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Nonintegrability of Hamiltonian system perturbed from integrable system with two singular points

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Abstract We give a Hamiltonian system which is nonintegrable in a domain containing two singular points and that is integrable in some neighborhood of a singular point. The system is an arbitrarily small nontrivial perturbation of an integrable Hamiltonian system given by confluence of regular singular points of a generalized hypergeometric system.

Keywords Nonintegrability · Hamiltonian system with two singular points · Hypergeometric system · Confluence · Okubo equation

1 Introduction

Let $n \geq 2$ be an integer, and consider the Hamiltonian system

$$\begin{cases} z^2 \dfrac{dq}{dz} = \nabla_p \mathcal{H}(z, q, p), \\ z^2 \dfrac{dp}{dz} = -\nabla_q \mathcal{H}(z, q, p), \end{cases} \tag{1}$$

where $q = (q_2, \ldots, q_n)$, $p = (p_2, \ldots, p_n)$. Here

$$\nabla_q := \left(\frac{\partial}{\partial q_2}, \ldots, \frac{\partial}{\partial q_n} \right), \quad \nabla_p := \left(\frac{\partial}{\partial p_2}, \ldots, \frac{\partial}{\partial p_n} \right).$$

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The system (1) is equivalent to an autonomous one

$$\begin{cases} \dot{q}_1 = H_{p_1}, & \dot{q} = \nabla_p H, \\ \dot{p}_1 = -H_{q_1}, & \dot{p} = -\nabla_q H, \end{cases} \tag{2}$$

where $q_1 = z$ and $H(q_1, q, p_1, p) := q_1^2 p_1 + \mathcal{H}(q_1, q, p)$ or $H(q_1, q, p_1, p) := p_1 + q_1^{-2}\mathcal{H}(q_1, q, p)$. We say that the Hamiltonian system (2) is C^ω-Liouville integrable if there exist first integrals $\phi_j \in C^\omega$ $(j = 1, \ldots, n)$ which are functionally independent on an open dense set and Poisson commuting, i.e., $\{\phi_j, \phi_k\} = 0$, $\{H, \phi_k\} = 0$, where $\{\cdot, \cdot\}$ denotes the Poisson bracket. The Hamiltonian H is a first integral of this autonomous system. We abbreviate C^ω-Liouville integrable to C^ω-integrable or integrable if there is no fear of confusion.

In [2] Bolsinov and Taimanov showed a non C^ω-integrability of some Hamiltonian system related with geodesic flow on a Riemannian manifold. Then Gorni and Zampieri showed similar results in the local setting, namely for a Hamiltonian system being singular at the origin they showed the non C^ω-integrability (cf. [3,5,6]). In this paper we study the nonintegrability from a semi-global point of view. Namely we consider Hamiltonian system which is singular at the origin $q_1 = 0$ as well as $q_1 = 1$. We shall show that the system is integrable near the origin, while it is not integrable in the domain containing both $q_1 = 0$ and $q_1 = 1$. The Hamiltonian function is given by the arbitrary small non zero perturbation of an integrable Hamiltonian of the confluent generalized hypergeometric system (cf. Sect. 2).

More precisely, we consider

$$H = \sum_{j \in J'} \mu_j q_j p_j + \frac{q_1^2}{(q_1 - 1)^2} \sum_{j \in J} \mu_j q_j p_j + q_1^2 p_1, \tag{3}$$

where μ_j are complex constants and J and J' are the sets of multi-indices such that

$$J \neq \emptyset,\ J' \neq \emptyset,\ J \cap J' = \emptyset,\ J \cup J' = \{2, \ldots, n\}. \tag{4}$$

The Hamiltonian is derived from the generalized hypergeometric system by confluence of singularities (cf. Sect. 2). The Hamiltonian system (2)–(3) determines the Hamiltonian vector field

$$\begin{aligned} \chi_H = {} & q_1^2 \frac{\partial}{\partial q_1} - 2 q_1 p_1 \frac{\partial}{\partial p_1} + \frac{2 q_1}{(q_1 - 1)^3} \left(\sum_{j \in J} \mu_j q_j p_j \right) \frac{\partial}{\partial p_1} \\ & + \sum_{j \in J'} \mu_j \left(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right) + \frac{q_1^2}{(q_1 - 1)^2} \sum_{j \in J} \mu_j \left(q_j \frac{\partial}{\partial q_j} - p_j \frac{\partial}{\partial p_j} \right). \end{aligned} \tag{5}$$

Let

$$H_1 := \sum_{j=2}^{n} p_j^2 B_j(q_1, p). \tag{6}$$

Note that H_1 does not depend on q. Suppose that the nonresonance condition (NRC) holds:

$$\forall \gamma = (\gamma_2, \ldots, \gamma_n) \in \mathbb{Z}^{n-1} \backslash \{0\},\ \sum_{j=2}^{n} \mu_j \gamma_j \neq 0, \tag{7}$$

i.e. μ_j's are linearly independent over $\mathbb{Z}^{n-1}$. Moreover, assume
(TC): For $k \in J'$, the equation

$$q_1^2 \frac{d}{dq_1} v - 2\mu_k v = B_k(q_1, 0) \tag{8}$$

has no solution v holomorphic at $q_1 = 0$, and for $k \in J$, the equation

$$q_1^2 \frac{d}{dq_1} w - 2\mu_k \frac{q_1^2 w}{(q_1-1)^2} = B_k(q_1, 0) + \mu_k \frac{q_1 B_k(0,0)}{(q_1-1)^2} + B_k(0,0) \tag{9}$$

has no solution w holomorphic at $q_1 = 1$.

Let $\Omega_1 \subset \mathbb{C}$ be a domain containing $\{q_1 = 0, 1\}$, and $\Omega_2 \subset \mathbb{C}^{2n-1}$ be a neighborhood of $(p_1, q, p) = (0, 0, 0)$ and define $\Omega := \Omega_1 \times \Omega_2$. Then we have

Theorem 1 *Assume that (NRC) and (TC) are satisfied. Then, there exists Ω such that the Hamiltonian system (2) is not C^ω-integrable in Ω. More precisely, for every first integral ϕ satisfying $\chi_{H+H_1}\phi = 0$ and holomorphic in Ω, there exists a holomorphic function ψ defined in some neighborhood of the origin $t = 0$ such that $\phi(q_1, q, p_1, p) = \psi(H + H_1)$ in some neighborhood of the origin.*

In spite of the non integrability shown in Theorem 1 we have the integrability about a singular point of χ_{H+H_1}. We recall that the Hamiltonian system corresponding to $H+H_1$ has irregular singularities at $q_1 = 0$ and $q_1 = 1$. We have

Proposition 1 *Suppose that $H_1(q_1, p)$ be independent of p_ν for every $\nu \in J'$. Then, χ_{H+H_1} is analytically Liouville-integrable in some neighborhood of the origin.*

Remark (i) In Sect. 5 we show that (TC) holds on an open dense set in the set of analytic functions. (TC) also implies that H_1 could be replaced by εH_1 with an arbitrary small $\varepsilon \neq 0$. On the other hand, it is necessary in Theorem 1 that H_1 does not vanish identically because H is integrable in view of Lemma 1 (cf. Sect. 3). Hence the non-integrability occurs by an arbitrary small non-zero generic perturbation.

