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An isomorphism theorem for Yokonuma–Hecke algebras and applications to link invariants

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Abstract We develop several applications of the fact that the Yokonuma–Hecke algebra of the general linear group GL is isomorphic to a direct sum of matrix algebras associated to Iwahori–Hecke algebras of type A . This includes a description of the semisimple and modular representation theory of the Yokonuma–Hecke algebras of GL and a complete classification of all the Markov traces for them. Finally, from these Markov traces, we construct 3-variables polynomials which are invariants for framed and classical knots and links, and investigate their properties with the help of the isomorphism. In particular, for classical knots, a consequence of the construction is that the obtained set of invariants is topologically equivalent to the HOMFLYPT polynomial. We thus recover results of Chlouveraki et al. (2015, arXiv:1505.06666) about the Juyumaya–Lambropoulou invariants.

1 Introduction

1.1. The Yokonuma–Hecke algebras have been introduced by Yokonuma in [22] as centraliser algebras of the permutation representation of Chevalley groups G with respect to a maximal unipotent subgroup of G. They are thus particular cases of unipotent Hecke algebras and they admit a natural basis indexed by double cosets (see [21] for more details on general unipotent Hecke algebras). For the Yokonuma–Hecke algebras, the natural description has been transformed into a simple presentation with generators and relations [9,11]. Assume that q is a power of a prime number then, from this presentation, one can observe that the Yokonuma–Hecke algebra of $G = \mathrm{GL}_n(\mathbb{F}_q)$ (sometimes called the Yokonuma–Hecke algebra of type A) is a deformation of the group algebra of the complex reflection group of type $G(d, 1, n)$, where $d = q - 1$.

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In this paper, we will consider the generic Yokonuma–Hecke algebras $Y_{d,n}$ ($n \in \mathbb{Z}_{\geq 0}$) depending on two indeterminates u and v and a positive integer d, over the ring $\mathbb{C}[u^{\pm 1}, v]$. The algebra $Y_{d,n}$ is also a deformation of the group algebra of the complex reflection group of type $G(d, 1, n)$, for any d, and the Yokonuma–Hecke algebra of $\mathrm{GL}_n(\mathbb{F}_q)$ is obtained by considering the specialization $u^2 = q$, $v = q - 1$ and the case $d = q - 1$.

There exist others well-known deformations of complex reflection groups of type $G(d, 1, n)$ which have been intensively studied during the last past years : the Ariki–Koike algebras. These algebras turn out to have a deep representation theory (in both semisimple and modular cases) which is now quite well-understood (see for example [6] for an overview).

For the Yokonuma–Hecke algebra $Y_{d,n}$, the set of simple modules has been explicitly described in the semisimple situation in combinatorial terms in [3] (see also [20] for general results on the semisimple representation theory of unipotent Hecke algebras). In addition, a criterion of semi simplicity has been deduced. Finally, a certain symmetrizing form has been defined and the associated Schur elements (which control a part of the representation theory of the algebra) have been calculated. They appear to have a particular simple form, namely products of Schur elements of Iwahori–Hecke algebras of type A. Thus, the study of the representation theory and the symmetric structure suggests a deep connection between the Yokonuma–Hecke algebra $Y_{d,n}$ and the Iwahori–Hecke algebra of type A.

In another way, a motivation for studying the Yokonuma–Hecke algebra comes from topology and more precisely the theory of knot and link invariants. Indeed, the algebra $Y_{d,n}$ is naturally a quotient of the framed braid group algebra, and in turn can be used to search for isotopy invariants for framed links in the same spirit as the Iwahori–Hecke algebra of type A is used to obtain an invariant for classical links (the HOMFLYPT polynomial).

In [10], Juyumaya introduced on $Y_{d,n}$ an analogue of the Ocneanu trace of the Iwahori–Hecke algebra of type A. This trace was subsequently used by Juyumaya and Lambropoulou to produce isotopy invariants for framed links [12,15]. Remarkably, they also produced isotopy invariants for classical links and singular links [13,14]. Even though the obtained invariants for classical links are different from the HOMFLYPT polynomial (excepted in some trivial cases), all the computed examples seem to indicate that the invariants for classical links obtained from $Y_{d,n}$ so far are topologically equivalent to the HOMFLYPT polynomial [2]. In fact, if we restrict to classical knots, such an equivalence has been announced in [1].

Again, it seems reasonable to expect an underlying connection between the algebra $Y_{d,n}$ and the Iwahori–Hecke algebra of type A which could explain this fact.

1.2. In this paper, we give several answers and new results in both directions: the representation theory and the knots and links theory. After recalling several results and detailing the structure of the algebras (in Sect. 2), in the third section, we indeed show that, over the ring $\mathbb{C}[u^{\pm 1}, v]$, there is an isomorphism between the Yokonuma–Hecke algebra $Y_{d,n}$ and a direct sum of matrix algebras over tensor products of Iwahori–Hecke algebras of type A. This result is in fact a special case of a result by Lusztig [18][1] but we give

The direct sum is naturally indexed by the set of compositions of n with d parts. Moreover, we provide explicit formulas for this isomorphism (and its inverse) which will allow us to concretely translate questions and problems from one side to the other.

Then, we first develop in Sect. 4 the applications of the isomorphism theorem concerning representation theory. Indeed, the isomorphism can be rephrased by saying that the Yokonuma–Hecke algebra $Y_{d,n}$ is Morita equivalent to a direct sum of tensor products of Iwahori–Hecke algebras of type A. As the result is valid in the generic situation (over the

[1] We thank G. Lusztig for pointing us this fact.

ring $\mathbb{C}[u^{\pm 1}, v]$), it passes to the specializations of the parameters u and v. This implies that both the semisimple and the modular representation theories of $Y_{d,n}$ can be deduced from the corresponding ones of the Iwahori–Hecke algebra of type A, which are well-studied (see e.g. [7]). In particular, the classification of simple modules of $Y_{d,n}$ and the decomposition matrices (in characteristic 0) follow.

In addition, the isomorphism theorem provides a natural symmetrizing form on $Y_{d,n}$ derived from the canonical symmetrizing form of the Iwahori–Hecke algebra of type A. As a first application of the explicit formulas, we show that this symmetrizing form actually coincides with the symmetrizing form defined in [3], which provides a direct proof and an explanation of the form of the Schur elements. We thus in particular recover the results of [3].

1.3. Another class of applications of the isomorphism theorem concerns the theory of classical and framed knots and links (Sects. 5 and 6). Indeed, we obtain a complete classification of the Markov traces on the family, on n, of the Yokonuma–Hecke algebras $Y_{d,n}$ (Theorem 5.3). This is done in two steps. First we translate, with the help of the isomorphism theorem, the Markov trace properties into properties of traces on tensor products of Iwahori–Hecke algebras of type A; then we fully characterize these traces using the known uniqueness of the Markov trace on the Iwahori–Hecke algebras of type A. In particular, we show that all the Markov traces on $Y_{d,n}$ are related with the unique Markov trace on the Iwahori–Hecke algebras of type A.

We note that we use a different definition of a Markov trace on $Y_{d,n}$ than in [10,12–15]. In there, the standard approach initiated by Jones for classical links was followed (see [8] and references therein). The first step is the construction on $Y_{d,n}$ of an analogue of the Ocneanu trace by Juyumaya [10]. Additional conditions were imposed in [10] in order to obtain the existence and unicity of this trace. Then, a rescaling procedure is necessary to construct invariants and, as it turned out, the trace does not rescale directly as in the classical case. A non-trivial rescaling procedure was implemented by Juyumaya and Lambropoulou in [13–15] by means of the so-called "E-system" and led to further restrictions on the parameters.

In the definition we use here for the Markov trace on $Y_{d,n}$, the imposed conditions are the minimal ones allowing to obtain link invariants, namely, the centrality and the so-called Markov condition (see Sect. 5.2). This will allow us to avoid any kind of rescaling procedure during the construction of invariants. This approach is explained in [7, section4.5] in the classical setting of the Iwahori–Hecke algebras of type A.

With the definition used here, the space of Markov traces has a structure of $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$-module. Our approach via the isomorphism theorem provides a distinguished basis, which is indexed by the set of compositions into d parts with all parts equal to 0 or 1. Thus the space of Markov traces on the Yokonuma–Hecke algebras has dimension $2^d - 1$.

1.4. Finally, the last section is devoted to the construction and the study of invariants for classical links and framed links. Following Juyumaya and Lambropoulou, we realize $Y_{d,n}$ as a quotient of the framed braid group. Actually we do more: we introduce a one-parameter family of homomorphisms from the group algebra of the framed braid group to the algebra $Y_{d,n}$ (deforming the canonical homomorphism). Then, for each homomorphism of this family and each Markov trace on $Y_{d,n}$, we construct an invariant for classical and framed links. As mentioned above, no rescaling is needed and the invariant is simply obtained by the composition of the homomorphism followed by the Markov trace.

The obtained invariants are Laurent polynomials in three variables: two of these variables are the parameters u and v in the definition of $Y_{d,n}$, and the third one is the parameter

appearing in the homomorphism from the group algebra of the framed braid group to $Y_{d,n}$. Among these invariants, we recover the ones obtained by Juyumaya and Lambropoulou by taking a particular value for this third parameter and some specific Markov traces.

Restricting to classical links, we use our construction via the isomorphism theorem to prove two results devoted to the comparison of the obtained invariants with the HOMFLYPT polynomial.

First we show that the HOMFLYPT polynomial is contained in them. More precisely, among the $2^d - 1$ basic Markov traces, there are d of them whose associated invariants coincide with the HOMFLYPT polynomial. These basic Markov traces are the ones indexed by compositions into d parts with only one part equal to 1 and all the others equal to 0 (in the particular case of the Juyumaya–Lambropoulou invariants, this result corrresponds to [2, Corollary1]).

Then, we show that the invariants obtained from the others basic Markov traces are always equal to 0 when applied to a classical knot. For classical knots, this solves completely the study of these invariants, which are thus shown to be topologically equivalent to the HOMFLYPT polynomial. This gives, in particular, a different proof of results of [1] about the Juyumaya–Lambropoulou invariants.

Notations

- We fix an integer $d \geq 1$, and we let $\{\xi_1, \dots, \xi_d\}$ be the set of roots of unity of order d. We will often use without mentioning that $\frac{1}{d}\sum_{0\leq s\leq d-1} \xi_a^s \xi_b^{-s}$ is equal to 1 if $a = b$ and is equal to 0 otherwise.
- Let $\mathcal{A}$ be an algebra defined over a commutative ring R. If R' is a commutative ring with a given ring homomorphism $\theta : R \to R'$, we will denote the specialized algebra $R'_\theta\mathcal{A} := R' \otimes_R \mathcal{A}$ where the tensor product is defined via θ. In particular, if R' is a commutative ring containing R as a subring, we denote simply by $R'\mathcal{A} := R' \otimes_R \mathcal{A}$ the algebra with ground ring extended to R'.
- We will denote by $\mathrm{M}_{i,j}$ an elementary matrix with 1 in position (i, j) and 0 everywhere else (the size of the matrix will always be given by the context).

2 Definitions and first properties

2.1 The Iwahori–Hecke algebra of type A

Let $n \in \mathbb{Z}_{\geq 1}$ and let u and v be indeterminates. The *Iwahori–Hecke algebra* $\mathcal{H}_n$ of type A_{n-1} is the associative $\mathbb{C}[u^{\pm 1}, v]$-algebra (with unit) with a presentation by generators :

$$T_1, \dots, T_{n-1},$$

and relations:

$$\begin{array}{rl} T_iT_j = T_jT_i & \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 1, \\ T_iT_{i+1}T_i = T_{i+1}T_iT_{i+1} & \text{for all } i = 1, \dots, n-2, \\ T_i^2 = u^2 + vT_i & \text{for all } i = 1, \dots, n-1. \end{array} \tag{1}$$

Note that $\mathcal{H}_1 = \mathbb{C}[u^{\pm 1}, v]$. It is convenient also to set $\mathcal{H}_0 := \mathbb{C}[u^{\pm 1}, v]$.

Let R be a ring and let $\theta : \mathbb{C}[u^{\pm 1}, v] \to R$ be a specialization such that $\theta(u^2) = 1$ and $\theta(v) = 0$ then the specialized algebra $R_\theta\mathcal{H}_n$ is naturally isomorphic to the group algebra $R\mathfrak{S}_n$ of the symmetric group.

Remark 2.1 Let q be an indeterminate. Another usual presentation of $\mathcal{H}_n$ is obtained by the specialization $\theta : \mathbb{C}[u^{\pm 1}, v] \to \mathbb{C}[q, q^{-1}]$ given by $\theta(u^2) = 1$ and $\theta(v) = q - q^{-1}$.

Let $w \in \mathfrak{S}_n$ and $s_{i_1} \dots s_{i_r}$ a reduced expression of w (where $(i_1, \dots, i_r) \in \{1, \dots, n-1\}^r$ and $s_i \in \mathfrak{S}_n$ denotes the transposition $(i, i+1)$ for $i = 1, \dots, n-1$). Then, by Matsumoto's lemma (see [7, §1.2]), the element $T_{i_1} \dots T_{i_r}$ does not depend of the choice of the reduced expression of w and thus, the element $T_w := T_{i_1} \dots T_{i_r}$ is well-defined. Then $\mathcal{H}_n$ is free as a $\mathbb{C}[u^{\pm 1}, v]$-module with basis $\{T_w \mid w \in \mathfrak{S}_n\}$ (see [7, Thm.4.4.6]). In particular it has dimension $n!$.

We also set $\widetilde{T}_i := u^{-1} T_i$, for $i \in \{1, \dots, n-1\}$ and,

$$\widetilde{T}_w := u^{-\ell(w)} T_w = \widetilde{T}_{i_1} \dots \widetilde{T}_{i_r}, \qquad \text{for } w \in \mathfrak{S}_n. \tag{2}$$

where $\ell(w)$ is the length of w. The set $\{\widetilde{T}_w \mid w \in \mathfrak{S}_n\}$ is also a $\mathbb{C}[u^{\pm 1}, v]$-basis of $\mathcal{H}_n$.

2.2 Compositions of *n*

Let $\mathrm{Comp}_d(n)$ be the set of *compositions* of n with d parts (or d-compositions of n), that is the set of d-tuples of non negative integers $\mu = (\mu_1, \dots, \mu_d)$ such that $\sum_{1 \le a \le d} \mu_a = n$. The set of d-compositions of n is denoted by $\mathrm{Comp}_d(n)$. We denote by $|\mu| := n$ the size of the composition μ.

For $\mu \in \mathrm{Comp}_d(n)$, the Young subgroup $\mathfrak{S}^\mu$ is the subgroup $\mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_d}$ of $\mathfrak{S}_n$, where $\mathfrak{S}_{\mu_1}$ acts on the letters $\{1, \dots, \mu_1\}$, $\mathfrak{S}_{\mu_2}$ acts on the letters $\{\mu_1 + 1, \dots, \mu_2\}$, and so on. The subgroup $\mathfrak{S}^\mu$ is generated by the transposition $(i, i+1)$ with $i \in I_\mu := \{1, \dots, n-1\} \backslash \{\mu_1, \mu_1 + \mu_2, \dots, \mu_1 + \cdots + \mu_{d-1}\}$.

