}_f^\tau$ we study solutions $w = (u, \eta) \in C^\infty(\mathbb{R} \times S^1, E) \times C^\infty(\mathbb{R}, \mathbb{R})$ to the Floer equations corresponding to *positive* gradient flow of $\mathscr{A}_f^\tau$

$$\begin{cases} \partial_s u + J_t(u)\big(\partial_t u - \eta X_{\mu_\tau}(u) - X_F(u)\big) = 0 \\ \partial_s \eta + \displaystyle\int_0^1 \mu_\tau(u)dt = 0\ . \end{cases} \tag{2.27}$$

Due to the assumption that B vanishes near critical points the Floer equation thought of as a differential operator is Fredholm. The main ingredients for defining Floer homology are transversality and compactness for solution spaces of the Floer equation. This needs some attention in our framework due the restriction of the class of almost complex structures we consider and due to potential bubbling-off of holomorphic spheres.

The projection $\wp$ maps critical points of the Rabinowitz Floer action functional $\mathscr{A}_f^\tau$ to those of the action functional of classical mechanics $\mathfrak{a}_f$ on (M, ω)

$$\begin{aligned} &\mathfrak{a}_f : \widetilde{\mathscr{L}}(M) \longrightarrow \mathbb{R} \\ &\mathfrak{a}_f\big([q, \bar{q}]\big) := \int_{D^2} \bar{q}^*\omega - \int_0^1 f\big(q(t)\big)dt\ , \end{aligned} \tag{2.28}$$

see Lemma 2.9. We recall that we chose the Morse function f in a C^2-small fashion, see Convention 2.5. This implies that all critical points of $\mathfrak{a}_f$ are critical points of f with some capping, i.e.

$$\mathrm{Crit}(\mathfrak{a}_f) \cong \mathrm{Crit}(f) \times \Gamma_M\ . \tag{2.29}$$

We use the following convention for the Conley–Zehnder index for $(x, A) \in \mathrm{Crit}(\mathfrak{a}_f) \cong \mathrm{Crit}(f) \times \Gamma_M$

$$\mu_{\mathrm{CZ}}^M(x, A) = -\mu_{\mathrm{Morse}}(x, f) + \tfrac{1}{2}\dim M + 2c_1^{TM}(A)\ . \tag{2.30}$$

Lemma 2.9 *The projection $\wp$ induces the map*

$$\begin{aligned} \Pi : \mathrm{Crit}(\mathscr{A}_f^\tau) &\longrightarrow \mathrm{Crit}(\mathfrak{a}_f) \cong \mathrm{Crit}(f) \times \Gamma_M \\ \big([u, A], \eta\big) &\longmapsto [\wp(u), A]\ . \end{aligned} \tag{2.31}$$

Proof This follows directly from the definition (2.6) of the action functional $\mathscr{A}_f^\tau$, see also Remark 2.1. □

After a choice of $j \in \mathfrak{j}$ the action functional $\mathfrak{a}_f$ gives rise to the following Floer equation for $q : \mathbb{R} \times S^1 \to M$

$$\partial_s q + j_t(q)\big(\partial_t q - X_f(q)\big) = 0\ . \tag{2.32}$$

We recall that solutions of either Floer equation is of finite energy if and only if it converges at $\pm\infty$ to critical points of $\mathscr{A}_f^\tau$ resp. $\mathfrak{a}_f$. That is, a solution $w = (u, \eta)$ of the Floer equation (2.27) has finite energy

$$\int_{\mathbb{R}} \int_{S^1} \Big(|\partial_s u|^2 + |\partial_s \eta|^2\Big)dtds < \infty \tag{2.33}$$

if and only if there exists $(u_\pm, \eta_\pm) \in \mathscr{L}(E) \times \mathbb{R}$ satisfying (2.7) such that

$$\lim_{s\to\pm\infty} \big(u(s, \cdot), \eta(s)\big) = (u_\pm, \eta_\pm) \tag{2.34}$$

and similarly for $\mathfrak{a}_f$. Following the usual Morse–Bott ideas we denote for $w_\pm \in \mathfrak{C} \subset \operatorname{Crit}(\mathscr{A}_f^\tau)$

$$\widehat{\mathcal{M}}(w_-, w_+) := \left\{w \text{ solves (2.27) with } \lim_{s\to\pm\infty} w(s) \in W^\pm(w_\pm, h)\right\} \tag{2.35}$$

the moduli space of finite energy solutions of the Floer equation of $\mathscr{A}_f^\tau$. Here $h : \operatorname{Crit}(\mathscr{A}_f^\tau) \to \mathbb{R}$ is the perfect Morse function from Definition 2.8 and $W^+(w_+, h)$ resp. $W^-(w_-, h)$ denotes the stable resp. unstable manifold of h on $\operatorname{Crit}(\mathscr{A}_f^\tau)$. Similarly, for $q_\pm \in \operatorname{Crit}(\mathfrak{a}_f)$ let

$$\widehat{\mathcal{N}}(q_-, q_+) := \left\{w \text{ solves (2.32) with } \lim_{s\to\pm\infty} q(s) = q_\pm\right\} . \tag{2.36}$$

Here, we abuse notation in the following sense. If $w_\pm = \big([u_\pm, \bar{u}_\pm], \eta_\pm\big) \in \operatorname{Crit}(\mathscr{A}_f^\tau)$ is given and $w = (u, \eta)$ is a finite energy solution of (2.27) then by $\lim_{s\to\pm\infty} w(s) = w_\pm$ we mean that

$$\lim_{s\to\pm\infty} \big(u(s,\cdot), \eta(s)\big) = (u_\pm, \eta_\pm) \tag{2.37}$$

and

$$\big[(-\bar{u}_-)\#u\#\bar{u}_+\big] = 0 \in \Gamma_E\ . \tag{2.38}$$

The same remark applies to $\mathfrak{a}_f$. Unless $w_- = w_+$ the moduli space $\widehat{\mathcal{M}}(w_-, w_+)$ carries a free $\mathbb{R}$-action by shifts. We denote the quotient by

$$\mathcal{M}(w_-, w_+) := \widehat{\mathcal{M}}(w_-, w_+)/\mathbb{R} \tag{2.39}$$

and similarly

$$\mathcal{N}(q_-, q_+) := \widehat{\mathcal{N}}(q_-, q_+)/\mathbb{R}\ . \tag{2.40}$$

All moduli spaces depend on additional data, e.g. an almost complex structure, which we suppress in the notation.

Lemma 2.10 *The projection $\wp$ induces the maps*

$$\begin{aligned} \Pi : \widehat{\mathcal{M}}(w_-, w_+) &\longrightarrow \widehat{\mathcal{N}}\big(\Pi(w_-), \Pi(w_+)\big) \\ w = (u,\eta) &\longmapsto \Pi(w) := \wp(u)\ , \\ \Pi : \mathcal{M}(w_-, w_+) &\longrightarrow \mathcal{N}\big(\Pi(w_-), \Pi(w_+)\big) \\ [w] &\longmapsto [\Pi(w)]\ . \end{aligned} \tag{2.41}$$

Proof This follows immediately from the fact that $\wp_*(X_F) = X_f$, $\wp_*(X_{\mu_\tau}) = 0$ and the specific form of $J_t = \begin{pmatrix} i & B_t \\ 0 & j_t \end{pmatrix}$, i.e. $\wp_* \circ J = j \circ \wp_*$ □

2.2 Transversality

We recall that $\mathfrak{j}$ is the space of S^1-families of compatible almost complex structures on (M, ω). We denote by

$$\mathfrak{j}_{\mathrm{reg}}(f) \subset \mathfrak{j} \tag{2.42}$$

the subset of $j \in \mathfrak{j}$ with the following two properties.

- All finite energy solutions of the Floer equation for $\mathfrak{a}_f$ with respect to j are regular, i.e. the operator obtained by linearizing the Floer equation is a surjective Fredholm operator for all finite energy solutions.
- For every $t \in S^1$ all *simple* j_t-holomorphic spheres are regular, i.e. the operator obtained by linearizing the holomorphic sphere equation is a surjective Fredholm operator.