By Proposition 1 we see that our class of Hamiltonians contains subclass for each of which the integrability at the origin holds. Hence the (non-) integrability in Theorem 1 is caused by the interference of singular points.

(ii) Of course, a globally integrable system is locally integrable. So, it is sufficient for the proof of Theorem 1 to prove the local non-integrability.

(iii) In these days, monodromy is usually treated from the point of view of the differential Galois theory (for example, see [7]) because of enrichment of the theory however, we treat it from another point of view.

2 Confluence of singularities

In this section we deduce (3) from the genelarized hypergeometric system

$$(z - C)\frac{dv}{dz} = Av, \tag{10}$$

where $C = \mathrm{diag}(\Lambda_1, {}^t\Lambda_1)$, Λ_1 being $(n-1)\times(n-1)$ matrix with eigenvalues $\lambda_2, \ldots, \lambda_n$ such that $\lambda_j \neq 0$ for all j (cf. [1,4]). For the sake of simplicity, we assume $\Lambda_1 = \mathrm{diag}(\lambda_2, \ldots, \lambda_n)$. We assume $A = \mathrm{diag}(A_1, A_1)$, where A_1 is an $(n-1)\times(n-1)$ constant matrix satisfying $\Lambda_1 A_1 = A_1 \Lambda_1$. For simplicity, we further assume $A_1 = \mathrm{diag}(\tau_2, \ldots, \tau_n)$.

Let $v = {}^t(q, p) \in \mathbb{C}^{2(n-1)}$. Define

$$H = \langle (z - \Lambda_1)^{-1} p, A_1 q \rangle, \tag{11}$$

where $\langle (x_2, \ldots, x_n), {}^t(y_2, \ldots, y_n) \rangle := \sum_{2 \le k \le n} x_k y_k$. Then, (10) is written in the Hamiltonian system

$$\frac{dq}{dz} = H_p(z, q, p), \ \frac{dp}{dz} = -H_q(z, q, p). \tag{12}$$

Now we operate the confluence of regular singularities. Let v_ν and $(Av)_\nu$ denote the νth entry of v and Av, respectively. Then we can write (12) in the form

$$(z - \lambda_\nu) \frac{dv_\nu}{dz} = (Av)_\nu.$$

Substituting $z = 1/\zeta$, we have

$$-\zeta^2 \frac{dv_\nu}{d\zeta} = (\zeta^{-1} - \lambda_\nu)^{-1} (Av)_\nu. \tag{13}$$

In the following, $a \mapsto b$ denotes the replacement of a by b.

Let $\zeta \mapsto \epsilon^{-1}\eta$; and $\lambda_\nu \mapsto \epsilon\lambda_\nu$ for $\nu \in J$, $\lambda_\nu \mapsto \lambda_\nu$ for $\nu \in J'$. Multiply the νth row of A in (13) by ϵ^{-1} if $\nu \in J'$ and take the limit $\epsilon \to 0$. Then (12) is reduced to the Hamiltonian system

$$-\eta^2 \frac{dq}{d\eta} = \mathfrak{A} A_1 q, \ -\eta^2 \frac{dp}{d\eta} = -{}^t A_1 \mathfrak{A} p, \tag{14}$$

where $\mathfrak{A} = \mathrm{diag}(\mathfrak{A}_2, \ldots, \mathfrak{A}_n)$ and

$$\mathfrak{A}_\nu := \begin{cases} -\lambda_\nu^{-1} & (\nu \in J'), \\ (\eta^{-1} - \lambda_\nu)^{-1} & (\nu \in J). \end{cases} \tag{15}$$

Note that (14) is irregular singular at $\eta = 0$.

In order to introduce another singular point, choose any $a \neq 0$ such that $a \neq \lambda_j^{-1}$ for all j and put $\zeta = \eta - a$. Let $\zeta \mapsto \epsilon^{-1}\zeta$ and $(A)_\nu \mapsto \epsilon^{-1}(A)_\nu$. Make substitution $a \mapsto \epsilon^{-1}a$ for $j \in J'$ and $a \mapsto a$ for $j \in J$ and take the limit $\epsilon \to 0$. Then (12) is reduced to a Hamiltonian system with irregular points at 0 and $-a$. Set $a = -1$. Finally, by transforming to the autonomous system and putting $\mu_j := \mu_j$, we obtain (3).

3 Proof of Proposition 1

Let H and H_1 be given by (3) and (6), respectively. First we show

Lemma 1 *If $k \in J$, then χ_H has first integrals*

$$q_k \exp\left(\frac{\mu_k}{q_1 - 1}\right), \quad p_k \exp\left(-\frac{\mu_k}{q_1 - 1}\right), \tag{16}$$

while, for $k \in J'$ it has

$$q_k \exp\left(\frac{\mu_k}{q_1}\right), \quad p_k \exp\left(-\frac{\mu_k}{q_1}\right). \tag{17}$$

Note that χ_H is analytically integrable at $q_1 = 0$ or $q_1 = 1$, because $q_k p_k$ is an analytic first integral about the singular point $q_1 = 0$ or $q_1 = 1$.

Proof of Lemma 1 The assertion is easily verified in view of the definition of first integrals.

Remark Lemma 1 says that in the C^∞ class the Hamiltonian is superintegrable. The perturbation in Proposition 1 breaks some first integrals, but not all of them. The remaining ones are not either sufficiently regular for integrability near both points.

Proof of Proposition 1 We have H_1 not depending on p_k, $k \in J'$, q_1, q_k, $k = 2, \ldots, n$ by hypothesis and (6). So the dynamical equations give that q_k, $k \in J'$, q_1, p_k, $k = 2, \ldots, n$ are first integrals of H_1. Thus in particular

$$p_k q_k, \ (k \in J'), \quad p_k \exp\left(-\frac{\mu_k}{q_1 - 1}\right), \quad (k \in J) \tag{18}$$

are first integrals of H_1, and are analytic at 0. As these are also first integrals of H, they are in involution and first integrals of $H + H_1$. This ends the proof.

4 Proof of Theorem 1

Let $\phi =: u$ be a holomorphic first integral in Ω and expand u at $p = 0$

$$u = \sum_\alpha u_\alpha(q_1, q, p_1) p^\alpha. \tag{19}$$

Substitute (19) into $\chi_{H+H_1} u = 0$ and compare the powers like $p^0 = 1$ of both sides. Then we have the equation of $u_0 = u_0(q_1, q, p_1)$

$$\{q_1^2 p_1, u_0\} + \sum_{j \in J'} \mu_j q_j \frac{\partial}{\partial q_j} u_0 + \frac{q_1^2}{(q_1 - 1)^2} \sum_{j \in J} \mu_j q_j \frac{\partial}{\partial q_j} u_0 = 0. \tag{20}$$

Indeed, no constant term in p appears from $\chi_{H_1} u$ in view of the definition of χ_{H_1}.