Similarly, we have an associated subalgebra $\mathcal{H}^\mu$ of $\mathcal{H}_n$ generated by $\{T_i \mid i \in I_\mu\}$. This is a free $\mathbb{C}[u^{\pm 1}, v]$-module with basis $\{T_w \mid w \in \mathfrak{S}^\mu\}$ (or $\{\widetilde{T}_w \mid w \in \mathfrak{S}^\mu\}$). The algebra $\mathcal{H}^\mu$ is naturally isomorphic to $\mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_r}$, where the tensor products are over $\mathbb{C}[u^{\pm 1}, v]$. Note that the defining relations of $\mathcal{H}^\mu$ in terms of the generators T_i with $i \in I_\mu$, are the relations from (1) involving only those generators.

Let $\mu \in \mathrm{Comp}_d(n)$. For $a \in \{1, \dots, d\}$, we denote by $\mu^{[a]}$ the composition in $\mathrm{Comp}_d(n+1)$ defined by

$$\mu^{[a]}_b := \mu_b \ \text{ if } b \neq a, \qquad \text{and} \qquad \mu^{[a]}_a := \mu_a + 1. \tag{3}$$

Conversely, if $\mu_a \ge 1$, we define $\mu_{[a]} \in \mathrm{Comp}_d(n-1)$ to be the unique composition such that

$$(\mu_{[a]})^{[a]} = \mu. \tag{4}$$

We also define the *base* of μ, denoted by $[\mu]$, to be the d-composition defined by

$$[\mu]_a = \begin{cases} 1 & \text{if } \mu_a \ge 1, \\ 0 & \text{if } \mu_a = 0, \end{cases} \qquad \text{for } a = 1, \dots, d. \tag{5}$$

The composition $[\mu]$ belongs to $\mathrm{Comp}_d(N)$ where N is the number of non-zero parts in μ. We denote by $\mathrm{Comp}^0_d(n)$ the set of d-compositions of n where all the parts belong to $\{0, 1\}$, and we set

$$\mathrm{Comp}^0_d := \bigcup_{n \ge 1} \mathrm{Comp}^0_d(n) = \{[\mu] \mid \mu \in \bigcup_{n \ge 1} \mathrm{Comp}_d(n)\}. \tag{6}$$

2.3 The Yokonuma–Hecke algebra

We define the Yokonuma–Hecke algebra $Y_{d,n}$ as the associative $\mathbb{C}[u^{\pm 1}, v]$-algebra (with unit) with a presentation by generators:

$$g_1, g_2, \dots, g_{n-1}, t_1, \dots, t_n,$$

and relations:

$$\begin{array}{rll} g_i g_j &= g_j g_i & \text{for all } i, j = 1, \dots, n-1 \text{ such that } |i-j| > 1, \\ g_i g_{i+1} g_i &= g_{i+1} g_i g_{i+1} & \text{for all } i = 1, \dots, n-2, \\ t_i t_j &= t_j t_i & \text{for all } i, j = 1, \dots, n, \\ g_i t_j &= t_{s_i(j)} g_i & \text{for all } i = 1, \dots, n-1 \text{ and } j = 1, \dots, n, \\ t_j^d &= 1 & \text{for all } j = 1, \dots, n, \\ g_i^2 &= u^2 + v e_i g_i & \text{for all } i = 1, \dots, n-1, \end{array} \tag{7}$$

where, for all $i = 1, \dots, n-1$,

$$e_i := \frac{1}{d} \sum_{0 \le s \le d-1} t_i^s t_{i+1}^{-s}.$$

Note that the elements e_i are idempotents and that we have $g_i e_i = e_i g_i$ for all $i = 1, \dots, n-1$. The elements g_i are invertible, with

$$g_i^{-1} = u^{-2} g_i - u^{-2} v e_i, \qquad \text{for all } i = 1, \dots, n-1. \tag{8}$$

We also set

$$\widetilde{g}_i := u^{-1} g_i, \qquad \text{for } i \in \{1, \dots, n-1\}.$$

We note that $\widetilde{g}_i^2 = 1 + u^{-1} v e_i \widetilde{g}_i$ and also that $\widetilde{g}_i^{-1} = \widetilde{g}_i - u^{-1} v e_i$, for $i = 1, \dots, n-1$.

Let R be a ring and let $\theta : \mathbb{C}[u^{\pm 1}, v] \to R$ be a specialization such that $\theta(u^2) = 1$ and $\theta(v) = 0$ then the specialized algebra $R_\theta Y_{d,n}$ is naturally isomorphic to the group algebra $RG(d, 1, n)$ of the complex reflection group $G(d, 1, n) \cong (\mathbb{Z}/d\mathbb{Z}) \wr \mathfrak{S}_n$. Note that in the case where $d = 1$ then $Y_{1,n}$ is nothing but the Iwahori–Hecke algebra of type A_{n-1} as defined in Sect. 2.1.

Remark 2.2 The presentation used in [3] of the Yokonuma–Hecke algebra is obtained, similarly to Remark 2.1, by a specialization $\theta : \mathbb{C}[u^{\pm 1}, v] \to \mathbb{C}[q, q^{-1}]$ such that q is an indeterminate, $\theta(u^2) = 1$ and $\theta(v) = q - q^{-1}$. The precise connections between the presentation above and the presentation used in [2,10,12–15] will be carefully investigated in Sect. 6 (see also [3, Remark1]).

Remark 2.3 Both the Iwahori-Hecke algebras and the Yokonuma-Hecke algebras can be defined over more general rings. However, for our purpose (see §2.4), we need to assume that the base ring contains the d-roots of the unity. Hence, for convenience, we here choose to work over the ring $\mathbb{C}[u^{\pm 1}, v]$.

We set, for $w \in \mathfrak{S}_n$ and $s_{i_1} \dots s_{i_r}$ a reduced expression for w,

$$g_w := g_{i_1} \dots g_{i_r} \qquad \text{and} \qquad \widetilde{g}_w := u^{-\ell(w)} g_w = \widetilde{g}_{i_1} \dots \widetilde{g}_{i_r}. \tag{9}$$

Again, by Matsumoto's lemma (see [7, §1.2]), the elements g_w and $\widetilde{g}_w$ are well-defined. The following multiplication rules in $Y_{d,n}$ follow directly from the definitions. For $w \in \mathfrak{S}_n$ and $i \in \{1, \dots, n-1\}$, we have

$$\widetilde{g}_w \widetilde{g}_i = \begin{cases} \widetilde{g}_{ws_i} & \text{if } l(ws_i) > l(w), \\ \widetilde{g}_{ws_i} + u^{-1} v \widetilde{g}_w e_i & \text{if } l(ws_i) < l(w); \end{cases} \tag{10}$$

$$\widetilde{g}_i \widetilde{g}_w = \begin{cases} \widetilde{g}_{s_i w} & \text{if } l(s_i w) > l(w), \\ \widetilde{g}_{s_i w} + u^{-1} v e_i \widetilde{g}_w & \text{if } l(s_i w) < l(w). \end{cases} \tag{11}$$

By [10] and Remark 2.2, $Y_{d,n}$ is a free $\mathbb{C}[u^{\pm 1}, v]$-module with basis

$$\{t_1^{k_1} \dots t_n^{k_n} \widetilde{g}_w \mid w \in \mathfrak{S}_n,\ k_1, \dots, k_n \in \mathbb{Z}/d\mathbb{Z}\} \tag{12}$$

and the rank of $Y_{d,n}$ is $d^n n!$. The algebra $Y_{d,n-1}$ naturally embeds in the algebra $Y_{d,n}$ in an obvious way.

Remark 2.4 For the isomorphism theorem, we will mainly use the elements $\widetilde{g}_i$ and $\widetilde{g}_w$ instead of g_i and g_w. Concerning this part, we could have given the presentation of $Y_{d,n}$ in terms of the generators $\widetilde{g}_i$ and thus remove one of the variables u or v. However, the generators g_i will be used systematically starting from Sect. 5 for applications to links theory.

2.4 A decomposition of $Y_{d,n}$

We consider the commutative subalgebra $\mathcal{A}_n := \langle t_1, \dots, t_n \rangle$ of $Y_{d,n}$. This algebra is naturally isomorphic to the group algebra of $(\mathbb{Z}/d\mathbb{Z})^n$ over $\mathbb{C}[u^{\pm 1}, v]$, and we will always implicitly make this identification in the following.

A complex character χ of the group $(\mathbb{Z}/d\mathbb{Z})^n$ is characterized by the choice of $\chi(t_j) \in \{\xi_1, \dots, \xi_d\}$ for each $j = 1, \dots, n$. We denote by $\mathrm{Irr}(\mathcal{A}_n)$ the set of complex characters of $(\mathbb{Z}/d\mathbb{Z})^n$, extended to $\mathcal{A}_n$.

Definition 2.5 For each $\chi \in \mathrm{Irr}(\mathcal{A}_n)$, we denote by E_χ the primitive idempotent of $\mathcal{A}_n$ associated to χ, that is, characterized by $\chi'(E_\chi) = 0$ if $\chi' \neq \chi$ and $\chi(E_\chi) = 1$.

The idempotent E_χ is explicitly written in terms of the generators as follows:

$$E_\chi = \prod_{1 \leq i \leq n} \left(\frac{1}{d} \sum_{0 \leq s \leq d-1} \chi(t_i)^s t_i^{-s} \right). \tag{13}$$

By definition, we have, for all $\chi \in \mathrm{Irr}(\mathcal{A}_n)$ and $i = 1, \dots, n$,

$$t_i E_\chi = E_\chi t_i = \chi(t_i) E_\chi. \tag{14}$$

The symmetric group $\mathfrak{S}_n$ acts by permutations on $(\mathbb{Z}/d\mathbb{Z})^n$ and in turn acts on $\mathrm{Irr}(\mathcal{A}_n)$. The action is given by the formula:

$$w(\chi)(t_i) = \chi(t_{w^{-1}(i)}), \qquad \text{for all } i = 1, \dots, n,\ w \in \mathfrak{S}_n \quad \text{and } \chi \in \mathrm{Irr}(\mathcal{A}_n).$$

In the algebra $Y_{d,n}$, due to the relation $g_w t_i = t_{w(i)} g_w$ for $i = 1, \dots, n$ and $w \in \mathfrak{S}_n$, we have

$$g_w E_\chi = E_{w(\chi)} g_w \qquad \text{and} \qquad \widetilde{g}_w E_\chi = E_{w(\chi)} \widetilde{g}_w. \tag{15}$$

Let $\chi \in \mathrm{Irr}(\mathcal{A}_n)$. For $a = 1, \dots, d$, denote by μ_a the cardinal of the set $\{j \in \{1, \dots, n\} \mid \chi(t_j) = \xi_a\}$. Then the sequence $(\mu_1, \dots, \mu_d)$ is a d-composition of n which is denoted by

$$\mathrm{Comp}(\chi) := (\mu_1, \dots, \mu_d) \in \mathrm{Comp}_d(n).$$

Let $\mu \in \mathrm{Comp}_d(n)$. Then we denote by

$$\mathcal{O}(\mu) := \{\chi \in \mathrm{Irr}(\mathcal{A}_n) \mid \mathrm{Comp}(\chi) = \mu\}$$

the orbit of the element $\chi \in \mathrm{Irr}(\mathcal{A}_n)$ under the action of the symmetric group and

$$m_\mu := \sharp\mathcal{O}_\mu = \frac{n!}{\mu_1!\mu_2!\dots\mu_d!}.$$

Definition 2.6 Let $\mu \in \mathrm{Comp}_d(n)$. We set

$$E_\mu := \sum_{\mathrm{Comp}(\chi)=\mu} E_\chi = \sum_{\chi\in\mathcal{O}(\mu)} E_\chi.$$

Due to the commutation relation (15), the elements E_μ, with $\mu \in \mathrm{Comp}_d(n)$, are central in $Y_{d,n}$. Moreover, as the set $\{E_\chi \mid \chi \in \mathrm{Irr}(\mathcal{A}_n)\}$ is a complete set of orthogonal idempotents, it follows at once that the set $\{E_\mu \mid \mu \in \mathrm{Comp}_d(n)\}$ forms a complete set of central orthogonal idempotents in $Y_{d,n}$. In particular, we have the following decomposition of $Y_{d,n}$ into a direct sum of two-sided ideals:

$$Y_{d,n} = \bigoplus_{\mu\in\mathrm{Comp}_d(n)} E_\mu Y_{d,n}. \tag{16}$$

2.5 Another basis for $Y_{d,n}$

We here give another basis for $Y_{d,n}$ using the idempotents we just defined. As the subalgebra $\mathcal{A}_n$ of $Y_{d,n}$ is isomorphic to the group algebra of $(\mathbb{Z}/d\mathbb{Z})^n$ over $\mathbb{C}[u^{\pm1}, v]$, the set $\{E_\chi \mid \chi \in \mathrm{Irr}(\mathcal{A}_n)\}$ is a $\mathbb{C}[u^{\pm1}, v]$-basis of $\mathcal{A}_n$, as well as the set $\{t_1^{k_1}\dots t_n^{k_n} \mid k_1, \dots, k_n \in \mathbb{Z}/d\mathbb{Z}\}$. So from the knowledge of the $\mathbb{C}[u^{\pm1}, v]$-basis (12) of $Y_{d,n}$, we also have that the set

$$\{E_\chi \widetilde{g}_w \mid \chi \in \mathrm{Irr}(\mathcal{A}_n),\ w \in \mathfrak{S}_n\} \tag{17}$$

is a $\mathbb{C}[u^{\pm1}, v]$-basis of $Y_{d,n}$. Moreover, this basis is compatible with the decomposition (16) of $Y_{d,n}$ since, for $\mu \in \mathrm{Comp}_d(n)$, we have $E_\chi \widetilde{g}_w \in E_\mu Y_{d,n}$ if and only if $\mathrm{Comp}(\chi) = \mu$. In other words, the set

$$\{E_\chi \widetilde{g}_w \mid \chi \in \mathrm{Irr}(\mathcal{A}_n) \text{ with } \mathrm{Comp}(\chi) = \mu,\ \ w \in \mathfrak{S}_n\}$$

is a $\mathbb{C}[u^{\pm1}, v]$-basis of $E_\mu Y_{d,n}$.

Now we will label the elements of $\mathrm{Irr}(\mathcal{A}_n)$ in a useful way for the following. This is done as follows. We first consider a distinguished element in each orbit $\mathcal{O}(\mu)$. Let $\mu \in \mathrm{Comp}_d(n)$. We denote

$$\chi_1^\mu \in \mathrm{Irr}(\mathcal{A}_n),$$

the character given by

$$\begin{cases} \chi_1^\mu(t_1) = \dots = \chi_1^\mu(t_{\mu_1}) = \xi_1, \\ \chi_1^\mu(t_{\mu_1+1}) = \dots = \chi_1^\mu(t_{\mu_1+\mu_2}) = \xi_2, \\ \vdots \quad\ \vdots \quad\ \vdots \quad\ \vdots \quad\ \vdots \quad\ \vdots \\ \chi_1^\mu(t_{\mu_1+\dots+\mu_{d-1}+1}) = \dots = \chi_1^\mu(t_{\mu_d}) = \xi_d. \end{cases} \tag{18}$$

Note that the stabilizer of χ_1^μ under the action of $\mathfrak{S}_n$ is the Young subgroup $\mathfrak{S}^\mu$. In each left coset in $\mathfrak{S}_n/\mathfrak{S}^\mu$, we take a representative of minimal length (such a representative is unique, see [7, §2.1]). We denote by

$$\{\pi_{1,\mu}, \dots, \pi_{m_\mu,\mu}\}$$

this set of distinguished left coset representatives of $\mathfrak{S}_n/\mathfrak{S}^\mu$ with the convention that $\pi_{1,\mu} = 1$ (recall that $m_\mu := \sharp\mathcal{O}(\mu)$). Then, if we set for all $k = 1, \ldots, m_\mu$:

$$\chi_k^\mu := \pi_{k,\mu}(\chi_1^\mu), \tag{19}$$

we have by construction that

$$\mathcal{O}(\mu) = \{\chi_1^\mu, \ldots, \chi_{m_\mu}^\mu\}.$$

To sum up, we have the following $\mathbb{C}[u^{\pm 1}, v]$-basis of $Y_{d,n}$:

$$\{E_{\chi_k^\mu}\widetilde{g}_w \mid w \in \mathfrak{S}_n,\ k = 1, \ldots, m_\mu,\ \mu \in \mathrm{Comp}_d(n)\}, \tag{20}$$

where, for each $\mu \in \mathrm{Comp}_d(n)$, the subset $\{E_{\chi_k^\mu}\widetilde{g}_w \mid w \in \mathfrak{S}_n,\ k = 1, \ldots, m_\mu\}$ is a $\mathbb{C}[u^{\pm 1}, v]$-basis of the two-sided ideal $E_\mu Y_{d,n}$.