According to Flore et al. [14] and McDuff and Salamon [20, Chapter 2] the subset $\mathfrak{j}_{\text{reg}}$ is of second category. For every $j \in \mathfrak{j}_{\text{reg}}(f)$ the moduli space $\widehat{\mathcal{N}}(q_-, q_+)$ is a smooth manifold of dimension

$$\dim \widehat{\mathcal{N}}(q_-, q_+) = \mu^M_{\text{CZ}}(q_+) - \mu^M_{\text{CZ}}(q_-) \ . \tag{2.43}$$

For $j \in \mathfrak{j}_{\text{reg}}(f)$ we denote by

$$\mathcal{B}_{\text{reg}}(j) \subset \mathcal{B}(j) \tag{2.44}$$

the subset of $B \in \mathcal{B}(j)$ with the following two properties

- All finite energy solutions of the Floer equation for $\mathscr{A}^\tau_f$ with respect to the corresponding J are regular.
- For every $t \in S^1$ all *simple* J_t-holomorphic spheres are regular.

We refer to [4] for details on the linearization of the Rabinowitz Floer equations. For $B \in \mathcal{B}_{\text{reg}}(j)$ the moduli space $\widehat{\mathcal{M}}(w_-, w_+)$ is a smooth manifold of dimension

$$\dim \widehat{\mathcal{M}}(w_-, w_+) = \mu(w_+) - \mu(w_-) \ . \tag{2.45}$$

The next proposition shows that this class of almost complex structures is sufficiently large.

Proposition 2.11 *For all $j \in \mathfrak{j}_{\text{reg}}(f)$ the set $\mathcal{B}_{\text{reg}}(j) \subset \mathcal{B}(j)$ is of second category.*

Proof We recall the splitting $TE \cong V \oplus H$, cf. Remark 2.2. Thus, we may consider the linearization of the Floer equation (2.27) in vertical resp. horizontal directions V resp. H. Since the projection $\wp$ induces an isomorphism $\wp_* : (H, d\alpha) \to (TM, \omega)$ and $j \in \mathfrak{j}_{\text{reg}}$, it follows from Lemma 2.10 that the linearization is already surjective in horizontal directions. To show that it is for generic choice of B also surjective in vertical directions we distinguish two cases for $w \in \widehat{\mathcal{M}}(w_-, w_+)$.

Case 1 $\Pi(w)$ is non-constant. We claim that $\Pi(w)$ necessarily leaves the neighborhood $\mathcal{U}$. We recall that $\mathcal{U}$ is the union of disjoint neighborhoods of all critical points of f where each such neighborhood contracts onto a critical point, see the discussion before Eq. (2.22). If $\Pi(w)$ is contained in $\mathcal{U}$ then it has to be a gradient trajectory connecting the same critical point of f with cappings A and $A\#\Pi(w)$. Since $\Pi(w)$ is contained in $\mathcal{U}$ the two cappings are homotopic to each other: $A = A\#\Pi(w) \in \Gamma_M$. Thus, $\Pi(w)$ is a gradient trajectory from a critical point of $\mathfrak{a}_f$ to itself (including cappings) and therefore $\Pi(w)$ is constant which is a contradiction. Therefore $\Pi(w)$ necessarily leaves the neighborhood $\mathcal{U}$. By [14, Theorem 4.3] the set of regular points for $\Pi(w)$ is open and dense. Since $\Pi(w)$ leaves $\mathcal{U}$ we may apply Lemma 2.15 below. Thus a standard argument, see for instance [14, Section 5] or [20, Chapter 3] establishes that for generic $B \in \mathcal{B}(j)$ the linearization of the gradient flow equation is also vertically surjective.

Case 2 $\Pi(w)$ is constant. But then w is a vortex and vortices are by Albers and Frauenfelder [3, Proposition A.1] always transverse. In fact vortices are independent of the perturbation $B \in \mathcal{B}(j)$.

It remains to prove that generically all *simple* J_t-holomorphic spheres, $t \in S^1$, are regular. If we were not to restrict to upper triangular J this is a standard result which relies on the fact that simple curves are somewhere injective, see [20, Chapter 2] for details.

We argue again as above. Due to the definition of $\mathfrak{j}_{\mathrm{reg}}$ the linearization of a simple J_t-holomorphic sphere is already surjective in horizontal directions. For vertical directions we use the notion of somewhere horizontally injective points, see Definition 2.13 below. The important observation is that horizontally injective points still form a dense subset, see Lemma 2.14. Therefore, we can apply again Lemma 2.15 to conclude that for generic $B \in \mathcal{B}(j)$ all simple J_t-holomorphic curves are regular. □

Remark 2.12 We recall that $\mathcal{B}^T(j) \subset \mathcal{B}(j)$ denotes the non-empty open convex subset consisting of those $B \in \mathcal{B}(j)$ for which the corresponding J is tame. From now on we always choose $j \in \mathfrak{j}_{\mathrm{reg}}$ and $B \in \mathcal{B}^T_{\mathrm{reg}}(j) := \mathcal{B}_{\mathrm{reg}}(j) \cap \mathcal{B}^T(j)$.

We recall the following notions and Lemmas considered in [3,15].

Definition 2.13 A J_t-holomorphic curve $u : S^2 \to E$ is called *somewhere horizontally injective* if there exists $z \in S^2$ such that

$$d^h u(z) := \wp_* \circ du(z) \neq 0, \quad u^{-1}(u(z)) = \{z\}\ . \tag{2.46}$$

It remains to prove the following two Lemmas.

Lemma 2.14 *Assume that $u : S^2 \to E$ is a simple J_t-holomorphic curve. Then u is horizontally injective on a dense set.*

Proof We denote by $I(u) \subset S^2$ the subset of injective points of u, by $R(\wp(u)) \subset S^2$ the subset of nonsingular points of $\wp(u)$ and by $S(u) \subset S^2$ the subset of horizontally injective points of u. Then

$$S(u) = I(u) \cap R(\wp(u)). \tag{2.47}$$

We first observe that $\wp(u)$ is j_t-holomorphic. We claim that $\wp(u) : S^2 \to M$ is not constant, since otherwise u would lie in one fiber and hence itself must be constant, contradicting the assumption that it is simple. Therefore, it follows from [20, Lemma 2.4.1] that the complement of $R(\wp(u))$ is finite. Moreover, it follows from [20, Proposition 2.5.1] that the complement of $I(u)$ is countable. Hence by (2.47) the complement of $S(u)$ is countable. In particular, $S(u)$ is dense in S^2. □

Lemma 2.15 *We fix $e \in E \backslash \wp^{-1}(\mathcal{U})$, $(v, h) \in T_e E = V_e \oplus H_e$ with $h \neq 0$ and $t_0 \in S^1$. Moreover, we fix $j \in \mathfrak{j}$ and $B \in \mathcal{B}(j)$. Then there exist $\widehat{B} \in \Gamma_0(S^1 \times E, \mathrm{L}(H, V))$ and $\widehat{j} \in T_j\mathfrak{j}$ with*

$$\begin{cases} \widehat{B}(t_0, e)h = v \\ i\widehat{B} + B\widehat{j} + \widehat{B}j = 0\ . \end{cases} \tag{2.48}$$

Remark 2.16 The second equation asserts that the pair $(\widehat{j}, \widehat{B})$ corresponds to a tangent vector of the space of almost complex structures we are considering.