Substituting the expansion $u_0 = \sum_\beta u_{0,\beta}(q_1, p_1) q^\beta$ into (20), we see that $U_0 := u_{0,0}$ satisfies $\{q_1^2 p_1, U_0\} = 0$, namely

$$\left(q_1 \frac{\partial}{\partial q_1} - 2 p_1 \frac{\partial}{\partial p_1}\right) U_0 = 0. \tag{21}$$

Substitute the expansion $U_0 = \sum_{\nu,\mu} c_{\nu\mu} q_1^\mu p_1^\nu$ into (21). Then we have $\sum_{\nu,\mu} c_{\nu,\mu}(\mu - 2\nu) q_1^\mu p_1^\nu = 0$. It follows that $c_{\nu,\mu} = 0$ for $\mu \neq 2\nu$. Hence we obtain

$$U_0 = \sum_\nu c_{\nu,2\nu} q_1^{2\nu} p_1^\nu = \sum_\nu c_{\nu,2\nu} (q_1^2 p_1)^\nu. \tag{22}$$

It follows that there exists a function of one variable t, $\phi_0(t)$ holomorphic in some neighborhood of $t = 0$ such that $U_0 = \phi_0(q_1^2 p_1)$.

Next, we focus on the equation of $u_{0,\beta}$ with $\beta \neq 0$

$$\{q_1^2 p_1, u_{0,\beta}\} + \sum_{j \in J'} \mu_j \beta_j u_{0,\beta} + \frac{q_1^2}{(q_1 - 1)^2} \sum_{j \in J} \mu_j \beta_j u_{0,\beta} = 0.$$

Expand

$$u_{0,\beta} = \sum_\nu \omega_{\beta,\nu}(q_1) p_1^\nu, \tag{23}$$

and consider the equation of $\omega_{\beta,\nu}$. If $\nu = 0$, then, by comparing the coefficients of $p_1^0 = 1$, we have

$$q_1^2 \frac{d}{dq_1}\omega_{\beta,0} + \left(\sum_{j\in J'} \mu_j \beta_j + \frac{q_1^2}{(q_1-1)^2} \sum_{j\in J} \mu_j \beta_j \right) \omega_{\beta,0} = 0. \tag{24}$$

Since $\beta \neq 0$, it follows from (NRC), (7), that either $A' := \sum_{j\in J'} \mu_j \beta_j \neq 0$ or $A := \sum_{j\in J} \mu_j \beta_j \neq 0$ is valid. If $A' \neq 0$, then we have $\omega_{\beta,0} = 0$ in some neighborhood of $q_1 = 0$. Indeed, by subsituting the expansion $\omega_{\beta,0} = \sum_{l=0}^{\infty} C_l q_1^l$ into (24) and by using the relations

$$q_1^2 \frac{d}{dq_1}\omega_{\beta,0} = \sum_{l=0}^{\infty} C_l l q_1^{l+1}$$

and

$$\frac{q_1^2}{(q_1-1)^2} \sum_{j\in J} \mu_j \beta_j \omega_{\beta,0} = \sum_{l=0}^{\infty} C_l' \, {q_1}^{l+2}$$

for some C_l', we obtain

$$\begin{aligned} &C_0 A' = 0 \quad \text{i.e. } C_0 = 0, \\ &C_1 A' + C_0 \cdot 0 = 0 \quad \text{i.e. } C_1 = 0, \\ &C_2 A' + C_0' + C_1 = 0 \quad \text{i.e. } C_2 = 0, \\ &\quad \cdots \end{aligned}$$

Note that $C_0' = 0$ since $C_0 = 0$. Hence we have $\omega_{\beta,0} = 0$.

In the case where $A' = 0$ and $A \neq 0$, (24) is written in

$$(q_1 - 1)^2 \frac{d}{dq_1}\omega_{\beta,0} + A\omega_{\beta,0} = 0. \tag{25}$$

Similarly to the case $A' \neq 0$, we obtain $\omega_{\beta,0} = 0$ in some neighborhood of $q_1 = 1$. Therefore, we have $\omega_{\beta,0} = 0$ in Ω_1.

Next, by comparing the coefficients of $p_1^1 = p_1$, we have the equation of $\omega_{\beta,1}(q_1)$

$$\left(q_1^2 \frac{d}{dq_1} - 2q_1 \right) \omega_{\beta,1} + \left(A' + \frac{q_1^2}{(q_1-1)^2} A \right) \omega_{\beta,1} = 0. \tag{26}$$

Similarly to the above, $A' \neq 0$ implies $\omega_{\beta,1} = 0$ near $q_1 = 0$, while $A' = 0$ and $A \neq 0$ imply $\omega_{\beta,1} = 0$ near $q_1 = 1$. Hence we have $\omega_{\beta,1} = 0$ in Ω_1. By the same argument we obtain $\omega_{\beta,\nu} = 0$ in Ω_1 for all $\nu \in \mathbb{N} \cup \{0\}$. It follows that $u_{0,\beta} = 0$ for all $\beta \neq 0$.

Therefore, we have

$$u_0 = u_{0,0}(q_1^2 p_1) + \sum_{\beta\neq 0} u_{0,\beta}(q_1^2 p_1) q^\beta = \phi_0(q_1^2 p_1) \tag{27}$$

for some $\phi_0(t)$ of one variable being analytic at $t = 0$. Note that

$$\begin{aligned} u|_{p=0} - \phi_0(H + H_1)|_{p=0} &= u_0(q_1, p_1) - \phi_0(H|_{p=0}) \\ &= \phi_0(q_1^2 p_1) - \phi_0(q_1^2 p_1) \equiv 0. \end{aligned}$$

Hence, without loss of generality, we may assume $u|_{p=0} = 0$.

Next we consider $u_\alpha = u_\alpha(q_1, p_1, q)$ for $|\alpha| = 1$. Write $\alpha = e_k$ $(2 \le k \le n)$ where $e_k := (0, \ldots, 0, 1, 0, \ldots, 0)$ is the kth unit vector. Then, u_α satisfies

$$\begin{aligned}\{q_1^2 p_1, u_\alpha\} &+ \sum_{j\in J'} \mu_j \left(q_j \frac{\partial}{\partial q_j} - \delta_{k,j}\right) u_\alpha \\ &+ \frac{q_1^2}{(q_1-1)^2} \sum_{j\in J} \mu_j \left(q_j \frac{\partial}{\partial q_j} - \delta_{k,j}\right) u_\alpha = 0,\end{aligned} \tag{28}$$

where $\delta_{k,j}$ is the Kronecker's delta, $\delta_{k,j} = 1$ if $k = j$, and =0 if otherwise. Note that, because $u_0 = 0$, χ_{H_1} gives no term.

Substitute the expansion $u_\alpha = \sum_\beta u_{\alpha,\beta}(q_1, p_1)q^\beta$ into (28), and compare the powers like $q^0 = 1$. Then we have the equation of $u_{\alpha,0}$

$$\{q_1^2 p_1, u_{\alpha,0}\} - \mu_k \left(\sum_{j\in J'} \delta_{k,j}\right) u_{\alpha,0} - \frac{q_1^2}{(q_1-1)^2} \left(\sum_{j\in J} \mu_j \delta_{k,j}\right) u_{\alpha,0} = 0. \tag{29}$$

If $k \in J'$, then

$$\{q_1^2 p_1, u_{\alpha,0}\} - \mu_k u_{\alpha,0} = 0.$$

Because $\mu_k \neq 0$ by (NRC) condition, we have $u_{\alpha,0} = 0$.

