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Relative Cartier divisors and Laurent polynomial extensions

Vivek Sadhu[1] · **Charles Weibel**[2]

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Abstract If $i : A \subset B$ is a commutative ring extension, we show that the group $\mathcal{I}(A, B)$ of invertible A-submodules of B is contracted in the sense of Bass, with $L\mathcal{I}(A, B) = H^0_{\text{et}}(A, i_*\mathbb{Z}/\mathbb{Z})$. This gives a canonical decomposition for $\mathcal{I}(A[t, \frac{1}{t}], B[t, \frac{1}{t}])$.

Keywords Invertible modules · Contracted functor · Étale sheaf

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1 Introduction

Let $A \subset B$ be a ring extension. The group $\mathcal{I}(A, B)$ of invertible A-submodules of B is related to the Picard groups and the units groups of A and B by the exact sequence

$$1 \to U(A) \to U(B) \to \mathcal{I}(A, B) \to \text{Pic } A \to \text{Pic } B.$$

(See [6, §2].) Replacing $A \subset B$ with $A[t] \subset B[t]$ and $A[t, 1/t] \subset B[t, 1/t]$ yields similar exact sequences. Following Bass [1], each functor F on rings defines functors NF and LF so that $F(A[t]) = F(A) \oplus NF(A)$ and for certain functors like $F = U$ and Pic, called

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✉ Charles Weibel
weibel@math.rutgers.edu
http://math.rutgers.edu/∼weibel

Vivek Sadhu
viveksadhu@imsc.res.in; viveksadhu@gmail.com

1 Institute of Mathematical Sciences, IV Cross Road, CIT Campus, Taramani, Chennai, Tamil Nadu 600113, India

2 Mathematics Department, Rutgers University, New Brunswick, NJ 08901, USA

contracted functors, we even have a natural decomposition

$$F(A[t, 1/t]) \cong F(A) \oplus NF(A) \oplus NF(A) \oplus LF(A).$$

The decompositions of $U(A[t, 1/t])$ and Pic $A[t, 1/t]$ are given in [1, XII.7.8] and [11]. We can define $N\mathcal{I}(A, B)$ and $L\mathcal{I}(A, B)$ in the same way. Here is our main result.

Theorem 1.1 *Given a commutative ring extension $f : A \subset B$, $\mathcal{I}$ is a contracted functor with $L\mathcal{I}(A, B) = H^0_{\text{et}}(\text{Spec}\, A, f_*\mathbb{Z}/\mathbb{Z})$. In particular, there is a natural decomposition*

$$\mathcal{I}(A[t, 1/t], B[t, 1/t]) \cong \mathcal{I}(A, B) \oplus N\mathcal{I}(A, B) \oplus N\mathcal{I}(A, B) \oplus L\mathcal{I}(A, B),$$

In addition, $L\mathcal{I}(A, B) = L\mathcal{I}(A[t], B[t]) = L\mathcal{I}(A[t, 1/t], B[t, 1/t])$.

This is proven in Theorem 5.1 and Proposition 3.4 below. Here $\mathbb{Z}$ is regarded as the constant étale sheaf on both Spec A and Spec B, and $f_*\mathbb{Z}$ is the direct image sheaf on Spec A. The group $L\mathcal{I}(A, B)$ also equals the Nisnevich cohomology group $H^0_{\text{nis}}(\text{Spec}\, A, f_*\mathbb{Z}/\mathbb{Z})$, but differs from the Zariski cohomology group $H^0_{\text{zar}}(\text{Spec}\, A, f_*\mathbb{Z}/\mathbb{Z})$; see Example 5.5.1.

For convenience, let us write $A[T]$ for $A[t_1, 1/t_1, \ldots, t_n, 1/t_n]$. As pointed out by Bass [1], we can iterate the operations N and L to get decompositions of $\mathcal{I}(A[T], B[T])$ using components $N^i L^j \mathcal{I}(A, B)$ for $1 \le i, j \le n$. Since our Main Theorem says that $NL\mathcal{I} = L^2\mathcal{I} = 0$, most of these terms are unnecessary.

Corollary 1.2 *For every ring extension $A \subset B$, $\mathcal{I}(A[T], B[T])$ is the direct sum of $\mathcal{I}(A, B)$, n terms of the form $L\mathcal{I}(A, B)$ and $2^i\binom{n}{i}$ terms of the form $N^i\mathcal{I}(A, B)$, $1 \le i \le n$.*

Since we know from [8] that $N\mathcal{I}(A, B) = 0$ is equivalent to A being seminormal in B (Definition 6.5) we can further conclude:

Corollary 1.3 *For $A \subset B$, the following are equivalent:*

(1) $\mathcal{I}(A, B) = \mathcal{I}(A[t, 1/t], B[t, 1/t])$;
(2) $H^0_{\text{et}}(\text{Spec}(A), f_*\mathbb{Z}/\mathbb{Z}) = 0$ and A is seminormal in B.

It is immediate from our Main Theorem that $L\mathcal{I}(A, B)$ is a torsionfree group (we give a simple proof in Corollary 3.6); it is free abelian of finite rank when A is pseudo-geometric and finite dimensional (Proposition 3.7).

A secondary goal of this paper is to give simple techniques for determining $L\mathcal{I}(A, B)$. For example, we may assume A and B are reduced as $L\mathcal{I}(A, B) \cong L\mathcal{I}(A_{\text{red}}, B_{\text{red}})$ (Theorem 4.1). The following special case of Proposition 6.2 gives an elementary criterion for the vanishing of $L\mathcal{I}(A, B)$. We say that an extension B/A is *connected* if for every prime ideal $\wp$ of A, the ring $B_\wp/\wp B_\wp$ is connected.

Proposition 1.4 *If B/A is finite and B is connected over A then $L\mathcal{I}(A, B) = 0$.*

We also show that $L\mathcal{I}(A, B) = 0$ can only happen if the extension $A \subset B$ is *anodal* in the sense of Asanuma (Theorem 6.8). The converse is true for integral, birational extensions of 1-dimensional domains, but Example 6.9 (taken from [11, 3.5]) shows that being integral, birational and anodal is not sufficient in higher dimension.

This paper is laid out as follows. In Sect. 2, we define contracted functors on extensions and recall some basic theory. In Sect. 3, we define $\mathcal{I}(A, B)$ and prove (in Proposition 3.4) that $NL\mathcal{I} = LL\mathcal{I} = 0$. Section 4 gives some basic properties of $\mathcal{I}(A, B)$. The rest of Theorem 1.1 is proven in Sect. 5, and Sect. 6 describes some conditions under which $L\mathcal{I}(A, B)$ vanishes.

2 Contracted functors

All of the rings we consider are commutative with 1, and all ring homomorphisms are unitary. The category of ring extensions has objects $f : A \hookrightarrow B$; a morphism from f to $f' : A' \hookrightarrow B'$ is a morphism $B \to B'$ sending A to A'.

In [1, XII], Bass defined the notion of a contracted functor from rings to abelian groups. This has a natural translation into the setting of ring extensions, which we now lay out. Given an indeterminate t, we write $f[t]$ for the polynomial ring extension $A[t] \hookrightarrow B[t]$ and write $f[t, 1/t]$ for the Laurent polynomial extension $A[t, 1/t] \hookrightarrow B[t, 1/t]$.

Definition 2.1 Let F be a functor from ring extensions to abelian groups. We write $LF(A, B)$ or $LF(f)$ for the cokernel of the map $F(f[t]) \times F(f[1/t]) \xrightarrow{\pm} F(f[t, 1/t])$ which is the difference of the maps induced by applying F to the morphisms $f \to f[t]$ and $f \to f[1/t]$. We write $Seq(F, f)$ for the following sequence (where Δ is the diagonal map):

$$1 \to F(f) \xrightarrow{\Delta} F(f[t]) \times F(f[1/t]) \xrightarrow{\pm} F(f[t, 1/t]) \to LF(f) \to 1.$$

We say F is *acyclic* if $Seq(F, f)$ is exact for every ring extension f. We say that F is *contracted* if $Seq(F, f)$ is naturally split exact, i.e., if there is a map $h(f) : LF(f) \to F(f[t, 1/t])$.

