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Lattice-type self-similar sets with pluriphase generators fail to be Minkowski measurable

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Abstract A long-standing conjecture of Lapidus states that under certain conditions, self-similar fractal sets fail to be Minkowski measurable if and only if they are of lattice type. It was shown by Falconer and Lapidus (working independently but both using renewal theory) that nonlattice self-similar subsets of $\mathbb{R}$ are Minkowski measurable, and the converse was shown by Lapidus and v. Frankenhuijsen a few years later, using complex dimensions. Around that time, Gatzouras used renewal theory to show that nonlattice self-similar subsets of $\mathbb{R}^d$ that satisfy the open set condition are Minkowski measurable for $d \geq 1$. Since then, much effort has been made to prove the converse. In this paper, we prove a partial converse by means of renewal theory. Our proof allows us to recover several previous results in this regard, but is much shorter and extends to a more general setting; several technical conditions appearing in previous work have been removed.

Keywords Self-similar set · Lattice and nonlattice case · Minkowski dimension · Minkowski measurability · Minkowski content

Mathematics Subject Classification Primary 28A75 · 28A80; Secondary 28A12

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1 Introduction

We address the Minkowski measurability (see Def. 2.1) for self-similar sets in $\mathbb{R}^d$. In particular, we attempt to characterize this property in terms of the lattice properties of the underlying iterated function system (IFS). Let $S = \{S_1, \ldots, S_N\}$ with $N \geq 2$ denote an IFS in which each S_i is a contractive similarity acting on $\mathbb{R}^d$, called a *self-similar system* in the sequel. Further, let $F \subseteq \mathbb{R}^d$ denote the self-similar set which is the unique non-empty compact set satisfying $F = \bigcup_{i=1}^N S_i F$; see [12]. If the scaling ratio of S_i is denoted by r_i, then the IFS is said to be *lattice* if there is an $r > 0$ such that each r_i can be written as r^{k_i} for some integer $k_i \in \mathbb{N}$ (see Def. 2.13), otherwise, the IFS is said to be *nonlattice*.

For $d = 1$, it was established independently by Falconer [7] and Lapidus [19] that self-similar sets generated by a nonlattice IFS (satisfying the strong separation condition, SSC) are Minkowski measurable. It was pointed out in [19] that SSC may be replaced by the weaker open set condition (OSC), see Def. 2.2. In both papers renewal theory arguments are used. The converse, that F is non-Minkowski measurable if the IFS is lattice, requires two more conditions, namely that the Minkowski dimension D of F is strictly less than 1, and that the OSC is satisfied with the interior of the convex hull of F as a feasible open set. Under these conditions the converse was established by Lapidus and van Frankenhuijsen in [20] by means of complex dimensions. Using a symbolic renewal theorem of Lalley [18], this result was recovered in [13], where the condition on the feasible open set was weakened. It was conjectured in [19, Conj. 3] that the equivalence statement should remain true for $d \geq 2$, when $d - 1 < D < d$. In [11], Gatzouras was able to prove and strengthen (under the OSC) one direction of this conjecture, namely, that for arbitrary $d \in \mathbb{N}$ and $D \in (0, d)$, the self-similar attractor $F \subset \mathbb{R}^d$ is Minkowski measurable when the IFS is nonlattice. It is an open problem to prove the converse, and this has been a very active area; see, for example, [3,16,22–24,28]. Our results in this paper give some further progress towards establishing the converse, i.e., showing that the attractor of a lattice self-similar system is not Minkowski measurable. At the same time we demonstrate that it is essential to exclude sets of integer Minkowski dimension D from the conjecture but that it is plausible to extend Lapidus' conjecture to the setting of non-integer $D \in (0, d)$; see Remark 1.2.

We work in a setting which includes—to the best of our knowledge—all the previous cases in which the Minkowski measurability of lattice self-similar sets has been addressed (see the detailed discussion at the very end of the introduction) and which extends the class of sets covered in several directions. For instance, we do not require the set F to possess a *compatible* feasible open set O satisfying the OSC, that is, one which satisfies bd $O \subset F$. This allows in particular to treat self-similar sets of any Minkowski dimension D and removes the assumption $D > d-1$, which is present in all previous work known to the authors. Instead of compatibility, we will assume throughout that the feasible open set O we work with satisfies the following additional conditions:

– (Strong OSC) $O \cap F \neq \varnothing$;
– (Projection condition) $S_i O \subseteq \overline{\pi_F^{-1}(S_i F)}$ for $i = 1, \ldots, N$.

Here π_F denotes the metric projection onto F (see Def. 2.7) and $\overline{A}$ denotes the closure of $A \subset \mathbb{R}^d$. It follows from results in [1] that one can always find a feasible open set satisfying both the strong OSC (SOSC) and the projection condition whenever OSC is satisfied (the "central open set"; see Remark 2.8). Therefore, these two conditions alone do not restrict at all the class of self-similar sets considered; they should be seen as a convenient choice of feasible open set we make in order to simplify the problem. The only further (rather restrictive) assumption we require is the following. We suppose that the ε-parallel set F_ε of F (see (2.1)) is well-behaved in the set

$$\Gamma = \Gamma(O) := O \setminus \bigcup_{i=1}^{N} S_i O. \tag{1.1}$$

More precisely, it is required that the parallel volume $\lambda_d(F_\varepsilon \cap \Gamma)$ of F restricted to the set Γ (where λ_d is Lebesgue measure) is piecewise polynomial in the variable ε. In this case we call the set F *pluriphase* with respect to Γ (and we call F *monophase* with respect to Γ if this parallel volume is a polynomial), see Def. 2.9 and the discussion afterwards for details. The pluriphase condition is a simplifying assumption on the geometry of F which would ideally be removed in future work. Note: the definitions of pluriphase and monophase are extended here to the present more general setting. In case of a compatible feasible set they reduce to the pluriphase/monophase assumption made in earlier work on this topic, e.g. in [3,16,24].). See also Remark 2.11.

The main results of this paper are summarized in the following statement.

Theorem 1.1 *Let $F \subset \mathbb{R}^d$ be a self-similar set which is the attractor of a lattice self-similar system $S = \{S_1, \ldots, S_N\}$, $N \geq 2$ satisfying the OSC. Let $D := \dim_{\mathcal{M}} F$ denote its Minkowski dimension.*

- (i) *If $D = \dim \operatorname{aff} F$ (where the latter is the dimension of the affine hull of F), then F is Minkowski measurable. In particular, this is true for $D = d$.*
- (ii) *Suppose $D < d$ is not an integer and there exists a strong feasible set O satisfying the projection condition such that F is pluriphase with respect to the set $\Gamma(O)$. Then F is not Minkowski measurable.*
- (iii) *Suppose $D < d$ is an integer and there exists a strong feasible set O satisfying the projection condition such that F is pluriphase with respect to the set $\Gamma(O)$. Then F is Minkowski measurable if and only if certain algebraic relations involving the data of the pluriphase representation are satisfied. In particular, these relations are never satisfied in the case when F is monophase with respect to $\Gamma(O)$.*

Part (ii) is reformulated and proved in Thm 3.4; a more precise formulation of part (iii), the case when D is an integer, is given in Thm 3.6. We stress that in the situation of part (iii) both cases are possible: lattice sets in $\mathbb{R}^d$ of integer Minkowski dimension $D < d$ can be Minkowski measurable or not; see Remark 1.2.

As for part (i), we can give a short proof immediately.

Proof of Theorem 1.1(i) As a function of ε, the tubular volume $\lambda_d(F_\varepsilon)$ is continuous and strictly increasing on $(0, \infty)$ for any compact set $F \subseteq \mathbb{R}^d$, and thus $\mathcal{M}_d(F) := \lim_{\varepsilon \searrow 0} \lambda_d(F_\varepsilon) = \lambda_d(F)$; see Def. 2.1 for the definition of the Minkowski content $\mathcal{M}_d$. In other words, the limit must exist and it is not difficult to see that it coincides with $\lambda_d(F)$. Now, if F is a self-similar set with $\dim_{\mathcal{M}} F = d$ satisfying OSC, then it is well known that F has interior points (see e.g. [27,30]) and therefore $\mathcal{M}_d(F) = \lambda_d(F) > 0$. Thus F is Minkowski measurable as claimed. Now, if $F \subseteq \mathbb{R}^d$ is a self-similar set such that $\dim_{\mathcal{M}} F = \dim \operatorname{aff} F$, then Minkowski measurability follows from working in the affine hull and observing that Minkowski measurability is independent of the dimension of the ambient space; cf. [14,29] and the references therein. □

Remark 1.2 The above proof indicates that if $\dim_{\mathcal{M}} F = \dim \operatorname{aff} F$, then F is Minkowski measurable regardless of whether it is lattice or nonlattice. The significance of this point is as follows: one may be naturally led to suppose that the original conjecture of Lapidus may be extended to include sets with Minkowski dimension $D \in (0, d)$ (instead of requiring $D \in (d-1, d)$, but Theorem 1.1(i) shows there are (somewhat trivial) counterexamples with $\dim_{\mathcal{M}} F = \dim \operatorname{aff} F$. For the case when $D < d$ is an integer (as in Theorem 1.1(iii)), a class of examples of Minkowski measurable lattice sets is discussed in Sect. 4, Example 4.1. At this point, all known examples of this type can be represented as the embedding in $\mathbb{R}^d$ of a self-similar set in $\mathbb{R}^D$. It may well be that this is the only way such a thing is possible.