On the other hand, if $k \in J$, then

$$\{q_1^2 p_1, u_{\alpha,0}\} - \frac{q_1^2}{(q_1-1)^2} \mu_k u_{\alpha,0} = 0.$$

By considering the equation around $q_1 = 1$ together with (NRC) condition we obtain $u_{\alpha,0} = 0$.

Next we consider $u_{\alpha,\beta} (\beta \neq 0)$ $(\alpha = (\alpha_2, \ldots, \alpha_n), \alpha_j = \delta_{j,k})$.

$$\begin{aligned}\{q_1^2 p_1, u_{\alpha,\beta}\} &+ \sum_{j\in J'} \mu_j(\beta_j - \alpha_j) u_{\alpha,\beta} \\ &+ \frac{q_1^2}{(q_1-1)^2} \sum_{j\in J} \mu_j(\beta_j - \alpha_j) u_{\alpha,\beta} = 0.\end{aligned} \tag{30}$$

If $\beta \neq \alpha$, then (NRC) condition yields $u_{\alpha,\beta} = 0$, by the similar argument as in the above. If $\beta = \alpha$, then we have $\{q_1^2 p_1, u_{\alpha,\alpha}\} = 0$. Hence, there exists $\phi_\alpha(t)$ of one variable t such that $u_{\alpha,\alpha} = \phi_\alpha(q_1^2 p_1)$. Therefore we obtain

$$u = \sum_{|\alpha|=1} \phi_\alpha(q_1^2 p_1) q^\alpha p^\alpha + O(|p|^2). \tag{31}$$

Now we consider the equation for u_α when $|\alpha| = 2$. We substitute (19) and (31) into the equation $\chi_{H+H_1} u = 0$ and compare the powers like p^α $(|\alpha| = 2)$. In order to get the expressions of the powers like p^α, we note that the following terms appear from $\chi_H u$:

$$\begin{aligned}\{q_1^2 p_1, u_\alpha\} &+ \sum_{j\in J'} \mu_j \left(q_j \frac{\partial}{\partial q_j} - \alpha_j\right) u_\alpha + \frac{q_1^2}{(q_1-1)^2} \sum_{j\in J} \mu_j \left(q_j \frac{\partial}{\partial q_j} - \alpha_j\right) u_\alpha \\ &+ \frac{2q_1}{(q_1-1)^3} \sum_{j\in J} \mu_j q^\alpha \frac{\partial}{\partial p_1} \phi_{\alpha - e_j}.\end{aligned} \tag{32}$$

On the other hand, the following terms appear from $\chi_{H_1} u$.

$$\sum_\nu \frac{\partial}{\partial p_\nu}\left(\sum_j p_j^2 B_j(q_1, p)\right)\frac{\partial}{\partial q_\nu}(\phi_{e_\nu} q_\nu p_\nu) - \frac{\partial}{\partial q_1}\left(\sum_j p_j^2 B_j(q_1, p)\right)\frac{\partial}{\partial p_1}(\sum_{|\alpha|=1}\phi_\alpha q^\alpha p^\alpha). \tag{33}$$

Note that the second term in (33) is $O(|p|^3)$. Hence it does not appear in the recurrence formula because $|\alpha| = 2$. Moreover, since we consider terms of $O(|p|^2)$, the first term yields

$$2\sum_\nu \phi_{e_\nu} B_\nu(q_1, 0)\delta_{\alpha, 2e_\nu}. \tag{34}$$

Therefore, by comparing the powers like p^α in $\chi_{H+H_1} u = 0$ we have

$$\begin{aligned}&\{q_1^2 p_1, u_\alpha\} + \sum_{j\in J'}\mu_j\left(q_j\frac{\partial}{\partial q_j} - \alpha_j\right)u_\alpha \\ &+\frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\mu_j\left(q_j\frac{\partial}{\partial q_j} - \alpha_j\right)u_\alpha \\ &+\frac{2q_1}{(q_1-1)^3}q^\alpha\sum_{j\in J}\mu_j\frac{\partial}{\partial p_1}\phi_{\alpha-e_j} + 2\sum_\nu \phi_{e_\nu} B_\nu(q_1, 0)\delta_{\alpha,2e_\nu} = 0.\end{aligned} \tag{35}$$

Expand u_α with respect to q, $u_\alpha = \sum_\beta u_{\alpha,\beta}(q_1, p_1)q^\beta$ and insert the expansion into (35). By comparing the power of q^β we obtain the recurrence relation for $u_{\alpha,\beta}(q_1, p_1)$. We consider 4 cases:

(i) $\alpha \neq 2e_\nu$ for every ν and $\beta \neq \alpha$.
(ii) $\alpha = 2e_k$ for some k and $\beta \neq \alpha, 0$.
(iii) $\alpha = 2e_k$ for some k and $\beta = 0$.
(iv) $\beta = \alpha$.

Case (i): We note that the fourth and the fifth terms of the left-hand side of (35) yield no term in the recurrence relation for $u_{\alpha,\beta}$. Indeed, the fourth term is a monomial of q^α. Hence, $u_{\alpha,\beta}$ satisfies

$$\{q_1^2 p_1, u_{\alpha,\beta}\} + \sum_{j\in J'}\mu_j(\beta_j - \alpha_j)u_{\alpha,\beta} + \frac{q_1^2}{(q_1-1)^2}\sum_{j\in J}\mu_j(\beta_j-\alpha_j)u_{\alpha,\beta} = 0. \tag{36}$$

By virtue of (NRC) and $\beta \neq \alpha$, either $\sum_{j\in J'}\mu_j(\beta_j - \alpha_j) \neq 0$ or $\sum_{j\in J}\mu_j(\beta_j - \alpha_j) \neq 0$ holds. One can easily show that $u_{\alpha,\beta} = 0$ by the holomorphy of $u_{\alpha,\beta}$.

Case (ii): Because the fourth and fifth terms of the left-hand side of (35) do not yield terms by the assumption $\beta \neq \alpha, 0$, we see that $u_{\alpha,\beta}$ satisfies (36). Therefore, we have $u_{\alpha,\beta} = 0$.

Case (iii): Let $k \in J'$. Because the fourth term of the left-hand side of (35) is a monomial q^α, $u_{\alpha,0}$ satisfies

$$\{q_1^2 p_1, u_{\alpha,0}\} - 2\mu_k u_{\alpha,0} + 2\phi_{e_k}(q_1^2 p_1)B_k(q_1, 0) = 0. \tag{37}$$

Expand $u_{\alpha,0}(q_1, p_1) = \sum_\nu u_{\alpha,0,\nu}(q_1) p_1^\nu$ and compare the constant terms in p_1 of both sides of (37). Then we have

$$q_1^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\mu_k u_{\alpha,0,0} + 2\phi_{e_k}(0) B_k(q_1, 0) = 0. \tag{38}$$

If $\phi_{e_k}(0) \neq 0$, then $v := u_{\alpha,0,0}/(-2\phi_{e_k}(0))$ satisfies

$$q_1^2 \frac{d}{dq_1} v - 2\mu_k v = B_k(q_1, 0),$$

which contradicts (TC). Hence, $\phi_{e_k}(0) = 0$ and (38) reduces to

$$q_1^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\mu_k u_{\alpha,0,0} = 0.$$

(NRC) condition implies $2\mu_k \neq 0$, and the holomorphcity of $u_{\alpha,0,0}$ at $q_1 = 0$ tells us $u_{\alpha,0,0} = 0$.