Following Bass [1, XII], we write $NF(f)$ or $N_tF(f)$ for the kernel of the map $F(f[t]) \to F(f)$ induced by $t \mapsto 1$. This map is split by the map $F(f \to f[t])$ induced by $B \subset B[t]$, and we have a natural decomposition $F(f[t]) = F(f) \oplus NF(f)$. Thus $Seq(F, f)$ is quasi-isomorphic to the sequence

$$1 \to F(f) \oplus N_tF(f) \oplus N_{1/t}F(f) \to F(f[t, 1/t]) \to LF(f) \to 1.$$

If F is a functor from rings to abelian groups, we can define functors $s^*F(f) = F(A)$ and $t^*F(f) = F(B)$ by composing with the source and target functors from ring extensions to rings sending $f : A \hookrightarrow B$ to $s(f) = A$ and $t(f) = B$. If F is contracted in Bass' sense then s^*F and t^*F are contracted in the sense of Definition 2.1.

It should be clear to the reader that the notion of contracted functor also makes sense for functors from many categories (such as commutative rings, schemes, ring extensions, ...) to any abelian category (such as abelian groups, sheaves, modules). When these choices are irrelevant, we will not specify them and merely refer to “contracted functors.”

Lemma 2.2 *Let $\eta : F \to G$ be a morphism of contracted functors. Then* $\ker(\eta)$ *and* $\mathrm{coker}\,(\eta)$ *are contracted functors, with* $L\ker(\eta) = \ker(L\eta)$ *and* $L\,\mathrm{coker}\,(\eta) = \mathrm{coker}\,(L\eta)$.

If $1 \to F \to G \to H \to 1$ is a short exact sequence of functors and F, H are acyclic then G is acyclic and there is a short exact sequence

$$1 \to LF \to LG \to LH \to 1.$$

Proof The first assertion is proven exactly as the corresponding assertion for contracted functors on rings in [1, XII.7.2] or [12, III.4.2]. The second assertion is proven exactly as Carter proved the corresponding assertion in [2, 1.2]. □

Corollary 2.3 *If $F_1 \to F_2 \to G \to H_1 \to H_2$ is an exact sequence of functors and the F_i, H_i are contracted then G is acyclic and there is an exact sequence for every f:*

$$LF_1(f) \to LF_2(f) \to LG(f) \to LH_1(f) \to LH_2(f).$$

Of course, Corollary 2.3 may be iterated to get exact sequences for $LLG(f)$, etc., because the LF_i and LH_i are contracted functors.

Example 2.4 Recall from [1, XII.7.8] [12, III.4.1.3] that the units U form a contracted functor on rings with $LU(A) = H^0(\mathrm{Spec}\, A, \mathbb{Z})$ and $LLU = NLU = 0$. Similarly, $F(A) = U(A_{\mathrm{red}})$ is a contracted functor and its contraction $LF(A) = LU(A_{\mathrm{red}})$ is isomorphic to $LU(A)$. Define $U_{\mathrm{nil}}(A)$ to be the kernel of $U(A) \to U(A_{\mathrm{red}})$; it is the multiplicative group $(1+\mathrm{nil}(A))^\times$. Since $1 \to U_{\mathrm{nil}}(A) \to U(A) \to U(A_{\mathrm{red}}) \to 1$ is an exact sequence, Lemma 2.2 implies that U_{nil} is a contracted functor with $LU_{\mathrm{nil}}(A) = 0$.

Given a commutative ring extension $f : A \hookrightarrow B$, define $U_{\mathrm{nil}}(f)$ to be the cokernel of $U_{\mathrm{nil}}(A) \to U_{\mathrm{nil}}(B)$. From the exact sequence $1 \to U_{\mathrm{nil}}(A) \to U_{\mathrm{nil}}(B) \to U_{\mathrm{nil}}(f) \to 1$ and Lemma 2.2, we see that $U_{\mathrm{nil}}(f)$ is a contracted functor with $LU_{\mathrm{nil}}(f) = 0$.

Remark 2.4.1 Since $NU_{\mathrm{nil}}(f) = (1 + t\,\mathrm{nil}(B)[t])^\times/(1 + t\,\mathrm{nil}(A)[t])^\times$, it follows that $NU_{\mathrm{nil}}(f) = 0$ if and only if $\mathrm{nil}(A) = \mathrm{nil}(B)$. This is trivial if B is reduced.

3 Relative Cartier divisors

Relative Cartier divisors are functors on ring extensions. Recall that the A-submodules of B form a monoid under multiplication, with identity A. An A-submodule L_1 of B is said to be *invertible* if $L_1L_2 = A$ for some L_2. In particular, L_1 is isomorphic to an invertible ideal of A. An invertible A-submodule is also said to be a *relative Cartier divisor*.

Definition 3.1 Given a ring extension $f : A \hookrightarrow B$, let $\mathcal{I}(f)$ denote the multiplicative group of all A-submodules of B which are invertible. We shall sometimes write $\mathcal{I}(A, B)$ for $\mathcal{I}(f)$. It is easily seen that $\mathcal{I}$ is a functor from the category of ring extensions to abelian groups.

The study of $\mathcal{I}(A, B)$ was initiated by Roberts and Singh in [6].

If $\mathcal{O}_A^\times$ is the Zariski sheaf of units on $\mathrm{Spec}(A)$ and $f_*\mathcal{O}_B^\times$ is the direct image sheaf on $\mathrm{Spec}(A)$ associated to the units on $\mathrm{Spec}(B)$, it is easy to see that

$$\mathcal{I}(A, B) \cong H^0_{\mathrm{zar}}(\mathrm{Spec}\, A, f_*\mathcal{O}_B^\times/\mathcal{O}_A^\times). \tag{3.2}$$

In effect, an invertible A-submodule L can be described by giving an open cover $\{U_i\}$, $U_i = \mathrm{Spec}(A[1/s_i])$ of $\mathrm{Spec}(A)$ and elements f_i of $B[1/s_i]^\times$ (defined modulo $A[1/s_i]^\times$) such that each f_i/f_j is in $A[1/s_is_j]^\times$.

For example, if A is a domain and K is the field of fractions, then $\mathcal{I}(A, K)$ is the group of Cartier divisors and the interpretation of $\mathcal{I}(A, K)$ as $H^0(\mathrm{Spec}\, A, f_*\mathcal{O}_K^\times/\mathcal{O}_A^\times)$ is standard. For this reason, we shall call $\mathcal{I}(f)$ the group of *relative Cartier divisors*.

Since $\mathrm{Pic}(A)$ (the Picard group of A) is $H^1(\mathrm{Spec}\, A, \mathcal{O}_A^\times)$, and $H^1(\mathrm{Spec}\, A, f_*\mathcal{O}_B^\times)$ is a subgroup of $\mathrm{Pic}(B)$, the (Zariski or étale) cohomology sequence associated to the exact sequence of sheaves on $\mathrm{Spec}\, A$, $1 \to \mathcal{O}_A^\times \to f_*\mathcal{O}_B^\times \to f_*\mathcal{O}_B^\times/\mathcal{O}_A^\times \to 1$, is the exact sequence mentioned in the Introduction:

$$1 \to U(A) \to U(B) \to \mathcal{I}(A, B) \to \mathrm{Pic}\ \ A \to \mathrm{Pic}\ \ B. \tag{3.3}$$

It is clear that this sequence is natural in f. (A more elementary proof of exactness is given in [6, Theorem 2.4]).