Proof of Lemma 2.15 First we extend h resp. v to sections also denoted by h resp. v supported in a small neighborhood of e. Then we define $\widehat{j}$ by

$$\widehat{j}h := v, \quad \widehat{j}jh := -jv, \quad \widehat{j}|_{\mathrm{span}\{h, jh\}^\perp} := 0, \tag{2.49}$$

where $\perp$ refers to the metric $\omega(\cdot, j\cdot)$ on H. We point out that $\text{span}\{h, jh\}^{\perp}$ is j-invariant. Then $\widehat{j}$ satisfies the equation

$$\widehat{j}j + j\widehat{j} = 0 \tag{2.50}$$

that is, $\widehat{j} \in T_j$j. Next we define $\widehat{B}$ by

$$\begin{aligned} \widehat{B}h &:= v \\ \widehat{B}jh &:= -\left(i\widehat{B} + B\widehat{j}\right)h \\ \widehat{B}|_{\text{span}\{h,jh\}^{\perp}} &:= 0. \end{aligned} \tag{2.51}$$

The first equation in (2.48) holds by construction. We show the second equation. We have

$$(i\widehat{B} + B\widehat{j} + \widehat{B}j)h = 0 \tag{2.52}$$

by the very definition of $\widehat{B}jh$. Moreover,

$$\begin{aligned} (i\widehat{B} + B\widehat{j} + \widehat{B}j)jh &= i\widehat{B}jh + B\widehat{j}jh - \widehat{B}h \\ &= -i\left(i\widehat{B} + B\widehat{j}\right)h - Bj\widehat{j}h - \widehat{B}h \\ &= \widehat{B}h - iB\widehat{j}h - Bj\widehat{j}h - \widehat{B}h \\ &= -iB\widehat{j}h - Bj\widehat{j}h \\ &= -(iB + Bj)\widehat{j}h \\ &= 0 \end{aligned} \tag{2.53}$$

where we used $jj = -1$, $\widehat{B}jh = -(i\widehat{B} + B\widehat{j})h$, $\widehat{j}j + j\widehat{j} = 0$, $ii = -1$ and finally $iB + Bj = 0$, see (2.22). Finally,

$$\left(i\widehat{B} + B\widehat{j} + \widehat{B}j\right)|_{\text{span}\{h,jh\}^{\perp}} = 0 \tag{2.54}$$

since $\text{span}\{h, jh\}^{\perp}$ is j-invariant and $\widehat{B}$ and $\widehat{j}$ vanish on it. □

This completes the discussion on transversality.

2.3 Compactness

In this subsection we discuss the appropriate compactness results for the moduli spaces $\mathcal{M}(w_-, w_+)$ of unparametrized gradient flow trajectories. This follows the usual scheme of Rabinowitz Floer homology, that is, we need to establish the following for a sequence $(u_\nu, \eta_\nu) \in \mathcal{M}(w_-, w_+)$, $\nu \in \mathbb{N}$.

(i) A uniform C^0-bound for the loops u_ν.
(ii) A uniform C^0-bound on the Lagrange multipliers η_ν.
(iii) A uniform bound on the derivatives of the loops u_ν.

The first two are proved in [15, Proposition 6.2 & 6.4]. We point out that the set-up in [15] is the same as ours except for the following. Frauenfelder's assumption of (E, Ω) being very negative is replaced by our assumption of semi-positivity. Moreover, the almost complex structures used are of the form $J = \begin{pmatrix} i & 0 \\ 0 & j \end{pmatrix}$, i.e. $B = 0$. The uniform C^0-bound for the loops is based on a maximum principle which continues to hold since in our setting B has compact support. The uniform C^0-bound on the Lagrange multipliers relies on a "fundamental lemma" which continues to hold verbatim.

To prove a uniform bound on the derivatives of the loops we argue by contradiction, i.e. by bubbling-off analysis. Indeed, since we already established uniform C^0-bounds for the loops and the Lagrange multipliers a blow-up of derivatives of u_ν leads to J_t-holomorphic spheres inside E. We claim that since we assume that E is semi-positive we can apply the results of Hofer-Salamon [18] and ensure that for generic S^1-family almost complex structure J of the form $J_t = \begin{pmatrix} i & B_t \\ 0 & j_t \end{pmatrix}$ with $B_t \in \mathcal{B}^T_{\text{reg}}(j)$, $t \in S^1$ the moduli spaces $\widehat{\mathcal{M}}(w_-, w_+)$ are compact up to breaking as long as $\mu_{\text{CZ}}(w_+) - \mu_{\text{CZ}}(w_-) \le 2$. In [18] Hofer-Salamon argue that bubbling-off of J_t-holomorphic spheres of Chern number at least 2 never occurs for index reasons. Moreover, they rule out bubbling-off of J_t-holomorphic sphere with Chern number less than 2 by carefully studying moduli spaces of J_t-holomorphic spheres. The crucial input is that for *simple* holomorphic spheres the linearized operator is a surjective Fredholm operator, see [18, Theorem 2.2]. We establishes the corresponding result for our restricted class of almost complex structures in Proposition 2.11. Therefore, the results in [18] apply to the Floer equation for $\mathscr{A}^\tau_f$ and we conclude that the moduli spaces $\widehat{\mathcal{M}}(w_-, w_+)$ are compact up to breaking as long as $\mu(w_+) - \mu(w_-) \le 2$.

2.4 Rabinowitz Floer homology

We define Rabinowitz Floer homology with the help of Novikov rings. Alternative approaches are via mixed direct/inverse limits. How these relate has been studied in [10]. The current approach is as in the original article [9].

The spaces $\mathfrak{C}_k$ and $\mathfrak{C}$ of critical point of $\mathscr{A}^\tau_f$ were defined in Definition 2.8. The vector space $\text{RFC}_*(\mathscr{A}^\tau_f)$, graded by μ (see (2.19)), is the set of all formal linear combinations

$$\xi = \sum_{w \in \mathfrak{C}} a_w w, \quad a_w \in \mathbb{Z}/2, \tag{2.55}$$

subject to the Novikov condition

$$\forall \kappa \in \mathbb{R}: \ \#\big\{w \in \mathfrak{C} \mid a_w \neq 0,\ \mathscr{A}^\tau_f(w) \ge \kappa\big\} < \infty\,. \tag{2.56}$$

It is a module over the Novikov ring

$$\Lambda_E := \left\{ \sum_{A \in \Gamma_E} n_A e^A \mid n_A \in \mathbb{Z}/2,\ \forall \kappa \in \mathbb{R}\colon \#\big\{A \in \Gamma_E \mid n_A \neq 0,\ \Omega(A) \ge \kappa\big\} < \infty \right\}. \tag{2.57}$$

The multiplicative structure on Λ_E is given by

$$\left(\sum_{A \in \Gamma_E} n_A e^A\right) \cdot \left(\sum_{B \in \Gamma_E} m_B e^B\right) := \sum_A \sum_B (n_A \cdot m_B)\, e^{A+B} = \sum_C \left(\sum_A n_A \cdot m_{C-A}\right) e^C$$

and the action of Λ_E on $\text{RFC}_k(\mathscr{A}^\tau_f)$ by

$$\left(\sum_{A \in \Gamma_E} n_A e^A\right) \cdot \left(\sum_{w \in \mathfrak{C}} a_w w\right) := \sum_w \left(\sum_A n_A \cdot a_{w\#-A}\right) w\,, \tag{2.58}$$

where we use the following notation. If $w = ([u, B], \eta)$ then $w\# - A = ([u, B - A], \eta)$. The differential ∂ on $\mathrm{RFC}_*(\mathscr{A}_f^\tau)$ is defined by

$$\begin{aligned} \partial : \mathrm{RFC}_k(\mathscr{A}_f^\tau) &\longrightarrow \mathrm{RFC}_{k-1}(\mathscr{A}_f^\tau) \\ \partial\, w &:= \sum_{z \in \mathfrak{C}_{k-1}} \#_2 \mathcal{M}(z, w)\, z\ . \end{aligned} \tag{2.59}$$

The compactness results described in Sect. 2.3 imply that $\mathcal{M}(z, w)$ is a finite set and $\#_2\mathcal{M}(z, w) \in \mathbb{Z}/2$ denotes its parity. Moreover, compactness up to breaking implies $\partial \circ \partial = 0$. The Rabinowitz Floer homology is then defined by

$$\mathrm{RFH}_k(\mathscr{A}_f^\tau) := \mathrm{H}_k\big(\mathrm{RFC}_*(\mathscr{A}_f^\tau), \partial\big), \quad k \in \tfrac{1}{2} + \mathbb{Z}\ . \tag{2.60}$$

Remark 2.17 To define $\mathrm{RFH}_*(\mathscr{A}_f^\tau)$ we made auxiliary choices, notably τ and f. The assumption that f is C^2-small is not necessary for defining $\mathrm{RFH}_*(\mathscr{A}_f^\tau)$, see [3] for more details. Nevertheless, we decided to make this assumption throughout this article. The choices of τ and f become relevant in the proof of Theorem 1.2. The methods of Cieliebak and Frauenfelder [9] show that $\mathrm{RFH}_*(\mathscr{A}_f^\tau)$ is independent of all these choices.