Next, $u_{\alpha,0,1}$ satisfies

$$\left(q_1^2 \frac{d}{dq_1} - 2q_1\right) u_{\alpha,0,1} - 2\mu_k u_{\alpha,0,1} + 2B_k(q_1, 0)\phi'_{e_k}(0) q_1^2 = 0. \tag{39}$$

Since $u_{\alpha,0,1}(q_1) = O(q_1^2)$, we put $u_{\alpha,0,1}(q_1) = q_1^2 \tilde{u}_{\alpha,0,1}(q_1)$ with $\tilde{u} := \tilde{u}_{\alpha,0,1}(q_1)$ satisfying

$$q_1^2 \frac{d}{dq_1} \tilde{u} - 2\mu_k \tilde{u} = -2B_k(q_1, 0)\phi'_{e_k}(0).$$

If $\phi'_{e_k}(0) \neq 0$, then, by putting $v = \tilde{u}/(-2\phi'_{e_k}(0))$, we have a contradiction to (TC). Therefore, $\phi'_{e_k}(0) = 0$ and $\tilde{u} = 0$.

Similarly we can show $u_{\alpha,0,\nu} = 0$ and $\phi_{e_k}^{(\nu)}(0) = 0$ for $\nu \in \mathbb{N} \cup \{0\}$, which implies $u_{\alpha,0} = 0$ and $\phi_{e_k} = 0$ for every $k \in J'$.

Let $k \in J$. Then $u_{\alpha,0}$ satisfies

$$\{q_1^2 p_1, u_{\alpha,0}\} - 2\mu_k \frac{q_1^2}{(q_1 - 1)^2} u_{\alpha,0} + 2\phi_{e_k}(q_1^2 p_1) B_k(q_1, 0) = 0.$$

Expand $u_{\alpha,0}(q_1, p_1) = \sum_\nu u_{\alpha,0,\nu}(q_1) p_1^\nu$. Then $u_{\alpha,0,0}$ satisfies

$$q_1^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\mu_k \frac{q_1^2}{(q_1 - 1)^2} u_{\alpha,0,0} + 2\phi_{e_k}(0) B_k(q_1, 0) = 0. \tag{40}$$

If $\phi_{e_k}(0) \neq 0$, then, by (40) we have $B_k(0, 0) = 0$. On the other hand, $v := u_{\alpha,0,0}/(-2\phi_{e_k}(0))$ satisfies

$$q_1^2 \frac{d}{dq_1} v - 2\mu_k \frac{q_1^2}{(q_1 - 1)^2} v = B_k(q_1, 0),$$

which contradicts (TC). So, $\phi_{e_k}(0) = 0$ and (40) reduces to

$$(q_1 - 1)^2 \frac{d}{dq_1} u_{\alpha,0,0} - 2\mu_k u_{\alpha,0,0} = 0.$$

Again we have $u_{\alpha,0,0} = 0$.

Next, consider the equation of $u_{\alpha,0,1}$

$$\left(q_1^2 \frac{d}{dq_1} - 2q_1\right) u_{\alpha,0,1} - 2\mu_k \frac{q_1^2}{(q_1 - 1)^2} u_{\alpha,0,1} = -2\phi'_{e_k}(0) q_1^2 B_k(q_1, 0). \tag{41}$$

Observing $u_{\alpha,0,1}(0) = 0$, we put $u_{\alpha,0,1}(q_1) = cq_1 + q_1^2 v$. Substituting it into (41), we have $c = -2\phi'_{e_k}(0)B_k(0,0)$ and v satisfies

$$-2\phi'_{e_k}(0)\left\{B_k(q_1,0) + B_k(0,0) + 2B_k(0,0)\mu_k \frac{q_1}{(q_1-1)^2}\right\}$$
$$= \left(q_1^2 \frac{d}{dq_1} - 2\mu_k \frac{q_1}{(q_1-1)^2}\right) v.$$

By use of (TC), we obtain $\phi'_{e_k}(0) = 0$ and $u_{\alpha,0,1} = 0$.

In general, $u_{\alpha,0,\nu}$ $(\nu \geq 2)$ satisfies

$$\left(q_1^2 \frac{d}{dq_1} - 2\nu q_1\right) u_{\alpha,0,\nu} - 2\mu_k \frac{q_1^2}{(q_1-1)^2} u_{\alpha,0,\nu} = -2\frac{\phi_{e_k}^{(\nu)}(0)}{\nu!} q_1^{2\nu} B_k(q_1,0). \tag{42}$$

Since we easily see $u_{\alpha,0,\nu} = O(q^{2\nu-1})$, we put $u_{\alpha,0,\nu} = cq_1^{2\nu-1} + q_1^{2\nu} w$. Then we have $c = -2\phi_{e_k}^{(\nu)}(0)B_k(0,0)/\nu!$ and w satisfies

$$-\frac{2\phi'_{e_k}(0)}{\nu!}\left\{B_k(q_1,0) + B_k(0,0) + 2B_k(0,0)\mu_k \frac{q_1}{(q_1-1)^2}\right\}$$
$$= \left(q_1^2 \frac{d}{dq_1} - 2\mu_k \frac{q_1^2}{(q_1-1)^2}\right) w.$$

By virtue of (TC), we obtain $\phi_{e_k}^{(\nu)}(0) = 0$ and $w = 0$. Therefore, $u_{\alpha,0,\nu} = 0$ for all $\nu \in \mathbb{N} \cup \{0\}$. Because of analyticity, we have $u_{\alpha,0} = 0$ and $\phi_{e_k} = 0$ for every $k \in J$. Consequently, $\phi_{e_k} = 0$ holds for all $k \in J' \cup J$.

Case (iv): Because $\phi_{e_k} = 0$ for every k by what we have proved in the above, the fourth and fifth terms of the left-hand side of (35) do not yield terms in the recurrence relation. Hence, $u_{\alpha,\alpha}$ satisfies $\{q_1^2 p_1, u_{\alpha,\alpha}\} = 0$. It follows that there exists a function of one variable $\phi_\alpha(t)$ such that $u_{\alpha,\alpha} = \phi_\alpha(q_1^2 p_1)$.