Proposition 3.4 *The functor $\mathcal{I}$ is acyclic, $NL\mathcal{I} = LL\mathcal{I} = 0$ and there is an exact sequence*

$$1 \to LU(A) \to LU(B) \to L\mathcal{I}(f) \to L\mathrm{Pic}(A) \to L\mathrm{Pic}(B).$$

Proof The units U and Picard group Pic are contracted functors on rings (see [11, 5.2]). Applying Corollary 2.3 to (3.3), we see that $\mathcal{I}$ is acyclic, and that there are exact sequences

$$1 \to LU(A) \to LU(B) \to L\mathcal{I}(f) \to L\text{Pic}(A) \to L\text{Pic}(B),$$
$$1 \to LLU(A) \to LLU(B) \to LL\mathcal{I}(f) \to LL\text{Pic}(A) \to LL\text{Pic}(B).$$

Now $NLU = LLU = 0$ by Example 2.4, and $LL\text{Pic} = NL\text{Pic} = 0$ by [11, 7.7]. This yields $LL\mathcal{I}(f) = 0$, $LU(A) = LU(A[t])$ and $L\text{Pic}(A[t]) = L\text{Pic}(A)$. It is immediate that $NL\mathcal{I}(f) = 0$. □

Since $L\,\text{Pic}$ vanishes on normal domains [11, 1.5.2], we see that (i) if A is a normal domain then $L\mathcal{I}(f) = 0$ if and only if B is connected, and (ii) If B is a normal domain then $L\mathcal{I}(f) = 0$ if and only if $L\text{Pic}(A) = 0$. More generally, we have:

Corollary 3.5 *If A is connected, and $f : A \hookrightarrow B$ is an extension, then $L\mathcal{I}(f) = 0$ if and only if (i) B is connected and (ii) $L\text{Pic}(A) \to L\text{Pic}(B)$ is an injection.*

Corollary 3.6 *The group $L\mathcal{I}(A, B)$ is always a torsion-free abelian group.*

Proof By [11, 2.3.1], $L\text{Pic}(A)$ is a torsion-free abelian group. In addition, the image of $LU(B)$ in $L\mathcal{I}(A, B)$ is free abelian by [11, Proposition 1.3]. The fact that $L\mathcal{I}(A, B)$ is torsionfree now follows from Proposition 3.4. □

Recall from [11] that a noetherian ring A is called **pseudo-geometric** if every reduced finite A-algebra B has finite normalization. For example, any finitely generated algebra over a field or over $\mathbb{Z}$ is pseudo-geometric.

Proposition 3.7 *If A is pseudo-geometric and* $\dim A < \infty$*, then $L\mathcal{I}(A, B)$ is a free abelian group.*

Proof When A is pseudo-geometric with $\dim A < \infty$, $L\text{Pic}\,A$ is a free abelian group by Proposition 2.3 of [11]. So the image of $L\mathcal{I}(f)$ in $L\text{Pic}(A)$ is a free abelian group. Again, Proposition 3.4 implies that $L\mathcal{I}(A, B)$ is a free abelian group. □

Remark 3.7.1 If A is a 1-dimensional domain, then $L\mathcal{I}(A, B)$ is a free abelian group. This follows from the sequence of Proposition 3.4, and the facts that (i) $L\text{Pic}(A)$ is a free abelian group [11, 3.4.1], (ii) subgroups of free abelian groups are free and (iii) the image of $LU(B)$ in $L\mathcal{I}(A, B)$ is free abelian [11, Prop. 1.3].

Question 3.8 *Is $L\mathcal{I}(A, B)$ always a free abelian group?*

For the rest of this paper, it is convenient to adopt scheme-theoretic language. Recall that a morphism of schemes $f : X \to S$ is *affine* if the inverse image $f^{-1}U$ of any affine open subset U of S is an affine open subset of X. We will say that an affine morphism is *faithful* if $\mathcal{O}_S \to f_*\mathcal{O}_X$ is an injection; if the inverse image of Spec(A) is $f^{-1}U = \text{Spec}(B)$, this implies that $A \to B$ is an injection.

Notation 3.9 The category of ring extensions embeds contravariantly into the category of faithful affine morphisms of schemes, $f : X \to S$; morphisms $f \to f'$ in this category are compatible pairs of maps $X \to X'$ and $S \to S'$. If f is a faithful affine morphism, $\mathcal{I}(f)$ will denote the multiplicative group of all $\mathcal{O}_S$-submodules of $f_*\mathcal{O}_X$ which are invertible.

It is clear that the formal yoga of Sects. 2 and 3 extend to the category of faithful affine morphisms $X \to S$. Given a faithful affine map $f : X \to S$, (3.2) easily generalizes to $\mathcal{I}(f) \cong H^0_{\text{zar}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)$. Proposition 3.4 implies that $\mathcal{I}$ is an acyclic functor with $NL\mathcal{I} = LL\mathcal{I} = 0$, Corollary 3.6 states that $L\mathcal{I}(f)$ is torsionfree. Remark 2.4.1 is replaced by: $NU_{\text{nil}}(f) = 0$ if and only if $H^0(S, \text{nil}\,\mathcal{O}_S) = H^0(X, \text{nil}\,\mathcal{O}_X)$.

4 Basic properties

In this short section, we give a few results that allow us to relate $L\mathcal{I}(f)$ to $L\mathcal{I}$ of other ring extensions. Given a map $f : A \hookrightarrow B$, we write f_{red} for the evident map $A_{\mathrm{red}} \hookrightarrow B_{\mathrm{red}}$.

Theorem 4.1 *The natural map* $L\mathcal{I}(A, B) \xrightarrow{\cong} L\mathcal{I}(A_{\mathrm{red}}, B_{\mathrm{red}})$ *is an isomorphism. In addition, there is a natural short exact sequence of functors on ring extensions*

$$1 \to U_{\mathrm{nil}}(f) \to \mathcal{I}(f) \to \mathcal{I}(f_{\mathrm{red}}) \to 0.$$

Proof Consider the commutative diagram

$$\begin{array}{ccccccccccc}
1 & \longrightarrow & U_{\mathrm{nil}}(A) & \longrightarrow & U_{\mathrm{nil}}(B) & \longrightarrow & U_{\mathrm{nil}}(f) & \longrightarrow & 1 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & & & \\
1 & \longrightarrow & U(A) & \longrightarrow & U(B) & \longrightarrow & \mathcal{I}(f) & \longrightarrow & \mathrm{Pic}(A) & \longrightarrow & \mathrm{Pic}(B) \\
 & & \downarrow & & \downarrow & & \downarrow & & \| & & \| \\
1 & \longrightarrow & U(A_{\mathrm{red}}) & \longrightarrow & U(B_{\mathrm{red}}) & \longrightarrow & \mathcal{I}(f_{\mathrm{red}}) & \longrightarrow & \mathrm{Pic}(A_{\mathrm{red}}) & \longrightarrow & \mathrm{Pic}(B_{\mathrm{red}}).
\end{array}$$

The groups $U_{\mathrm{nil}}(A)$ and $U_{\mathrm{nil}}(f)$ were defined in Example 2.4, where we observed that the two left columns and the top row are short exact sequences; the bottom two rows are the exact sequences (3.3). Since $U_{\mathrm{nil}}(A)$ is the intersection of $U_{\mathrm{nil}}(B)$ and $U(A)$ in $U(B)$, a diagram chase shows that the third column is exact.