Remark 2.18 We recall that we restrict ourselves to the class of almost complex structures J of the form $J = \begin{pmatrix} i & B \\ 0 & j \end{pmatrix}$ with $B \in \mathcal{B}^T_{\mathrm{reg}}(j)$. It is unclear to us whether it is possible to extend the definition of $\mathrm{RFH}_*(\mathscr{A}_f^\tau)$ beyond this class of almost complex structures. We crucially rely on Frauenfelder's result, namely that the fact that the projection of the Floer equation of $\mathscr{A}_f^\tau$ gives the Floer equation of $\mathfrak{a}_f$ on M can be used to obtain uniform C^0-bounds for the Lagrange multiplier. For this $\wp$ needs to be J-j-holomorphic.

3 A filtration and the proof of vanishing

We use the fact that $\mathrm{RFC}_*(\mathscr{A}_f^\tau)$ admits a filtration. For $l \in \mathbb{Z}$ we set

$$\mathrm{RFC}_k^l(\mathscr{A}_f^\tau) := \left\{ \sum_w a_w w \in \mathrm{RFC}_k(\mathscr{A}_f^\tau) \mid \mu_{\mathrm{CZ}}^M\big(\Pi(w)\big) = l \right\} \tag{3.1}$$

and

$$\mathrm{RFC}_k^{\le l}(\mathscr{A}_f^\tau) := \left\{ \sum_w a_w w \in \mathrm{RFC}_k(\mathscr{A}_f^\tau) \mid \mu_{\mathrm{CZ}}^M\big(\Pi(w)\big) \le l \right\}, \tag{3.2}$$

where we recall that $A \in \Gamma_E \cong \Gamma_M$, see Remark 2.1.

Lemma 3.1

$$\partial\Big(\mathrm{RFC}_k^{\le l}(\mathscr{A}_f^\tau)\Big) \subset \mathrm{RFC}_{k-1}^{\le l}(\mathscr{A}_f^\tau) \tag{3.3}$$

hence we can decompose

$$\partial = \sum_{i \ge 0} \partial_i = \partial_0 + \partial_1 + \cdots \tag{3.4}$$

with

$$\begin{aligned} \partial_i : \mathrm{RFC}_k^l(\mathscr{A}_f^\tau) &\longrightarrow \mathrm{RFC}_{k-1}^{l-i}(\mathscr{A}_f^\tau) \\ \partial_i w &:= \sum_{\substack{z\in\mathfrak{C}_{k-1} \\ \mu_{\mathrm{CZ}}^M(\Pi(z))=\mu_{\mathrm{CZ}}^M(\Pi(w))-i}} \#_2\mathcal{M}(z,w)z\,. \end{aligned} \tag{3.5}$$

Proof This is a direct consequence of Lemma 2.10 together with (2.43). □

Remark 3.2 From $\partial^2 = 0$ and the filtration we derive for every $i \geq 0$ the equation

$$\sum_{j=0}^{i} \partial_j \partial_{i-j} = 0. \tag{3.6}$$

E.g. $\partial_0\partial_0 = 0$, $\partial_0\partial_1 + \partial_1\partial_0 = 0$ etc. In particular, ∂_0 is a differential.

The main idea for proving Theorem 1.2 is that ∂_0 counts solutions of the Floer equation (2.27) which are entirely contained inside fibers of $\wp$ over critical points of f. Thus the homology of ∂_0 is the sum of $\mathrm{Crit}(f)\times\Gamma_M$-many copies of the Rabinowitz Floer homology of $\big(\Sigma_\tau\cap\wp^{-1}(q),\wp^{-1}(q)\big)\cong(S^1,\mathbb{C})$, $q\in\mathrm{Crit}(f)$, each of which vanishes.

Proposition 3.3 *The differential ∂_0 counts precisely the solutions $w=(u,\eta)$ of the Floer equation (2.27) with image contained entirely in a fiber over some critical point of f. That is, there exists $q\in\mathrm{Crit}(f)$ such that $u(\mathbb{R}\times S^1)\subset\wp^{-1}(q)$. Moreover, if $w_\pm=([u_\pm,A_\pm],\eta_\pm)\in\mathrm{Crit}(\mathscr{A}_f^\tau)$ are the asymptotic limits of w then*

$$A_- = A_+ \in \Gamma_E\,. \tag{3.7}$$

Proof Let $w=(u,\eta)$ be a gradient flow line from $w_-=([u_-,A_-],\eta_-)$ to $w_+=([u_+,A_+],\eta_+)$ with

$$\mu_{\mathrm{CZ}}^M\big(\Pi(w_+)\big)=\mu_{\mathrm{CZ}}^M\big(\Pi(w_-)\big)\,. \tag{3.8}$$

Using Lemma 2.10 we see that

$$\wp(u)\in\widehat{\mathcal{N}}\big(\Pi(w_+),\Pi(w_-)\big)\,. \tag{3.9}$$

According to (2.43), equation (3.8) implies that that

$$\dim\widehat{\mathcal{N}}\big(\Pi(w_+),\Pi(w_-)\big)=0\,, \tag{3.10}$$

which in turn implies that $\wp(u)$ is s-independent, i.e. constant $\wp(u)=q\in\mathrm{Crit}(f)$, see Lemma 2.9. In other words, $u(\mathbb{R}\times S^1)\subset\wp^{-1}(q)$. Moreover, in view of (2.38), we have

$$A_- = A_+ \in \Gamma_M \cong \Gamma_E\,. \tag{3.11}$$

This finishes the proof. □

Corollary 3.4

$$\mathrm{H}_k\big(\mathrm{RFC}_*(\mathscr{A}_f^\tau),\partial_0\big)=0 \quad \forall k\in\tfrac{1}{2}+\mathbb{Z}\,. \tag{3.12}$$

Proof For $q \in \mathrm{Crit}(f)$ we fix an identification

$$\big(\Sigma_\tau \cap \wp^{-1}(q), \wp^{-1}(q)\big) \cong (S^1_\tau, \mathbb{C}) \tag{3.13}$$

together with the symplectic form, its primitive and the complex structure i. Here S^1_τ is the circle bounding a disk of area $\pi\tau^2$. For $A \in \Gamma_M$ we denote by

$$\mathrm{RFC}_*(q, A) \tag{3.14}$$

the vector space generated over $\mathbb{Z}/2$ by critical points of the form $\big([u, A]^\pm, \eta\big) \in \mathfrak{C} \subset \mathrm{Crit}(\mathscr{A}^\tau_f)$ with $\wp(u) = q$. Proposition 3.3 implies that $\mathrm{RFC}_*(q, A)$ is a ∂_0-subcomplex of $\mathrm{RFC}_*(\mathscr{A}^\tau_f)$. With the above identification we see that

$$\big(\mathrm{RFC}_{k+2c_1^{TE}(A)+\frac{1}{2}\dim M}(q, A), \partial_0\big) = \big(\mathrm{RFC}_k(S^1, \mathbb{C}), \partial\big)\ . \tag{3.15}$$

Let v be the primitive Reeb orbit over q then all generators are of the form $\big([v^n, A]^\pm, \eta = n - f(q)\big)$. Since $\mathrm{RFH}_*(S^1, \mathbb{C}) = 0$ due to [1,9] and $\mu\big([v^{n-1}, A]^\pm, n-1-f(q)\big) + 2 = \mu\big([v^n, A]^\pm, n - f(q)\big)$ from Lemma 2.7, we know that

$$\begin{aligned} \partial_0\big([v^n, A]^-, n - f(q)\big) &= \big([v^{n-1}, A]^+, n-1-f(q)\big) \\ \partial_0\big([v^n, A]^+, n - f(q)\big) &= 0\ . \end{aligned} \tag{3.16}$$

Let $\xi = \sum_w a_w w \in \mathrm{RFC}_k(\mathscr{A}^\tau_f)$ with $\partial_0\xi = 0$, i.e. $\sum_w a_w \partial_0 w = 0$. If $a_w \neq 0$ then w is of the form $\big([v^n, A]^+, n - f(q)\big)$ and we let w' be the corresponding element $\big([v^{n+1}, A]^-, n + 1 - f(q)\big)$. Then

$$\xi' := \sum_w a_w w' \tag{3.17}$$

satisfies the Novikov condition, i.e. $\xi' \in \mathrm{RFC}_{k+1}(\mathscr{A}^\tau_f)$, since $\mathscr{A}^\tau_f(w') = \mathscr{A}^\tau_f(w) + \tau$ due to (2.11). From (3.16), $\partial_0\xi' = \xi$ and this completes the proof. □

Lemma 3.5 *We assume now that* $c_1^{TM} = c\omega : \pi_2(M) \to \mathbb{Z}$.