Therefore we have proved

$$u = \sum_{|\alpha|=2} \phi_\alpha(q_1^2 p_1)\, q^\alpha p^\alpha + O(|p|^3).$$

Finally we shall prove

Lemma 2 *Suppose*

$$u = \sum_{|\alpha|=\nu} \phi_\alpha(q_1^2 p_1) q^\alpha p^\alpha + O(|p|^{\nu+1}) \tag{43}$$

for some $\nu \geq 1$. *Then we have*

(i) $\phi_\alpha = 0$ *for all* α *satisfying* $|\alpha| = \nu$.
(ii) *For every* α *satisfying* $|\alpha| = \nu+1$, *there exists a holomorphic function* ϕ_α *of one variable such that*

$$u = \sum_{|\alpha|=\nu+1} \phi_\alpha(q_1^2 p_1) q^\alpha p^\alpha + O(|p|^{\nu+2}). \tag{44}$$

We have already proved (43) for $\nu = 1, 2$. Note that the lemma ends the proof of Theorem 1 because we have $u = 0$ as an analytic function of q and p.

Proof of Lemma 2 By comparing the coefficients of p^α in $\chi_{H+H_1}u = 0$ we have

$$\begin{aligned}
&\{q_1^2 p_1, u_\alpha\} + \sum_{J'} \mu_j \left(q_j \frac{\partial}{\partial q_j} - \alpha_j \right) u_\alpha \\
&+ \frac{q_1^2}{(q_1-1)^2} \sum_J \mu_j \left(q_j \frac{\partial}{\partial q_j} - \alpha_j \right) u_\alpha \\
&+ \frac{2q_1}{(q_1-1)^3} \left(\sum_J \mu_j q_j p_j \right) \frac{\partial}{\partial p_1} u_\gamma + \sum_{j,\gamma} \frac{\partial H_1}{\partial p_j} \frac{\partial}{\partial q_j} u_\gamma = 0, \qquad (45)
\end{aligned}$$

where $|\gamma| < |\alpha|$ and $\alpha = \gamma + e_j$.

Let $|\alpha| = \nu + 1$. Substituting the expansion $u_\alpha = \sum_\beta u_{\alpha,\beta}(q_1, p_1) q^\beta$ into (45) and by using (43), we obtain the relation for $u_{\alpha,\beta}$

$$\begin{aligned}
&\{q_1^2 p_1, u_{\alpha,\beta}\} + \sum_{J'} \mu_j (\beta_j - \alpha_j) u_{\alpha,\beta} + \frac{q_1^2}{(q_1-1)^2} \sum_J \mu_j (\beta_j - \alpha_j) u_{\alpha,\beta} \\
&+ 2 \frac{q_1}{(q_1-1)^3} \sum_J \mu_j \frac{\partial}{\partial p_1} \phi_{\alpha-e_j}(q_1^2 p_1) \delta_{\alpha,\beta} \\
&+ 2 \sum_{j \in J' \cup J} \delta_{\alpha-2e_j,\beta} B_j(q_1, 0) \phi_{\alpha-e_j}(\alpha_j - 1) = 0. \qquad (46)
\end{aligned}$$

Indeed, because it is easy to show the expressions up to the fourth term in the left-hand side of (46), we consider the fifth term, which corresponds to the fifth term in the left-hand side of (45). In view of (43) we may consider $2\sum_j p_j B_j(q_1, 0)$ in $\frac{\partial H_1}{\partial p_j}$ because other terms have no effect to (45). Hence we may consider terms containing $p^{\alpha-e_j}$ in $\frac{\partial}{\partial q_j} u_\gamma$. By (43) the coefficient of the term containing $p^{\alpha-e_j}$ is $(\alpha_j - 1) q^{\alpha-2e_j} B_j(q_1, 0) \phi_{\alpha-e_j}$. Hence we have the desired expression.

Set $B' := \sum_{\in J'} \mu_j (\beta_j - \alpha_j)$ and $B := \sum_{j \in J} \mu_j (\beta_j - \alpha_j)$. We consider 4 cases.

Case (1) The case where $\alpha - 2e_j \neq \beta$ for $j = 2, \ldots, n$ and $B' \neq 0$. Clearly we have $\beta \neq \alpha$. It follows that the fourth and the fifth terms in the left-hand side of (46) vanish. Hence we have $u_{\alpha,\beta} = 0$ by considering (46) at $q_1 = 0$.

Case (2) The case where $\alpha - 2e_j \neq \beta$ for $j = 2, \ldots, n$, $\beta \neq \alpha$ and $B' = 0$. By (NRC) we have $B \neq 0$. Hence the fourth and the fifth terms in the left-hand side of (46) vanish. We have $u_{\alpha,\beta} = 0$ by considering (46) at $q_1 = 1$.

Case (3) The case where $\alpha - 2e_k = \beta$ for some k. Clearly, we have $\beta \neq \alpha$. Assume $k \in J$. Then, for every $j \in J'$ we have $j \neq k$, and hence $\alpha_j = \beta_j$, which implies $B' = 0$. Equation (46) is reduced to

$$\{q_1^2 p_1, u_{\alpha,\beta}\} - 2\mu_k \frac{q_1^2}{(q_1-1)^2} u_{\alpha,\beta} + 2(\alpha_k - 1)\phi_{\alpha-e_k} B_k(q_1, 0) = 0.$$

Expand $u_{\alpha,\beta} = \sum_{\nu=0}^\infty u_{\alpha,\beta,\nu}(q_1) p_1^\nu$. We will show that $\phi_{\alpha-e_k}$ vanishes.

Indeed, $v := u_{\alpha,\beta,0}$ satisfies

$$q_1^2 \frac{dv}{dq_1} - 2\mu_k \frac{q_1^2}{(q_1-1)^2} v = -2(\alpha_k - 1)\phi_{\alpha-e_k}(0) B_k(q_1, 0).$$

Note that $\alpha_k = 2 + \beta_k \geq 2$. If $\phi_{\alpha - e_k}(0) \neq 0$, then $w := v/(-2(\alpha_k - 1)\phi_{\alpha - e_k}(0))$ is a holomorphic solution at $q_1 = 0$ of the equation

$$q_1^2 \frac{dw}{dq_1} - 2\mu_k \frac{q_1^2}{(q_1 - 1)^2} w = B_k(q_1, 0).$$

Because one can verify $B_k(0, 0) = 0$, we have a contradiction to (TC). Hence we have $\phi_{\alpha - e_k}(0) = 0$ and $u_{\alpha,\beta,0} = 0$.

Next, $v = u_{\alpha,\beta,1}$ satisfies

$$q_1^2 \frac{dv}{dq_1} - 2\mu_k \frac{q_1^2}{(q_1 - 1)^2} v - 2q_1 v = -2(\alpha_k - 1)\phi'_{\alpha - e_k}(0) q_1^2 B_k(q_1, 0).$$

By comparing the coefficients of q_1^2 of both sides we see that $v = O(q_1^2)$. Similarly to the above, $w := vq_1^{-2}$ leads to a contradiction to (TC). Hence, we have $\phi'_{\alpha - e_k}(0) = 0$ and $u_{\alpha,\beta,1} = 0$.