Since $LU_{\mathrm{nil}}(f) = 0$ by Example 2.4, the isomorphism $L\mathcal{I}(f) \cong L\mathcal{I}(f_{\mathrm{red}})$ follows from the second part of Lemma 2.2, applied to the third column. □

Remark 4.1.1 The first part of Theorem 4.1 extends to $X \to S$ by our Main Theorem 5.1 below; see 5.3. The second part of Theorem 4.1 can fail for $X \to S$ as $U(S) \to U(S_{\mathrm{red}})$ need not be onto.

Corollary 4.2 $N\mathcal{I}(A, B) \xrightarrow{\cong} N\mathcal{I}(A_{\mathrm{red}}, B_{\mathrm{red}})$ *if and only if* $\mathrm{nil}(A) = \mathrm{nil}(B)$.
Moreover, if $\mathrm{nil}(A) \neq \mathrm{nil}(B)$ *then* $N\mathcal{I}(f) \neq 0$.

Proof Replacing f by $f[t]$ in Theorem 4.1, we have the exact sequence

$$1 \to NU_{\mathrm{nil}}(f) \to N\mathcal{I}(f) \to N\mathcal{I}(f_{\mathrm{red}}) \to 0.$$

By Remark 2.4.1, the first term vanishes if and only if $\mathrm{nil}(A) = \mathrm{nil}(B)$. □

Lemma 4.3 ([9, §3]) *Suppose that* $f : A \hookrightarrow B$ *and* $g : B \hookrightarrow C$ *are extensions. Then there is a short exact sequence*

$$1 \to \mathcal{I}(A, B) \to \mathcal{I}(A, C) \to \mathcal{I}(B, C).$$

Proof We have an exact sequence of sheaves on $\mathrm{Spec}(A)$:

$$1 \to f_*\mathcal{O}_B^\times/\mathcal{O}_A^\times \to (fg)_*\mathcal{O}_C^\times/\mathcal{O}_A^\times \to f_*(g_*\mathcal{O}_C^\times/\mathcal{O}_B^\times)$$

Now apply the left exact global sections functor and use (3.2). □

Lemma 4.4 *Let* $\mathfrak{a}$ *be an ideal of* B *contained in* A*. Then* $L\mathcal{I}(A, B) \cong L\mathcal{I}(A/\mathfrak{a}, B/\mathfrak{a})$.

Proof Write $\bar{f}$ for $A/\mathfrak{a} \hookrightarrow B/\mathfrak{a}$. By Proposition 2.6 of [6], $\mathcal{I}(A, B) \cong \mathcal{I}(\bar{f})$. Since $\mathfrak{a}[t]$ is an ideal of $B[t]$ contained in $A[t]$, and $\mathfrak{a}[t, 1/t]$ is an ideal of $B[t, 1/t]$ in $A[t, 1/t]$, the same is true for $\mathcal{I}(A[t], B[t])$ and $\mathcal{I}(A[t, 1/t], B[t, 1/t])$. The result follows from a comparison of $Seq(\mathcal{I}, f)$ and $Seq(\mathcal{I}, \bar{f})$. □

Here is another elementary result about $\mathcal{I}$, which allows us to assume for example that A is noetherian and B is of finite type over A.

Lemma 4.5 *$\mathcal{I}$ commutes with filtered colimits. That is, if $A \subset B$ is the filtered union of extensions $A_\lambda \subset B_\lambda$ then $\mathcal{I}(A, B) = \varinjlim \mathcal{I}(A_\lambda, B_\lambda)$ and $L\mathcal{I}(A, B) = \varinjlim L\mathcal{I}(A_\lambda, B_\lambda)$.*

Proof Since $U(B) = \cup U(B_\lambda)$ and $\mathrm{Pic}(B) = \varinjlim \mathrm{Pic}(B_\lambda)$, this lemma follows from (3.3), $Seq(\mathcal{I}, f)$ and the fact that filtered direct limits are exact. □

Proposition 4.6 *Suppose that $A = \prod_1^n A_i$ and $B = \prod_1^n B_i$, where $A_i \subset B_i$. Then $\mathcal{I}(A, B) = \prod \mathcal{I}(A_i, B_i)$, $N\mathcal{I}(A, B) = \prod N\mathcal{I}(A_i, B_i)$ and $L\mathcal{I}(A, B) = \prod L\mathcal{I}(A_i, B_i)$.*

Proof Every A-submodule of B has the form $M = \prod M_i$, where each M_i is an A_i-submodule of B_i. If M is invertible with inverse $N = \prod N_i$, then there are $m_j = (m_{ij}) \in M$, $n_j = (n_{ij}) \in N$ so that $\sum_j m_{ij} n_{ij} = 1$ for all i. This shows that each M_i is an invertible A_i-submodule of B_i, and hence that the natural map from $\mathcal{I}(A, B)$ to $\prod \mathcal{I}(A_i, B_i)$ is an injection. To see that it is a surjection, suppose that M_i are invertible A_i-submodules of B_i with inverses N_i. Then for each i there are m_{ij} and n_{ij} so that $\sum_j m_{ij} n_{ij} = 1$. Thus $\prod M_i$ is an invertible A-submodule of B.

Since $(\prod_1^n A_i)[t] = \prod_1^n (A_i[t])$, the assertions about $N\mathcal{I}$ and $L\mathcal{I}$ follow by replacing A_i with $A_i[t]$ and $A_i[t, 1/t]$, and similarly for B_i. □

5 Main theorem

The goal of this section is to show that $\mathcal{I}$ is a contracted functor, whose contraction is an étale cohomology group. We refer the reader to [3] for basic properties of étale sheaves and étale cohomology.

Theorem 5.1 *$\mathcal{I}$ is a contracted functor on ring extensions, and its contraction is*

$$L\mathcal{I}(A, B) = H^0_{\mathrm{et}}(\mathrm{Spec}\, A, (f_*\mathbb{Z})/\mathbb{Z}) = H^0_{\mathrm{nis}}(\mathrm{Spec}\, A, (f_*\mathbb{Z})/\mathbb{Z}).$$

Here $(f_*\mathbb{Z})/\mathbb{Z}$ denotes the quotient sheaf in the étale topology. Theorem 5.1 is just the special case $S = \mathrm{Spec}(A)$ and $X = \mathrm{Spec}(B)$ of the following result.

Theorem 5.2 *$\mathcal{I}$ is a contracted functor on faithful affine maps, with contraction*

$$L\mathcal{I}(f) = H^0_{\mathrm{et}}(S, (f_*\mathbb{Z})/\mathbb{Z}) = H^0_{\mathrm{nis}}(S, (f_*\mathbb{Z})/\mathbb{Z}).$$

Corollary 5.3 *$L\mathcal{I}(f) \cong L\mathcal{I}(f_{\mathrm{red}})$.*

We begin the proof of Theorem 5.2 by generalizing (3.2) to the étale and Nisnevich topologies on S. Recall that if $\mathcal{F}$ is an étale sheaf on S (a sheaf on S_{et}) then it is also a Nisnevich and a Zariski sheaf, and $H^0_{\mathrm{et}}(S, \mathcal{F}) = H^0_{\mathrm{nis}}(S, \mathcal{F}) = H^0_{\mathrm{zar}}(S, \mathcal{F}) = \mathcal{F}(S)$. This remark applies for example to the sheaves of units. To avoid confusion, it will be convenient to write $\mathcal{O}_S^\times$ and $f_*\mathcal{O}_X^\times$ for the étale sheaves $U \mapsto \Gamma(U, \mathcal{O}_U)^\times$ and $U \mapsto \Gamma(f^{-1}U, \mathcal{O}_{f^{-1}U})^\times$, instead of the traditional $\mathbb{G}_m$ and $f_*(\mathbb{G}_m|_{X_{\mathrm{et}}})$. Of course, they are also sheaves for the Nisnevich topology on S.