The proofs of the other parts of Theorem 1.1 are obtained using the elementary tools of probabilistic renewal theory. These are combined with recent results in [32], where the projection condition was observed to be essential for deriving renewal equations in terms of the generator of an associated tiling. Based on the renewal theorem, we obtain in Theorem 3.1 and Corollary 3.2 a characterization of Minkowski measurability in terms of a periodic function p (see (3.2)) which can be expressed completely in terms of the parallel volume $\lambda_d(F_\varepsilon \cap \Gamma)$ of F restricted to Γ. This result may be of independent interest for future work as it does not require the pluriphase condition and thus applies to all (nontrivial) self-similar sets. We use this result to prove our main results, the non-Minkowski measurability in the case when F is pluriphase w.r.t. the set Γ, by exploiting and refining an idea in [16].

Before moving on to the results, we explain in more detail the improvements obtained here compared to previous results from [3,16,24]. The assumptions of [16] and [24] only differ slightly, and combining [16, Theorem 2.38] with [24, Theorem 5.4] the resulting nonmeasurability result applies to the case when the following requirements are met:

(a) The open set condition (see Def. 2.2) holds for a feasible open set O with $\operatorname{bd} O \subseteq F$, which in particular implies $d - 1 < D < d$.
(b) The D-dimensional outer Minkowski content of $\overline{O}$ is finite (see [24, Def. 5.2]).
(c) The *generator* $G = O \setminus \bigcup_{i=1}^{N} \overline{S_i O}$ (see Def. 2.5) has only finitely many connected components.
(d) Each connected component of the generator G is *monophase* (see Def. 2.9).

In our main result, Theorem 1.1, we remove (a)–(c), and replace (d) with the more general condition that F is pluriphase w.r.t. Γ. The significance of removing (a) is that the results of the present paper (in particular, Theorem 1.1) cover sets of any Minkowski dimensions $D \leq d$. Other notable improvements of the present article are the comparatively shorter and simpler proofs based on probabilistic renewal theory. In [24], the more powerful (but also more complicated) apparatus of *fractal sprays and complex dimensions* was used; the approach taken in [16] uses renewal theory in symbolic dynamics (motivated by [18]) yielding results for the more general class of self-conformal sets. When the renewal theorem for symbolic dynamics is restricted to the self-similar setting, it boils down to the probabilistic renewal theorem (as used in [17]) which we apply directly here. This direct application makes the proofs significantly shorter and simpler. It should be noted that under the additional assumption that O coincides with the interior of the convex hull of F, a result similar to [24, Theorem 5.4]/[16, Theorem 2.38] was independently proven in [3] using Mellin transforms. See [4–6] for further interesting and related results.

The structure of the article is as follows. In Sect. 2 we lay the foundations for stating and proving our main results in Sect. 3. The final section, Sect. 4, is devoted to examples demonstrating our findings.

2 Preliminaries

We present the terminology required to state and prove our main theorems.

2.1 Minkowski measurability

Let A be a compact subset in Euclidean space $\mathbb{R}^d$ and $\varepsilon \geq 0$. The *ε-parallel set* of A (or *ε-tubular neighborhood* of A) is

$$A_\varepsilon := \left\{x \in \mathbb{R}^d : d(x, A) \leq \varepsilon\right\}, \tag{2.1}$$

where $d(x, A) := \inf\{\|x - a\| : a \in A\}$ is the Euclidean distance of x to the set A.

A *tube formula* for A is an explicit formula for $\lambda_d(A_\varepsilon)$, as a function of ε, where λ_d denotes the d-dimensional Lebesgue measure; see [22–24] for a discussion of fractal tube formulas. The volume $\lambda_d(A_\varepsilon)$ is referred to as the *ε-parallel volume* of A and we call $\lambda_d(A_\varepsilon \cap B)$ the *ε-parallel volume of A inside B* for any Borel set $B \subseteq \mathbb{R}^d$.

Definition 2.1 Let A be a compact subset of Euclidean space $\mathbb{R}^d$. For $0 \leq \alpha \leq d$, we denote by

$$\mathcal{M}_\alpha(A) := \lim_{\varepsilon \to 0} \varepsilon^{\alpha - d} \lambda_d(A_\varepsilon) \tag{2.2}$$

the *α-dimensional Minkowski content* of A whenever this limit exists (as a value in $[0, \infty]$). If $\mathcal{M}_\alpha(A)$ exists and satisfies $0 < \mathcal{M}_\alpha(A) < \infty$, then A is called *Minkowski measurable (of dimension α)*, and $\dim_{\mathcal{M}} A := \alpha$ is the *Minkowski dimension* of A. If the limit in (2.2) does not exist, one may consider the logarithmic Cesàro average known as the *(α-dimensional) average Minkowski content* (which always exists in the case of self-similar sets A, see [11]). It is defined by

$$\overline{\mathcal{M}}_\alpha(A) := \lim_{\delta \to 0} \frac{1}{|\ln \delta|} \int_\delta^1 \varepsilon^{\alpha - d} \lambda_d(A_\varepsilon) \frac{\mathrm{d}\varepsilon}{\varepsilon}.$$

whenever this limit exists.

2.2 Self-similar tilings and their generators

Let $S = \{S_1, \ldots, S_N\}$, $N \geq 2$ be an iterated function system (IFS), where each S_i is a similarity mapping of $\mathbb{R}^d$ with scaling ratio r_i, where $0 < r_i < 1$. Then we call S a *self-similar system*. For $A \subseteq \mathbb{R}^d$, we write

$$\mathbf{S}A := \bigcup_{i=1}^N S_i(A). \tag{2.3}$$

The *self-similar set F* generated by the IFS S is the unique compact and nonempty solution of the fixed-point equation $F = \mathbf{S}F$; cf. [12], also called the *attractor* of S.

We study the parallel volume of the attractor by studying the parallel volume inside a certain tiling of its complement, which is constructed via the IFS as described below. The tiling construction was introduced in [26] and developed in [27], where tilings by open sets were studied; see also [22–25]. In this paper we consider self-similar tilings in a generalized sense, with the tiles not necessarily being open (see Def. 2.5). The construction of a self-similar tiling requires the IFS to satisfy the *open set condition* and a *nontriviality condition*, as described in the following two definitions.

Definition 2.2 A self-similar system $S = \{S_1, \dots, S_N\}$ satisfies the *open set condition* (OSC) if and only if there is a nonempty open set $O \subseteq \mathbb{R}^d$ such that

$$\begin{aligned} &S_i(O) \subseteq O, && i = 1, \dots, N \quad \text{and} \\ &S_i(O) \cap S_j(O) = \varnothing, && i \neq j. \end{aligned} \tag{2.4}$$

In this case, O is called a *feasible open set* for $\{S_1, \dots, S_N\}$; see [1,9,12]. If additionally $O \cap F \neq \varnothing$, then O is called a *strong* feasible open set.

It was shown in [30] that if a self-similar system satisfies OSC, then it possesses a strong feasible open set.

Definition 2.3 A self-similar set F, which is the attractor of a self-similar system $S = \{S_1, \dots, S_N\}$ satisfying OSC, is said to be *nontrivial* if there exists a feasible open set O such that

$$O \not\subseteq \overline{\mathbf{S}O}, \tag{2.5}$$

where $\overline{\mathbf{S}O}$ denotes the closure of $\mathbf{S}O$; otherwise, F is called *trivial*.

This condition is needed to ensure that the set $\Gamma = O \backslash \mathbf{S}O$ in Def. 2.5 has nonempty interior. It turns out that nontriviality is independent of the particular choice of the set O. It is shown in [27] that F is trivial if and only if it has nonempty interior, which amounts to the following characterization of nontriviality:

Proposition 2.4 ([27, Corollary 5.4]) *Let $F \subseteq \mathbb{R}^d$ be a self-similar set which is the attractor of a self-similar system satisfying OSC. Then F is nontrivial if and only if F has Minkowski dimension strictly less than d.*

Unless explicitely stated otherwise, all self-similar sets considered here are assumed to be nontrivial, and the discussion of a self-similar tiling $\mathcal{T}$ implicitly assumes that the corresponding attractor F is nontrivial and that the corresponding system satisfies OSC.

Denote the set of all finite *words* formed by the alphabet $\{1, \dots, N\}$ by

$$\mathcal{W} := \bigcup_{k=0}^{\infty} \{1, \dots, N\}^k . \tag{2.6}$$

For any word $w = w_1 w_2 \cdots w_n \in \mathcal{W}$, let $r_w := r_{w_1} \cdot \dots \cdot r_{w_n}$ and $S_w := S_{w_1} \circ \dots \circ S_{w_n}$. In particular, if $w \in \mathcal{W}$ is the *empty word*, then $r_w = 1$ and $S_w = \mathrm{Id}$.

Definition 2.5 Let O be a feasible open set for $\{S_1, \dots, S_N\}$. The *self-similar tiling $\mathcal{T}(O)$* associated with the IFS $\{S_1, \dots, S_N\}$ is the collection of open sets

$$\mathcal{T}(O) := \{S_w(G) \colon w \in \mathcal{W}\}, \tag{2.7}$$

where the open set $G := O \backslash \overline{\mathbf{S}O}$ is called the *generator* of the tiling. We call the tiling $\widetilde{\mathcal{T}}(O) := \{S_w(\Gamma) \colon w \in \mathcal{W}\}$ generated by

$$\Gamma = \Gamma(O) := O \backslash \mathbf{S}O, \tag{2.8}$$

a *self-similar tiling in a generalized sense*, with the tiles not necessarily being open.