3 The isomorphism theorem

The aim of this part is to prove that $Y_{d,n}$ and $\bigoplus_{\mu\in\mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu)$ are isomorphic as $\mathbb{C}[u^{\pm 1}, v]$-algebras. We will exhibit an explicit isomorphism between the two algebras.

3.1 The statement

Let $\mu \in \mathrm{Comp}_d(n)$. We recall that $E_\mu Y_{d,n}$ is a two-sided ideal of $Y_{d,n}$ and is also a unital subalgebra with unit E_μ. We define a linear map

$$\Phi_\mu : \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu) \to E_\mu Y_{d,n},$$

by setting, for any matrix consisting of basis elements $\widetilde{T}_{w_{i,j}}$ of $\mathcal{H}^\mu$ (that is, with $w_{i,j} \in \mathfrak{S}^\mu$),

$$\Phi_\mu\big((\widetilde{T}_{w_{i,j}})_{1\le i,j\le m_\mu}\big) = \sum_{1\le i,j\le m_\mu} E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu} w_{i,j} \pi_{j,\mu}^{-1}}\, E_{\chi_j^\mu}. \tag{21}$$

We also define a linear map

$$\Psi_\mu : E_\mu Y_{d,n} \to \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu),$$

as follows. Let $k \in \{1, \ldots, m_\mu\}$ and $w \in \mathfrak{S}_n$, and let $j \in \{1, \ldots, m_\mu\}$ be uniquely defined (given k) by the relation $w(\chi_j^\mu) = \chi_k^\mu$. Note that we thus have $\pi_{k,\mu}^{-1} w \pi_{j,\mu} \in \mathfrak{S}^\mu$. We then set

$$\Psi_\mu(E_{\chi_k^\mu}\widetilde{g}_w) = \widetilde{T}_{\pi_{k,\mu}^{-1} w \pi_{j,\mu}}\, \mathrm{M}_{k,j}, \tag{22}$$

where we recall that $\mathrm{M}_{k,j}$ denotes the elementary matrix with 1 in position (k, j).

Now we can state the main result of this section. Recall the decomposition (16) of $Y_{d,n}$.

Theorem 3.1 *Let $\mu \in \mathrm{Comp}_d(n)$. The linear map Φ_μ is an isomorphism of $\mathbb{C}[u^{\pm 1}, v]$-algebra with inverse map Ψ_μ. In turn,*

$$\Phi_n := \bigoplus_{\mu\in\mathrm{Comp}_d(n)} \Phi_\mu : \bigoplus_{\mu\in\mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu) \to Y_{d,n}$$

is an isomorphism with inverse map:

$$\Psi_n := \bigoplus_{\mu\in\mathrm{Comp}_d(n)} \Psi_\mu : Y_{d,n} \to \bigoplus_{\mu\in\mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu).$$

The rest of this section is devoted to the proof of the theorem.

3.2 Preliminary results

We first prove a series of useful lemmas.

Lemma 3.2 *Let $\mu \in \mathrm{Comp}_d(n)$ and $i \in \{1, \dots, m_\mu\}$. We consider a reduced expression $s_{i_1} \dots s_{i_k}$ of $\pi_{i,\mu}$. Then for all $l \in \{1, \dots, k\}$, we have:*

$$e_{i_l} E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)} = E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)} e_{i_l} = 0$$

Proof By definition, $\pi_{i,\mu}$ is the (unique) element of $\mathfrak{S}_n$ with minimal length satisfying $\pi_{i,\mu}(\chi_1^\mu) = \chi_i^\mu$. As a consequence, we have for all $l = 1, \dots, k$:

$$s_{i_l} \cdot s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu) \neq s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)$$

which is equivalent to

$$s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)(t_{i_l}) \neq s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)(t_{i_l+1}).$$

Thus by (14), we have

$$t_{i_l} E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)} \neq t_{i_l+1} E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)}.$$

This discussion shows that

$$t_{i_l} t_{i_{l+1}}^{-1} E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)} = \xi_j E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)}$$

for a d-root of unity $\xi_j \neq 1$. We conclude that

$$E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)} e_{i_l} = e_{i_l} E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)} = \left(\sum_{0 \leq s \leq d-1} \xi_j^s \right) E_{s_{i_{l+1}} \dots s_{i_k}(\chi_1^\mu)} = 0$$

where we note, for the first equality, that e_{i_l} commutes with any E_χ. □

Lemma 3.3 *For all $\mu \in \mathrm{Comp}_d(n)$, $1 \leq i, j \leq m_\mu$ and $w \in \mathfrak{S}_n$, we have:*

(i) $E_{\chi_1^\mu} \widetilde{g}_{\pi_{i,\mu}}^{-1} \widetilde{g}_w \widetilde{g}_{\pi_{j,\mu}} E_{\chi_1^\mu} = E_{\chi_1^\mu} \widetilde{g}_{\pi_{i,\mu}^{-1} w \pi_{j,\mu}} E_{\chi_1^\mu}$;
(ii) $E_{\chi_i^\mu} \widetilde{g}_{\pi_{i,\mu}} \widetilde{g}_w \widetilde{g}_{\pi_{j,\mu}}^{-1} E_{\chi_j^\mu} = E_{\chi_i^\mu} \widetilde{g}_{\pi_{i,\mu} w \pi_{j,\mu}^{-1}} E_{\chi_j^\mu}$.

Proof Let us denote a reduced expression of $\pi_{i,\mu}$ by $s_{i_1} \dots s_{i_k}$. We have :

$$\begin{aligned} E_{\chi_1^\mu} \widetilde{g}_{\pi_{i,\mu}}^{-1} \widetilde{g}_w &= E_{\chi_1^\mu} \widetilde{g}_{i_k}^{-1} \dots \widetilde{g}_{i_1}^{-1} \widetilde{g}_w \\ &= \widetilde{g}_{i_k}^{-1} \dots \widetilde{g}_{i_2}^{-1} E_{s_{i_2} \dots s_{i_k}(\chi_1^\mu)} \widetilde{g}_{i_1}^{-1} \widetilde{g}_w. \end{aligned}$$

Recall that for all $j = 1, \dots, n$, we have $\widetilde{g}_j^{-1} = \widetilde{g}_j - u^{-1} v e_j$. Thus, with repeated applications of Lemma 3.2 together with the multiplication rule (11), we deduce that:

$$\begin{aligned} E_{\chi_1^\mu} \widetilde{g}_{\pi_{i,\mu}}^{-1} \widetilde{g}_w &= \widetilde{g}_{i_k}^{-1} \dots \widetilde{g}_{i_2}^{-1} E_{s_{i_2} \dots s_{i_k}(\chi_1^\mu)} \widetilde{g}_{s_{i_1} w} \\ &= \widetilde{g}_{i_k}^{-1} \dots \widetilde{g}_{i_3}^{-1} E_{s_{i_3} \dots s_{i_k}(\chi_1^\mu)} \widetilde{g}_{i_2}^{-1} \widetilde{g}_{s_{i_1} w} \\ &= \dots \\ &= E_{\chi_1^\mu} \widetilde{g}_{\pi_{i,\mu}^{-1} w}. \end{aligned}$$

Now let us denote by $s_{j_1} \dots s_{j_l}$ a reduced expression of $\pi_{j,\mu}$, we have:

$$\begin{aligned} E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}}^{-1}\, \widetilde{g}_w\, \widetilde{g}_{\pi_{j,\mu}}\, E_{\chi_1^\mu} &= E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}^{-1} w}\, \widetilde{g}_{\pi_{j,\mu}}\, E_{\chi_1^\mu} \\ &= E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}^{-1} w}\, \widetilde{g}_{j_1} \dots \widetilde{g}_{j_l}\, E_{\chi_1^\mu} \\ &= E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}^{-1} w}\, \widetilde{g}_{j_1}\, E_{s_{j_2} \dots s_{j_l}(\chi_1^\mu)}\, \widetilde{g}_{i_2} \dots \widetilde{g}_{i_l}. \end{aligned}$$

As above, with repeated applications of Lemma 3.2 together with the multiplication rule (10), we obtain:

$$\begin{aligned} E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}}^{-1}\, \widetilde{g}_w\, \widetilde{g}_{\pi_{j,\mu}}\, E_{\chi_1^\mu} &= E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}^{-1} w s_{j_1}}\, E_{s_{j_2} \dots s_{j_l}(\chi_1^\mu)}\, \widetilde{g}_{j_2} \dots \widetilde{g}_{j_l} \\ &= E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}^{-1} w s_{j_1}}\, \widetilde{g}_{j_2}\, E_{s_{j_3} \dots s_{j_l}(\chi_1^\mu)}\, \widetilde{g}_{j_3} \dots \widetilde{g}_{j_l} \\ &= \dots \\ &= E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}^{-1} w \pi_{j,\mu}}\, E_{\chi_1^\mu}, \end{aligned}$$

which proves item (i). Let us now prove item (ii). We have by (19) and (15) :

$$\begin{aligned} E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu}}\, \widetilde{g}_w\, \widetilde{g}_{\pi_{j,\mu}}^{-1}\, E_{\chi_j^\mu} &= \widetilde{g}_{\pi_{i,\mu}}\, E_{\chi_1^\mu}\, \widetilde{g}_w\, E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{j,\mu}}^{-1} \\ &= \widetilde{g}_{\pi_{i,\mu}}\, E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}^{-1} \pi_{i,\mu} w \pi_{j,\mu}^{-1} \pi_{j,\mu}}\, E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{j,\mu}}^{-1} \\ &= \widetilde{g}_{\pi_{i,\mu}}\, E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}}^{-1}\, \widetilde{g}_{\pi_{i,\mu} w \pi_{j,\mu}^{-1}}\, \widetilde{g}_{\pi_{j,\mu}}\, E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{j,\mu}}^{-1}, \end{aligned}$$

where the last equality comes from item (i). The proof is concluded using that $\widetilde{g}_{\pi_{i,\mu}} E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{i,\mu}}^{-1} = E_{\chi_i^\mu}$ and $\widetilde{g}_{\pi_{j,\mu}} E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{j,\mu}}^{-1} = E_{\chi_j^\mu}$. □

Lemma 3.4 *Let $\mu \in \mathrm{Comp}_d(n)$. The map*

$$\phi_\mu : \mathcal{H}^\mu \to E_{\chi_1^\mu} Y_{d,n} E_{\chi_1^\mu},$$

defined on the generators by

$$\forall i \in I_\mu,\ \phi_\mu(T_i) = E_{\chi_1^\mu} g_i E_{\chi_1^\mu},$$

extends to an homomorphism of algebras.

Proof Recall that the subspace $E_{\chi_1^\mu} Y_{d,n} E_{\chi_1^\mu}$ is a unital subalgebra of $Y_{d,n}$ with unit $E_{\chi_1^\mu}$.

We first note that if $i \in I_\mu$ then $s_i(\chi_1^\mu) = \chi_1^\mu$. We thus have

$$g_i E_{\chi_1^\mu} = E_{\chi_1^\mu} g_i$$

and this relation easily implies that the elements $E_{\chi_1^\mu} g_i E_{\chi_1^\mu}$ with $i \in I_\mu$ satisfy the braid relations. It remains to check the "quadratic relation". We have

$$\begin{aligned} (E_{\chi_1^\mu} g_i E_{\chi_1^\mu})^2 &= E_{\chi_1^\mu} g_i^2 E_{\chi_1^\mu} \\ &= u^2 E_{\chi_1^\mu} + v E_{\chi_1^\mu} e_i g_i E_{\chi_1^\mu} \\ &= u^2 E_{\chi_1^\mu} + v E_{\chi_1^\mu} g_i E_{\chi_1^\mu} \end{aligned}$$

The last equality comes from the fact that for $i \in I_\mu$ we have $t_i E_{\chi_1^\mu} = t_{i+1} E_{\chi_1^\mu}$, and thus

$$e_i E_{\chi_1^\mu} = E_{\chi_1^\mu} e_i = E_{\chi_1^\mu}.$$

Thus all the defining relations of $\mathcal{H}^\mu = \mathcal{H}_{\mu_1} \otimes \dots \otimes \mathcal{H}_{\mu_d}$ are satisfied so that ϕ_μ can be extended to a homomorphism of algebras. □

Remark 3.5 One can actually show that the morphism ϕ_μ is an isomorphism. Indeed the lemma implies that ϕ_μ is given on the standard basis of $\mathcal{H}^\mu$ by $\phi_\mu(T_w) = E_{\chi_1^\mu} g_w E_{\chi_1^\mu}$, $w \in \mathfrak{S}^\mu$. Moreover, if $w \in \mathfrak{S}^\mu$ then $w(\chi_1^\mu) = \chi_1^\mu$ and therefore $\phi_\mu(T_w) = E_{\chi_1^\mu} g_w$. So it remains to check that $\{E_{\chi_1^\mu} g_w \mid w \in \mathfrak{S}^\mu\}$ is a basis of $E_{\chi_1^\mu} Y_{d,n} E_{\chi_1^\mu}$. The linear independence is immediate, while the spanning property follows from the following calculation, for a basis element $E_{\chi_i^\nu} g_w$ of $Y_{d,n}$ and $\nu \in \mathrm{Comp}_d(n)$,

$$E_{\chi_1^\mu} \cdot E_{\chi_i^\nu} g_w \cdot E_{\chi_1^\mu} = E_{\chi_1^\mu} E_{\chi_i^\nu} E_{w(\chi_1^\mu)} g_w = \begin{cases} E_{\chi_1^\mu} g_w & \text{if } \mu = \nu, i = 1 \text{ and } w \in \mathfrak{S}^\mu; \\ 0 & \text{otherwise.} \end{cases}$$

In the following, we will not use the fact that ϕ_μ is actually an isomorphism (actually, it is a consequence of Theorem 3.1 below).

3.3 Proof of the main result

Proof of Theorem 3.1 We are now in position to prove Theorem 3.1. Let $\mu \in \mathrm{Comp}_d(n)$.