Lemma 5.4 *The Zariski quotient sheaf $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ is an étale sheaf. Consequently, $\mathcal{I}(f) \cong H^0_{\text{et}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) \cong H^0_{\text{nis}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)$.*

Proof Since $H^1_{\text{zar}}(S, \mathcal{O}_S^\times) = H^1_{\text{et}}(S, \mathcal{O}_S^\times)$ and $H^1_{\text{et}}(S, f_*\mathcal{O}_X^\times)$ is a subgroup of $H^1_{\text{et}}(X, \mathcal{O}_X^\times)$, we have a commutative diagram:

$$\begin{array}{ccccccccc}
H^0_{\text{zar}}(S, \mathcal{O}_S^\times) & \to & H^0_{\text{zar}}(S, f_*\mathcal{O}_X^\times) & \to & H^0_{\text{zar}}(S, f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times) & \to & H^1_{\text{zar}}(S, \mathcal{O}_S^\times) & \to & H^1_{\text{zar}}(S, f_*\mathcal{O}_X^\times) \\
\| & & \| & & \downarrow & & \| & & \downarrow \text{into} \\
H^0_{\text{et}}(S, \mathcal{O}_S^\times) & \to & H^0_{\text{et}}(S, f_*\mathcal{O}_X^\times) & \to & H^0_{\text{et}}(S, (f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times)_{\text{et}}) & \to & H^1_{\text{et}}(S, \mathcal{O}_S^\times) & \to & H^1_{\text{et}}(S, f_*\mathcal{O}_X^\times).
\end{array}$$

From the 5-lemma, we see that the middle vertical map is an isomorphism, i.e., that $f_*\mathcal{O}_X^\times/\mathcal{O}_S^\times$ is an étale sheaf, and hence a Nisnevich sheaf. The final assertion follows from (3.2). □

Notation 5.5 Given a scheme S, we write $S[t]$ for $S \times \text{Spec}(\mathbb{Z}[t])$; there is a natural map $p^{S,t} : S[t] \to S$. When the base S is clear we simply write p^t, so that $p^t_*\mathcal{O}_{S[t]}^\times$ denotes the direct image sheaf on S; it is both a Zariski and an étale sheaf on S. Similarly, we write $S[t, 1/t]$ for $S \times \text{Spec}(\mathbb{Z}[t, 1/t])$, with projection $p : S[t, 1/t] \to S$, and also write $p_*\mathcal{O}_{S[t,1/t]}^\times$ for the direct image sheaf on S. Given $f : X \to S$ then, by abuse of notation, we will also write $f_* p^t_*\mathcal{O}_{X[t]}^\times$ for the direct image sheaf on S associated to the composition $X[t] \to X \to S$, etc.

For notational simplicity, we shall write $\mathcal{O}^\times$ and $f^T_*\mathcal{O}^\times$ for the étale sheaves $\mathcal{O}_{S[t,1/t]}^\times$ and $f[t, 1/t]_*\mathcal{O}_{X[t,1/t]}^\times$ on $S[t, 1/t]$. Thus Lemma 5.4 yields the formula

$$\mathcal{I}(f[t, 1/t]) \cong H^0_{\text{et}}\left(S[t, 1/t], f^T_*\mathcal{O}^\times/\mathcal{O}^\times\right) \cong H^0_{\text{et}}\left(S, p_*\left(f^T_*\mathcal{O}^\times/\mathcal{O}^\times\right)\right).$$

Replacing H^0_{et} with H^0_{nis} yields an analogous formula.

Example 5.5.1 The analogue of the formulas $\mathcal{I}(f[t, 1/t]) \cong H^0_{\text{et}}(S, p_*(f^T_*\mathcal{O}^\times/\mathcal{O}^\times))$ and $L\mathcal{I}(A, B) = H^0_{\text{et}}(\text{Spec}\, A, (f_*\mathbb{Z}/\mathbb{Z}))$ fail for the Zariski cohomology. To see this, consider the subring A of $B = k[x]$ defining the node. It is not hard to see that

$$\mathcal{I}(f[t, 1/t]) \cong \text{Pic}(A[t, 1/t]) = \text{Pic}(A) \oplus \mathbb{Z} \quad \text{and} \quad L\mathcal{I}(f) \cong L\text{Pic}(A) \cong \mathbb{Z}$$

(see [11, 2.2]), yet $H^0_{\text{zar}}(S, p_*(f^T_*\mathcal{O}^\times/\mathcal{O}^\times)_{\text{zar}}) = \text{Pic}(A)$ and $H^0_{\text{zar}}(S, (f_*\mathbb{Z}/\mathbb{Z})_{\text{zar}}) = 0$.

A similar calculation for the local ring $A_\wp$ of the node and $B_\wp$ the (semilocal) normalization of $A_\wp$ shows that $L\mathcal{I}(A_\wp, B_\wp) = \mathbb{Z}$.

Recall that a local ring A is *hensel* if every finite A-algebra B is a direct product of local rings. A Nisnevich sheaf on $\text{Spec}(A)$ is zero if and only if it is zero on $\text{Spec}(A^h_\wp)$ for every prime ideal $\wp$, where $A^h_\wp$ is the henselization of the local ring $A_\wp$.

Lemma 5.6 *If A is a hensel local ring then $L\mathcal{I}(A, B) = H^0(\text{Spec}\, B, \mathbb{Z})/\mathbb{Z}$.*

Proof By [11, 2.5], $L\text{Pic}(A) = 0$. Since $LU(A) = \mathbb{Z}$ and $LU(B) = H^0(\text{Spec}\, B, \mathbb{Z})$, the sequence of Proposition 3.4 yields the result. □

Remark 5.6.1 As noted in [11, 1.2.1], $H^0(\text{Spec}\, B, \mathbb{Z})/\mathbb{Z}$ is a free abelian group for every B. We saw in Corollary 3.6 that $L\mathcal{I}(f)$ is always torsionfree.

If we fix $f : X \to S$ and view $L\mathcal{I}$ as the presheaf $U \mapsto L\mathcal{I}(U, f^{-1}U)$ on the étale site of S, Lemma 5.6 says that the associated étale sheaf is $f_*\mathbb{Z}/\mathbb{Z}$. Therefore we have a canonical map $a_f : L\mathcal{I}(f) \to H^0_{\text{et}}(S, f_*\mathbb{Z}/\mathbb{Z})$.

Theorem 5.7 *The canonical map* $a_f : L\mathcal{I}(f) \to H^0_{\text{et}}(S, f_*\mathbb{Z}/\mathbb{Z})$ *is an isomorphism.*

Proof We claim there is a commutative diagram whose rows are the exact sequence of Proposition 3.4 and the cohomology sequence associated to $\mathbb{Z} \to f_*\mathbb{Z} \to f_*\mathbb{Z}/\mathbb{Z}$:

$$\begin{array}{ccccccccc} LU(S) & \longrightarrow & LU(X) & \longrightarrow & L\mathcal{I}(f) & \longrightarrow & L\,\mathrm{Pic}(S) & \longrightarrow & L\,\mathrm{Pic}(X) \\ \| & & \| & & \downarrow{\scriptstyle a_f} & & \| & & \| \\ H^0_{\text{et}}(S,\mathbb{Z}) & \longrightarrow & H^0_{\text{et}}(X,\mathbb{Z}) & \longrightarrow & H^0_{\text{et}}(S, f_*\mathbb{Z}/\mathbb{Z}) & \longrightarrow & H^1_{\text{et}}(S,\mathbb{Z}) & \longrightarrow & H^1_{\text{et}}(X,\mathbb{Z}). \end{array}$$

Given this claim, the theorem follows from the 5-lemma.

The left three vertical maps are the canonical maps from the evident presheaves to the global sections of the associated sheaves, so the left two squares commute. Since the right two vertical maps are the natural isomorphisms of [11, 5.5], the right square also commutes. Thus we only need to show that the remaining square commutes.