- *In case* $c = 0$ *we have for all* $\tau > 0$

$$\mathrm{RFC}_k(\mathscr{A}^\tau_f) = \bigoplus_{l=-\frac{1}{2}\dim M}^{\frac{1}{2}\dim M} \mathrm{RFC}^l_k(\mathscr{A}^\tau_f) \tag{3.18}$$

and

$$\partial_n = 0 \quad \forall n \geq \dim M + 1\ . \tag{3.19}$$

- *In case* $c \geq 1$ *we assume* $(c-1)\tau < 1$. *Then a formal sum* $\xi = \sum_w a_w w$, $a_w \in \mathbb{Z}/2$, $w \in \mathfrak{C}_k$ *satisfies the Novikov condition*

$$\forall \kappa \in \mathbb{R}:\ \#\big\{w \in \mathfrak{C}_k \mid a_w \neq 0,\ \mathscr{A}^\tau_f(w) \geq \kappa\big\} < \infty \tag{3.20}$$

if and only if

$$\forall \kappa \in \mathbb{R}:\ \#\big\{w \in \mathfrak{C}_k \mid a_w \neq 0,\ \mu^M_{CZ}\big(\wp(w)\big) \geq \kappa\big\} < \infty. \tag{3.21}$$

In particular, for all $\xi \in \mathrm{RFC}_k(\mathscr{A}^\tau_f)$ *there exists* $l(\xi) \in \mathbb{Z}$ *with*

$$\xi \in \mathrm{RFC}^{\leq l(\xi)}_k(\mathscr{A}^\tau_f)\ . \tag{3.22}$$

Proof If we write $w = \big([v^n, A]^{\pm}, \eta\big) \in \mathfrak{C}_k$ then according to (2.11) and Definition 2.8

$$\begin{aligned} \mathscr{A}_f^\tau(w) &= n\tau + \omega(A) - (\tau+1) f(\wp(v)) \\ k = \mu(w) &= 2n + 2c_1^{TE}(A) - \mu_{\text{Morse}}(\wp(v); f) + \tfrac{1}{2}\dim M\big(\pm\tfrac{1}{2}\big) \\ &= 2n + 2(c-1)\omega(A) - \mu_{\text{Morse}}(\wp(v); f) + \tfrac{1}{2}\dim M\big(\pm\tfrac{1}{2}\big)\ . \end{aligned} \tag{3.23}$$

We solve the second equation for n

$$n = \tfrac{1}{2}k - (c-1)\omega(A) + \tfrac{1}{2}\big[\mu_{\text{Morse}}(\wp(v); f) - \tfrac{1}{2}\dim M\big(\pm\tfrac{1}{2}\big)\big] \tag{3.24}$$

and abbreviate $e := \mu_{\text{Morse}}(\wp(v); f) - \frac{1}{2}\dim M\big(\pm\frac{1}{2}\big)$. In particular, $|e| \leq \frac{1}{2}\dim M + \frac{1}{2}$. Thus, we can rewrite the action value as

$$\begin{aligned} \mathscr{A}_f^\tau(w) &= n\tau + \omega(A) - (\tau+1) f(\wp(v)) \\ &= \big(\tfrac{1}{2}k - (c-1)\omega(A) + \tfrac{1}{2}e\big)\tau + \omega(A) - (\tau+1) f(\wp(v)) \\ &= \big(1 - (c-1)\tau\big)\omega(A) + \tfrac{1}{2}\big(k+e\big)\tau - (\tau+1) f(\wp(v))\ . \end{aligned} \tag{3.25}$$

Next we observe that

$$\begin{aligned} \mu_{\text{CZ}}^M(\Pi(w)) &= -\mu_{\text{Morse}}(\wp(v); f) + \tfrac{1}{2}\dim M + 2c_1^{TM}(A) \\ &= -\mu_{\text{Morse}}(\wp(v); f) + \tfrac{1}{2}\dim M + 2c\omega(A). \end{aligned} \tag{3.26}$$

In case $c \geq 1$ and $(c-1)\tau < 1$ equations (3.25) and (3.26) imply the if-and-only-if statement of the Lemma. The statement (3.22) follows from the if-part of the if-and-only-if statement since $\xi \in \text{RFC}_k(\mathscr{A}_f^\tau)$ satisfies the Novikov condition by the very definition of RFC.

The case $c = 0$ follows immediately from equation (3.26). □

We are now in the position to prove Theorem 1.2. We treat the symplectically aspherical case last and assume now that $c_1^{TM} = c\omega$. We first consider the case $c \geq 0$. If $c = 0$ we assume that (E, Ω) is semi-positive.

Proof of Theorem 1.2 for $c \geq 0$ We fix $\xi \in \text{RFC}_k(\mathscr{A}_f^\tau)$ with

$$\partial\xi = 0\ . \tag{3.27}$$

Our aim is to construct $\theta \in \text{RFC}_{k+1}(\mathscr{A}_f^\tau)$ with $\partial\theta = \xi$. We split ξ as follows.

$$\xi = \sum_{l=-\infty}^{l(\xi)} \xi_l \quad \text{with} \quad \xi_l \in \text{RFC}_k^l(\mathscr{A}_f^\tau)\ , \tag{3.28}$$

where $l(\xi) \in \mathbb{Z}$ is taken from Lemma 3.5. If $c = 0$ we set $l(\xi) := \frac{1}{2}\dim M$. We expand $\partial\xi = 0$ according to $\partial = \sum_{i\geq 0}\partial_i$ and collect terms in $\text{RFC}_{k-1}^{l(\xi)-I}(\mathscr{A}_f^\tau)$ for all $I \geq 0$. We recall from Lemma 3.1 that ∂_i drops the upper degree by i. This leads to

$$\sum_{i=0}^{I} \partial_i \xi_{l(\xi)+i-I} = 0\ , \tag{3.29}$$

since $\partial_i\xi_m \in \text{RFC}_{k-1}^{l(\xi)-I}(\mathscr{A}_f^\tau)$ if and only if $m - i = l(\xi) - I$.

Claim 1 For all $l \leq l(\xi)$ there exists $\theta_l \in \text{RFC}_{k+1}^l(\mathscr{A}_f^\tau)$ such that

$$\sum_{i=0}^{I} \partial_i \theta_{l(\xi)+i-I} = \xi_{l(\xi)-I} \tag{3.30}$$

holds for $I \geq 0$.

Proof of Claim 1 We inductively construct θ_l. For $I = 0$ equation (3.29) reduces to

$$\partial_0 \xi_{l(\xi)} = 0. \tag{3.31}$$

Corollary 3.4 implies that there exists $\theta_{l(\xi)} \in \mathrm{RFC}_{k+1}^{l(\xi)}(\mathscr{A}_f^\tau)$ with

$$\partial_0 \theta_{l(\xi)} = \xi_{l(\xi)}. \tag{3.32}$$

Now assume that we already constructed $\theta_{l(\xi)}, \ldots, \theta_{l(\xi)-(I-1)}$ satisfying equation (3.30). Then we compute

$$\begin{aligned}
\partial_0\left(\xi_{l(\xi)-I} - \sum_{i=1}^{I} \partial_i \theta_{l(\xi)+i-I}\right) &= \partial_0 \xi_{l(\xi)-I} - \sum_{i=1}^{I} \partial_0 \partial_i \theta_{l(\xi)+i-I} \\
&\overset{(*)}{=} \partial_0 \xi_{l(\xi)-I} + \sum_{i=1}^{I} \sum_{j=1}^{i} \partial_j \partial_{i-j} \theta_{l(\xi)+i-I} \\
&\overset{(**)}{=} \partial_0 \xi_{l(\xi)-I} + \sum_{j=1}^{I} \partial_j \left(\sum_{i=j}^{I} \partial_{i-j} \theta_{l(\xi)+i-I} \right) \\
&= \partial_0 \xi_{l(\xi)-I} + \sum_{j=1}^{I} \partial_j \left(\sum_{i=0}^{I-j} \partial_i \theta_{l(\xi)+i-(I-j)} \right) \\
&\overset{(***)}{=} \partial_0 \xi_{l(\xi)-I} + \sum_{j=1}^{I} \partial_j \xi_{l(\xi)+j-I} \\
&= \sum_{j=0}^{I} \partial_j \xi_{l(\xi)+j-I} \\
&= 0\,.
\end{aligned} \tag{3.33}$$

Here we used equation (3.6) in $(*)$, the usual relabeling $\sum_{i=1}^{I} \sum_{j=1}^{i} = \sum_{j=1}^{I} \sum_{i=j}^{I}$ in $(**)$, the induction hypothesis (3.30) in $(***)$ and (3.29) at the end. Using again Corollary 3.4 we find $\theta_{l(\xi)-I} \in \mathrm{RFC}_{k+1}^{l(\xi)-I}(\mathscr{A}_f^\tau)$ with

$$\partial_0 \theta_{l(\xi)-I} = \xi_{l(\xi)-I} - \sum_{i=1}^{I} \partial_i \theta_{l(\xi)+i-I}, \tag{3.34}$$

in other words

$$\sum_{i=0}^{I} \partial_i \theta_{l(\xi)+i-I} = \xi_{l(\xi)-I}. \tag{3.35}$$

This proves Claim 1. □

Now we consider

$$\theta := \sum_{l=-\infty}^{l(\xi)} \theta_l\,. \tag{3.36}$$

We will show that θ satisfies the Novikov condition and $\partial\theta = \xi$. We need slightly different arguments for the cases $c = 0$ and $c \geq 1$.