1. We first prove that Φ_μ is a morphism. Before this, we note that by Lemma 3.3(ii), for all $1 \leq i, j \leq m_\mu$ and $w \in \mathfrak{S}^\mu$, we have:

$$\begin{aligned} E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu} w \pi_{j,\mu}^{-1}}\, E_{\chi_j^\mu} &= E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu}}\, \widetilde{g}_w\, \widetilde{g}_{\pi_{j,\mu}}^{-1}\, E_{\chi_j^\mu} \\ &= \widetilde{g}_{\pi_{i,\mu}}\, E_{\chi_1^\mu}\, \widetilde{g}_w\, E_{\chi_1^\mu}\, \widetilde{g}_{\pi_{j,\mu}}^{-1} \\ &= \widetilde{g}_{\pi_{i,\mu}}\, \phi_\mu(\widetilde{T}_w)\, \widetilde{g}_{\pi_{j,\mu}}^{-1}. \end{aligned} \tag{23}$$

Now, let $i, j, k, l \in \{1, \ldots, m_\mu\}$ and $w, w' \in \mathfrak{S}^\mu$. We have:

$$\Phi_\mu\big(\widetilde{T}_w\, \mathrm{M}_{i,j}\big)\, \Phi_\mu\big(\widetilde{T}_{w'}\, \mathrm{M}_{k,l}\big) = E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu} w \pi_{j,\mu}^{-1}}\, E_{\chi_j^\mu}\, E_{\chi_k^\mu}\, \widetilde{g}_{\pi_{k,\mu} w' \pi_{l,\mu}^{-1}}\, E_{\chi_l^\mu}.$$

As $E_{\chi_j^\mu}$ and $E_{\chi_k^\mu}$ belong to a family of pairwise orthogonal idempotents, this is equal to 0 if $j \neq k$. On the other hand, we also have that $\widetilde{T}_w\, \mathrm{M}_{i,j} \cdot \widetilde{T}_{w'}\, \mathrm{M}_{k,l}$ is equal to 0 if $j \neq k$.

So it remains only to consider the situation $j = k$. If $j = k$, we obtain

$$\begin{aligned} \Phi_\mu\big(\widetilde{T}_w\, \mathrm{M}_{i,j}\big)\, \Phi_\mu\big(\widetilde{T}_{w'}\, \mathrm{M}_{j,l}\big) &= E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu} w \pi_{j,\mu}^{-1}}\, E_{\chi_j^\mu} \cdot E_{\chi_j^\mu}\, \widetilde{g}_{\pi_{j,\mu} w' \pi_{l,\mu}^{-1}}\, E_{\chi_l^\mu} \\ &= \widetilde{g}_{\pi_{i,\mu}}\, \phi_\mu(\widetilde{T}_w)\, \widetilde{g}_{\pi_{j,\mu}}^{-1} \cdot \widetilde{g}_{\pi_{j,\mu}}\, \phi_\mu(\widetilde{T}_{w'})\, \widetilde{g}_{\pi_{l,\mu}}^{-1} \\ &= \widetilde{g}_{\pi_{i,\mu}}\, \phi_\mu(\widetilde{T}_w \cdot \widetilde{T}_{w'})\, \widetilde{g}_{\pi_{l,\mu}}^{-1}, \end{aligned} \tag{24}$$

where we used successively (23) and Lemma 3.4. On the other hand, we have that $\widetilde{T}_w\, \mathrm{M}_{i,j} \cdot \widetilde{T}_{w'}\, \mathrm{M}_{j,l}$ is equal to $\widetilde{T}_w\, \widetilde{T}_{w'}\, \mathrm{M}_{i,l}$. The product $\widetilde{T}_w\, \widetilde{T}_{w'}$ can be written uniquely as

$$\widetilde{T}_w\, \widetilde{T}_{w'} = \sum_{x \in \mathfrak{S}^\mu} c_x \widetilde{T}_x,$$

for some coefficients $c_x \in \mathbb{C}[u^{\pm 1}, v]$. We have now

$$\begin{aligned} \Phi_\mu\big(\widetilde{T}_w\, \widetilde{T}_{w'}\, \mathrm{M}_{i,l}\big) &= \sum_{x \in \mathfrak{S}^\mu} c_x\, E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu} x \pi_{j,\mu}^{-1}}\, E_{\chi_j^\mu} \\ &= \sum_{x \in \mathfrak{S}^\mu} c_x\, \widetilde{g}_{\pi_{i,\mu}}\, \phi_\mu(\widetilde{T}_x)\, \widetilde{g}_{\pi_{l,\mu}}^{-1}, \end{aligned}$$

by (23) again. Comparing with (24) concludes the verification that Φ_μ is a morphism of algebras.

2. We now prove that Φ_μ and Ψ_μ are inverse maps. Let $w \in \mathfrak{S}_n$ and let $i \in \{1, \dots, m_\mu\}$. Let also $j \in \{1, \dots, m_\mu\}$ be uniquely defined by $\chi_i^\mu = w(\chi_j^\mu)$. By definition of Φ_μ and Ψ_μ, we have

$$\Phi_\mu \circ \Psi_\mu(E_{\chi_i^\mu}\, \widetilde{g}_w) \;=\; \Phi_\mu(\widetilde{T}_{\pi_{i,\mu}^{-1} w \pi_{j,\mu}}\, \mathrm{M}_{i,j}) \;=\; E_{\chi_i^\mu}\, \widetilde{g}_w\, E_{\chi_j^\mu}.$$

As $\chi_j^\mu = w^{-1}(\chi_i^\mu)$ and $E_{\chi_i^\mu}$ is an idempotent, we conclude that this is indeed equal to $E_{\chi_i^\mu} \widetilde{g}_w$.

On the other hand, let $w \in \mathfrak{S}^\mu$ and $i, j \in \{1, \dots, m_\mu\}$.

$$\begin{aligned}\Psi_\mu \circ \Phi_\mu\big(\widetilde{T}_w\, \mathrm{M}_{i,j}\big) &= \Psi_\mu\big(E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu} w \pi_{j,\mu}^{-1}}\, E_{\chi_j^\mu}\big)\\ &= \Psi_\mu\big(E_{\chi_i^\mu}\, \widetilde{g}_{\pi_{i,\mu} w \pi_{j,\mu}^{-1}}\big) \qquad \text{(because } \pi_{i,\mu} w_{i,j} \pi_{j,\mu}^{-1}(\chi_j^\mu) = \chi_i^\mu\text{)}\\ &= \widetilde{T}_{\pi_{i,\mu}^{-1}\pi_{i,\mu} w \pi_{j,\mu}^{-1}\pi_{j',\mu}}\, \mathrm{M}_{i,j'},\end{aligned}$$

where the integer $j' \in \{1, \dots, m_\mu\}$ is uniquely defined by $\pi_{i,\mu} w \pi_{j,\mu}^{-1}(\chi_{j'}^\mu) = \chi_i^\mu$. As $w \in \mathfrak{S}^\mu$, this condition yields $j' = j$, which concludes the proof. □

Example 3.6 Let $d = 2$ and $n = 4$. We will give explicitly in this example the images of $g_1, \dots, g_{n-1}, t_1, \dots, t_n$ and $e_1, \dots, e_{n-1}$ of $Y_{d,n}$ under the isomorphism Ψ_n of Theorem 3.1. In the matrices below, the dots stand for coefficients equal to 0.

First, we note that, for any $\mu \in \mathrm{Comp}_d(n)$, the matrix $\Psi_\mu(t_i) (i \in \{1, \dots, n\})$ is diagonal, more precisely, we have:

$$\begin{aligned}\Psi_\mu(t_i) = \Psi_\mu(E_\mu t_i) = \Psi_\mu\left(\sum_{1\le k\le m_\mu} E_{\chi_k^\mu} t_i\right) &= \Psi_\mu\left(\sum_{1\le k\le m_\mu} \chi_k^\mu(t_i) E_{\chi_k^\mu}\right)\\ &= \sum_{1\le k\le m_\mu} \chi_k^\mu(t_i)\, \mathrm{M}_{k,k}.\end{aligned}$$

We will denote by $\mathrm{Diag}(x_1, \dots, x_N)$ a diagonal matrix with coefficients $x_1, \dots, x_N$ on the diagonal. We also recall that, for $i = 1, \dots, n-1$, we have $g_i = u\widetilde{g}_i$ and

$$\Psi_\mu(\widetilde{g}_i) = \Psi_\mu(E_\mu \widetilde{g}_i) = \Psi_\mu\left(\sum_{1\le k\le m_\mu} E_{\chi_k^\mu} \widetilde{g}_i\right) = \sum_{1\le k\le m_\mu} \widetilde{T}_{\pi_{k,\mu}^{-1} s_i \pi_{j_k,\mu}}\, \mathrm{M}_{k,j_k},$$

where, for each $k \in \{1, \dots, m_\mu\}$, the integer $j_k \in \{1, \dots, m_\mu\}$ is uniquely determined by $s_i(\chi_{j_k}^\mu) = \chi_k^\mu$.

- Let $\mu = (4, 0)$ or $\mu = (0, 4)$. Then $m_\mu = 1$ and $\mathcal{H}^\mu \cong \mathcal{H}_4$. There is only one character in the orbit $\mathcal{O}(\mu)$, which is $\chi_1^\mu = (\xi_a, \xi_a, \xi_a, \xi_a)$, where $a = 1$ if $\mu = (4, 0)$ and $a = 2$ if $\mu = (0, 4)$. In this situation, we have

$$g_i \mapsto (T_i), \qquad t_j \mapsto (\xi_a), \qquad e_i \mapsto (1), \qquad \text{for } i = 1, 2, 3 \text{ and } j = 1, 2, 3, 4,$$

  where $a = 1$ if $\mu = (4, 0)$ and $a = 2$ if $\mu = (0, 4)$.
- Let $\mu = (3, 1)$. Then $m_\mu = 4$ and $\mathcal{H}^\mu \cong \mathcal{H}_3 \otimes \mathcal{H}_1$ and we identify it below with $\mathcal{H}_3$. We order the characters in the orbit $\mathcal{O}(\mu)$ as follows:

$$\begin{aligned}&\chi_1^\mu = (\xi_1, \xi_1, \xi_1, \xi_2), \quad \chi_2^\mu = (\xi_1, \xi_1, \xi_2, \xi_1),\\ &\chi_3^\mu = (\xi_1, \xi_2, \xi_1, \xi_1) \quad \text{and} \quad \chi_4^\mu = (\xi_2, \xi_1, \xi_1, \xi_1).\end{aligned}$$

Thus we have $\pi_{1,\mu} = 1$, $\pi_{2,\mu} = s_3$, $\pi_{3,\mu} = s_2 s_3$ and $\pi_{4,\mu} = s_1 s_2 s_3$. The map Ψ_μ is given by:

$$g_1 \mapsto \begin{pmatrix} T_1 & \cdot & \cdot & \cdot \\ \cdot & T_1 & \cdot & \cdot \\ \cdot & \cdot & \cdot & u \\ \cdot & \cdot & u & \cdot \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} T_2 & \cdot & \cdot & \cdot \\ \cdot & \cdot & u & \cdot \\ \cdot & u & \cdot & \cdot \\ \cdot & \cdot & \cdot & T_1 \end{pmatrix}, \quad g_3 \mapsto \begin{pmatrix} \cdot & u & \cdot & \cdot \\ u & \cdot & \cdot & \cdot \\ \cdot & \cdot & T_2 & \cdot \\ \cdot & \cdot & \cdot & T_2 \end{pmatrix};$$

$$t_1 \mapsto \mathrm{Diag}(\xi_1, \xi_1, \xi_1, \xi_2), \quad t_2 \mapsto \mathrm{Diag}(\xi_1, \xi_1, \xi_2, \xi_1),$$
$$t_3 \mapsto \mathrm{Diag}(\xi_1, \xi_2, \xi_1, \xi_1), \quad t_4 \mapsto \mathrm{Diag}(\xi_2, \xi_1, \xi_1, \xi_1);$$
$$e_1 \mapsto \mathrm{Diag}(1, 1, 0, 0), \quad e_2 \mapsto \mathrm{Diag}(1, 0, 0, 1), \quad e_3 \mapsto \mathrm{Diag}(0, 0, 1, 1).$$

- Let $\mu = (1, 3)$. Then $m_\mu = 4$ and $\mathcal{H}^\mu \cong \mathcal{H}_1 \otimes \mathcal{H}_3$ and we identify it below with $\mathcal{H}_3$. We order the characters in the orbit $\mathcal{O}(\mu)$ as follows:

$$\chi_1^\mu = (\xi_1, \xi_2, \xi_2, \xi_2), \quad \chi_2^\mu = (\xi_2, \xi_1, \xi_2, \xi_2),$$
$$\chi_3^\mu = (\xi_2, \xi_2, \xi_1, \xi_2) \quad \text{and} \quad \chi_4^\mu = (\xi_2, \xi_2, \xi_2, \xi_1).$$

Thus we have $\pi_{1,\mu} = 1$, $\pi_{2,\mu} = s_1$, $\pi_{3,\mu} = s_2 s_1$ and $\pi_{4,\mu} = s_3 s_2 s_1$. The map Ψ_μ is given by:

$$g_1 \mapsto \begin{pmatrix} \cdot & u & \cdot & \cdot \\ u & \cdot & \cdot & \cdot \\ \cdot & \cdot & T_1 & \cdot \\ \cdot & \cdot & \cdot & T_1 \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} T_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & u & \cdot \\ \cdot & u & \cdot & \cdot \\ \cdot & \cdot & \cdot & T_2 \end{pmatrix}, \quad g_3 \mapsto \begin{pmatrix} T_2 & \cdot & \cdot & \cdot \\ \cdot & T_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & u \\ \cdot & \cdot & u & \cdot \end{pmatrix};$$

$$t_1 \mapsto \mathrm{Diag}(\xi_1, \xi_2, \xi_2, \xi_2), \quad t_2 \mapsto \mathrm{Diag}(\xi_2, \xi_1, \xi_2, \xi_2),$$
$$t_3 \mapsto \mathrm{Diag}(\xi_2, \xi_2, \xi_1, \xi_2), \quad t_4 \mapsto \mathrm{Diag}(\xi_2, \xi_2, \xi_2, \xi_1);$$
$$e_1 \mapsto \mathrm{Diag}(0, 0, 1, 1), \quad e_2 \mapsto \mathrm{Diag}(1, 0, 0, 1), \quad e_3 \mapsto \mathrm{Diag}(1, 1, 0, 0).$$

- Let $\mu = (2, 2)$. Then $m_\mu = 6$ and $\mathcal{H}^\mu \cong \mathcal{H}_2 \otimes \mathcal{H}_2$. We order the characters in the orbit $\mathcal{O}(\mu)$ as follows:

$$\chi_1^\mu = (\xi_1, \xi_1, \xi_2, \xi_2), \quad \chi_2^\mu = (\xi_1, \xi_2, \xi_1, \xi_2), \quad \chi_3^\mu = (\xi_2, \xi_1, \xi_1, \xi_2),$$
$$\chi_4^\mu = (\xi_1, \xi_2, \xi_2, \xi_1), \quad \chi_5^\mu = (\xi_2, \xi_1, \xi_2, \xi_1), \quad \chi_6^\mu = (\xi_2, \xi_2, \xi_1, \xi_1).$$

Thus we have $\pi_{1,\mu} = 1$, $\pi_{2,\mu} = s_2$, $\pi_{3,\mu} = s_1 s_2$, $\pi_{4,\mu} = s_3 s_2$, $\pi_{5,\mu} = s_1 s_3 s_2$ and $\pi_{6,\mu} = s_2 s_1 s_3 s_2$. The map Ψ_μ is given by (where $T_1' := T_1 \otimes 1$ and $T_1'' := 1 \otimes T_1$):

$$g_1 \mapsto \begin{pmatrix} T_1' & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & u & \cdot & \cdot & \cdot \\ \cdot & u & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & u & \cdot \\ \cdot & \cdot & \cdot & u & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & T_1'' \end{pmatrix}, \quad g_2 \mapsto \begin{pmatrix} \cdot & u & \cdot & \cdot & \cdot & \cdot \\ u & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & T_1' & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & T_1'' & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & u \\ \cdot & \cdot & \cdot & \cdot & u & \cdot \end{pmatrix},$$

$$g_3 \mapsto \begin{pmatrix} T_1'' & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & u & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & u & \cdot \\ \cdot & u & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & u & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & T_1' \end{pmatrix};$$

$$t_1 \mapsto \mathrm{Diag}(\xi_1,\xi_1,\xi_2,\xi_1,\xi_2,\xi_2),\ t_2 \mapsto \mathrm{Diag}(\xi_1,\xi_2,\xi_1,\xi_2,\xi_1,\xi_2),$$
$$t_3 \mapsto \mathrm{Diag}(\xi_2,\xi_1,\xi_1,\xi_2,\xi_2,\xi_1),\quad t_4 \mapsto \mathrm{Diag}(\xi_2,\xi_2,\xi_2,\xi_1,\xi_1,\xi_1);$$
$$e_1 \mapsto \mathrm{Diag}(1,0,0,0,0,1),\quad e_2 \mapsto \mathrm{Diag}(0,0,1,1,0,0), e_3 \mapsto \mathrm{Diag}(1,0,0,0,0,1).$$

3.4 Natural inclusions of subalgebras

We recall that, for any $n \geq 1$, the algebra $Y_{d,n}$ is naturally embedded into $Y_{d,n+1}$, as the subalgebra generated by $t_1,\dots,t_n, g_1,\dots,g_{n-1}$. If $x \in Y_{d,n}$, we will abuse notation and write also x for the corresponding element of $Y_{d,n+n'}$, $n' \geq 1$. Very often the context will make clear where x lives, and otherwise we will specify it explicitly.