In general, $v = u_{\alpha,\beta,\nu} (\nu \geq 2)$ satisfies

$$q_1^2 \frac{dv}{dq_1} - 2\mu_k \frac{q_1^2}{(q_1 - 1)^2} v - 2q_1 \nu v = -\frac{2(\alpha_k - 1)}{\nu!} \phi^{(\nu)}_{\alpha - e_k}(0) q_1^{2\nu} B_k(q_1, 0).$$

Similarly to the above, we have $\phi^{(\nu)}_{\alpha - e_k}(0) = 0$ and $u_{\alpha,\beta,\nu} = 0$. Therefore, $\phi_{\alpha - e_k} = 0$ and $u_{\alpha,\beta} = 0$ for $k \in J$.

Let $k \in J'$. Equation (46) is reduced to

$$\left\{q_1^2 p_1, u_{\alpha,\beta}\right\} - 2\mu_k u_{\alpha,\beta} + 2(\alpha_k - 1)\phi_{\alpha - e_k} B_k(q_1, 0) = 0.$$

The holomorphicity of $u_{\alpha,\beta}$ at $q_1 = 0$ and (TC) implies $\phi_{\alpha - e_k}(0) = 0$ and $u_{\alpha,\beta} = 0$ for $k \in J'$. Therefore, $\phi_\alpha = 0$ for $k \in J'$. Because $\phi_\alpha = 0$ for $k \in J$, we have $\phi_\alpha = 0$ for all α with $|\alpha| = \nu$.

Case (4) The case $\beta = \alpha$. We have $\{q_1^2 p_1, u_{\alpha,\alpha}\} = 0$, since we have proved $\phi_\gamma = 0$ for $|\gamma| = \nu$. Hence, there exists ϕ_α such that $u_{\alpha,\alpha} = \phi_\alpha(q_1^2 p_1)$.

Consequently, we have proved the lemma. □

5 Properties of (TC)

We will show that (TC) holds for almost all $B_k(q_1, 0)$. Set $q_1 = t$, $B_k(t, 0) =: a(t)$ and $c := \mu_k$, and write (8) in the form

$$t^2 \frac{d}{dt} v - 2cv = a(t). \tag{47}$$

Clearly, if $a(t)$ is a constant function, then (TC) does not hold since (47) has a constant solution $v = -a(0)/(2c)$. We first prove

Proposition 2 *Suppose that $a(t)$ is a polynomial of degree $\ell \geq 1$. Then (47) has an analytic solution at $t = 0$ if and only if (47) has a polynomial solution v of degree $\ell - 1$. The set of $a(t)$ for which (47) has a polynomial solution is contained in the set of codimension one of the set of polynomials of degree ℓ.*

Remark For a given polynomial v of degree $\ell - 1$, define $a(t)$ by (47). Clearly the set of a's such that (47) has a polynomial solution is an infinite set.

Proof of Proposition 2 Let $a(t) = \sum_{j=0}^{\ell} a_j t^j$ ($a_\ell \neq 0$) and let $v(t) = \sum_{j=0}^{\infty} v_j t^j$ be the analytic solution of (47). By inserting the expansions into (47) and by comparing the powers of t we obtain

$$v_0 = -a_0/(2c), \quad v_n = (n-1)v_{n-1}/(2c) - a_n/(2c), \; n = 1, 2, \ldots \tag{48}$$

If $n > \ell$, then we have $v_n = (n-1)v_{n-1}/(2c)$. Therefore, if $v_\ell = 0$, then $v_n = 0$ for $n > \ell$. Hence v is a polynomial. On the other hand, if $v_\ell \neq 0$, then $v_n = (2c)^{\ell-n}(n-1)(n-2)\cdots \ell v_\ell$. It follows that $v(t)$ is not analytic in any neighborhood of the origin, which contradicts to the assumption. Hence v is a polynomial of degree $\ell - 1$. The converse statement is trivial.

We will show the latter half. By the recurrence formula (48), one easily sees that v_ℓ is a nontrivial linear function of $a_0, \ldots, a_\ell$. Hence the condition $v_\ell = 0$ is satisfied for a polynomial $a(t)$ on the set of codimension 1. This completes the proof. □

Example We give an example of $B_k(q_1, 0)$'s satisfying the condition (TC) in Theorem 1. We use the notation in Proposition 2. If $k \in J'$, then we look for $a(t) \equiv B_k(t, 0)$ such that $a(t) = \alpha t + \beta t^2$ for some complex constants α and β. In order to verify that (47) has no solution v being analytic at $t = 0$, we expand $v(t) = \sum_{j=0}^{\infty} v_j t^j$ and consider the recurrence relation (48). We assume that $c = \mu_k \neq 0$. Clearly, we have $v_1 = -\alpha/(2c)$ and $v_1 - 2cv_2 = \beta$. It follows that $v_2 = -(\alpha/(2c) + \beta)/(2c)$. For $n \geq 3$, we have $v_n = (n-1)v_{n-1}/(2c)$, which implies $v_n = (n-1)!(2c)^{2-n} v_2$. Therefore, if $v_2 \neq 0$, then v does not converge. Hence (47) has no analytic solution. We observe that $v_2 \neq 0$ holds if $\alpha/(2c) + \beta \neq 0$.

Next we assume $k \in J$, and we consider (9) in (TC). (9) is rewritten in (53) which follows. We look for $b(t)$ such that $b(t) = \gamma t^2 + \delta t^3$ for some complex constants γ and δ. We set $q_1 = t + 1$. Since $b(0) = 0$, we have $a(0) = 0$. Hence, by (53) we have the relation

$$a(t+1) = a(q_1) = (\gamma + \delta t)(t+1)^2 = q_1^2(\gamma - \delta + \delta q_1).$$

In order to verify (TC) we argue as in the above. We expand $w(t)$ in the series $w(t) = w_2 t^2 + w_3 t^3 + \cdots$ and we subsitutute it into (53). By comparing the powers of t^2 of both sides we have $w_2 = -\gamma/(2c)$. Similarly, we have $w_3 = -(\gamma/c + \delta)/(2c)$. If $\gamma + c\delta \neq 0$, then we have $w_3 \neq 0$ and we see that the formal power series expansion of $w(t) = w_2 t^2 + w_3 t^3 + \cdots$ diverges. Hence we have the desired property. Consequently, we choose $B_k(q_1, 0) = \alpha q_1 + \beta q_1^2$ with $\alpha/(2c) + \beta \neq 0$ for $k \in J'$, and $B_k(q_1, 0) = q_1^2(\gamma - \delta + \delta q_1)$ with $\gamma + c\delta \neq 0$ for $k \in J$. Then we see that (TC) is satisfied.