If $c = 0$ then $\xi = \sum_{l=-\frac{1}{2}\dim M}^{\frac{1}{2}\dim M} \xi_l$, see Lemma 3.5, is the sum of finitely many non-zero ξ_l each of which satisfies the Novikov condition. Since $\partial = \partial_0 + \cdots \partial_{\dim M}$ we obtain only finitely many non-zero θ_l each of which satisfies the Novikov condition by (the proof of) Corollary 3.4. Thus, $\theta = \sum \theta_l$ satisfies the Novikov condition, too.

If $c \geq 1$ then Lemma 3.5 implies that each $\theta_i \in \mathrm{RFC}^i_{k+1}(\mathscr{A}^\tau_f)$ is a finite sum of elements in $\mathfrak{C}$ (as opposed to a general Novikov sum.) Thus, using again Lemma 3.5 we see that $\theta = \sum_{l=-\infty}^{l(\xi)} \theta_l$ satisfies the Novikov condition.

In both cases the equation $\partial\theta = \xi$ holds by construction. Indeed, the part of

$$\partial\theta = \sum_{l=-\infty}^{l(\xi)} \partial\theta_l = \sum_{l=-\infty}^{l(\xi)} \sum_{i=0}^{\infty} \underbrace{\partial_i\theta_l}_{\in \mathrm{RFC}_k^{l-i}} \in \mathrm{RFC}_k(\mathscr{A}^\tau_f) \tag{3.37}$$

in $\mathrm{RFC}^r_k(\mathscr{A}^\tau_f)$ is $\sum_{i=0}^{l(\xi)-r} \partial_i\theta_{i+r}$. By relabeling $I = l(\xi) - r$ we compute

$$\begin{aligned}\partial\theta &= \sum_{r=-\infty}^{l(\xi)} \sum_{i=0}^{l(\xi)-r} \partial_i\theta_{i+r}\\ &= \sum_{I=0}^{\infty}\sum_{i=0}^{I} \partial_i\theta_{l(\xi)+i-I}\\ &= \sum_{I=0}^{\infty} \xi_{l(\xi)-I}\\ &= \sum_{l=-\infty}^{l(\xi)} \xi_l\\ &= \xi,\end{aligned} \tag{3.38}$$

where we used Eq. (3.35) in the third equality. Thus, for every $\xi \in \mathrm{RFC}_k(\mathscr{A}^\tau_f)$ with $\partial\xi = 0$ we constructed $\theta \in \mathrm{RFC}_{k+1}(\mathscr{A}^\tau_f)$ with $\partial\theta = \xi$. This finishes the proof. □

Proof of Theorem 1.2 for $2c\nu \leq -\dim M$

In this proof we make the assumption that the Morse function $f : M \to \mathbb{R}$ additionally satisfies $f(M) \subset (0, 1)$. We fix $\xi \in \mathrm{RFC}_k(\mathscr{A}^\tau_f)$ with

$$\partial\xi = 0\,. \tag{3.39}$$

We will again construct $\theta \in \mathrm{RFC}_{k+1}(\mathscr{A}^\tau_f)$ with $\partial\theta = \xi$. This time we split ξ as follows.

$$\xi = \sum_{A\in\Gamma_M} \xi_A \quad \text{with} \quad \xi_A = \sum_{\substack{w\in\mathfrak{C}_k\\ [\Pi(w)]\in \mathrm{Crit}(f)\times\{A\}}} a_w w\,. \tag{3.40}$$

Claim 2 If $\mathcal{M}\Big(\big([v, B], \hat\eta\big), \big([u, A], \eta\big)\Big) \neq \emptyset$ then $A = B \in \Gamma_M \cong \Gamma_E$.

Proof of Claim 2 We compare action and Conley–Zehnder index of $\Pi\big([u,A],\eta\big)$ and $\Pi\big([v,B],\hat\eta\big)$. Using that the moduli space is non-empty we conclude

$$\mathfrak{a}_f\big(\Pi\big([u,A],\eta\big)\big)=\omega(A)-f(\wp(u))\geq\omega(B)-f(\wp(v))=\mathfrak{a}_f\big(\Pi\big([v,B],\hat\eta\big)\big) \tag{3.41}$$

and

$$\begin{aligned}\mu^M_{\mathrm{CZ}}\big(\Pi\big([u,A],\eta\big)\big)&=-\mu_{\mathrm{Morse}}\big(\wp(u),f\big)+\tfrac12\dim M+2c_1^{TM}(A)\\&\geq-\mu_{\mathrm{Morse}}\big(\wp(v),f\big)+\tfrac12\dim M+2c_1^{TM}(B)\\&=\mu^M_{\mathrm{CZ}}\big(\Pi\big([v,B],\hat\eta\big)\big)\ .\end{aligned} \tag{3.42}$$

We assume now that $A\neq B$. We recall that $f(M)\subset(0,1)$. Thus, the first inequality simplifies to

$$\omega(A)>\omega(B) \tag{3.43}$$

since $\omega\big(\pi_2(M)\big)=\nu\mathbb{Z}$. From $c_1^{TM}=c\omega$ with $c<0$ we conclude then $c_1^{TM}(A)<c_1^{TM}(B)$. The minimal Chern number of M equals $-c\nu$. Thus, we have

$$c_1^{TM}(A)\leq c_1^{TM}(B)+c\nu \tag{3.44}$$

and from (3.42)

$$\begin{aligned}2c_1^{TM}(A)&\geq-\mu_{\mathrm{Morse}}\big(\wp(v),f\big)+\mu_{\mathrm{Morse}}\big(\wp(u),f\big)+2c_1^{TM}(B)\\&\geq-\dim M+2c_1^{TM}(B).\end{aligned} \tag{3.45}$$

We conclude that

$$2c\nu\geq-\dim M. \tag{3.46}$$

In case $2c\nu<-\dim M$ we arrive at a contradiction. It remains to treat the case $2c\nu=-\dim M$. In this case we claim that the inequality (3.44) necessarily becomes the equality

$$c_1^{TM}(A)=c_1^{TM}(B)+c\nu. \tag{3.47}$$

Otherwise (3.44) is actually of the form $c_1^{TM}(A)\leq c_1^{TM}(B)+2c\nu$ since $-c\nu$ is the minimal Chern number of M. As above this implies then that $4c\nu\geq-\dim M$, i.e. $2\dim M\leq\dim M$, and thus $\dim M=0$. I.e. we are left with the case $\Sigma=S^1\subset\mathbb{C}=E$ in which Theorem 1.2 is true: $\mathrm{RFH}_*(S^1,\mathbb{C})=0$, [1,9]

We combine $c_1^{TM}(A)=c_1^{TM}(B)+c\nu$ with (3.42) and arrive at

$$\mu_{\mathrm{Morse}}\big(\wp(v),f\big)-\mu_{\mathrm{Morse}}\big(\wp(u),f\big)\geq-2c\nu=\dim M \tag{3.48}$$

which turns the inequality (3.42) into an equality:

$$\mu^M_{\mathrm{CZ}}\big(\Pi\big([u,A],\eta\big)\big)=\mu^M_{\mathrm{CZ}}\big(\Pi\big([v,B],\hat\eta\big)\big). \tag{3.49}$$