Let $n \geq 1$ and $\mu = (\mu_1,\dots,\mu_d) \in \mathrm{Comp}_d(n)$. For any $\mu' \geq \mu$ for the natural order on compositions (namely, $\mu'_a \geq \mu_a$ for $a = 1,\dots,d$), we have a natural embedding of $\mathcal{H}^\mu$ into $\mathcal{H}^{\mu'}$. Explicitly, using the isomorphisms

$$\mathcal{H}^\mu \simeq \mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_d} \quad \text{and} \quad \mathcal{H}^{\mu'} \simeq \mathcal{H}_{\mu'_1} \otimes \cdots \otimes \mathcal{H}_{\mu'_d},$$

the embedding is given by

$$\mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_d} \ni x_1 \otimes \cdots \otimes x_d \mapsto x_1 \otimes \cdots \otimes x_d \in \mathcal{H}_{\mu'_1} \otimes \cdots \otimes \mathcal{H}_{\mu'_d}.$$

When $\mu' = \mu^{[a]}$ for $a \in \{1,\dots,d\}$, see (3), the natural embedding $\mathcal{H}^\mu \subset \mathcal{H}^{\mu^{[a]}}$ is expressed on the basis $\{\widetilde{T}_w,\ w \in \mathfrak{S}^\mu\}$ of $\mathcal{H}^\mu$ by

$$\mathcal{H}^\mu \ni \widetilde{T}_w \mapsto \widetilde{T}_{(\mu_1+\cdots+\mu_a+1,\dots,n-1,n)\, w\, (\mu_1+\cdots+\mu_a+1,\dots,n-1,n)^{-1}} \in \mathcal{H}^{\mu^{[a]}}, \tag{25}$$

where $(\mu_1+\cdots+\mu_a+1,\dots,n-1,n)$ is the cyclic permutation on $\mu_1+\cdots+\mu_a+1,\dots,n-1,n$.

Inclusion of basis elements Let $E_{\chi_k^\mu}\widetilde{g}_w$ be an element of the basis of $Y_{d,n}$, where $\mu \in \mathrm{Comp}_d(n)$, $k \in \{1,\dots,m_\mu\}$ and $w \in \mathfrak{S}_n$.

For $a \in \{1,\dots,d\}$, denote by k_a the integer in $\{1,\dots,m_{\mu^{[a]}}\}$ such that $\chi_{k_a}^{\mu^{[a]}} \in \mathrm{Irr}(\mathcal{A}_{n+1})$ is the character given by

$$\chi_{k_a}^{\mu^{[a]}}(t_i) = \chi_k^\mu(t_i),\ \text{ if } i = 1,\dots,n, \qquad \text{and} \qquad \chi_{k_a}^{\mu^{[a]}}(t_{n+1}) = \xi_a.$$

The characters $\{\chi_{k_a}^{\mu^{[a]}},\ a = 1,\dots,d\}$ are all the irreducible characters of $\mathcal{A}_{n+1}$ containing χ_k^μ in their restriction to $\mathcal{A}_n$, and therefore we have $E_{\chi_k^\mu} = \sum_{1\leq a\leq d} E_{\chi_{k_a}^{\mu^{[a]}}}$ in $\mathcal{A}_{n+1}$. Thus, in $Y_{d,n+1}$, we have:

$$E_{\chi_k^\mu}\widetilde{g}_w = \sum_{1\leq a\leq d} E_{\chi_{k_a}^{\mu^{[a]}}}\widetilde{g}_w. \tag{26}$$

A formula for $\pi_{k_a,\mu^{[a]}}$ Let $a \in \{1,\dots,d\}$. We recall that $\pi_{k,\mu}$ is defined as the element of $\mathfrak{S}_n$ of minimal length such that $\pi_{k,\mu}(\chi_1^\mu) = \chi_k^\mu$, and similarly, $\pi_{k_a,\mu^{[a]}}$ is the element of $\mathfrak{S}_{n+1}$ of minimal length such that $\pi_{k_a,\mu^{[a]}}(\chi_1^{\mu^{[a]}}) = \chi_{k_a}^{\mu^{[a]}}$. Writing symbolically a character $\chi \in \mathcal{A}_{n+1}$ as the collection $(\chi(t_1),\dots,\chi(t_{n+1}))$, we have

$$\chi_1^{\mu^{[a]}} = (\underbrace{\xi_1,\dots,\xi_1}_{\mu_1},\dots,\underbrace{\xi_a,\dots,\xi_a}_{\mu_a+1},\dots,\underbrace{\xi_d\dots,\xi_d}_{\mu_d}) \qquad \text{and} \qquad \chi_{k_a}^{\mu^{[a]}} = (\chi_k^\mu,\xi_a),$$

so that the last occurrence of ξ_a in $\chi_1^{\mu^{[a]}}$ is in position $\mu_1+\cdots+\mu_a+1$. Also, χ_1^{μ} is obtained from $\chi_1^{\mu^{[a]}}$ by removing this last ξ_a. It is thus straightforward to see that

$$\pi_{k_a,\mu^{[a]}} = \pi_{k,\mu} \cdot (\mu_1+\cdots+\mu_a+1,\ldots,n,n+1)^{-1}. \tag{27}$$

In the remaining of the paper, to simplify notations, we will often write $\pi_k = \pi_{k,\mu}$ if there is no ambiguity on the choice of μ and also $\pi_{k_a} := \pi_{k_a,\mu^{[a]}}$, for any $k \in \{1,\ldots,m_\mu\}$ and $a \in \{1,\ldots,d\}$.

Inclusion of matrix algebras The successive compositions of the isomorphism Φ_n, the natural embedding of $Y_{d,n}$ in $Y_{d,n+1}$ and the isomorphism $\Psi_{n+1} = \Phi_{n+1}^{-1}$ gives the embedding ι of the following diagram:

$$\begin{array}{ccc} Y_{d,n+1} & \xrightarrow{\Psi_{n+1}} & \displaystyle\bigoplus_{\mu\in \mathrm{Comp}_d(n+1)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu) \\ \bigcup & & \uparrow \iota \\ Y_{d,n} & \xleftarrow{\Phi_n} & \displaystyle\bigoplus_{\mu\in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu) \end{array}$$

In the formula below, an element $x \in \mathcal{H}^\mu$ is also seen as an element of $\mathcal{H}^{\mu^{[a]}}$, for any $a \in \{1,\ldots,d\}$, via the natural embeddings recalled above. We keep the same notation x.

Proposition 3.7 *The embedding ι is given by $\iota = \bigoplus_{\mu\in\mathrm{Comp}_d(n)} \iota_\mu$, where the injective morphisms ι_μ are given by:*

$$\begin{array}{rccc} \iota_\mu\ : \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu) & \to & \displaystyle\bigoplus_{1\le a\le d} \mathrm{Mat}_{m_{\mu^{[a]}}}(\mathcal{H}^{\mu^{[a]}}) \\ x\,\mathrm{M}_{i,j} & \mapsto & \displaystyle\sum_{1\le a\le d} x\,\mathrm{M}_{i_a,j_a} \end{array},$$

for any $\mu \in \mathrm{Comp}_d(n)$, any $x \in \mathcal{H}^\mu$ and any $i,j \in \{1,\ldots,m_\mu\}$.

Proof Let $E_{\chi_i^\mu}\widetilde{g}_w$ be an element of the basis of $Y_{d,n}$, where $\mu \in \mathrm{Comp}_d(n)$, $i \in \{1,\ldots,m_\mu\}$ and $w \in \mathfrak{S}_n$. Let $j \in \{1,\ldots,m_\mu\}$ be uniquely determined by $w(\chi_i^\mu) = \chi_j^\mu$. We have

$$\Phi_n^{-1}(E_{\chi_i^\mu}\widetilde{g}_w) = \widetilde{T}_{\pi_i^{-1}w\pi_j}\,\mathrm{M}_{i,j}.$$

On the other hand, we have, using (26),

$$\begin{aligned} \Psi_{n+1}(E_{\chi_i^\mu}\,\widetilde{g}_w) &= \sum_{1\le a\le d} \Psi_{n+1}(E_{\chi_{i_a}^{\mu^{[a]}}}\,\widetilde{g}_w) \\ &= \sum_{1\le a\le d} \widetilde{T}_{\pi_{i_a}^{-1}w\pi_{j_a}}\,\mathrm{M}_{i_a,j_a}, \end{aligned}$$

since, for any $a \in \{1,\ldots,d\}$, the integer $j_a \in \{1,\ldots,m_{\mu^{[a]}}\}$ is indeed such that $w(\chi_{j_a}^{\mu^{[a]}}) = \chi_{i_a}^{\mu^{[a]}}$ (as $w \in \mathfrak{S}_n$). So we only have to check that $\widetilde{T}_{\pi_{i_a}^{-1}w\pi_{j_a}}$ is the image in $\mathcal{H}^{\mu^{[a]}}$ of $\widetilde{T}_{\pi_i^{-1}w\pi_j} \in \mathcal{H}^\mu$ under the natural embedding. This is an immediate consequence of (25) and (27). □

4 Applications to representation theory

This section presents the first applications of the isomorphism theorem obtained in the preceding section. First we study the consequences on the representation theory of $Y_{d,n}$ and then we concentrate on the symmetric structure of this algebra.

4.1 Simple modules

The case of an isomorphism between an algebra and a matrix algebra is a classical example of Morita equivalence which will be discussed in the next subsection. In fact, in this case, the equivalence is explicit. In particular, due to the isomorphism, any simple $Y_{d,n}$-modules is of the form

$$(M_1 \otimes \cdots \otimes M_d)^{m_\mu},$$

where $\mu = (\mu_1, \ldots, \mu_d)$ is a d-composition of n and M_a is a simple module of $\mathcal{H}_{\mu_a}$, for each $a = 1, \ldots, d$ (see for example [17, §17.B]).

As the representation theory of the Iwahori–Hecke algebra is quite well understood (at least in characteristic 0, see for example [7, ch. 8, 9, 10] for the semisimple case and [6] for the modular case), we can deduce from Theorem 3.1 the following results:

- Let $\theta : A \to k$ be a specialization to a field k such that $\theta(u^2) = 1$ and $\theta(v) = q - q^{-1}$ for an element in $q \in k^\times$. Let $k_\theta Y_{d,n}$ be the specialized algebra then $k_\theta Y_{d,n}$ is split semisimple if and only if for all $\mu \in \mathrm{Comp}_d(n)$, the algebra $k_\theta \mathcal{H}^\mu$ is split semisimple. By [6, Ex.3.1.19], this happens if and only if:

$$\prod_{1 \le m \le n} (1 + q^2 + \cdots + q^{2m-2}) \neq 0$$

  we thus recover the semisimplicity criterion found in [3, §6].
- The simple $k_\theta Y_{d,n}(q)$-modules are naturally labelled by the set of d-partitions of rank n when the algebra is split semisimple. Moreover, in the non semisimple case, if we set

$$e := \min\{i > 0 \mid 1 + q^2 + \cdots + q^{2i-2} = 0\}$$

  then the simple modules are labelled by the d-tuples of partitions such that each partition is e-regular.
- The irreducible characters are completely determined by the irreducible characters of the Iwahori–Hecke algebra of type A. For $M \in \mathrm{Irr}(k_\theta \mathcal{H}^\mu)$ with character χ_M, the character of the simple $k_\theta Y_{d,n}$-module $(M)^{m_\mu}$ is given by:

$$\chi(h) = \chi_M \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(h), \qquad \text{for } h \in k_\theta Y_{d,n},$$

  where $\mathrm{Tr}_{\mathrm{Mat}_{m_\mu}}$ is the usual trace function on the matrix algebra. In particular, the decomposition matrices of the Yokonuma–Hecke algebra are entirely determined by the decomposition matrices of the Iwahori–Hecke algebra of type A.

4.2 A Morita equivalence

From Theorem 3.1, we can thus deduce (see for example [17, Ch.17]) a Morita equivalence between the Yokonuma–Hecke algebra and a direct sum of Iwahori–Hecke algebras of type A over any ring.

Proposition 4.1 *Let R be a commutative ring and $\theta : \mathbb{C}[u^{\pm 1}, v] \to R$ be a specialization. Then the algebra $R_\theta Y_{d,n}$ is Morita equivalent to $\bigoplus_{\mu \in \mathrm{Comp}_d(n)} R_\theta \mathcal{H}^\mu$.*

In addition, consider the Hecke algebra of the complex reflection group $G(d, 1, n)$ (also known as Ariki–Koike algebra). Let R be a commutative ring with unit and let $\mathbf{Q} := (Q_0, \dots, Q_{d-1}) \in R^d$ and $x \in R^\times$. The Hecke algebra of $G(d, 1, n)$ is the R-algebra $\mathcal{H}_n^{\mathbf{Q},x}$ with generators

$$T_0, T_1, \dots, T_{n-1},$$

and relations :

$$\begin{aligned}
&T_i T_{i+1} T_i = T_{i+1} T_i T_{i+1} \ (i = 1, \dots, n-2),\\
&T_i T_j = T_j T_i \ (|j - i| > 1),\\
&(T_i - x)(T_i + x^{-1}) = 0 \ (i = 1, \dots, n-1).\\
&(T_0 - Q_0)(T_0 - Q_1) \dots (T_0 - Q_{d-1}) = 0
\end{aligned}$$

Remark 4.2 Note that if $d = 1$ and $R = \mathbb{C}[x, x^{-1}]$ then $\mathcal{H}_n^{\mathbf{Q},x}$ is nothing but the Iwahori–Hecke algebra of type A_{n-1} with parameter x of Remark 2.1.