Recall from 5.5 that $\mathcal{O}^\times$ and $f_*^T\mathcal{O}^\times$ are the étale sheaves $\mathcal{O}^\times_{S[t,1/t]}$ and $f[t,1/t]_*\mathcal{O}^\times_{X[t,1/t]}$ on $S[t,1/t]$. The sheafification of $A[t,1/t]^\times \to H^0(S,\mathbb{Z})$ on S is a map $\partial_S : p_*\mathcal{O}^\times \to \mathbb{Z}$; it induces a map $Rp_*\mathcal{O}^\times \to \mathbb{Z}$ in the derived category of étale sheaves. Similarly, the sheafification of $B[t,1/t]^\times \to H^0(S, f_*\mathbb{Z})$ on S induces a map $f_*\partial_X : Rp_*(f_*^T\mathcal{O}^\times) \to f_*\mathbb{Z}$. Thus we have a morphism of triangles in the derived category.

$$\begin{array}{ccccccc} Rp_*(\mathcal{O}^\times) & \longrightarrow & Rp_*(f_*^T\mathcal{O}^\times) & \longrightarrow & Rp_*(f_*^T\mathcal{O}^\times/\mathcal{O}^\times) & \longrightarrow & Rp_*(\mathcal{O}^\times)[1] \\ \downarrow{\scriptstyle \partial_S} & & \downarrow{\scriptstyle f_*\partial_X} & & \downarrow & & \downarrow{\scriptstyle \partial_S} \\ \mathbb{Z} & \longrightarrow & f_*\mathbb{Z} & \longrightarrow & (f_*\mathbb{Z})/\mathbb{Z} & \longrightarrow & \mathbb{Z}[1]. \end{array} \qquad (5.7.1)$$

Note that $H^0_{\text{et}}(S, Rp_*(\mathcal{O}^\times)[1]) = H^1_{\text{et}}(S[t,1/t],\mathcal{O}^\times) = \mathrm{Pic}(S[t,1/t])$ and, by Lemma 5.4,

$$H^0_{\text{et}}\left(S, Rp_*\left(f_*^T\mathcal{O}^\times/\mathcal{O}^\times\right)\right) = H^0_{\text{et}}\left(S[t,1/t], f_*^T\mathcal{O}^\times/\mathcal{O}^\times\right) = \mathcal{I}(f[t,1/t]). \qquad (5.7.2)$$

Thus applying H^0_{et} to the right-hand square in (5.7.1) yields the commutative square

$$\begin{array}{ccc} \mathcal{I}(f[t,1/t]) & \longrightarrow & \mathrm{Pic}(S[t,1/t]) \\ \downarrow & & \downarrow \\ H^0_{\text{et}}(S, f_*\mathbb{Z}/\mathbb{Z}) & \longrightarrow & H^1_{\text{et}}(S,\mathbb{Z}). \end{array}$$

The left map factors as $\mathcal{I}(f[t,1/t]) \to L\mathcal{I}(f) \xrightarrow{a_f} H^0_{\text{et}}(S, f_*\mathbb{Z}/\mathbb{Z})$, and the right map factors as $\mathrm{Pic}(S[t,1/t]) \to L\,\mathrm{Pic}(S) \cong H^1_{\text{et}}(S,\mathbb{Z})$. The top map is the map in (3.3), fitting into the commutative square with surjective vertical maps

$$\begin{array}{ccc} \mathcal{I}(f[t,1/t]) & \longrightarrow & \mathrm{Pic}(S[t,1/t]) \\ \downarrow & & \downarrow \\ L\mathcal{I}(f) & \longrightarrow & L\,\mathrm{Pic}(S), \end{array}$$

which is implicit in Proposition 3.4. The claim follows. □

Corollary 5.8 *The Nisnevich quotient sheaf* $f_*\mathbb{Z}/\mathbb{Z}$ *is an étale sheaf.*

Proof By Lemma 5.6, it suffices to observe that if S is hensel local we have $L\mathcal{I}(f) = H^0_{\mathrm{et}}(S, f_*\mathbb{Z}/\mathbb{Z})$. □

It remains to show that $\mathcal{I}$ is a contracted functor. Sheafifying the sequence $Seq(U, 1_S)$ yields the sequence of sheaves on S:

$$1 \to \mathcal{O}_S^\times \xrightarrow{\Delta} p^t_*\left(\mathcal{O}^\times_{S[t]}\right) \times p^{1/t}_*\left(\mathcal{O}^\times_{S[1/t]}\right) \xrightarrow{\pm} p_*\left(\mathcal{O}^\times_{S[t,1/t]}\right) \xrightarrow{\partial_S} \mathbb{Z} \to 1. \tag{5.9}$$

In addition, Bass' contraction $t_A : H^0(\mathrm{Spec}\, A, \mathbb{Z}) \to A[t, 1/t]^\times$ is natural in A, so we may sheafify it to obtain a morphism $t_S : \mathbb{Z} \to p_*(\mathcal{O}^\times_{S[t,1/t]})$ of (Zariski or étale) sheaves on S.

Lemma 5.10 *The sequence* (5.9) *of sheaves on S is split exact, with splitting* t_S.

Proof On an affine open $\mathrm{Spec}(R)$ of S, this is just the sequence

$$1 \to R^\times \xrightarrow{\Delta} R[t]^\times \times R[1/t]^\times \xrightarrow{\pm} R[t, 1/t]^\times \xrightarrow{\partial_R} \mathbb{Z} \to 1.$$

The fact that it is exact, and naturally split by t_S is proven in [1, XII.7.8]; see [11, 7.2]. □

Corollary 5.11 *Given a faithful affine map* $f : X \to S$*, the direct image of the sequence* (5.9) *on X is a split exact sequence of (Zariski or étale) sheaves on S, with splitting* f_*t_X*:*

$$1 \to f_*\mathcal{O}_X^\times \xrightarrow{\Delta} f_*p^t_*\left(\mathcal{O}^\times_{X[t]}\right) \times f_*p^{1/t}_*\left(\mathcal{O}^\times_{X[1/t]}\right) \xrightarrow{\pm} f_*p_*\left(\mathcal{O}^\times_{X[t,1/t]}\right) \xrightarrow{f_*\partial_X} f_*\mathbb{Z} \to 1.$$

The global sections of the sequences in 5.10 and 5.11 are of course Bass' sequences $Seq(U, S)$ and $Seq(U, X)$. By 5.5, the sheafification of $\mathcal{I}(f[t, 1/t])$ on S is $p_*(f^T_*\mathcal{O}^\times/\mathcal{O}^\times)$. Theorem 5.7 says that the sheafification of $\mathcal{I}(f[t, 1/t]) \to L\mathcal{I}(f)$ on S is a canonical map $\partial_f : p_*(f^T_*\mathcal{O}^\times/\mathcal{O}^\times) \to f_*\mathbb{Z}/\mathbb{Z}$.