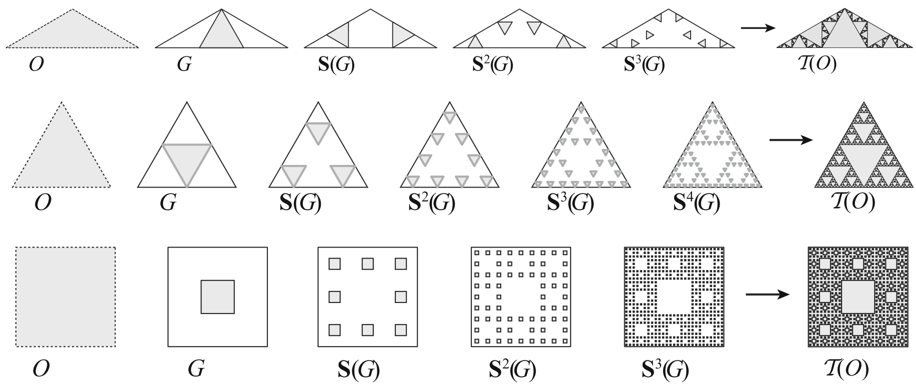

Fig. 1 From top to bottom: a Koch curve tiling, a Sierpinski gasket tiling, and a Sierpinski carpet tiling. In each of these examples, the set O is the interior of the convex hull of F. In case of Sierpinski gasket and Sierpinski carpet, the set F is monophase w.r.t. Γ (see Def. 2.9), while this is not true in case of the Koch curve. The Koch curve tiling does not satisfy the compatibility criterion bd $O \subseteq F$ but the other two examples do

Remark 2.6 Self-similar tilings generated by G were introduced in [25–27] and further studied in [2,3,6,15,21–24]. The nomenclature stems from the fact (proved in [27, Theorem 5.7]) that $\mathcal{T}(O)$ is an *open tiling* of O in the sense that

$$\overline{O} = \overline{\bigcup_{w \in \mathcal{W}} S_w(G)}, \tag{2.9}$$

where the *tiles* $S_w(G)$ are pairwise disjoint open sets.

We will find it more useful to work in terms of the set Γ instead of G for most of the sequel. Observe that $G \subseteq \Gamma$ and that $\Gamma \setminus G \subset \bigcup_i \operatorname{bd} S_i O$. If F is assumed to be nontrivial, then the set Γ has nonempty interior and we let

$$g := \sup\{d(x, F) \colon x \in \Gamma\} = \sup\{d(x, F) \colon x \in G\} \tag{2.10}$$

denote the maximal distance of a point in Γ to F. The reason for the use of Γ is that Lebesgue measure is not stable with respect to the closure operation: one may have $\lambda_d(U) < \lambda_d(\overline{U})$ for an open set $U \subseteq \mathbb{R}^d$. We remark that

$$O = \Gamma \cup \mathbf{S}O = \Gamma \cup \mathbf{S}\Gamma \cup \mathbf{S}^2 O = \cdots = \bigcup_{k=0}^{n} \mathbf{S}^k \Gamma \cup \mathbf{S}^{n+1} O, \tag{2.11}$$

where all the unions are disjoint, and hence (2.9) implies

$$\overline{O} = \overline{\bigcup_{w \in \mathcal{W}} S_w \Gamma}. \tag{2.12}$$

Therefore, $\widetilde{\mathcal{T}}(O)$ from Def. 2.5 gives a tiling of O, where the tiles $S_w \Gamma$ are pairwise disjoint but not necessarily open, justifying the term self-similar tiling in a generalized sense. Also, (2.11) allows for the following nice decomposition of the ε-parallel volume of the attractor which is used in the proof of Theorem 3.1:

$$\lambda_d(F_\varepsilon) = \sum_{i=1}^{N} \lambda_d(F_\varepsilon \cap S_i O) + \lambda_d(F_\varepsilon \cap \Gamma) + \lambda_d(F_\varepsilon \setminus O). \tag{2.13}$$

This representation is particularly useful for sets O satisfying the projection condition.

Definition 2.7 For a compact set $A \subseteq \mathbb{R}^d$, we let π_A denote the *metric projection* onto A. It is defined on the set of points $x \in \mathbb{R}^d$ which have a unique nearest neighbour y in A by $\pi_A(x) = y$. Let O be a feasible open set of the self-similar system $\{S_1, \ldots, S_N\}$ with attractor F. Then O is said to satisfy the *projection condition* if

$$S_i O \subseteq \overline{\pi_F^{-1}(S_i F)} \quad \text{for } i = 1, \ldots, N. \tag{2.14}$$

If the projection condition is satisfied then

$$F_\varepsilon \cap S_i O = (S_i F)_\varepsilon \cap S_i O \tag{2.15}$$

for each $\varepsilon > 0$ and $i = 1, \ldots, N$ (see [32, Lem. 3.19]).

Remark 2.8 The *central open set* is a particular choice of feasible open set that exists for any IFS satisfying the OSC; it is defined and studied in [1]. It is rather easy to see that the central open set will always satisfy the projection condition. It is also clear that F is contained in the central open set, so the *strong* OSC is also automatically satisfied. Therefore, it is always possible to find a strong feasible open set which satisfies the projection condition, as long as the OSC holds. A proof of these facts is given in [32, Prop. 3.17].

Definition 2.9 For a given IFS and a fixed feasible open set O, we call the attractor F *pluriphase with respect to the set* $\Gamma = \Gamma(O)$ (as defined in (2.8)) if and only if there exists a finite partition of the interval $(0, \infty)$ with partition points $0 =: a_0 < a_1 < \cdots < a_{M-1} < a_M := g$ such that, for $\varepsilon > 0$,

$$\lambda_d(F_\varepsilon \cap \Gamma) = \sum_{m=1}^{M} \mathbb{1}_{(a_{m-1}, a_m]}(\varepsilon) \sum_{k=0}^{d} \kappa_{m,k} \varepsilon^{d-k} + \mathbb{1}_{(g,\infty)}(\varepsilon) \lambda_d(\Gamma), \tag{2.16}$$

for some constants $\kappa_{m,k} \in \mathbb{R}$, where $\mathbb{1}$ denotes a characteristic function and g is as in (2.10). We assume that the representation in (2.16) is given with M minimal, so that for each $m = 1, \ldots, M$, there exists a $k \in \{0, \ldots, d\}$ with $\kappa_{m,k} \neq \kappa_{m-1,k}$. Imposing minimality of M, we call F *monophase with respect to* Γ if and only if $M = 1$ in the above representation.

Remark 2.10 At the time of writing, there is no known characterization of the pluriphase or monophase conditions in terms of the self-similar system $\{S_i\}_{i=1}^N$. However, it is known from [15] that a convex polytope in $\mathbb{R}^d$ is monophase (with Steiner-like function of class C^{d-1}) iff it admits an inscribed d-dimensional Euclidean ball (i.e., a d-ball tangent to each facet). This includes regular polygons in $\mathbb{R}^2$ and regular polyhedra in $\mathbb{R}^d$, as well as all triangles and higher-dimensional simplices. Furthermore, it was recently shown in [15] that (under mild conditions), any convex polyhedron in $\mathbb{R}^d$ ($d \geq 1$) is pluriphase, thereby resolving in the affirmative a conjecture made in [21–23]. We refer to [15] for further relevant interesting results.

Remark 2.11 In [21,22] the notions monophase, pluriphase and the symbol "g" were introduced for the generator of a self-similar tiling satisfying the compatibility condition bd $O \subseteq F$. The reader should be aware that these terms in the present paper only match previous usage in the literature for the case when bd $O \subseteq F$. In this context, the upper endpoint g of the relevant interval in Def. 2.9 was defined as the *inradius* $\tilde{g}$ of G, i.e., the maximal radius of an open metric ball contained in the set G. Moreover, the generator G (as a set) was called monophase if $\lambda_d(G_{-\varepsilon})$ is polynomial in ε for $\varepsilon \in (0, \tilde{g})$, where

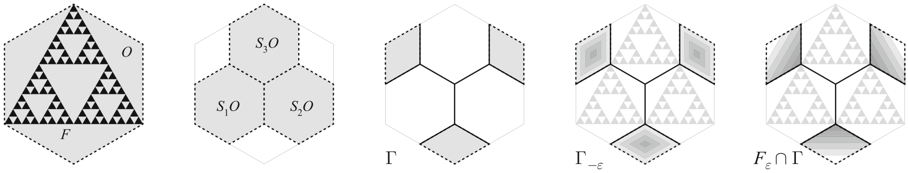

Fig. 2 A Sierpinski gasket tiling alternative to the one from Fig. 1. Here, O is not the interior of the convex hull of F, but rather the *central open set* discussed in [1]. The set F is pluriphase (but not monophase) w.r.t. Γ, while the set Γ (as a set) is monophase. At right, the sets $\Gamma_{-\varepsilon} := (\text{bd}\,\Gamma)_\varepsilon \cap \Gamma$ and $F_\varepsilon \cap \Gamma$ are shown for several values of ε. For this example, $g = (2\sqrt{3})^{-1}$ and $\tilde{g} = 1/8$; see Remark 2.11. This example will be further studied in Sect. 4, Example 4.2

$G_{-\varepsilon} := \{x \in G : d(x, G^c) \leq \varepsilon\}$ and pluriphase if $\lambda_d(G_{-\varepsilon})$ is piecewise polynomial in ε for $\varepsilon \in (0, \tilde{g})$. However, Fig. 2 shows an example where $g \neq \tilde{g}$ and where G (as a set) is monophase but F is not monophase w.r.t. Γ. We return to this example in Sect. 4, Example 4.2. It is clear that the inradius and the notions mono- and pluriphase for sets are not the proper concept for the situation where bd $O \nsubseteq F$. It is also clear from this observation that (2.10) and Def. 2.9 are the natural extensions to the present (more general) setting.

Remark 2.12 Some examples of self-similar tilings associated to familiar fractal sets are shown in Fig. 1. In each case, there is a connected monophase generator. In Fig. 2 an alternative tiling associated with the Sierpinski gasket is provided. Here, the generator is not connected.