Now assume in addition that for all $a \neq b$ and $-n < i < n$ the element $x^{2i} Q_a - Q_b$ is an invertible element of R. By [4,5], over R, $\mathcal{H}_n^{\mathbf{Q},x}$ is Morita equivalent to $\bigoplus_{\mu \in \mathrm{Comp}_d(n)} R_\theta \mathcal{H}^\mu$. We thus deduce the following result.

Corollary 4.3 *Under the above hypothesis, assume that $\theta : A \to R$ is a specialization such that $\theta(u^2) = 1$ and $\theta(v) = x - x^{-1}$ then $R_\theta Y_{d,n}$ is Morita equivalent to $\mathcal{H}_n^{\mathbf{Q},x}$.*

4.3 Symmetrizing form and Schur elements

We now study the symmetric structure of the Yokonuma–Hecke algebra. The algebra $Y_{d,n}$ is symmetric and thus it has a symmetrizing form which controls part of its representation theory. We will in particular recover results obtained in [3] concerning the symmetric structure of $Y_{d,n}$. In fact, the isomorphism theorem will also give an explanation and a new interpretation of these results.

Preliminaries on symmetric algebras We recall that a symmetric algebra H over a commutative ring R is an R-algebra equipped with a trace function

$$\boldsymbol{\tau} : H \to R$$

such that the bilinear form

$$\begin{aligned}
H \times H &\to R\\
(h_1, h_2) &\mapsto \boldsymbol{\tau}(h_1 h_2)
\end{aligned}$$

is non-degenerate. We refer to [7, Ch.7] for a study of the properties of the symmetric algebras. In particular, if K is a field containing R and such that KH is split semisimple, then for all $V \in \mathrm{Irr}(KH)$, there exists s_V in the integral closure of R in K such that

$$\boldsymbol{\tau} = \sum_{\chi \in \mathrm{Irr}(H)} (1/s_V)\, \chi_V,$$

where $\boldsymbol{\tau}$ is extended to KH and χ_V is the character associated to V. The elements s_V are called the Schur elements associated to $\boldsymbol{\tau}$ and they are known to control part of the representation theory of H. We will use the following general result.

Lemma 4.4 (i) *Let $N \in \mathbb{Z}_{>0}$. The algebra $\mathrm{Mat}_N(H)$ is a symmetric algebra with symmetrizing form $\boldsymbol{\tau}^{\mathrm{mat}} := \boldsymbol{\tau} \circ \mathrm{Tr}_{\mathrm{Mat}_N}$, where $\mathrm{Tr}_{\mathrm{Mat}_N}$ is the usual trace function on $\mathrm{Mat}_N(H)$.*
(ii) *Let M be a simple KH-module and s_M its Schur element associated to $\boldsymbol{\tau}$. Then the Schur element s_M^{mat} of the simple $\mathrm{Mat}_N(KH)$-module M^N associated to $\boldsymbol{\tau}^{\mathrm{mat}}$ is equal to s_M.*

Proof (i) The form $\boldsymbol{\tau}^{\mathrm{mat}}$ is clearly a trace function. All we have to do is to check that this is non-degenerate. Let $b_1 \in \mathrm{Mat}_N(H)$ and assume that for all $b_2 \in \mathrm{Mat}_N(H)$, we have $\boldsymbol{\tau}^{\mathrm{mat}}(b_1.b_2) = 0$. Let $h \in H$ and consider the element $b_3 := h.\mathrm{Id}_N \in \mathrm{Mat}_N(H)$ where Id_N is the identity matrix in $\mathrm{Mat}_N(H)$. Then we have

$$\begin{aligned}\boldsymbol{\tau}^{\mathrm{mat}}(b_1.b_2b_3) &= \boldsymbol{\tau} \circ \mathrm{Tr}_{\mathrm{Mat}_N}(hb_1.b_2)\\ &= \boldsymbol{\tau}(h.\,\mathrm{Tr}_{\mathrm{Mat}_N}(b_1.b_2))\end{aligned}$$

As this element is zero for all $h \in H$ and as $\boldsymbol{\tau}$ is a symmetrizing trace, we have $\mathrm{Tr}_{\mathrm{Mat}_N}(b_1.b_2) = 0$ for all $b_2 \in \mathrm{Mat}_N(H)$. This implies that $b_1 = 0$.

(ii) Let E_M be a primitive idempotent of KH associated to the simple module M. Then, by definition, we have $\boldsymbol{\tau}(E_M) = 1/s_M$. Now let $E_M^{\mathrm{mat}} \in \mathrm{Mat}_N(KH)$ be the matrix with E_M in position (1, 1) and 0 everywhere else. Then E_M^{mat} is a primitive idempotent of $\mathrm{Mat}_N(KH)$ associated to the simple module M^N. Thus, we calculate

$$1/s_M^{\mathrm{mat}} = \boldsymbol{\tau}^{\mathrm{mat}}(E_M^{\mathrm{mat}}) = \boldsymbol{\tau} \circ \mathrm{Tr}_{\mathrm{Mat}_N}(E_M^{\mathrm{mat}}) = \boldsymbol{\tau}(E_M) = 1/s_M,$$

which is the desired result. □

Symmetric structure of $Y_{d,n}$ The Iwahori–Hecke algebra of type A is a symmetric algebra with symmetrizing form $\boldsymbol{\tau}_n : \mathcal{H}_n \to \mathbb{C}[u^{\pm 1}, v]$ given on the basis elements by

$$\boldsymbol{\tau}_n(\widetilde{T}_w) = \begin{cases} 1 & \text{if } w = 1,\\ 0 & \text{otherwise.}\end{cases}$$

This, in turn, implies the existence of a symmetrizing form for all $\mu = (\mu_1, \ldots, \mu_d) \in \mathrm{Comp}_d(n)$ on $\mathcal{H}^\mu$ by restriction: $\boldsymbol{\tau}^\mu : \mathcal{H}^\mu \to \mathbb{C}[u^{\pm 1}, v]$. Seeing as usual $\mathcal{H}^\mu$ as $\mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_d}$, this linear form satisfies, for all $(w_1, \ldots, w_d) \in \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_d}$,

$$\boldsymbol{\tau}^\mu(\widetilde{T}_{w_1} \otimes \cdots \otimes \widetilde{T}_{w_d}) = \boldsymbol{\tau}_{\mu_1}(\widetilde{T}_{w_1}) \ldots \boldsymbol{\tau}_{\mu_d}(\widetilde{T}_{w_d}). \tag{28}$$

For any $n \geq 1$ and λ a partition of n, let M_λ be the simple module of the split semisimple algebra $\mathbb{C}(u, v)\mathcal{H}_n$. We denote by $s_\lambda := s_{M_\lambda}$ the Schur element of M_λ associated to $\boldsymbol{\tau}$. We also set $s_\emptyset := 1$.

Now let $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^d)$ be a d-tuple of partitions such that $\mu = (|\lambda^1|, \ldots, |\lambda^d|)$. Then $M_{\lambda_1} \otimes \cdots \otimes M_{\lambda_d}$ is a simple $\mathbb{C}(u, v)\mathcal{H}^\mu$-module and, from (28), its Schur element associated to $\boldsymbol{\tau}^\mu$, denoted by $s_{\boldsymbol{\lambda}}$, is given by:

$$s_{\boldsymbol{\lambda}} = s_{\lambda^1} \ldots s_{\lambda^d}. \tag{29}$$

Finally, from Lemma 4.4(i), we obtain a symmetrizing form on the algebra $\bigoplus_{\mu \in \mathrm{Comp}_d(n)} \mathrm{Mat}_{m_\mu}(\mathcal{H}^\mu)$ given by $\bigoplus_{\mu \in \mathrm{Comp}_d(n)} \boldsymbol{\tau}^\mu \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}}$. Moreover, from Lemma 4.4(ii), the associated Schur element of the simple module indexed by a d-partition $\boldsymbol{\lambda} = (\lambda^1, \ldots, \lambda^d)$ of n is given by Formula (29).

Let us come back to the Yokonuma–Hecke algebra $Y_{d,n}$. By the discussion in Sect. 4.1, we know that the simple $\mathbb{C}(u, v)Y_{d,n}$-modules are labelled by the set of d-partitions of n.

From the preceding discussion together with the isomorphism theorem (Theorem 3.1), we obtain naturally a symmetrizing form on $Y_{d,n}$ given explicitly by

$$\boldsymbol{\rho}_n := \bigoplus_{\mu \in \mathrm{Comp}_d(n)} \boldsymbol{\tau}^{\mu} \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu,$$

and moreover, the associated Schur elements are given by Formula (29).

Alternative formula for $\boldsymbol{\rho}_n$ In [3], it is proved that the following formula defines a symmetrizing form $\widetilde{\boldsymbol{\rho}}_n : Y_{d,n} \to \mathbb{C}[u^{\pm 1}, v]$ on $Y_{d,n}$:

$$\widetilde{\boldsymbol{\rho}}_n(t_1^{a_1} \dots t_n^{a_n} \widetilde{g}_w) = \begin{cases} d^n & \text{if } a_1 = \dots = a_n = 1 \text{ and } w = 1, \\ 0 & \text{otherwise.} \end{cases}$$

It turns out that the form $\widetilde{\boldsymbol{\rho}}_n$ actually coincides with the natural symmetrizing form $\boldsymbol{\rho}_n$.

Proposition 4.5 *The form $\widetilde{\boldsymbol{\rho}}_n$ coincides with the symmetrizing form $\boldsymbol{\rho}_n$ on $Y_{d,n}$.*

Proof We study the values taken by the two traces on the basis given by

$$\{E_\chi \widetilde{g}_w \mid \chi \in \mathrm{Irr}(\mathcal{A}_n),\ w \in \mathfrak{S}_n\}.$$

So let us fix $\mu \in \mathrm{Comp}_d(n)$, $k \in \{1, \dots, m_\mu\}$ and $w \in \mathfrak{S}_n$. We have, using Formula (13) for $E_{\chi_k^\mu}$,

$$\begin{aligned} \widetilde{\boldsymbol{\rho}}_n(E_{\chi_k^\mu} \widetilde{g}_w) &= \widetilde{\boldsymbol{\rho}}_n \left(\left(\prod_{1 \le i \le n} \frac{1}{d} \sum_{0 \le s \le d-1} \chi_k^\mu(t_i)^s \, t_i^{-s} \right) \widetilde{g}_w \right) \\ &= \widetilde{\boldsymbol{\rho}}_n \left(\left(\prod_{1 \le i \le n} \frac{1}{d} \right) \widetilde{g}_w \right) \\ &= \begin{cases} 1 & \text{if } w = 1, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

On the other hand we have

$$\begin{aligned} \boldsymbol{\rho}_n(E_{\chi_k^\mu} \widetilde{g}_w) &= \boldsymbol{\tau}^{\mu} \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(E_{\chi_k^\mu} \widetilde{g}_w) \\ &= \boldsymbol{\tau}^{\mu} \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} (\widetilde{T}_{\pi_{k,\mu}^{-1} w \pi_{j,\mu}} \mathrm{M}_{k,j}) \end{aligned}$$

where $j \in \{1, \dots, m_\mu\}$ satisfies $w(\chi_j^\mu) = \chi_k^\mu$. We have $j = k$ if and only if $\pi_{k,\mu}^{-1} w \pi_{k,\mu} \in \mathfrak{S}^\mu$. We obtain :

$$\boldsymbol{\rho}_n(E_{\chi_k^\mu} \widetilde{g}_w) = \begin{cases} 1 & \text{if } \pi_{k,\mu}^{-1} w \pi_{k,\mu} = 1 \iff w = 1, \\ 0 & \text{otherwise.} \end{cases}$$

and this concludes the proof. □

Remark 4.6 The Schur elements associated to $\widetilde{\rho}$ were obtained in [3] by a direct calculation. From Proposition 4.5 and the discussion before it, we recover the result, namely that the Schur elements associated to $\widetilde{\rho}$ are given by Formula (29). Furthermore, we note that Proposition 4.5 implies immediately the centrality and the non-degeneracy of $\widetilde{\rho}$ (which was also proved by direct calculations in [3]).

5 Classification of Markov traces on Yokonuma–Hecke algebras

In this section, we use the isomorphism theorem to obtain a complete classification of Markov traces on $Y_{d,n}$. We use a definition of Markov traces analogous to the one in [7, section4.5] for the Iwahori–Hecke algebras of type A. From now on, we extend the ground ring to $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ and keep the same notations ($\mathcal{H}_n, \mathcal{H}^{\mu}, Y_{d,n}, ...$) for the extended algebras.

5.1 Markov traces on Iwahori–Hecke algebras of type A

A Markov trace on the family of algebras $\{\mathcal{H}_n\}_{n\geq 1}$ is a family of linear functions $\tau_n : \mathcal{H}_n \to \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ $(n \geq 1)$ satisfying:

$$\begin{array}{ll} \text{(M1) } \tau_n(xy) = \tau_n(yx), & \text{for } n \geq 1 \text{ and } x, y \in \mathcal{H}_n; \ (Trace\ condition) \\ \text{(M2) } \tau_{n+1}(xT_n) = \tau_{n+1}(xT_n^{-1}) = \tau_n(x), & \text{for } n \geq 1 \text{ and } x \in \mathcal{H}_n. \quad (Markov\ condition) \end{array} \tag{30}$$

It is a normalized Markov trace if it satisfies in addition

$$\text{(M0) } \tau_1(1) = 1. \ (Normalization\ condition) \tag{31}$$

In (M2) and in the following as well, we keep the same notation x for an element of $\mathcal{H}_n$ and the corresponding element of $\mathcal{H}_{n+n'}$, $n' \geq 1$, using the natural embedding of $\mathcal{H}_n$ in $\mathcal{H}_{n+n'}$.

It is a classical result that a normalized Markov trace on $\{\mathcal{H}_n\}_{n\geq 1}$ exists and is unique [7, section 4.5]. From now on, $\{\tau_n\}_{n\geq 1}$ will be this unique normalized Markov trace. For later use, we also set $\tau_0 : \mathcal{H}_0 := \mathbb{C}[u^{\pm 1}, v^{\pm 1}] \to \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ to be the identity map on $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$.

As $T_n - u^2 T_n^{-1} = v$ for any $n \geq 1$, we have, using the Markov condition, that

$$\tau_{n+1}(x) = v^{-1}(1 - u^2)\tau_n(x) \quad \text{for any } n \geq 1 \text{and any } x \in \mathcal{H}_n, \tag{32}$$

and by induction on n, using that $\tau_1(1) = 1$, we obtain

$$\tau_n(1) = \left(v^{-1}(1 - u^2)\right)^{n-1} \quad \text{for any } n \geq 1. \tag{33}$$

We will need later the following properties of the Markov trace $\{\tau_n\}_{n\geq 1}$. For the second item, we recall that $\mathcal{H}^{\mu} \simeq \mathcal{H}_{\mu_1} \otimes \cdots \otimes \mathcal{H}_{\mu_d}$ and these two algebras are identified. Recall also that this algebra is naturally embedded in $\mathcal{H}_n$ for any $\mu = (\mu_1, \ldots, \mu_d) \in \mathrm{Comp}_d(n)$. See (5) for the definition of $[\mu]$.