Proposition 5.12 *The map* $\partial_f : p_*(f^T_*\mathcal{O}^\times/\mathcal{O}^\times) \to f_*\mathbb{Z}/\mathbb{Z}$ *is split by a natural map of sheaves on S:*

$$t_f : f_*\mathbb{Z}/\mathbb{Z} \to p_*\left(f^T_*\mathcal{O}^\times/\mathcal{O}^\times\right).$$

Proof Applying the left exact functor p_* to $1 \to \mathcal{O}^\times \to f^T_*\mathcal{O}^\times \to f^T_*\mathcal{O}^\times/\mathcal{O}^\times \to 1$, we get exactness of the middle row in the following commutative diagram of sheaves on S.

$$\begin{array}{ccccccccc}
0 & \to & \mathbb{Z} & \to & f_*\mathbb{Z} & \to & (f_*\mathbb{Z})/\mathbb{Z} & \to & 0 \\
 & & \downarrow{\scriptstyle t_S} & & \downarrow{\scriptstyle f_*t_X} & & \vdots{\scriptstyle t_f} & & \\
0 & \to & p_*\mathcal{O}^\times & \to & p_*f^T_*\mathcal{O}^\times & \to & p_*(f^T_*\mathcal{O}^\times/\mathcal{O}^\times) & \to & R^1p_*\mathcal{O}^\times \\
 & & \downarrow{\scriptstyle \partial_S} & & \downarrow{\scriptstyle f_*\partial_X} & & \downarrow{\scriptstyle \partial_f} & & \\
0 & \to & \mathbb{Z} & \to & f_*\mathbb{Z} & \to & (f_*\mathbb{Z})/\mathbb{Z} & \to & 0
\end{array}$$

(The top and bottom rows are tautologically exact.) The maps t_S, f_*t_X induce the map t_f; since $\partial_S\, t_S$ and $\partial_X\, t_X$ are the identity, so are $(f_*\partial_X)(f_*t_X)$ and $\partial_f\, t_f$. □

Proof of Theorem 5.2 By Theorem 5.7, we have $L\mathcal{I}(f) \cong H^0_{\mathrm{et}}(S, f_*\mathbb{Z}/\mathbb{Z})$. By Proposition 5.12, we have a natural section t_f of the sheaf map ∂_f. By 5.5, the global sections of the map ∂_f is the map $\mathcal{I}(f[t, 1/t]) \to L\mathcal{I}(f)$ in $Seq(\mathcal{I}, f)$. Hence the global sections of t_f provide the required natural splitting. □

Here is an easy consequence of Theorem 5.2, which is related to Lemma 4.3.

Corollary 5.13 *Suppose that $f : A \hookrightarrow B$ and $g : B \hookrightarrow C$ are extensions. Then there is a short exact sequence*

$$1 \to L\mathcal{I}(A, B) \to L\mathcal{I}(A, C) \to L\mathcal{I}(B, C).$$

More generally, given faithful affine maps $X \xrightarrow{g} T \xrightarrow{f} S$, there is an exact sequence

$$1 \to L\mathcal{I}(f) \to L\mathcal{I}(fg) \to L\mathcal{I}(g).$$

Proof Applying f_* to the exact sequence $0 \to \mathbb{Z} \to g_*\mathbb{Z} \to (g_*\mathbb{Z})/\mathbb{Z} \to 0$ on T (or Spec(B)) yields the exact sequence of Nisnevich sheaves on S (or Spec(A)):

$$1 \to (f_*\mathbb{Z})/\mathbb{Z} \to (fg)_*\mathbb{Z}/\mathbb{Z} \to f_*(g_*\mathbb{Z}/\mathbb{Z}).$$

Now apply the left exact global sections functor and use Theorem 5.1. □

6 The vanishing of $L\mathcal{I}(A, B)$

In this section, we discuss some conditions on $A \subset B$ under which $L\mathcal{I}(A, B) = 0$. We begin by noting two elementary consequences of the sheaf property of $f_*\mathbb{Z}/\mathbb{Z}$: (i) if $s, t \in A$ are comaximal then $L\mathcal{I}(A, B) \subset L\mathcal{I}(A[\frac{1}{s}], B[\frac{1}{s}]) \oplus L\mathcal{I}(A[\frac{1}{t}], B[\frac{1}{t}])$, and (ii) if $L\mathcal{I}(A_\wp, B_\wp) = 0$ for every prime $\wp$ of A then $L\mathcal{I}(A, B) = 0$. The converse does not hold:

Example 6.1 If $A = \mathbb{C}[x]$ and $B = \mathbb{C}[x, y]/(y^2 - x^2)$ then $L\mathcal{I}(A, B) = 0$, but if $\wp \neq xA$ we have $L\mathcal{I}(A_\wp, B_\wp) = \mathbb{Z}$ (use Proposition 3.4).

If $A = k[s, s^{-1}]$ and $B = k[x, x^{-1}]$ with $s = x^2$ then $L\mathcal{I}(f) = 0$ but $L\mathcal{I}(A_\wp, B_\wp) = \mathbb{Z}$ for every nonzero prime $\wp$ of A. Thus $(f_*\mathbb{Z})/\mathbb{Z}$ is nonzero; its stalk is $\mathbb{Z}$ at any closed point, but is 0 at the generic point.

A simple necessary condition for $L\mathcal{I}(f)$ to vanish is for the Nisnevich sheaf $f_*\mathbb{Z}/\mathbb{Z}$ to vanish. It is not enough for the Zariski sheaf $f_*\mathbb{Z}/\mathbb{Z}$ to vanish; Example 5.5.1 shows that even if A is a local ring we can have $(f_*\mathbb{Z}/\mathbb{Z})_{\text{zar}} = 0$ and $H^0_{\text{zar}}(A, f_*\mathbb{Z}/\mathbb{Z}) = 0$ but $L\mathcal{I}(f) = H^0_{\text{nis}}(A, f_*\mathbb{Z}/\mathbb{Z}) \neq 0$.

For finite morphisms, we have a simple criterion. We say that a map $f : X \to S$ is *connected* if it for every point s of S, the fiber $X_s = f^{-1}(s)$ is connected or empty. For a map Spec$(B) \to$ Spec(A), this means that for each prime ideal $\wp$ of A either there is no prime of B over $\wp$ or else the fiber ring $B \otimes_A k(\wp)$ is connected.

Proposition 6.2 *Suppose that $f : X \to S$ is finite. Then*

(a) the Nisnevich sheaf $(f_*\mathbb{Z})/\mathbb{Z}$ is zero if and only if f is connected.

(b) If f is connected then $L\mathcal{I}(f) = 0$.

Proof Since the problem is local in S, we may suppose that $S =$ Spec(A) and $X =$ Spec(B), with A a local ring. Let A^h be the henselization of $A_\wp$, and set $B' = B \otimes_A A^h$. Since f is finite, B' is a product of $n \geq 1$ hensel local rings B_i, each finite over A^h; see [3, 1.4.2]. Since $B/\wp B = B'/\wp B' = \prod B_i/\wp B_i$, the fiber of f at $\wp$ has n components, and is connected iff $n = 1$, i.e., iff B' is hensel local. Since the stalk of $f_*\mathbb{Z}/\mathbb{Z}$ at $\wp$ is zero iff B' is hensel local, the result follows. □

Examples 6.3 The hypothesis in 6.2 that f be finite is necessary.

(a) If $A = \mathbb{Z}$ and $B = \mathbb{Z}[\frac{1}{p}] \times \mathbb{Z}/p$ then $f :$ Spec$(B) \to$ Spec(A) is quasi-finite and connected, but $L\mathcal{I}(f) = \mathbb{Z}$ by Corollary 3.5. Here $f_*\mathbb{Z}/\mathbb{Z}$ is a skyscraper sheaf (at p).

(b) If A is the coordinate ring $k[x,y]/(y^2-x^3-x^2)$ of the node, and $B=A[1/x]$ then $A\subset B$ is étale and connected, yet $L\mathcal{I}(A,B)\cong L\mathrm{Pic}(A)\neq 0$ by Corollary 3.5. In this case, $f_*\mathbb{Z}/\mathbb{Z}$ is the skyscraper sheaf $\mathbb{Z}$ at the nodal point.

(c) If $A=k[x]$ and $B=A[b,e]/(e^2-e-bx)$ then f is not connected, as $B/xB\cong k[b]\times k[b]$. On the other hand, $f_*\mathbb{Z}/\mathbb{Z}=0$ and hence $L\mathcal{I}(f)=0$. In fact, if $\wp\neq xA$ then $B\otimes A^h\cong A^h[e]$. In this case, f has relative dimension 1.