Now we proceed as in the proof of Proposition 3.3 in order to conclude that all element in $\mathcal{M}\Big(\big([v,B],\hat\eta\big),\big([u,A],\eta\big)\Big)$ are actually differentials which are entirely contained in fibers of E and thus $A=B$, again by Proposition 3.3. □

We recall that we split the cycle ξ as

$$\xi=\sum_{A\in\Gamma_M}\xi_A\ . \tag{3.50}$$

It follows from the Claim 2 and $\partial\xi = 0$ that

$$\partial\xi_A = 0 \quad \forall A \in \Gamma_M \ . \tag{3.51}$$

Observe that for every A the sum

$$\xi_A = \sum_{\substack{w\in\mathfrak{C}_k \\ [\Pi(w)]\in\mathrm{Crit}(f)\times\{A\}}} a_w w \tag{3.52}$$

is finite. Indeed, we know that w is of the form $w = ([u, A], \eta)$ with fixed A and u being an l-fold cover of a simple Reeb orbit over a critical point of f. Moreover, the index of w is fixed: $\mu(w) = k$. Therefore, Definition 2.8 of the index μ and the index formula Lemma 2.7 allow only for finitely many combinations. The number of possibilities is bounded by $\frac{1}{2}\dim M$. In particular, the number of possibilities does *not* depend on A.

Now, we apply again the inductive procedure (3.30) from the proof in case $c \geq 0$ to obtain $\theta_A \in \mathrm{RFC}_{k+1}(\mathscr{A}_f^\tau)$ with

$$\partial\theta_A = \xi_A. \tag{3.53}$$

If we set

$$\theta := \sum_{A\in\Gamma_M} \theta_A \tag{3.54}$$

then Claim 2 implies $\partial\theta = \xi$. As above it remains to check that θ satisfies the Novikov condition (2.56). For this we express for some A

$$\theta_A = \sum_{\substack{z\in\mathfrak{C}_{k+1} \\ [\Pi(z)]\in\mathrm{Crit}(f)\times\{A\}}} b_z z \ . \tag{3.55}$$

The same argument we used to conclude that each ξ_A is a finite sum gives the same for θ_A. Moreover, since $\mu(\theta_A) = k + 1$, $\partial\theta_A = \xi_A$ again the claim, the index formula Lemma 2.7 and the computation of the action (2.11) implies that exists $C > 0$ such that

$$|\mathscr{A}_f^\tau(z) - \mathscr{A}_f^\tau(w)| \leq C \tag{3.56}$$

whenever $\mathcal{M}(w, z) \neq \emptyset$ for some w appearing in ξ_A and z in θ_A. The constant C does *not* depend on A, indeed we may choose $C = \frac{\tau}{2}\dim M + \max f - \min f$.

Thus, ξ satisfying the Novikov condition implies that θ satisfies the Novikov condition since their actions are of bounded distance. This completes the proof.

Proof of Theorem 1.2 *for* $\omega\big(\pi_2(M)\big) = 0$

We follow the proof of the case $2c\nu \leq -\dim M$. We first establish Claim 2, i.e. that $\mathcal{M}\big(([v, B], \hat{\eta}), ([u, A], \eta)\big) \neq \emptyset$ implies $A = B \in \Gamma_M$ holds without assuming $c_1^{TM} = c\omega$ under the assumption that the Morse function $f : M \to \mathbb{R}$ is sufficiently small.

We assume otherwise. Then we find a sequence $\epsilon_n \to 0$ and a sequence of elements $w_n \in \mathcal{M}\big(([v_n, B_n], \hat{\eta}_n), ([u_n, A_n], \eta_n)\big)$ where w_n satisfies the Floer equation for $\mathscr{A}_{f_n}^\tau$ with $f_n := \epsilon_n f$. By definition of $\mathcal{M}$ we have $c_1^{TE}(-A_n\#w_n\#B_n) = 0$. Since $\omega\big(\pi_2(M)\big) = 0$ we can identify $c_1^{TE} = c_1^{TM} : \pi_2(E) \cong \pi_2(M) \to \mathbb{Z}$. We recall that $\Pi(w_n)$ are solutions of the Floer equation of $\mathfrak{a}_{f_n}$ with

$$c_1^{TM}(-A_n\#\Pi(w_n)\#B_n) = c_1^{TE}(-A_n\#w_n\#B_n) = 0 \ . \tag{3.57}$$

We point out that the Floer cylinders $\Pi(w_n)$ topologically form spheres since their asymptotic limits lie in $\mathrm{Crit}(f_n)$. Therefore, we can rewrite

$$c_1^{TM}(-A_n\#\Pi(w_n)\#B_n) = -c_1^{TM}(A_n) + c_1^{TM}(\Pi(w_n)) + c_1^{TM}(B_n) = 0\,. \tag{3.58}$$

Moreover, they have uniformly bounded energy since their energy is given by the action difference of $\mathfrak{a}_{f_n}$ which, in turn, is bounded by $\max f_n - \min f_n$ thanks to our assumption $\omega\big(\pi_2(M)\big) = 0$. Therefore, we can take the Floer-Gromov limit of $\Pi(w_n)$. Floer-Gromov compactness implies that we find a bubble tree of holomorphic spheres in (M, ω). Since we assume that $\omega\big(\pi_2(M)\big) = 0$ all holomorphic spheres are constant and therefore

$$c_1^{TM}(\Pi(w_n)) = 0 \tag{3.59}$$

for sufficiently large n from which we conclude

$$c_1^{TM}(A_n) = c_1^{TM}(B_n) \tag{3.60}$$

for sufficiently large n. That is, the above claim indeed holds for sufficiently small $f : M \to \mathbb{R}$. We now can proceed as in the proof of the case $2c\nu \leq -\dim M$.

Remark 3.6 In the latter two cases of the proof of Theorem 1.2 we assume that the auxiliary Morse function f is very small. This is an echo of the 'true' proof of Theorem 1.2 in the full Morse–Bott setting, i.e. the case of $\mathscr{A}^{\tau}_{f=0}$. Indeed, in both cases $2c\nu \leq -\dim M$ and $\omega\big(\pi_2(M)\big) = 0$ the Morse–Bott differential is of the form $\partial = \partial_0 +$ auxiliary Morse trajectories which immediately implies the Theorem.

4 A conjectural explanation

Let V be a Liouville domain, i.e. a compact exact symplectic manifold with contact type boundary. We recall one of the main theorems by Cieliebak–Frauenfelder–Oancea in [11]. There is a long exact sequence between symplectic (co-)homology SH and Rabinowitz Floer homology RFH as follows.

$$\cdots \longrightarrow \mathrm{SH}^{-*}(V) \longrightarrow \mathrm{SH}_*(V) \longrightarrow \mathrm{RFH}_*(\partial V, V) \longrightarrow \mathrm{SH}_{-*-1}(V) \longrightarrow \cdots\,. \tag{4.1}$$

Moreover, the map $\mathrm{SH}^{-*}(V) \to \mathrm{SH}_*(V)$ splits as

$$\begin{array}{ccccc} \mathrm{SH}^{-*}(V) & \longrightarrow & & \longrightarrow & \mathrm{SH}_*(V) \\ \downarrow & & & & \uparrow \\ \mathrm{H}^{-*+d}(V, \partial V) & \xrightarrow{\mathrm{PD}} & \mathrm{H}_{*+d}(V) & \xrightarrow{\mathrm{incl}_*} & \mathrm{H}_{*+d}(V, \partial V) \end{array} \tag{4.2}$$

where PD denotes Poincare duality and $d = \frac{1}{2}\dim V$. As observed by Ritter in [22] this long exact sequence together with the fact that SH is a ring with unity leads to the statement

$$\mathrm{SH}^*(V) = 0 \iff \mathrm{SH}_*(V) = 0 \iff \mathrm{RFH}_*(\partial V, V) = 0. \tag{4.3}$$

Note that our (co-)homology and grading conventions match with the ones in [11].