Lemma 5.1 (i) *For any $n \geq 1$, we have:*

$$\tau_{n+1}(xT_k y) = \tau_{n+1}(xT_k^{-1} y) = \tau_n(xy), \quad \textit{for any } k \in \{1, \ldots, n\} \textit{ and } x, y \in \mathcal{H}_k. \tag{34}$$

(ii) *For any $n \geq 1$ and any $\mu \in \mathrm{Comp}_d(n)$, we have:*

$$\tau_n(x_1 \otimes \cdots \otimes x_d) = \left(v^{-1}(1-u^2)\right)^{|[\mu]|-1} \tau_{\mu_1}(x_1) \ldots \tau_{\mu_d}(x_d), \quad \textit{for any } x_1 \otimes \cdots \otimes x_d \in \mathcal{H}^{\mu}. \tag{35}$$

Proof (i) Let $n \geq 1$, $k \in \{1, \ldots, n\}$ and $x, y \in \mathcal{H}_k$. We proceed by induction on $n - k$. For $k = n$, Equation (34) follows directly from Conditions (M1) and (M2). Assume $k < n$. Then T_{k+1} exists in $\mathcal{H}_{n+1}$ and commutes with x and y. By centrality of τ_{n+1}, we have

$$\tau_{n+1}(xT_k^{\pm 1} y) = \tau_{n+1}(xT_{k+1}T_k^{\pm 1}T_{k+1}^{-1} y).$$

Using the braid relation $T_{k+1}T_k^{\pm 1}T_{k+1}^{-1} = T_k^{-1}T_{k+1}^{\pm 1}T_k$ and the induction hypothesis, we conclude that

$$\tau_{n+1}(xT_k^{\pm 1}y) = \tau_{n+1}(xT_k^{-1}T_{k+1}^{\pm 1}T_k y) = \tau_n(xT_k^{-1}T_k y) = \tau_n(xy).$$

(ii) Let $n \geq 1$, $\mu \in \mathrm{Comp}_d(n)$ and $x = x_1 \otimes \cdots \otimes x_d \in \mathcal{H}^\mu$. First assume that $x = 1$. We have, using (33) and the convention $\tau_0(1) = 1$,

$$\tau_{\mu_1}(1)\ldots\tau_{\mu_d}(1) = \prod_{\mu_a \geq 1} \left(v^{-1}(1-u^2)\right)^{\mu_a - 1} = \left(v^{-1}(1-u^2)\right)^{n-|[\mu]|},$$

which yields, together with (33), Formula (35) for $x = 1$.

Let now $x \neq 1$. We proceed by induction on n (the case $n = 1$ being covered by the case $x = 1$). Using that $x \neq 1$, we take $a \in \{1, \ldots, d\}$ to be such that $x_{a+1} = \cdots = x_d = 1$ and $x_a = h_1 T_k h_2 \in \mathcal{H}_{\mu_a}$ with $k \in \{1, \ldots, \mu_a - 1\}$ and $h_1, h_2 \in \mathcal{H}_k$ (in particular, $\mu_a \geq 2$). We set $\nu = \mu_{[a]} \in \mathrm{Comp}_d(n-1)$ (that is, $\nu_a = \mu_a - 1$ and $\nu_b = \mu_b$ if $b \neq a$). We calculate, using item (i),

$$\begin{aligned}\tau_n(x) &= \tau_n(x_1 \otimes \cdots \otimes x_{a-1} \otimes h_1 T_k h_2 \otimes 1 \cdots \otimes 1)\\ &= \tau_{n-1}(x_1 \otimes \cdots \otimes x_{a-1} \otimes h_1 h_2 \otimes 1 \cdots \otimes 1);\end{aligned}$$

using induction hypothesis, we then obtain

$$\tau_n(x) = \left(v^{-1}(1-u^2)\right)^{|[\nu]|-1} \tau_{\nu_1}(x_1)\ldots\tau_{\nu_{a-1}}(x_{a-1})\tau_{\nu_a}(h_1h_2)\tau_{\nu_{a+1}}(1)\ldots\tau_{\nu_d}(1);$$

finally, we have $[\nu] = [\mu]$, since $\mu_a \geq 2$, and moreover, using item (i), $\tau_{\nu_a}(h_1h_2) = \tau_{\mu_a - 1}(h_1h_2) = \tau_{\mu_a}(h_1T_kh_2)$. So we conclude that Formula (35) is satisfied. □

5.2 Markov traces on Yokonuma–Hecke algebras

A Markov trace on the family of algebras $\{Y_{d,n}\}_{n\geq 1}$ is a family of linear functions $\rho_n : Y_{d,n} \to \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ $(n \geq 1)$ satisfying:

$$\begin{array}{ll} \text{(M1)}\ \rho_n(xy) = \rho_n(yx), & \text{for any } n \geq 1 \text{ and } x, y \in Y_{d,n}; \quad \textit{(Trace condition)} \\ \text{(M2)}\ \rho_{n+1}(xg_n) = & \\ \quad \rho_{n+1}(xg_n^{-1}) = \rho_n(x), & \text{for any } n \geq 1 \text{ and } x \in Y_{d,n}. \quad \textit{(Markov condition)} \end{array} \tag{36}$$

Remark 5.2 For the Markov traces on the Iwahori–Hecke algebras, there is no loss of generality in considering the normalized Markov trace. Indeed, it is straightforward to see that if $\{\tau_n\}_{n\geq 1}$ is a Markov trace on $\{\mathcal{H}_n\}_{n\geq 1}$ such that $\tau_1(1) = 0$, then all the linear functions τ_n are identically 0. So we can assume that $\tau_1(1) \neq 0$ and normalize it so that $\tau_1(1) = 1$.

This remark is no longer valid for the Yokonuma–Hecke algebras, for which a Markov trace $\{\rho_n\}_{n\geq 1}$ may satisfy $\rho_1(1) = 0$ without being trivial (see the classification below). Therefore, we will work in the general setting of non-normalized Markov traces.

Let $n \geq 1$ and let κ be any linear function from $Y_{d,n}$ to $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ satisfying the trace condition $\kappa(xy) = \kappa(yx)$, $\forall x, y \in Y_{d,n}$. Recall the isomorphisms Φ_μ and $\Psi_\mu = \Phi_\mu^{-1}$ between $\mathrm{Mat}_{m_\mu}(\mathcal{H}_\mu)$ and $E_\mu Y_{d,n}$ given by (21)-(22). For each $\mu \in \mathrm{Comp}_d(n)$, the composed map $\kappa \circ \Phi_\mu$ is a linear map from $\mathrm{Mat}_{m_\mu}(\mathcal{H}_\mu)$ to $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ also satisfying the trace condition. As the usual trace of a matrix is the only trace function on a matrix algebra (up to normalization), the map $\kappa \circ \Phi_\mu$ is of the form:

$$\kappa \circ \Phi_\mu = \kappa^\mu \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}},$$

for some trace function $\kappa^\mu : \mathcal{H}^\mu \to \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$. In other words, we have

$$\kappa(x) = \sum_{\mu \in \mathrm{Comp}_d(n)} \kappa^\mu \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(E_\mu x), \quad \text{for } x \in Y_{d,n},$$

(where we wrote $x = \sum_{\mu \in \mathrm{Comp}_d(n)} E_\mu x$) and we refer to the maps κ^μ as the trace functions associated to κ.

Classification of Markov traces on $\{Y_{d,n}\}_{n \geq 1}$ We are now ready to give the classification of Markov traces on the Yokonuma–Hecke algebras $\{Y_{d,n}\}_{n \geq 1}$ which is the main result of this section. We refer to Sect. 2, (5) and (6), for the definition of the base $[\mu]$ of a composition μ and of the set Comp_d^0.

Theorem 5.3 *A family $\{\rho_n\}_{n \geq 1}$ of linear functions, $\rho_n : Y_{d,n} \to \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, is a Markov trace on the family of algebras $\{Y_{d,n}\}_{n \geq 1}$ if and only if there is a set of parameters $\{\alpha_{\mu^0}, \ \mu^0 \in \mathrm{Comp}_d^0\} \subset \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ such that*

$$\rho_n(x) = \sum_{\mu \in \mathrm{Comp}_d(n)} \rho^\mu \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(E_\mu x), \quad \text{for } n \geq 1 \text{ and } x \in Y_{d,n}, \tag{37}$$

where the associated trace functions $\rho^\mu : \mathcal{H}^\mu \to \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ are given by

$$\rho^\mu = \alpha_{[\mu]} \cdot \tau_{\mu_1} \otimes \cdots \otimes \tau_{\mu_d}, \quad \text{for any } \mu \in \mathrm{Comp}_d(n). \tag{38}$$

The remaining of this section is devoted to the proof of the Theorem.

Preliminary lemmas Let $\{\rho_n\}_{n \geq 1}$ be a family of linear functions $\rho_n : Y_{d,n} \to \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ satisfying the trace condition (M1).

Lemma 5.4 *The family $\{\rho_n\}_{n \geq 1}$ satisfies the Markov condition (M2) if and only if the associated traces ρ^μ satisfy, for any $\mu \in \bigcup_{n \geq 1} \mathrm{Comp}_d(n)$ and any $a \in \{1, \ldots, d\}$ such that $\mu_a \geq 1$,*

$$\begin{aligned} \rho^{\mu^{[a]}}(x_1 \otimes \cdots \otimes x_a T_{\mu_a} \otimes \cdots \otimes x_d) &= \rho^{\mu^{[a]}}(x_1 \otimes \cdots \otimes x_a T_{\mu_a}^{-1} \otimes \cdots \otimes x_d) \\ &= \rho^\mu(x_1 \otimes \cdots \otimes x_a \otimes \cdots \otimes x_d), \end{aligned} \tag{39}$$

for any $x_1 \otimes \cdots \otimes x_d \in \mathcal{H}^\mu$.

Proof Let $n \geq 1$ and $x \in Y_{d,n}$. In the proof, we will often use the notations and the results explained in Sect. 3.4. First, note that it is enough to take $x = E_{\chi_i^\mu} \widetilde{g}_w$, an element of the basis of $Y_{d,n}$, where $\mu \in \mathrm{Comp}_d(n)$, $i \in \{1, \ldots, m_\mu\}$ and $w \in \mathfrak{S}_n$. For later use, we denote by b the integer in $\{1, \ldots, d\}$ such that $\chi_i^\mu(t_n) = \xi_b$.

We recall that $\Psi_\mu(x) = \widetilde{T}_{\pi_i^{-1} w \pi_j} \mathrm{M}_{i,j}$, where $j \in \{1, \ldots, m_\mu\}$ is uniquely determined by $w(\chi_j^\mu) = \chi_i^\mu$. Thus, we have:

$$\begin{aligned} \rho_n(x) &= \rho^\mu \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(x) \\ &= \begin{cases} 0 & \text{if } w(\chi_i^\mu) \neq \chi_i^\mu; \\ \rho^\mu(\widetilde{T}_{\pi_i^{-1} w \pi_i}) & \text{if } w(\chi_i^\mu) = \chi_i^\mu. \end{cases} \end{aligned} \tag{40}$$

Now, in $Y_{d,n+1}$, we have (due to defining formulas (9) for g_w and $\widetilde{g}_w$)

$$x g_n = u \sum_{1 \leq a \leq d} E_{\chi_{i_a}^{\mu^{[a]}}} \widetilde{g}_{w s_n},$$

and we note that, for $1 \leq a \leq d$, we have $ws_n(\chi_{i_a}^{\mu^{[a]}}) = \chi_{i_a}^{\mu^{[a]}}$ if and only if $s_n(\chi_{i_a}^{\mu^{[a]}}) = \chi_{i_a}^{\mu^{[a]}}$ and $w(\chi_{i_a}^{\mu^{[a]}}) = \chi_{i_a}^{\mu^{[a]}}$ (because $w \in \mathfrak{S}_n$). It means that $ws_n(\chi_{i_a}^{\mu^{[a]}}) = \chi_{i_a}^{\mu^{[a]}}$ if and only if $a = b$ and $w(\chi_i^{\mu}) = \chi_i^{\mu}$. Thus we obtain:

$$\begin{aligned}\rho_{n+1}(xg_n) &= u \sum_{1\leq a\leq d} \rho^{\mu^{[a]}} \circ \mathrm{Tr}_{\mathrm{Mat}_{m_{\mu^{[a]}}}} \circ \Psi_{\mu^{[a]}}(xg_n) \\ &= \begin{cases} 0 & \text{if } w(\chi_i^{\mu}) \neq \chi_i^{\mu}; \\ u\,\rho^{\mu^{[b]}}(\widetilde{T}_{\pi_{i_b}^{-1} ws_n \pi_{i_b}}) & \text{if } w(\chi_i^{\mu}) = \chi_i^{\mu}. \end{cases}\end{aligned}$$

We write $\pi_{i_b}^{-1} ws_n \pi_{i_b} = \pi_{i_b}^{-1} w \pi_{i_b} \cdot \pi_{i_b}^{-1} s_n \pi_{i_b}$. Recall that $\pi_{i_b} = \pi_i \cdot (\mu_1 + \cdots + \mu_b + 1, \ldots, n, n+1)^{-1}$, see (27); moreover,

$$\pi_i(n+1) = n+1, \quad \text{and} \quad \pi_i(\mu_1 + \cdots + \mu_b) = n,$$

since $\pi_i \in \mathfrak{S}_n$ and π_i is the element of minimal length sending χ_1^{μ} on χ_i^{μ}. Therefore we have

$$\begin{aligned}\pi_{i_b}^{-1} s_n \pi_{i_b} &= (\mu_1 + \cdots + \mu_b,\ \mu_1 + \cdots + \mu_b + 1) \quad \text{and} \\ \pi_{i_b}^{-1} w \pi_{i_b}(\mu_1 + \cdots + \mu_b + 1) &= \mu_1 + \cdots + \mu_b + 1,\end{aligned}$$

We conclude that $\widetilde{T}_{\pi_{i_b}^{-1} ws_n \pi_{i_b}} = \widetilde{T}_{\pi_{i_b}^{-1} w \pi_{i_b}} \widetilde{T}_{\mu_1+\cdots+\mu_b} = u^{-1} \widetilde{T}_{\pi_{i_b}^{-1} w \pi_{i_b}} T_{\mu_1+\cdots+\mu_b}$, and in turn

$$\rho_{n+1}(xg_n) = \begin{cases} 0 & \text{if } w(\chi_i^{\mu}) \neq \chi_i^{\mu}; \\ \rho^{\mu^{[b]}}(\widetilde{T}_{\pi_{i_b}^{-1} w \pi_{i_b}} T_{\mu_1+\cdots+\mu_b}) & \text{if } w(\chi_i^{\mu}) = \chi_i^{\mu}. \end{cases} \tag{41}$$

Now we will calculate $\rho_{n+1}(xg_n^{-1})$ using $g_n^{-1} = u^{-2}(g_n - ve_n)$. First we note that $e_n x = \sum_{1\leq a\leq d} e_n E_{\chi_{i_a}^{\mu^{[a]}} \widetilde{g}_w}$, which gives $e_n x = E_{\chi_{i_b}^{\mu^{[b]}} \widetilde{g}_w}$, since $e_n E_\chi = 0$ whenever $\chi(t_n) \neq \chi(t_{n+1})$. In addition we have $w(\chi_{i_b}^{\mu^{[b]}}) = \chi_{i_b}^{\mu^{[b]}}$ if and only if $w(\chi_i^{\mu}) = \chi_i^{\mu}$. Therefore, we obtain, using first the centrality of ρ_{n+1},

$$\begin{aligned}\rho_{n+1}(xe_n) &= \rho_{n+1}(e_n x) \\ &= \rho^{\mu^{[b]}} \circ \mathrm{Tr}_{\mathrm{Mat}_{m_{\mu^{[b]}}}} \circ \Psi_{\mu^{[b]}}(E_{\chi_{i_b}^{\mu^{[b]}} \widetilde{g}_w}) \\ &= \begin{cases} 0 & \text{if } w(\chi_i^{\mu}) \neq \chi_i^{\mu}; \\ \rho^{\mu^{[b]}}(\widetilde{T}_{\pi_{i_b}^{-1} w \pi_{i_b}}) & \text{if } w(\chi_i^{\mu}) = \chi_i^{\mu}. \end{cases}\end{aligned}$$