Examples 6.4 Even if f is finite but not connected we may still have $L\mathcal{I}(f)=0$.

a) $L\mathcal{I}(\mathbb{R}[x],\mathbb{C}[x])=0$, but $\mathbb{R}[x]\subset\mathbb{C}[x]$ is not connected. In fact, $f_*\mathbb{Z}/\mathbb{Z}$ is nonzero exactly at those primes $\wp$ with $\mathbb{R}[x]/\wp\cong\mathbb{C}$. This example shows that the rank of $f_*\mathbb{Z}/\mathbb{Z}$ is not semicontinuous.

b) If $A=k[x]$ and $B=k[x,y]/(y^2=x^3+x^2)$ is the coordinate ring of the node then $L\mathcal{I}(f)=H^0(\mathrm{Spec}(A),f_*\mathbb{Z}/\mathbb{Z})=0$ but $(f_*\mathbb{Z})/\mathbb{Z}$ is nonzero because the stalk is $\mathbb{Z}$ at every point except at $x=0,-1$ and at the generic point (where the stalks are 0).

We now turn to the connection between $L\mathcal{I}$ and seminormalization.

Definition 6.5 (Swan [10, §2]) An extension $A\subset B$ is *subintegral* if B is integral over A, and $\mathrm{Spec}(B)\to\mathrm{Spec}(A)$ is a bijection inducing isomorphisms on all residue fields.

We say that A is *seminormal* in B if whenever $b\in B$ and $b^2,b^3\in A$ then $b\in A$. The *seminormalization* of A in B is the largest subring ${}^+\!A_B$ of B which is subintegral over A. By [10, 2.5], ${}^+\!A_B$ is seminormal in B.

These notions extend to faithful affine maps of schemes in the evident way; the seminormalization of S in X may be constructed by gluing together the seminormalizations on each affine open. We omit the details.

Remark 6.5.1 The condition that $N\mathcal{I}(A,B)=0$ is equivalent to A being seminormal in B, and implies that $N^i\mathcal{I}(A,B)=0$ for all $i>0$; this was proven in [8, 1.5,1.7]. More generally, a faithful affine map $f:X\to S$ is seminormal if and only if $N\mathcal{I}(f)=0$. This follows from the affine case, since both $N\mathcal{I}$ and seminormality are Zariski-local on S.

Proposition 6.6 $L\mathcal{I}(A,{}^+\!A_B)=0$ *and* $L\mathcal{I}(A,B)\cong L\mathcal{I}({}^+\!A_B,B)$.

Proof The first assertion follows from Proposition 3.4, since $LU(A)=LU({}^+\!A_B)$ (because $\mathrm{Spec}(A)\to\mathrm{Spec}({}^+\!A_B)$ is a bijection) and $L\mathrm{Pic}(A)=L\mathrm{Pic}({}^+\!A_B)$, by [11, 5.4]. (The hypothesis in [11, 5.4] that A be reduced is not needed in its proof.)

Now $\mathcal{I}(A[t,1/t],B[t,1/t])\to\mathcal{I}({}^+\!A_B[t,1/t],B[t,1/t])$ is onto by [7, 4.1]. Hence the map $L\mathcal{I}(A,B)\to L\mathcal{I}({}^+\!A_B,B)$ is onto. By Corollary 5.13, the kernel is $L\mathcal{I}(A,{}^+\!A_B)=0$. □

Definition 6.7 (Asanuma) A ring extension $A\subset B$ is called *anodal* if every $b\in B$ such that $(b^2-b)\in A$ and $(b^3-b^2)\in A$ belongs to A.

If $A\subset B$ is anodal then every idempotent of B belongs to A, so $H^0(A,\mathbb{Z})=H^0(B,\mathbb{Z})$. In particular, every anodal extension of a domain is connected. If A is a field then $A\subset B$ is anodal if and only if B is connected. The first author proved that the composition of anodal extensions is anodal; see [7, 3.1].

The following result generalizes a result of Asanuma (see [11, 3.4]), who considered the case $B=\mathrm{frac}(A)$, as well as several results of the first author in [7].

Theorem 6.8 *Let $A \subset B$ be an extension.*

(1) If $L\mathcal{I}(A, B) = 0$ then $A \subset B$ is anodal.
(2) If A is a 1-dimensional domain, and $A \subset B$ is an integral, birational and anodal extension, then $L\mathcal{I}(A, B) = 0$.

Example 6.3(b) shows that the integral hypothesis is necessary in Theorem 6.8(2). Example 6.9 shows that not all integral, birational anodal extensions have $L\mathcal{I}(f) = 0$.

Proof (cf. Onoda-Yoshida [4, 1.10]) Let $b \in B$ be such that $b^2 - b, b^3 - b^2$ are in A; we need to show that $b \in A$. Consider the finite subring $C = A[b]$ of B. If $L\mathcal{I}(A, B) = 0$, then $L\mathcal{I}(A, C) = 0$ by Corollary 5.13. Let $\mathfrak{a}$ denote the ideal $(b^2 - b)C$ of C; it is also an ideal of A, so $L\mathcal{I}(A/\mathfrak{a}, C/\mathfrak{a}) = 0$ by Lemma 4.4. By Proposition 3.4, this implies that $H^0(A/\mathfrak{a}, \mathbb{Z}) \cong H^0(C/\mathfrak{a}, \mathbb{Z})$. Since the image $\bar{b}$ of b is idempotent in $C/\mathfrak{a}$, this forces $\bar{b} \in A/\mathfrak{a}$ and hence $b \in A$, proving (1).

(2) Now suppose that A is a 1-dimensional domain, and that $A \subset B$ is an integral, birational and anodal extension. Since B is the union of finite A-algebras B_λ, all of which are anodal over A, we may assume that B is a finite A-algebra by Lemma 4.5. Since $A \subset B$ is finite and birational, the conductor ideal $\mathfrak{c}$ is nonzero, so $\dim A/\mathfrak{c} = 0$. By [11, 3.6], the extension $\bar{f} : A/\mathfrak{c} \subset B/\mathfrak{c}$ is anodal because $A \subset B$ is. In particular, $\bar{f}$ is connected. Since $\dim(A/\mathfrak{c}) = 0$, $L\mathrm{Pic}(A/\mathfrak{c}) = 0$ (by [11, 1.6.1]) and hence $L\mathcal{I}(\bar{f}) = 0$ by Proposition 3.4. By Lemma 4.4, $L\mathcal{I}(A, B) = L\mathcal{I}(A/\mathfrak{c}, B/\mathfrak{c}) = 0$. □

Here is an example of a 2-dimensional integral, birational and anodal extension with $L\mathcal{I}(A, B) \neq 0$, Thus Theorem 6.8(2) does not extend to $\dim(A) > 1$.

Example 6.9 Let X be the coordinate axes in the plane, and $f : X \to S$ the quotient identifying each axis with the normalization of the node S. Consider the pushout $S \to S' = \mathrm{Spec}(A)$ of the tautological inclusion of X in $\mathbb{A}^2 = \mathrm{Spec}(B)$ along f.

The map $\mathbb{A}^2 \to S'$ is the map $\mathrm{Spec}(B) \to \mathrm{Spec}(A)$ of Example 3.5 in [11]. By construction, A is a 2-dimensional domain whose integral closure is $B = k[x, y]$, so $A \subset B$ is an integral, birational extension. It is shown in [11, 3.5.2] that $A \subset B$ is anodal and $L\mathrm{Pic}\, A = \mathbb{Z}$. Since B is normal, $L\mathrm{Pic}\, B = 0$. Since A, B are domains, $LU(A) = LU(B) = \mathbb{Z}$ (by Example 2.4). By Proposition 3.4, $L\mathcal{I}(A, B) \cong L\mathrm{Pic}(A) \cong \mathbb{Z}$.

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