In [21] Oancea proves $\mathrm{SH}_*(E) = 0$ for negative line bundles $\wp : E \to M$ under the condition that (E, Ω) is symplectically aspherical. We point out that even in the symplectically aspherical case E is not a Liouville manifold since $[\Omega] \neq 0 \in \mathrm{H}^2(E)$. Ritter computes in [23, Theorem 1] for more general negative line bundles

$$\mathrm{SH}_*(E) \cong \mathrm{QH}_{*+d}(E, \Sigma)/\ker r^k \tag{4.4}$$

where $d = \frac{1}{2}\dim E$, $\mathrm{QH}_*(E, \Sigma)$ is the relative quantum homology of the disk bundle inside E with boundary Σ and $r : \mathrm{QH}_*(E) \to \mathrm{QH}_{*-2}(E)$ is the map given by quantum intersection product with $\mathrm{PD}(\wp^* c_1^E) \in \mathrm{QH}_{2d-2}(E)$. Finally (4.4) holds for any $k \geq \dim \mathrm{H}_*(M)$. In particular,

$$\mathrm{SH}_*(E) = 0 \quad \Longleftrightarrow \quad \wp^* c_1^E \text{ is nilpotent in } \mathrm{QH}^*(E) \tag{4.5}$$

which generalizes Oancea's computation. Considering for instance the bundle $\wp : \mathcal{O}(-n) \to \mathbb{C}\mathrm{P}^m$ it follows that

$$\mathrm{SH}_*(\mathcal{O}(-n)) \neq 0, \tag{4.6}$$

for $n \leq m$, see [23, Section 1.5]. On the other hand, Theorem 1.2 applies since $(\mathbb{C}\mathrm{P}^m, n\omega_{\mathrm{FS}})$ is monotone with $c = \frac{m+1}{n}$ with corresponding negative line bundle $\mathcal{O}(-n)$, i.e. we conclude

$$\mathrm{RFH}_*(\Sigma, \mathcal{O}(-n)) = 0. \tag{4.7}$$

In this case Σ is a Lens space. This is, of course, no contradiction to (4.3) since the space $\mathcal{O}(-n)$ is *not* a Liouville manifold. Also, $(\mathbb{C}\mathrm{P}^m, n\omega_{\mathrm{FS}})$ is not symplectically aspherical.

We offer the following conjectural explanation of Ritter's result (4.4) in terms of the long exact sequence (4.1) from [11] and Theorem 1.2. We claim that the long exact sequence (4.1) remains valid for negative line bundles E (and probably even more generally) but the splitting of the map $\mathrm{SH}^{-*}(V) \to \mathrm{SH}_*(V)$ needs to be corrected as follows.

$$\begin{array}{ccccc} \mathrm{SH}^{-*}(E) & \longrightarrow & & & \mathrm{SH}_*(E) \longrightarrow \mathrm{RFH}_*(\Sigma, E) \\ \big\downarrow & & & & \big\uparrow c_* \\ \mathrm{QH}^{-*+d}(E, \Sigma) & \xrightarrow{\mathrm{PD}} & \mathrm{QH}_{*+d}(E) & \xrightarrow{\mathrm{incl}_*} & \mathrm{QH}_{*+d}(E, \Sigma) \end{array} \tag{4.8}$$

Here, as in [23], we identify $\mathrm{QH}_*(E, \Sigma)$ as Floer homology of a Hamiltonian with very small slope at infinity or equivalently as symplectic homology in the action window $(-\varepsilon, \varepsilon)$. Then c_* is just a continuation homomorphism induced by a canonical inclusion map. We refer to [23] for details. In particular, if $\mathrm{RFH}_*(\Sigma, E) = 0$ then the map c_* is surjective and

$$\mathrm{SH}_*(E) \cong \mathrm{QH}_{*+d}(E, \Sigma)/\ker c_*. \tag{4.9}$$

Ritter's important observation in [23] is that c_* is indeed surjective and can be identified with r^k for large k under his assumptions.

As mentioned above Ritter's and the present result holds for the bundle $\mathcal{O}(-n) \to \mathbb{C}\mathrm{P}^m$. In fact, from inspection of Ritter's article [23] it seems that Theorem 1.2 applies to all examples Ritter considers.

Acknowledgements We thank Urs Frauenfelder for illuminating discussion on the present article. PA is supported by SFB 878. JK is supported by DFG Grant KA 4010/1-1.

References

1. Albers, P., Frauenfelder, U.: Leaf-wise intersections and Rabinowitz Floer homology. J. Topol. Anal. **2**(1), 77–98 (2010)

2. Albers, P., Frauenfelder, U.: Rabinowitz Floer homology: a survey. In: Global Differential Geometry. Springer Proceedings of Mathematics, vol. 17, pp. 437–461. Springer, Heidelberg (2012)
3. Albers, P., Frauenfelder, U.: Bubbles and Onis. J. Fixed point theory appl. (2014, to appear). arXiv:1412.4360
4. Abbondandolo, A., Merry, W.J.: Floer Homology on the Time-Energy Extended Phase Space (2014). J. Symplectic Geom. arXiv:1411.4669
5. Abbondandolo, A., Schwarz, M.: Estimates and computations in Rabinowitz–Floer homology. J. Topol. Anal. **1**(4), 307–405 (2009)
6. Bourgeois, F., Oancea, A.: Symplectic homology, autonomous Hamiltonians, and Morse–Bott moduli spaces. Duke Math. J. **146**(1), 71–174 (2009)
7. Borman, M.S.: Quasi-states, quasi-morphisms, and the moment map. Int. Math. Res. Not. **11**, 2497–2533 (2013)
8. Borman, M.S., Zapolsky, F.: Quasimorphisms on contactomorphism groups and contact rigidity. Geom. Topol. **19**(1), 365–411 (2015)
9. Cieliebak, K., Frauenfelder, U.: A Floer homology for exact contact embeddings. Pac. J. Math. **293**(2), 251–316 (2009)
10. Cieliebak, K., Frauenfelder, U.: Morse homology on noncompact manifolds. J. Korean Math. Soc. **48**(4), 749–774 (2011)
11. Cieliebak, K., Frauenfelder, U., Oancea, A.: Rabinowitz Floer homology and symplectic homology. Ann. Sci. Éc. Norm. Supér. (4) **43**(6), 957–1015 (2010)
12. Eliashberg, Y., Polterovich, L.: Partially ordered groups and geometry of contact transformations. Geom. Funct. Anal. **10**(6), 1448–1476 (2000)
13. Fauck, A.: Rabinowitz-Floer homology on Brieskorn spheres. Int. Math. Res. Not. **14**, 5874–5906 (2015)
14. Floer, A., Hofer, H., Salamon, D.A.: Transversality in elliptic Morse theory for the symplectic action. Duke Math. J. **80**(1), 251–292 (1995)
15. Frauenfelder, U.: Rabinowitz Action Functional on Very Negative Line Bundles. Habilitationsschrift, Munich/München (2008)
16. Geiges, H.: An Introduction to Contact Topology, Cambridge Studies in Advanced Mathematics, vol. 109. Cambridge University Press, Cambridge (2008)
17. Givental, A.B. : The nonlinear Maslov index. In: Geometry of Low-Dimensional Manifolds, 2 (Durham, 1989). LondonMathematical Society, Lecture Note Series, vol. 151, pp. 35–43. Cambridge University Press, Cambridge (1990)
18. Hofer, H., Salamon, D.A.: Floer homology and Novikov rings. In: The Floer Memorial Volume. Progr. Math., vol. 133, pp. 483–524. Basel, Birkhäuser (1995)
19. Hofer, H., Zehnder, E.: Symplectic invariants and Hamiltonian dynamics. In: Birkhäuser Advanced Texts: Basler Lehrbücher. Birkhäuser, Basel (1994)
20. McDuff, D., Salamon, D.A.: J-holomorphic curves and symplectic topology. In: American Mathematical Society Colloquium Publications, vol. 52. American Mathematical Society, Providence (2004)
21. Oancea, A.: Fibered symplectic cohomology and the Leray–Serre spectral sequence. J. Symplectic Geom. **6**(3), 267–351 (2008)
22. Ritter, A.: Topological quantum field theory structure on symplectic cohomology. J. Topol. **6**(2), 391–489 (2013)
23. Ritter, A.: Floer theory for negative line bundles via Gromov–Witten invariants. Adv. Math. **262**, 1035–1106 (2014)
24. Robbin, J., Salamon, D.A.: The Maslov index for paths. Topology **32**(4), 827–844 (1993)
25. Sandon, S.: Equivariant homology for generating functions and orderability of lens spaces. J. Symplectic Geom. **9**(2), 123–146 (2011)