2.3 Lattice, nonlattice and the renewal theorem

Definition 2.13 Consider a family of similarity mappings $S = \{S_1, \ldots, S_N\}$, and let r_i denote the scaling ratio of S_i. The family is said to be of *lattice type* iff there is an $r > 0$ such that each scaling ratio r_i can be written as $r_i = r^{k_i}$ for some integer k_i, and to be of *nonlattice type* otherwise. There is a smallest number $r > 0$ for which the aforementioned representation can be found. We always use this minimal r, and we say that S is *lattice with base r*.

An extended discussion of the implications of the lattice/nonlattice dichotomy may be found in [20, Theorem 3.6]. The lattice/nonlattice dichotomy also appears in probabilistic renewal theory, where the usual nomenclature is "arithmetic/non-arithmetic". For more details, see [10, Section XIII], [8, Section 7] or [31, §4].

For use in the sequel, we include here a version of the renewal theorem formulated for a discrete probability distribution $\sum_{i=1}^N p_i \delta_{y_i}$, where δ_y is a point mass (Dirac measure) concentrated at $y \in \mathbb{R}$. This theorem will be applied to the distribution

$$\sum_{i=1}^{N} r_i^D \delta_{y_i}, \tag{2.17}$$

where D is the *similarity dimension* of F. The similarity dimension is the unique positive real number α that satisfies the Moran equation $r_1^\alpha + r_2^\alpha + \cdots + r_N^\alpha = 1$, i.e., the unique $D > 0$ that makes (2.17) into a discrete probability distribution.

Theorem 2.14 (Renewal Theorem, see [8, Corollary 7.3] or [31, §4]) *Let $p_1, \ldots, p_N \in (0, 1)$ satisfy $\sum_{i=1}^N p_i = 1$, and let $y_1, \ldots, y_N > 0$. Let $z : \mathbb{R} \to \mathbb{R}$ be a function with a* *discrete set of discontinuities which satisfies*

$$|z(t)| \leq c_1 \,\mathrm{e}^{-c_2|t|}, \qquad \textit{for all } t \in \mathbb{R}, \tag{2.18}$$

for some constants $0 < c_1, c_2 < \infty$. *Also, let* $Z : \mathbb{R} \to \mathbb{R}$ *be the unique solution of the renewal equation*

$$Z(t) = z(t) + \sum_{i=1}^{N} p_i Z(t - y_i) \tag{2.19}$$

which satisfies $\lim_{t\to-\infty} Z(t) = 0$. *Then the following holds:*

(i) If $\{y_1, \dots, y_N\} \subseteq h \cdot \mathbb{Z}$ *and* $h > 0$ *is maximal as such, then*

$$Z(t) \sim \frac{h}{\eta} \sum_{\ell\in\mathbb{Z}} z(t - \ell h), \qquad \textit{as } t \to \infty. \tag{2.20}$$

(ii) If there does not exist $h > 0$ *such that* $\{y_1, \dots, y_N\} \subseteq h \cdot \mathbb{Z}$, *then*

$$\lim_{t\to\infty} Z(t) = \frac{1}{\eta} \int_{-\infty}^{\infty} z(\tau)\, \mathrm{d}\tau. \tag{2.21}$$

Here, $\eta := \sum_{i=1}^{N} y_i p_i$. *Moreover, in both cases, we have*

$$\lim_{T\to\infty} \frac{1}{T} \int_0^T Z(t)\, \mathrm{d}t = \frac{1}{\eta} \int_{-\infty}^{\infty} z(t)\, \mathrm{d}t. \tag{2.22}$$

In (2.20), the notation $g \sim f$, as $t \to \infty$, means that g is *asymptotic to* f as $t \to \infty$ in the sense that for any $\delta > 0$, there is a number $s = s(\delta)$ such that $g(t)$ lies between $(1-\delta)f(t)$ and $(1+\delta)f(t)$ for all $t \geq s$.

Remark 2.15 If the self-similar system $\{S_1, \dots, S_N\}$ is lattice, then there exist $r > 0$ and $k_i \in \mathbb{N}$ such that $r_i = r^{k_i}$, where r_i denotes the scaling ratio of S_i for $i = 1, \dots, N$. In this case $\{-\ln r_1, \dots, -\ln r_N\} = \{-k_1 \ln r, \dots, -k_N \ln r\} \subseteq -\ln r \cdot \mathbb{Z}$ and thus, we are in case (i) of the renewal theorem. On the other hand, if $\{S_1, \dots, S_N\}$ is nonlattice, then we are in case (ii) of the renewal theorem.

Remark 2.16 (*A brief dictionary*) The renewal theorem above is given in terms of the additive variable $t \in \mathbb{R}$ but will be applied in the context of the multiplicative variable $\varepsilon \in (0, g]$. For the reader's convenience, we offer the following translation of symbols corresponding to the change of variables $\varepsilon = \mathrm{e}^{-t}$:

e^{-t}	$t \to \infty$	$-\ln g \leq t < \infty$	e^{-h}	$t - \ell h$	$\frac{1}{h}(t + \ln g)$	p_i
ε	$\varepsilon \to 0$	$0 < \varepsilon \leq g$	r	$-\ln(r^{-\ell}\varepsilon)$	$\log_r(g^{-1}\varepsilon)$	r_i^D.

3 Statement and proof of the main results

In order to prove Theorem 1.1, we first prove a theorem which provides information on the asymptotic behavior of the parallel volume of the self-similar set $F \subseteq \mathbb{R}^d$ under weaker conditions. This intermediate result is of independent interest as it does not require the pluriphase condition to be satisfied, and thus may provide an avenue for eventually removing this hypothesis. In analogy with the notation above, we say that $f \sim g$ as $\varepsilon \to 0$ iff $f \circ h \sim g \circ h$ as $t \to \infty$, where $h(t) := \exp(-t)$. If $f, g : (0, \infty) \to (0, \infty)$ this is equivalent to assuming that $\lim_{\varepsilon\to 0} f(\varepsilon)/g(\varepsilon) = 1$.

Theorem 3.1 *Let $F \subset \mathbb{R}^d$ be the attractor of a self-similar system $S = \{S_1, \dots, S_N\}$ satisfying the OSC and let r_i denote the contraction ratio of S_i for $i = 1, \dots, N$. Assume F is nontrivial (i.e. $D := \dim_{\mathcal{M}}(F) < d$). Let O be an arbitrary strong feasible set satisfying the projection condition, $\Gamma := O \setminus \mathbf{S}O$ and g as in (2.10). If S is lattice with base r then*

$$\varepsilon^{D-d}\lambda_d(F_\varepsilon) \sim \frac{\ln r}{\sum_{i=1}^N r_i^D \ln r_i} p(\varepsilon), \qquad \text{as } \varepsilon \to 0, \tag{3.1}$$

where $p : (0, g] \to \mathbb{R}$ is defined by

$$p(\varepsilon) := \varepsilon^{D-d} \sum_{\ell \in \mathbb{Z}} r^{\ell(D-d)} \lambda_d(F_{r^\ell \varepsilon} \cap \Gamma), \qquad \text{for } \varepsilon > 0. \tag{3.2}$$

Moreover, for $\varepsilon \in (rg, g]$, p has the alternative representation

$$p(\varepsilon) = \varepsilon^{D-d} \left[\frac{\lambda_d(\Gamma)}{r^{D-d} - 1} + \sum_{\ell=0}^{\infty} r^{\ell(D-d)} \lambda_d(F_{r^\ell \varepsilon} \cap \Gamma) \right]. \tag{3.3}$$

If one can show that the periodic function p is non-constant, then Theorem 3.1 implies that $\limsup_{\varepsilon \to 0} \varepsilon^{D-d}\lambda_d(F_\varepsilon) > \liminf_{\varepsilon \to 0} \varepsilon^{D-d}\lambda_d(F_\varepsilon)$ and hence that F is not Minkowski measurable. On the other hand, if the function p is constant, then (3.1) implies immediately that F is Minkowski measurable. Note that in both cases p is a strictly positive function. This is for instance obvious from (3.3) since the first term is strictly positive and all terms in the second summation are non-negative. We save these important observations for later use:

Corollary 3.2 *Under the hypothesis of Theorem 3.1, the self-similar set F is Minkowski measurable if and only if the function p (given by (3.2) or (3.3)) is constant, that is, $p(\varepsilon) = C$ for some constant $C > 0$ and all $\varepsilon > 0$. Moreover, in this case the D-dimensional Minkowski content of F is given by*

$$\mathcal{M}_D(F) = \frac{\ln r}{\sum_{i=1}^N r_i^D \ln r_i} \cdot C.$$

Note that Theorem 3.1 and Corollary 3.2 apply to all nontrivial self-similar sets satisfying OSC. There is no monophase or pluriphase condition present and the projection condition on its own does not impose any restrictions. There exists always a strong feasible set O for which it is satisfied, see Remark 2.8. Only trivial self-similar sets are excluded. In this case, the statement of Theorem 3.1 does not make sense, since $\Gamma = \emptyset$ and thus $p \equiv 0$. But such sets are always Minkowski measurable and there is no need for a statement like this. (Note that for a trivial self-similar set $F \subset \mathbb{R}^d$, the d-dimensional Minkowski content is given by $\mathcal{M}_d(F) = \lambda_d(F)$.)