Next we study (TC) when $a(t)$ is an analytic function. By replacing $v(t)$ and $a(t)$ with $v(t) - v(0)$ and $a(t) - a(0)$, $(2cv(0) = -a(0))$, respectively, we may assume that $v(0) = 0$ and $a(0) = 0$ in (47). Then we have

Proposition 3 *The set of analytic functions $a(t)$'s at the origin such that (47) has an analytic solution v is contained in the set of codimension 1 of the set of germs of analytic functions at $t = 0$.*

Proof Let v be the analytic solution of (47) at $t = 0$. Set $v(t) = t\tilde{v}(t)$ and $a(t) = t\tilde{a}(t)$. Then

$$t^2 \frac{d}{dt}\tilde{v} + t\tilde{v} - 2c\tilde{v} = \tilde{a}(t). \tag{49}$$

We make the (formal) Borel transform $\mathcal{B}(\tilde{v})$ to (49)

$$\mathcal{B}(\tilde{v})(z) \equiv \widehat{\tilde{v}}(z) := \sum_{n=1}^{\infty} v_n \frac{z^{n-1}}{(n-1)!}. \tag{50}$$

Because $\tilde{v}(t)$ and $\tilde{a}(t)$ are analytic at $t = 0$, it follows that $\mathcal{B}(\tilde{v})(z)$ and $\mathcal{B}(\tilde{a})(z)$ are entire functions of exponential type of order 1. Recalling that $\mathcal{B}\left((t^2\frac{d}{dt} + t)\tilde{v}\right)(z) = z\mathcal{B}(\tilde{v})(z)$ we have

$$(z - 2c)\mathcal{B}(\tilde{v}) = \mathcal{B}(\tilde{a})(z). \tag{51}$$

It follows that

$$\mathcal{B}(\tilde{a})(2c) = 0. \tag{52}$$

This shows that the germ $\{a_n\}_{n=1}^{\infty}$ of $a(t)$ at $t = 0$ is contained in the hyperplane. This ends the proof. □

Next we consider (9) in (TC). We set $t = q_1 - 1$, $a(t+1) := B_k(t+1, 0)$, $c = \mu_k$ and $a(0) = B_k(0, 0)$. Then (9) can be written in

$$\left(t^2\frac{d}{dt} - 2c\right)w = \frac{t^2}{(t+1)^2}a(t+1) + \frac{a(0)}{(t+1)^2}(t^2 + c(t+1)) =: b(t). \tag{53}$$

This equation has the same form as (47). We determine $w(0)$ by $-2cw(0) = b(0)$. If we make the appropriate change of unknown functions w and b as before, one may assume that $w(0) = 0$ and $b(0) = 0$. In view of the definition of $b(t)$ we have $ca(0) = 0$. Hence we have $a(0) = 0$. It follows that $b(t) = t^2 a(t+1)/(t+1)^2$. In the following we assume $w(0) = 0$ and $a(0) = 0$. Then we have

Proposition 4 *Suppose that $a(t)$ is holomorphic in a connected domain containing $t = 0$ and $t = 1$. Then the set of $a(t)$ for which (53) has an analytic solution is contained in the set of codimension one of the set of germs of analytic functions at $t = 0$.*

Proof Let $w(t)$ be an analytic solution of (53) at $t = 0$. We set $\alpha := a'(0)$ and $a(z) = \alpha z + A(z)z^2$ for some analytic function $A(z)$. Then, by the general formula w is given by

$$w = \exp\left(-\frac{2c}{t}\right)\left(K + \int_{\tau}^{t}\exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right)ds\right), \tag{54}$$

where K and $\tau \neq 0$ are some constants. We take a smooth curve γ which connects τ and the origin such that it stays in the half space, $\Re(c/t) < 0$ near the origin. Then the limit

$$\begin{aligned}&\int_{\tau}^{0}\exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right)ds \\ &:= \lim_{t\in\gamma, t\to 0}\int_{\tau}^{t}\exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right)ds\end{aligned} \tag{55}$$

exists and it is a non-constant analytic function of τ. If the condition

$$K + \int_{\tau}^{0}\exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right)ds \neq 0 \tag{56}$$

holds, then, by taking the limit $t \to 0$, $\Re(c/t) < 0$ in (54) we see that $w(t)$ tends to infinity, which contradicts to the analyticity of w at the origin. Hence we have

$$K = \int_{0}^{\tau}\exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right)ds. \tag{57}$$

By substituting (57) to (54) we have

$$w(t) = \exp\left(-\frac{2c}{t}\right)\int_{0}^{t}\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right)ds. \tag{58}$$

We take t sufficiently close to the origin such that the Taylor expansion $A(s+1) = \sum_{n=0}^{\infty} a_n s^n$ converges for $|s| \leq |t|$. Because $w(te^{2\pi i}) = w(t)$ holds by the analyticity of w, it follows that

$$\int_t^{te^{2\pi i}} \exp\left(\frac{2c}{s}\right)\left(\frac{\alpha}{s+1} + A(s+1)\right) ds = 0. \tag{59}$$

By calculating the residue we have $\int_t^{te^{2\pi i}} \exp\left(\frac{2c}{s}\right) \frac{\alpha}{s+1} ds = 2\pi i\alpha(1 - e^{-2c})$. The non-resonance condition implies $c = \mu_k \neq 0$, and hence $1 - e^{-2c} \neq 0$. Hence, by (59) the germ of $A(z)/\alpha$ at $z = 1$ (in case $\alpha \neq 0$) or that of $A(z)$ at $z = 1$(in case $\alpha = 0$) is contained in some hyperplane of the set of germs of analytic functions.

We recall that $A(z)$ is analytic in some domain containing $z = 0$ and $z = 1$. We will show that by the analytic continuation from $z = 1$ to $z = 0$ the germ of $A(z)$ at $z = 1$ is transformed to that of $A(z)$ at $z = 0$ by an infinite matrix. If we can prove this, then the germ of $A(z)$ or $A(z)/\alpha$ at $z = 0$ is contained in some hyperplane. In view of $a(z) = \alpha z + A(z)z^2$, the germ of $a(z)$ at $z = 0$ is contained in some hyperplane.

We take a rectifiable curve which connects $z = 1$ and $z = 0$. First we consider the analytic continuation from $z = 1$ to $z = z_0$, where z_0 is contained in the disk centered at $z = 1$ in which $A(z)$ is analytic. Let $A(z) = \sum_{n=0}^{\infty} a_n(z-1)^n$ be the expansion at $z = 1$. Then the Taylor expansion of $A(z)$ at $z = z_0$ is given by

$$\sum_{k=0}^{\infty} \frac{(z-z_0)^k}{k!} \sum_{n=k}^{\infty} a_n(z_0-1)^{n-k} \frac{n!}{(n-k)!}. \tag{60}$$

It follows that the germ at $z = z_0$ is given by

$$\left(\sum_{n=k}^{\infty} a_n \binom{n}{k} (z_0-1)^{n-k}\right)_{k=0}^{\infty}. \tag{61}$$

Hence the germ at $z = 1$ is transformed to the one in (60) by the infinite matrix

$$\mathcal{A} := \left((z_0-1)^{n-k}\binom{n}{k}\right)_{k\downarrow 0,1,\ldots;n\rightarrow 0,1,\ldots}, \tag{62}$$

where we set the (k, n)-component $(k > n)$ to be zero. Note that if $|z_0 - 1|$ is sufficiently small, then $\mathcal{A}$ defines a continuous linear operator on the space of sequences with an appropriate norm. Therefore, if the germ of $A(z)$ at $z = 1$ is contained in the hyperplane, then the germ of $A(z)$ at $z = z_0$ is contained in some hyperplane. By finite times of analytic continuation we see that the germ of $A(z)$ at $z = 0$ is contained in some hyperplane. This completes the proof. □

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