Proof of Theorem 3.1 We decompose the parallel volume of F through

$$\lambda_d(F_\varepsilon) = \lambda_d(F_\varepsilon \setminus O) + \lambda_d(F_\varepsilon \cap O). \tag{3.4}$$

For the first summand on the right hand side of (3.4), we note that $\mathbf{S}O \subseteq O$ so that $O^c \subseteq \mathbf{S}O^c \subseteq (\mathbf{S}O^c)_\varepsilon$. By [31, Corollary 5.6.3] we know that there exist $c, \gamma > 0$ such that

$$\lambda_d(F_\varepsilon \cap (\mathbf{S}O^c)_\varepsilon) \le c\varepsilon^{d-D+\gamma}, \qquad \text{for } \varepsilon \in (0, 1). \tag{3.5}$$

Note that it is this estimate which requires the hypothesis that O is a strong feasible set. Equation (3.5) implies $\lambda_d(F_\varepsilon \setminus O) \le c\varepsilon^{d-D+\gamma}$, whence

$$\lim_{\varepsilon \to 0} \varepsilon^{D-d}\lambda_d(F_\varepsilon \setminus O) = 0. \tag{3.6}$$

Now we turn to the more interesting second summand on the right hand side of (3.4). From (2.13), we have

$$\lambda_d(F_\varepsilon \cap O) = \sum_{i=1}^{N} \lambda_d(F_\varepsilon \cap S_i O) + \lambda_d(F_\varepsilon \cap \Gamma), \qquad \text{for } \varepsilon > 0. \tag{3.7}$$

We deduce from (2.15) that

$$\lambda_d(F_\varepsilon \cap S_i O) = \lambda_d((S_i F)_\varepsilon \cap S_i O) = r_i^d \cdot \lambda_d(F_{\varepsilon/r_i} \cap O). \tag{3.8}$$

Note that it is (2.15) which uses the hypothesis concerning the projection condition. We multiply (3.7) by $\varepsilon^{D-d}\mathbb{1}_{(0,g]}(\varepsilon)$ to obtain

$$\begin{aligned}\varepsilon^{D-d}\mathbb{1}_{(0,g]}(\varepsilon)\lambda_d(F_\varepsilon \cap O) = &\sum_{i=1}^{N} r_i^D\,(\varepsilon/r_i)^{D-d}\,\mathbb{1}_{(0,g]}\,(\varepsilon/r_i)\,\lambda_d(F_{\varepsilon/r_i} \cap O)\\ &+ \sum_{i=1}^{N} r_i^d \varepsilon^{D-d}\mathbb{1}_{(r_i g, g]}(\varepsilon)\lambda_d(F_{\varepsilon/r_i} \cap O)\\ &+ \varepsilon^{D-d}\mathbb{1}_{(0,g]}(\varepsilon)\lambda_d(F_\varepsilon \cap \Gamma).\end{aligned}$$

Setting

$$\tilde{Z}(\varepsilon) := \varepsilon^{D-d}\mathbb{1}_{(0,g]}(\varepsilon)\lambda_d(F_\varepsilon \cap O), \tag{3.9}$$

the previous equation can be rewritten as

$$\tilde{Z}(\varepsilon) = \sum_{i=1}^{N} r_i^D \tilde{Z}(\varepsilon/r_i) + \tilde{z}(\varepsilon)$$

where we have introduced

$$\tilde{z}(\varepsilon) := \sum_{i=1}^{N} r_i^d \varepsilon^{D-d}\mathbb{1}_{(r_i g, g]}(\varepsilon)\lambda_d(F_{\varepsilon/r_i} \cap O) + \varepsilon^{D-d}\mathbb{1}_{(0,g]}(\varepsilon)\lambda_d(F_\varepsilon \cap \Gamma) \tag{3.10}$$

The definition of g yields $\Gamma \subseteq F_g$, which implies for all $w \in \mathcal{W}$ that $S_w\Gamma \subseteq S_w(F_g) \subseteq (S_w F)_g \subseteq F_g$. Consequently, (2.12) yields $O \subseteq \overline{O} \subseteq F_g$ and we have $F_\varepsilon \cap O = O$ for any $\varepsilon \geq g$. Since $\varepsilon/r_i > g$ for $\varepsilon \in (r_i g, g]$, we can thus reduce $\tilde{z}(\varepsilon)$ to

$$\tilde{z}(\varepsilon) = \varepsilon^{D-d} \cdot \left(\lambda_d(O)\sum_{i=1}^{N} r_i^d \mathbb{1}_{(r_i g, g]}(\varepsilon) + \mathbb{1}_{(0,g]}(\varepsilon)\lambda_d(F_\varepsilon \cap \Gamma)\right). \tag{3.11}$$

In order to be able to apply the renewal theorem (Theorem 2.14), we make the variable transformation $\varepsilon = \mathrm{e}^{-t}$ with $t \in \mathbb{R}$ and write $Z(t) := \tilde{Z}(\mathrm{e}^{-t})$ and $z(t) := \tilde{z}(\mathrm{e}^{-t})$. This gives

$$Z(t) = \sum_{i=1}^{N} r_i^D Z(t + \ln r_i) + z(t). \tag{3.12}$$

Since $\Gamma \subseteq \mathbf{S}O^c$, the function z satisfies (2.18) by (3.5). Moreover, z clearly has a finite set of discontinuities. A consequence of $\mathbb{1}_{(0,g]}(\varepsilon)$ being a factor of $\tilde{Z}(\varepsilon)$ and thus $\mathbb{1}_{[-\ln g,\infty)}(t)$ being a factor of $Z(t)$ is that $\lim_{t\to-\infty} Z(t) = 0$ holds true. By Moran's equation, $\sum_{i=1}^{N} r_i^D = 1$. Thus, Theorem 2.14 is applicable to Z with $p_i := r_i^D$ and $y_i := -\ln r_i$

for $i = 1, \ldots, N$. Since the lattice condition gives $r_i = r^{k_i}$ for $i = 1, \ldots, N$, we have $\{y_1, \ldots, y_N\} = \{-k_1 \ln r, \ldots, -k_N \ln r\} \subseteq -\ln r \cdot \mathbb{Z}$. Therefore, Theorem 2.14(i) yields

$$Z(t) \sim \frac{\ln r}{\sum_{i=1}^N r_i^D \ln r_i} \sum_{\ell \in \mathbb{Z}} z(t - \ell h) \qquad \text{as } t \to \infty.$$

In ε-notation, this is

$$\tilde{Z}(\varepsilon) \sim \frac{\ln r}{\sum_{i=1}^N r_i^D \ln r_i} \sum_{\ell \in \mathbb{Z}} \tilde{z}(\varepsilon \cdot r^{-\ell}) \quad \text{as } \varepsilon \to 0. \tag{3.13}$$

Noting that the above sum is absolutely convergent (all terms are positive) and using (3.11), we have

$$\begin{aligned} \widetilde{p}(\varepsilon) := \sum_{\ell \in \mathbb{Z}} \tilde{z}(\varepsilon \cdot r^{-\ell}) = \varepsilon^{D-d} \Bigg[& \lambda_d(O) \sum_{i=1}^N r_i^d \sum_{\ell \in \mathbb{Z}} r^{-\ell(D-d)} \mathbb{1}_{(r_i g, g]}(r^{-\ell}\varepsilon) \\ & + \sum_{\ell \in \mathbb{Z}} r^{-\ell(D-d)} \mathbb{1}_{(0,g]}(r^{-\ell}\varepsilon) \lambda_d(F_{r^{-\ell}\varepsilon} \cap \Gamma) \Bigg]. \end{aligned} \tag{3.14}$$

Define

$$L(\varepsilon) := \left\lfloor \log_r \tfrac{\varepsilon}{g} \right\rfloor \quad \text{and} \quad L^i(\varepsilon) := \left\lfloor \log_r \tfrac{\varepsilon}{g} - k_i \right\rfloor = L(\varepsilon) - k_i,$$

where $\lfloor x \rfloor$ denotes the floor function of $x \in \mathbb{R}$, so that $x \mapsto \lfloor x \rfloor + 1$ rounds x up to the nearest integer strictly larger than x. Upon noting that $r_i g < r^{-\ell}\varepsilon \le g$ iff $\log_r \frac{\varepsilon}{r_i g} < \ell \le \log_r \frac{\varepsilon}{g}$ iff $L^i(\varepsilon) + 1 \le \ell \le L(\varepsilon)$ for $\ell \in \mathbb{Z}$, we see that (3.14) becomes

$$\widetilde{p}(\varepsilon) = \varepsilon^{D-d} \left[\lambda_d(O) \sum_{i=1}^N r_i^d \sum_{\ell = L^i(\varepsilon)+1}^{L(\varepsilon)} r^{-\ell(D-d)} + \sum_{\ell=-\infty}^{L(\varepsilon)} r^{-\ell(D-d)} \lambda_d(F_{r^{-\ell}\varepsilon} \cap \Gamma) \right]. \tag{3.15}$$

However, we know that $\tilde{p}$ is by definition multiplicatively periodic with multiplicative period r, so it suffices to work with $\varepsilon \in (rg, g]$, in which case $L(\varepsilon) = 0$ and $L^i(\varepsilon) = -k_i$. Using the geometric series, the Moran equation $\sum_{i=1}^N r_i^D = 1$ and that $r_i = r^{k_i}$, especially the first summand in (3.15) simplifies significantly and we obtain

$$\widetilde{p}(\varepsilon) = \varepsilon^{D-d} \left[\frac{\lambda_d(O)}{1 - r^{D-d}} \left(\sum_{i=1}^N r_i^d - 1 \right) + \sum_{\ell=0}^{\infty} r^{\ell(D-d)} \lambda_d(F_{r^{\ell}\varepsilon} \cap \Gamma) \right], \qquad \text{for } \varepsilon \in (rg, g]. \tag{3.16}$$

Note that

$$\lambda_d(\Gamma) = \lambda_d\left(O \setminus \bigcup_{i=1}^N S_i O \right) = \lambda_d(O) - \sum_{i=1}^N \lambda_d(S_i O) = \lambda_d(O) \left(1 - \sum_{i=1}^N r_i^d \right).$$

Therefore, (3.16) becomes

$$\widetilde{p}(\varepsilon) = \varepsilon^{D-d} \left[\frac{\lambda_d(\Gamma)}{r^{D-d} - 1} + \sum_{\ell=0}^{\infty} r^{\ell(D-d)} \lambda_d(F_{r^{\ell}\varepsilon} \cap \Gamma) \right], \qquad \text{for } \varepsilon \in (rg, g], \tag{3.17}$$

which shows that the asymptotic relation (3.1) in Theorem 3.1 holds indeed with the (periodic continuation of the) function p given by (3.3). Using that $\lambda_d(F_{r^{\ell_\varepsilon}} \cap \Gamma) = \lambda_d(\Gamma)$ for $\varepsilon \in (rg, g]$ and $\ell \leq -1$ we obtain from the geometric series expansion that

$$\widetilde{p}(\varepsilon) = \varepsilon^{D-d} \sum_{\ell \in \mathbb{Z}} r^{\ell(D-d)} \lambda_d(F_{r^{\ell_\varepsilon}} \cap \Gamma), \qquad \text{for } \varepsilon \in (rg, g]. \tag{3.18}$$

Thus $\widetilde{p}(\varepsilon) = p(\varepsilon)$ for $\varepsilon \in (rg, g]$, where p is as defined in (3.2), and the periodicity of p implies now that the representation (3.2) (resp. (3.18)) is in fact valid for all $\varepsilon > 0$. This shows that (3.1) holds also with p given by (3.2) and concludes the proof of Theorem 3.1.□

Remark 3.3 Under the additional assumptions that bd $O \subseteq F$ and that $\overline{O} \backslash \overline{\mathbf{S}O}$ consists of a finite number of connected components, Theorem 3.1 was proven in [16, Theorem 2.38], the Ph. D. thesis of the first author. There, the proof builds on results which were shown for the more general class of self-conformal sets by means of renewal theory in symbolic dynamics. Consequently, it contains arguments to overcome difficulties in the conformal setting which do not occur in the self-similar situation. The proof of Theorem 3.1 presented here is shorter and more direct for the self-similar situation. Moreover, the statement of Theorem 3.1 is more general; neither does it require bd $O \subseteq F$, nor the assumption that $\overline{O} \backslash \overline{\mathbf{S}O}$ possesses a finite number of connected components.

3.1 The case when $D = \dim_{\mathcal{M}} F$ is not an integer.

Now we restate and prove part (ii) of Theorem 1.1, the case when D is not an integer; the integer case is discussed in Theorem 3.6.

Theorem 3.4 *Let F be a self-similar set which is the attractor of the lattice self-similar system $S = \{S_1, \ldots, S_N\}$, $N \geq 2$ satisfying the OSC. Assume the Minkowski dimension $D = \dim_{\mathcal{M}} F$ is not an integer and that we can find a strong feasible open set O satisfying the projection condition (see Def. 2.7) such that F is pluriphase with respect to $\Gamma(O)$. In this situation, F is not Minkowski measurable.*

Remark 3.5 Note that the hypothesis that D is not an integer implies that $D < d$. Therefore, Prop. 2.4 ensures that the nontriviality condition will be met for any feasible open set. Therefore, the set Γ is always nonempty and we do not need to include nontriviality in the hypothesis of Theorem 3.4.

Proof of Theorem 3.4 We assume the partition we work with is minimal (see Def. 2.9). In the case $M \geq 2$ we will use the fact that for $\ell \in \mathbb{Z}$ and $m = 2, \ldots, M$ we have the equivalences

$$r^\ell \varepsilon \in (a_{m-1}, a_m] \Leftrightarrow \ell \in \left[\log_r \tfrac{a_m}{\varepsilon}, \log_r \tfrac{a_{m-1}}{\varepsilon}\right) \cap \mathbb{Z} \quad \Leftrightarrow \quad \ell \in \Big\{L_m(\varepsilon), \ldots, L_{m-1}(\varepsilon) - 1\Big\}.$$

Here

$$L_m(\varepsilon) := \left\lceil \log_r \frac{a_m}{\varepsilon} \right\rceil, \qquad m = 1, \ldots, M\,, \tag{3.19}$$

where $\lceil x \rceil$ is the ceiling function. In the case $M \geq 1$ and $m = 1$ we have

$$r^\ell \varepsilon \in (0, a_1] \quad \Leftrightarrow \quad \ell \in [L_1(\varepsilon), \infty) \cap \mathbb{Z}.$$

See Fig. 3 and Example 4.2 in Sect. 4 for examples of the form the functions L_m take.

Substituting the pluriphase representation (2.16) into (3.2) and using the funtions L_m, we obtain

$$\begin{aligned} p(\varepsilon) &= \varepsilon^{D-d} \sum_{\ell \in \mathbb{Z}} r^{\ell(D-d)} \left[\sum_{m=1}^{M} \mathbb{1}_{(a_{m-1}, a_m]}(r^\ell \varepsilon) \sum_{k=0}^{d} \kappa_{m,k} (r^\ell \varepsilon)^{d-k} + \mathbb{1}_{(g,\infty)}(r^\ell \varepsilon) \lambda_d(\Gamma) \right] \\ &= \varepsilon^{D-d} \Bigg[\sum_{\ell = L_1(\varepsilon)}^{\infty} r^{\ell(D-d)} \sum_{k=0}^{d} \kappa_{1,k} (r^\ell \varepsilon)^{d-k} + \sum_{m=2}^{M} \sum_{\ell = L_m(\varepsilon)}^{L_{m-1}(\varepsilon)-1} r^{\ell(D-d)} \sum_{k=0}^{d} \kappa_{m,k} (r^\ell \varepsilon)^{d-k} \\ &\quad + \sum_{\ell=-\infty}^{L_M(\varepsilon)-1} r^{\ell(D-d)} \lambda_d(\Gamma) \Bigg]. \end{aligned} \tag{3.20}$$

Since $\lim_{\varepsilon \to 0} \lambda_d(F_\varepsilon \cap \Gamma) = 0$, we know that $\kappa_{1,d} = 0$, but in fact more is true: the strong feasibility of O allows us to again invoke (3.5) and thereby deduce that $\kappa_{1,k} = 0$ for all $k \geq D$. This remark is crucial since it ensures absolute convergence of the first series. Since also the other series are absolutely convergent we may change the order of summation and evaluate the series over ℓ by means of the geometric series to obtain

$$\begin{aligned} p(\varepsilon) &= \sum_{k=0}^{d} \kappa_{1,k} \frac{(r^{L_1(\varepsilon)} \varepsilon)^{(D-k)}}{1 - r^{D-k}} \\ &\quad + \sum_{k=0}^{d} \sum_{m=2}^{M} \kappa_{m,k} \varepsilon^{D-k} \frac{r^{L_m(\varepsilon)(D-k)} - r^{L_{m-1}(\varepsilon)(D-k)}}{1 - r^{D-k}} - \lambda_d(\Gamma) \frac{(r^{L_M(\varepsilon)} \varepsilon)^{(D-d)}}{1 - r^{D-d}} \\ &= \sum_{m=1}^{M-1} \sum_{k=0}^{d} \frac{(r^{L_m(\varepsilon)} \varepsilon)^{D-k}}{1 - r^{D-k}} (\kappa_{m,k} - \kappa_{m+1,k}) \\ &\quad + \sum_{k=0}^{d} \kappa_{M,k} \frac{(r^{L_M(\varepsilon)} \varepsilon)^{(D-k)}}{1 - r^{D-k}} - \frac{\lambda_d(\Gamma) (r^{L_M(\varepsilon)} \varepsilon)^{(D-d)}}{1 - r^{D-d}}. \end{aligned}$$

Setting

$$\kappa_{M+1,k} := 0 \quad \text{for } k = 0, \ldots, d-1, \qquad \text{and} \qquad \kappa_{M+1,d} := \lambda_d(\Gamma), \tag{3.21}$$

we may write

$$p(\varepsilon) = \sum_{k=0}^{d} \frac{\varepsilon^{D-k}}{1 - r^{D-k}} \eta_k(\varepsilon), \tag{3.22}$$

where

$$\eta_k(\varepsilon) := \sum_{m=1}^{M} r^{L_m(\varepsilon)(D-k)} (\kappa_{m,k} - \kappa_{m+1,k}). \tag{3.23}$$

Our aim is to show that p is non-constant. For this, we restrict our investigations to an interval whose length aligns with the multiplicative period r of p, namely the interval $(rg, g]$. On this interval, each L_m is piecewise constant with at most one point of discontinuity. Therefore each η_k is piecewise constant on $(rg, g]$ with at most M points of discontinuity. Moreover, these points of discontinuity coincide for each $k = 0, \ldots, d$. Therefore, there is a finite number of disjoint intervals of strictly positive length

$$I_1, \dots, I_Q \quad \text{with} \quad (rg, g] = \bigcup_{q=1}^{Q} I_q, \tag{3.24}$$

such that all η_k are constant on each I_q. We denote the constant value of η_k on I_q by $\beta_{k,q}$.

The strict positivity of p on $(rg, g]$ implies, that for each $q \in \{1, \dots, Q\}$ there exists some $k \in \{0, 1, \dots, d\}$ such that $\beta_{k,q} \neq 0$. Indeed, if all $\beta_{k,q}$ are zero for some q, then obviously p would be identically zero on I_q, in contradiction with its positivity.

Now fix some q_0 and assume p is constant on I_{q_0}. Then $p'(\varepsilon) = 0$ on the interior $\mathrm{int}(I_{q_0})$ of I_{q_0}, which would imply

$$0 = \sum_{k=0}^{d} \frac{D-k}{1-r^{D-k}} \beta_{k,q_0} \varepsilon^{D-k-1}, \qquad \text{for each } \varepsilon \in \mathrm{int}(I_{q_0}). \tag{3.25}$$

However, this contradicts the linear independence of the functions $\{\varepsilon^{D-1}, \dots, \varepsilon^{D-d-1}\}$ on I_{q_0}. Hence p is not a constant function and it follows now by Corollary 3.2 that F is not Minkowski measurable. □

3.2 The case when $D = \dim_{\mathcal{M}} F$ is an integer.

When the Minkowski dimension D of F is an integer then both are possible: F can be Minkowski measurable or non-Minkowski measurable. In our main theorem of this section, Theorem 3.6, we provide equivalent characterizations of Minkowski measurability in the current setting.

Suppose that F is pluriphase with respect to Γ. We consider the functions L_m as defined in (3.19). Examples of these functions appear in Figs. 3 and 4. In the proof of Theorem 3.4, we used the facts that each L_m is piecewise constant on $(rg, g]$ with at most one point of discontinuity to deduce that there is a finite number of disjoint intervals $\{I_q\}_{q=1}^{Q}$ of strictly positive length such that $(rg, g] = \bigcup_{q=1}^{Q} I_q$ and such that every L_m is constant on each interval I_q; see (3.24).

For the formulation of one of the equivalent characterizations of Minkowski measurability in Theorem 3.6, it is convenient to group together those indices m for which L_m has the same point of discontinuity in $(rg, g]$:

$$U_q := \Big\{m \in \{1, \dots, n\} : L_m(\varepsilon_q) \neq L_m(\varepsilon_{q+1}) \text{ for } \varepsilon_q \in I_q,\ \varepsilon_{q+1} \in I_{q+1}\Big\}, \quad q < Q, \tag{3.26}$$

$$U_Q := \{1, \dots, M\} \setminus \bigcup_{q=1}^{Q-1} U_q. \tag{3.27}$$

Note that for each $m \in U_Q$ the function L_m is a constant function on $(rg, g]$.

We now give a precise statement of part (iii) of Theorem 1.1, that is for the case when D is an integer.

Theorem 3.6 *Let $F \subset \mathbb{R}^d$ be a self-similar set which is the attractor of the lattice self-similar system $S = \{S_1, \dots, S_N\}$, $N \geq 2$ satisfying the OSC. Assume the Minkowski dimension $D = \dim_{\mathcal{M}} F$ is an integer different from d and that we can find a strong feasible open set O satisfying the projection condition (see Def. 2.7) such that F is pluriphase with respect to $\Gamma(O)$. Let $C > 0$. Then the following assertions are equivalent:*

(1) F is Minkowski measurable with Minkowski content given by

$$\mathcal{M}_D(F) = \frac{\ln r}{\sum_{i=1}^N r_i^D \ln r_i} \cdot C\,.$$

(2) The function p from (3.2) is a constant function taking the value C.
(3) For $\varepsilon \in (rg, g]$,

$$C = \sum_{m=1}^{M} L_m(\varepsilon)(\kappa_{m+1,D} - \kappa_{m,D}) \quad and \tag{3.28}$$

$$0 = \sum_{m=1}^{M} r^{L_m(\varepsilon)(D-k)}(\kappa_{m,k} - \kappa_{m+1,k}) \quad for\ k \neq D. \tag{3.29}$$

(4) With U_q defined as in (3.26)–(3.27) and g as in (2.10),

$$0 = \sum_{m \in U_q} a_m^{D-k}(\kappa_{m,k} - \kappa_{m+1,k}) \tag{3.30}$$

for all pairs $(k, q) \in (\{0, \ldots, d\} \times \{1, \ldots, Q\}) \backslash \{(D, Q)\}$, and

$$C = \sum_{m \in U_Q} L_m(g)(\kappa_{m+1,D} - \kappa_{m,D}). \tag{3.31}$$

Moreover, if O is such that F is monophase w.r.t. Γ, then these assertions are never met and so in particular F is not Minkowski measurable.

Remark 3.7 Nonmeasurability arising from lattice-type iterated function systems is due to geometric oscillations arising from the alignments of multiplicative periods in the scaling factors of the various mappings in the IFS; see [20]. In some sense, the algebraic conditions formulated in (3) and more clearly in (4) describe the situation when there are extra geometric oscillations induced by the pluriphase representation of the volume function $\lambda_d(F_\varepsilon \cap \Gamma)$ that cancel out the geometric oscillations intrinsic to the IFS. These conditions should be viewed as a kind of *latticeness* of the representation (and thus of the set Γ in relation with F).

Remark 3.8 Note that the hypothesis $D \neq d$ is equivalent to the hypothesis that F is non-trivial, by Prop. 2.4.

Proof of Theorem 3.6 Equivalence of the assertions (1) and (2) is clear from Corollary 3.2. To show the equivalence (2) $\Leftrightarrow$ (3), we go again through the steps of the proof of Theorem 3.4 and point out the modifications necessary in the case when D is an integer. Up to equation (3.20) there is no difference in the derivation, but from this equation onwards, the Dth terms in all the summations over k have to be treated differently. First, notice that $\kappa_{1,D} = 0$ according to the discussion just after (3.20) (since $D \leq D$). So there is no concern regarding the summation from $L_1(\varepsilon)$ to ∞ for the Dth terms in the first summand in (3.20), they just vanish. The second summand in (3.20) changes; we rewrite it here with the term for $k = D$ extracted:

$$\varepsilon^{D-d} \sum_{m=2}^{M} \sum_{\ell=L_m(\varepsilon)}^{L_{m-1}(\varepsilon)-1} r^{\ell(D-d)} \sum_{\substack{k=0 \\ k \neq D}}^{d} \kappa_{m,k}(r^\ell \varepsilon)^{d-k} + \varepsilon^{D-d} \sum_{m=2}^{M} \sum_{\ell=L_m(\varepsilon)}^{L_{m-1}(\varepsilon)-1} r^{\ell(D-d)} \kappa_{m,D}(r^\ell \varepsilon)^{d-D}$$

The first expression is dealt with exactly as in the proof of Theorem 3.4. For the second term, we have

$$\eta_D(\varepsilon) := \varepsilon^{D-d} \sum_{m=2}^{M} \sum_{\ell=L_m(\varepsilon)}^{L_{m-1}(\varepsilon)-1} r^{\ell(D-d)} \kappa_{m,D} (r^\ell \varepsilon)^{d-D}$$
$$= \sum_{m=2}^{M} \sum_{\ell=L_m(\varepsilon)}^{L_{m-1}(\varepsilon)-1} \kappa_{m,D} = \sum_{m=1}^{M} L_m(\varepsilon)(\kappa_{m+1,D} - \kappa_{m,D}),$$

where we have used the notation introduced in (3.21). Thus we obtain the following modification of (3.23)

$$\eta_k(\varepsilon) := \begin{cases} \sum_{m=1}^{M} r^{L_m(\varepsilon)(D-k)}(\kappa_{m,k} - \kappa_{m+1,k}), & k \neq D, \\ \sum_{m=1}^{M} L_m(\varepsilon)(\kappa_{m+1,k} - \kappa_{m,k}), & k = D, \end{cases} \tag{3.32}$$

and (3.22) is replaced by

$$p(\varepsilon) = \eta_D(\varepsilon) + \sum_{\substack{k=0 \\ k \neq D}}^{d} \frac{\varepsilon^{D-k}}{1 - r^{D-k}} \eta_k(\varepsilon), \tag{3.33}$$

where still all the functions $\eta_k(\varepsilon)$ are piecewise constant with finitely many pieces in $(rg, g]$.

Now assume that (2) holds, i.e. $p(\varepsilon) = C$ for $\varepsilon \in (rg, g]$ and some $C > 0$. Restricting to an arbitrary subinterval on which the functions η_k are all constant, we can use again the linear independence of the functions ε^{D-k}, $k = 0, \dots, d$ to conclude that $\eta_k(\varepsilon) = 0$ for $k \neq D$ and thus $\eta_D(\varepsilon) = C$ on this subinterval. Since this applies to all such subintervals, it holds on $(rg, g]$, which shows that (2) implies (3). The reverse implication is obvious from equation (3.33).

Our next step is to show the equivalence of (3) and (4).

(i) First, we consider the case $k = D$ and show that (3.31) together with (3.30) holding for pairs $(k, q) \in \{D\} \times \{1, \dots, Q-1\}$ is equivalent to (3.28).

Observe that (3.28) can be rewritten as

$$C = \sum_{m=1}^{M} L_m(\varepsilon)(\kappa_{m+1,D} - \kappa_{m,D}) = \sum_{q=1}^{Q} \underbrace{\sum_{m \in U_q} L_m(\varepsilon)(\kappa_{m+1,D} - \kappa_{m,D})}_{=:A_{D,q}(\varepsilon)} \tag{3.34}$$

for all $\varepsilon \in (rg, g]$. Note that for $q = Q$ the sum $A_{D,Q}(\varepsilon) = A_{D,Q}(g)$ is independent of $\varepsilon \in (rg, g]$ by construction. Thus, if $Q = 1$ then (3.28) is equivalent to (3.31). Now consider the case $Q \geq 2$ and fix $q \in \{1, \dots, Q-1\}$. By construction $A_{D,q'}$ is constant on $I_q \cup I_{q+1}$ when $q' \in \{1, \dots, Q-1\} \setminus \{q\}$. Moreover, for $m \in U_q$ we have $L_m(\varepsilon_{q+1}) - L_m(\varepsilon_q) = 1$ for $\varepsilon_{q+1} \in I_{q+1}$ and $\varepsilon_q \in I_q$, yielding

$$\left| A_{D,q}(\varepsilon_{q+1}) - A_{D,q}(\varepsilon_q) \right| = \left| \sum_{m \in U_q} (\kappa_{m+1,D} - \kappa_{m,D}) \right|.$$

Thus, (3.34) holds if and only if

$$0 = \sum_{m \in U_q} (\kappa_{m+1,D} - \kappa_{m,D}) \quad \text{for } q < Q, \qquad \text{and} \qquad C = \sum_{m \in U_Q} L_m(\varepsilon)(\kappa_{m+1,D} - \kappa_{m,D})$$

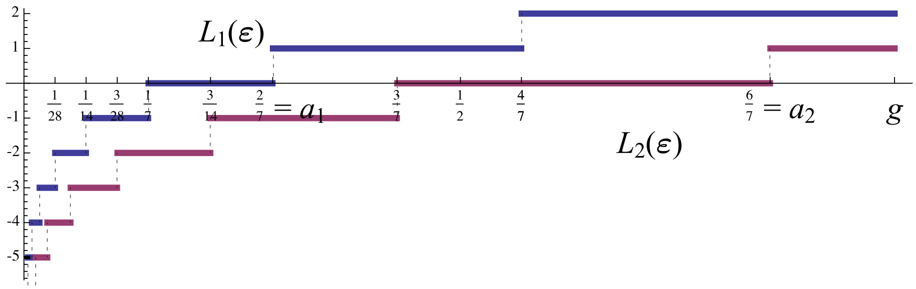

Fig. 3 The functions $L_1(\varepsilon)$ and $L_2(\varepsilon)$ as defined in (3.19) for an example where $a_1 = \frac{2}{7}, a_2 = \frac{6}{7}, r = \frac{1}{2}$, and $g = 1$. (For clarity, $L_3(\varepsilon)$ is not depicted here.)

for $\varepsilon \in (rg, g]$. Noting that $L_m(\varepsilon) = L_m(g)$ holds for all $\varepsilon \in (rg, g]$ whenever $m \in U_Q$, the statement (i) is verified.

(ii) Second, we consider the case $k \neq D$ and show that (3.30) holding for pairs $(k, q) \in (\{0, \dots, d\} \backslash \{D\}) \times \{1, \dots, Q\}$ is equivalent to (3.29).

Observe that, $\eta_k(\varepsilon) = 0$ on $(rg, g]$ if and only if

$$0 = \eta_k(\varepsilon) = \sum_{q=1}^{Q} \underbrace{\sum_{m \in U_q} r^{L_m(\varepsilon)(D-k)}(\kappa_{m,k} - \kappa_{m+1,k})}_{=:A_{q,k}(\varepsilon)}, \quad \text{on } (rg, g].$$

In the same way as in the case of $k = D$, one can deduce that $0 = \sum_{q=1}^{Q} A_{q,k}(\varepsilon)$ on $(rg, g]$ if and only if

$$0 = A_{q,k}(\varepsilon) = \sum_{m \in U_q} r^{L_m(\varepsilon)(D-k)}(\kappa_{m,k} - \kappa_{m+1,k}) \quad \text{on } (rg, g], \quad \text{for } q = 1, \dots, Q. \tag{3.35}$$

By definition of the sets U_q, we know that $\{\log_r \frac{a_m}{\varepsilon}\} = \{\log_r \frac{a_{m'}}{\varepsilon}\}$ for $m, m' \in U_q$. Denoting this common value by $g_q(\varepsilon)$, Equation (3.35) is equivalent to

$$0 = r^{(1-g_q(\varepsilon))(D-k)} \sum_{m \in U_q} \left(\frac{a_m}{\varepsilon}\right)^{D-k} (\kappa_{m,k} - \kappa_{m+1,k}) \quad \text{on } (rg, g], \quad \text{for } q = 1, \dots, Q.$$

This is equivalent to (3.30), since $k \neq D$ and the power functions $\{\varepsilon^{-D}, \dots, \varepsilon^{d-D}\}$ are linearly independent. Thus, (3) is equivalent to assertion (4).

Finally, if F is monophase w.r.t. Γ, then $M = 1$. The discussion directly after (3.20) gives $\kappa_{1,d} = 0$, and (3.21) gives $\kappa_{2,d} = \lambda_d(\Gamma) \neq 0$. Therefore, (3.29) cannot be satisfied for $k = d$ and the last assertion in Theorem 3.6 follows. □

4 Examples

In the case $D \in \mathbb{N}$ there are examples of self-similar sets arising from lattice IFS which are Minkowski measurable. In Example 4.1 we provide a simple example of a lattice IFS with common scaling ratio $r = \frac{1}{2}$ for which the attractor has integer Minkowski dimension

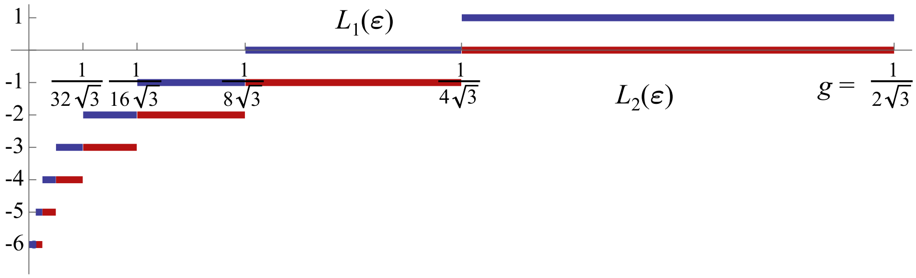

Fig. 4 The functions $L_1(\varepsilon)$ and $L_2(\varepsilon)$ as defined in (3.19) for the Sierpinski gasket with O and Γ as in Fig. 2, see Example 4.2 for details. Here, $a_1 = \sqrt{3}/12$, $a_2 = g = \sqrt{3}/6$ and $r = 1/2$

$D = \dim_{\mathcal{M}} F = 2$. Note that this example comes by "cheating" the nontriviality condition via embedding in a higher-dimensional Euclidean space; see Remark 1.2.

Example 4.1 Let $S_1, \ldots, S_4 : \mathbb{R}^3 \to \mathbb{R}^3$ be given by

$$S_1(x) := x/2, \qquad S_3(x) := x/2 + (0, 1/2, 0),$$
$$S_2(x) := x/2 + (1/2, 0, 0), \qquad S_4(x) := x/2 + (1/2, 1/2, 0).$$

It is not difficult to see that $F := [0, 1] \times [0, 1] \times \{0\}$ is the associated invariant set and that $D = 2$, so we are in the case $D \in \mathbb{N}$. Define $O := (0, 1) \times (0, 1) \times (-1/2, 1/2)$. Then O is a strong feasible open set for $\{S_1, \ldots, S_4\}$. Moreover the nontriviality condition is satisfied, since $\bigcup_{i=1}^4 \overline{S_i(O)} = [0, 1] \times [0, 1] \times [-1/4, 1/4]$. With $\Gamma := O \setminus \mathbf{S}O$ as before,

$$\lambda_3(F_\varepsilon \cap \Gamma) = \begin{cases} 0, & \varepsilon < 1/4, \\ 2(\varepsilon - 1/4), & 1/4 \le \varepsilon < 1/2, \\ 1/2, & 1/2 \le \varepsilon. \end{cases} \tag{4.1}$$

Thus, $a_0 = 0$, $a_1 = 1/4$ and $a_2 = g = 1/2$, which implies $\{\log_r(a_m)\} = 0$ for $m = 1, 2$, since $r = 1/2$. Hence, $Q = 1$ with Q as in (3.24), implying $U_Q = \{1, 2\}$. Moreover, $\kappa_{2,2} = 2$, $\kappa_{2,3} = -1/2$, $\kappa_{3,3} = 1/2$ and $\kappa_{m,k} = 0$ for all other pairs m, k. Using Theorem 3.6 (4), we conclude that F is Minkowski measurable with Minkowski content 2. This can also be deduced directly by evaluating the function p

Example 4.2 In this example, we return to the example which we provided in Fig. 2. Here, $S = \{S_1, S_2, S_3\}$ is the standard IFS which generates the Sierpinski gasket F, and O and Γ are as in Fig. 2. Then,

$$\lambda_2(F_\varepsilon \cap \Gamma) = \begin{cases} 6\sqrt{3}\varepsilon^2, & 0 \le \varepsilon < \sqrt{3}/12, \\ 6\sqrt{3}\varepsilon^2 - 3\varepsilon + \sqrt{3}/4, & \sqrt{3}/12 \le \varepsilon < \sqrt{3}/6, \\ \sqrt{3}/4, & \sqrt{3}/6 \le \varepsilon. \end{cases} \tag{4.2}$$

Thus, F is pluriphase w.r.t. Γ. Moreover, $D = \dim_{\mathcal{M}}(F) = \log_2(3) \notin \mathbb{N}$ and O is a strong feasible open set satisfying the projection condition. Therefore, we can apply Theorem 3.4 and deduce that F is not Minkowski measurable. For this example we want to visualize the functions L_m from (3.19), which the proof of Theorem 3.4 heavily uses. From (4.2) we conclude that $a_0 = 0$, $a_1 = \sqrt{3}/12$ and $a_2 = g = \sqrt{3}/6$. Moreover, $r = 1/2$. Therefore,

$$L_1(\varepsilon) = \left\lceil \log_r \left(\frac{a_1}{\varepsilon}\right) \right\rceil = \left\lceil \frac{\ln(\sqrt{3}/12) - \ln(\varepsilon)}{-\ln(2)} \right\rceil \quad \text{and} \quad L_2(\varepsilon) = \left\lceil \frac{\ln(\sqrt{3}/6) - \ln(\varepsilon)}{-\ln(2)} \right\rceil.$$

The plot of L_1 and L_2 is provided in Fig. 4.

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