As we have $T_{\mu_1+\cdots+\mu_b}^{-1} = u^{-2}(T_{\mu_1+\cdots+\mu_b} - v)$ in $\mathcal{H}^{\mu^{[b]}}$, we conclude that

$$\rho_{n+1}(xg_n^{-1}) = \begin{cases} 0 & \text{if } w(\chi_i^{\mu}) \neq \chi_i^{\mu}; \\ \rho^{\mu^{[b]}}(\widetilde{T}_{\pi_{i_b}^{-1} w \pi_{i_b}} T_{\mu_1+\cdots+\mu_b}^{-1}) & \text{if } w(\chi_i^{\mu}) = \chi_i^{\mu} \end{cases} \tag{42}$$

To sum up, in (40)–(42), we obtained first that

$$\rho_n(x) = \rho_{n+1}(xg_n) = \rho_{n+1}(xg_n^{-1}) = 0, \quad \text{if } w(\chi_i^{\mu}) \neq \chi_i^{\mu}.$$

Furthermore, if $w(\chi_i^{\mu}) = \chi_i^{\mu}$, then we write $\widetilde{T}_{\pi_i^{-1} w \pi_i} = x_1 \otimes \cdots \otimes x_d \in \mathcal{H}^{\mu}$, and we note that, due to (25) and (27), $\widetilde{T}_{\pi_{i_b}^{-1} w \pi_{i_b}}$ is the image in $\mathcal{H}^{\mu^{[b]}}$ of $\widetilde{T}_{\pi_i^{-1} w \pi_i}$ under the natural

inclusion $\mathcal{H}^{\mu} \subset \mathcal{H}^{\mu^{[b]}}$. So, if $w(\chi_i^{\mu}) = \chi_i^{\mu}$, Formulas (40)–(42) read

$$\rho_n(x) = \rho^{\mu}(x_1 \otimes \cdots \otimes x_b \otimes \cdots \otimes x_d) \quad \text{and}$$
$$\rho_{n+1}(x g_n^{\pm 1}) = \rho^{\mu^{[b]}}(x_1 \otimes \cdots \otimes x_b T_{\mu_b}^{\pm 1} \otimes \cdots \otimes x_d).$$

We conclude the proof of the Lemma by noticing that, when i runs through $\{1, \ldots, m_{\mu}\}$, every b such that $\mu_b \geq 1$ is obtained, and moreover every element of $\mathcal{H}^{\mu}$ can be written as $\widetilde{T}_{\pi_i^{-1} w \pi_i}$ for some $w \in \mathfrak{S}_n$ satisfying $w(\chi_i^{\mu}) = \chi_i^{\mu}$. □

Lemma 5.5 *The family $\{\rho_n\}_{n\geq 1}$ satisfies the Markov condition (M2) if and only if the associated traces ρ^{μ} satisfy, for any $\mu \in \bigcup_{n\geq 1} \mathrm{Comp}_d(n)$ and any $a \in \{1, \ldots, d\}$ such that $\mu_a \geq 1$,*

$$\begin{aligned} \rho^{\mu^{[a]}}(x_1 \otimes \cdots \otimes x_a T_k \otimes \cdots \otimes x_d) &= \rho^{\mu^{[a]}}(x_1 \otimes \cdots \otimes x_a T_k^{-1} \otimes \cdots \otimes x_d) \\ &= \rho^{\mu}(x_1 \otimes \cdots \otimes x_a \otimes \cdots \otimes x_d), \end{aligned} \tag{43}$$

for any $k \in \{1, \ldots, \mu_a\}$ and any $x_1 \otimes \cdots \otimes x_d \in \mathcal{H}^{\mu}$ such that $x_a \in \mathcal{H}_k \subset \mathcal{H}_{\mu_a}$.

Proof The "if" is a direct consequence of Lemma 5.4, using the assumption with $k = \mu_a$.

To prove the "only if", we assume that the family $\{\rho_n\}_{n\geq 1}$ satisfies the Markov condition (M2), and we proceed by induction on $\mu_a - k$ (it is very similar to the proof of Lemma 5.1(i), so we only sketch it). The case $\mu_a - k = 0$ is Lemma 5.4. So let $k < \mu_a$. Then T_{k+1} exists in $\mathcal{H}_{\mu_a+1}$ and commutes with x_a. By centrality of $\rho^{\mu^{[a]}}$, we have

$$\rho^{\mu^{[a]}}(x_1 \otimes \cdots \otimes x_a T_k^{\pm 1} \otimes \cdots \otimes x_d) = \rho^{\mu^{[a]}}(x_1 \otimes \cdots \otimes x_a T_{k+1} T_k^{\pm 1} T_{k+1}^{-1} \otimes \cdots \otimes x_d).$$

Using $T_{k+1} T_k^{\pm 1} T_{k+1}^{-1} = T_k^{-1} T_{k+1}^{\pm 1} T_k$ and the induction hypothesis, we obtain Formula (43). □

Proof of Theorem 5.3 We are now ready to prove Theorem 5.3. Let $\{\rho_n\}_{n\geq 1}$ be a Markov trace on $\{Y_{d,n}\}_{n\geq 1}$. As explained before Theorem 5.3, the existence of associated traces ρ^{μ}, such that Formula (37) holds, follows from the trace condition for ρ_n. We set $\alpha_{\mu} := \rho^{\mu}(1)$ for any $\mu \in \mathrm{Comp}_d^0$.

Let $n \geq 1$, $\mu \in \mathrm{Comp}_d(n)$ and $x = x_1 \otimes \cdots \otimes x_d \in \mathcal{H}^{\mu}$. We will prove that

$$\rho^{\mu}(x) = \alpha_{[\mu]} \cdot \tau_{\mu_1}(x_1) \ldots \tau_{\mu_d}(x_d). \tag{44}$$

First assume that $\mu = [\mu]$ (which is always true if $n = 1$), so that every μ_a is 0 or 1. Then we have $x = 1$ and Formula (44) follows from $\tau_1(1) = \tau_0(1) = 1$.

Assume now that $\mu \neq [\mu]$. We proceed by induction on n. First let $x \neq 1$, so that we have $a \in \{1, \ldots, d\}$ such that $x_a \neq 1$ (in particular, $\mu_a \geq 2$). We set $x_a = h_1 T_k h_2$, where $k \in \{1, \ldots, \mu_a - 1\}$ and $h_1, h_2 \in \mathcal{H}_k \subset \mathcal{H}_{\mu_a}$. We denote $\nu := \mu_{[a]} \in \mathrm{Comp}_d(n-1)$ (that is, $\nu_a = \mu_a - 1$ and $\nu_b = \mu_b$ if $b \neq a$), so that we have

$$\begin{aligned} \rho^{\mu}(x) &= \rho^{\mu}(x_1 \otimes \cdots \otimes h_1 T_k h_2 \otimes \cdots \otimes x_d) \\ &= \rho^{\nu}(x_1 \otimes \cdots \otimes h_1 h_2 \otimes \cdots \otimes x_d) \\ &= \alpha_{[\nu]} \tau_{\mu_1}(x_1) \ldots \tau_{\mu_a - 1}(h_1 h_2) \ldots \tau_{\mu_d}(x_d) \\ &= \alpha_{[\mu]} \tau_{\mu_1}(x_1) \ldots \tau_{\mu_a}(h_1 T_k h_2) \ldots \tau_{\mu_d}(x_d), \end{aligned}$$

where we first used (43) from Lemma 5.5, then the induction hypothesis and finally the property of τ_{μ_a} stated in item (i) of Lemma 5.1 (we also noted that $[\nu] = [\mu]$ since $\mu_a \geq 2$).

Finally let $x = 1$. As $\mu \neq [\mu]$, we can choose $a \in \{1, \dots, d\}$ such that $\mu_a \geq 2$. We recall that, in $\mathcal{H}_{\mu_a}$, we have $1 = v^{-1}(T_1 - u^2 T_1^{-1})$. Setting again $\nu := \mu_{[a]} \in \mathrm{Comp}_d(n-1)$, we calculate (below $v^{-1}(T_1 - u^2 T_1^{-1})$ is inserted in the a-th factor of the tensor product):

$$\rho^\mu(1) = \rho^\mu(1 \otimes \cdots 1 \otimes v^{-1}(T_1 - u^2 T_1^{-1}) \otimes 1 \cdots \otimes 1) = v^{-1}(1-u^2)\rho^\nu(1),$$

where we used (43) in Lemma 5.5. Using the induction hypothesis, together with the fact that $[\nu] = [\mu]$ (since $\mu_a \geq 2$), we obtain

$$\rho^\mu(1) = \alpha_{[\mu]} v^{-1}(1-u^2)\tau_{\mu_1}(1)\dots\tau_{\mu_a-1}(1)\dots\tau_{\mu_d}(1) = \alpha_{[\mu]}\tau_{\mu_1}(1)\dots\tau_{\mu_a}(1)\dots\tau_{\mu_d}(1),$$

where we used that $\tau_{\mu_a}(1) = v^{-1}(1-u^2)\tau_{\mu_a-1}(1)$ (Formula (33)). This concludes the proof of Formula (44).

For the converse part of the theorem, we only have to check that, given a set of parameters $\{\alpha_\mu,\ \mu \in \mathrm{Comp}_d^0\} \subset \mathbb{C}[u^{\pm 1}, v^{\pm 1}]$, the family $\{\rho_n\}_{n\geq 1}$ of linear functions given by (37) and (38) is a Markov trace. The trace condition is obviously satisfied as well as Equation (39). The proof is concluded using Lemma 5.4. □

Remark 5.6 In view of Lemma 5.1, item (ii), the associated traces of a Markov trace $\{\rho_n\}_{n\geq 1}$ described by Theorem 5.3 can be formally expressed as

$$\rho^\mu = \frac{\alpha_{[\mu]}}{\left(v^{-1}(1-u^2)\right)^{|[\mu]|-1}} \cdot \tau_n, \qquad \text{for any } \mu \in \mathrm{Comp}_d(n),$$

where τ_n acts on $\mathcal{H}^\mu$ by restriction from $\mathcal{H}_n$ (note that Lemma 5.1, item (ii), asserts in particular that the right hand side evaluated on $x \in \mathcal{H}^\mu$ indeed belongs to $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$).

Basis of the space of Markov traces The classification of Markov traces on $\{Y_{d,n}\}_{n\geq 1}$ given by Theorem 5.3 can be formulated by saying that the space of Markov traces is a $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$-module (for pointwise addition and scalar multiplication), which is free and of rank the cardinal of the set Comp_d^0. We have

$$\sharp\,\mathrm{Comp}_d^0 = \sum_{1\leq k\leq d} \binom{d}{k} = 2^d - 1.$$

Further, Theorem 5.3 provides a natural basis for this module. Indeed, for any $\mu^0 \in \mathrm{Comp}_d^0$, let $\{\rho_{\mu^0,n}\}_{n\geq 1}$ be the family of linear functions given by Formulas (37)-(38), for the following choice of parameters:

$$\alpha_{\mu^0} := 1 \qquad \text{and} \qquad \alpha_{\nu^0} := 0, \quad \text{for } \nu^0 \in \mathrm{Comp}_d^0 \text{ such that } \nu^0 \neq \mu^0.$$

Then, $\{\rho_{\mu^0,n}\}_{n\geq 1}$ is a Markov trace on $\{Y_{d,n}\}_{n\geq 1}$ and it is given by

$$\rho_{\mu^0,n}(x) = \sum_{\substack{\mu\in\mathrm{Comp}_d(n)\\ [\mu]=\mu^0}} (\tau_{\mu_1}\otimes\cdots\otimes\tau_{\mu_d}) \circ \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(E_\mu x), \qquad \text{for } n \geq 1 \text{ and } x \in Y_{d,n}. \tag{45}$$

It follows from the classification that the following set is a $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$-basis of the space of Markov traces on $\{Y_{d,n}\}_{n\geq 1}$:

$$\left\{ \{\rho_{\mu^0,n}\}_{n\geq 1} \mid \mu^0 \in \mathrm{Comp}_d^0 \right\}. \tag{46}$$

6 Invariants for links and $\mathbb{Z}/d\mathbb{Z}$-framed links

Now that we have obtained a complete description of the Markov traces for $Y_{d,n}$, we will use them to deduce invariants for both framed and classical knots and links. In addition, we compare these invariants with the one coming from the study of the Iwahori-Hecke algebra of type A: the HOMFLYPT polynomial.

6.1 Classical braid group and HOMFLYPT polynomial

Let $n \in \mathbb{Z}_{\geq 1}$. The braid group B_n (of type A_{n-1}) is generated by elements $\sigma_1, \ldots, \sigma_{n-1}$, with defining relations:

$$\begin{array}{rl} \sigma_i\sigma_j = \sigma_j\sigma_i & \text{for all } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } i = 1, \ldots, n-2. \end{array} \tag{47}$$

With the presentation (1), the Iwahori–Hecke algebra $\mathcal{H}_n$ is a quotient of the group algebra over $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ of the braid group B_n. We denote by $\delta_{\mathcal{H},n}$ the associated surjective morphism:

$$\delta_{\mathcal{H},n} \,:\, \mathbb{C}[u^{\pm 1}, v^{\pm 1}]\big[B_n\big] \to \mathcal{H}_n, \quad \sigma_i \mapsto T_i, \;\; i = 1, \ldots, n-1\,.$$

The classical Alexander's theorem asserts that any link can be obtained as the closure of some braid. Next, the classical Markov's theorem gives necessary and sufficient conditions for two braids to have the same closure up to isotopy (see, e.g., [8]). The condition is that the two braids are equivalent under the equivalence relation generated by the conjugation and the so-called Markov move, namely, generated by

$$\alpha\beta \sim \beta\alpha \;\; (\alpha, \beta \in B_n, \; n \geq 1) \qquad \text{and} \qquad \alpha\sigma_n^{\pm 1} \sim \alpha \;\; (\alpha \in B_n, \; n \geq 1). \tag{48}$$

Note that, in the Markov move, we consider α alternatively as an element of B_n or of B_{n+1} by the natural embedding $B_n \subset B_{n+1}$.

The conditions (M1) and (M2) in (30) for the Markov trace $\{\tau_n\}_{n\geq 1}$ on the algebras $\mathcal{H}_n$ reflect this equivalence relation and, as a consequence, we obtain an isotopy invariant for links as follows. Let K be a link and $\beta_K \in B_n$ a braid on n strands having K as its closure. The map $\Gamma_{\mathcal{H}}$ from the set of links to the ring $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ defined by

$$\Gamma_{\mathcal{H}}(K) = \tau_n \circ \delta_{\mathcal{H},n}(\beta_K),$$

only depends on the isotopy class of K (this is immediate, comparing (30) and (48)), and thus provides an isotopy invariants for links.

Remark 6.1 The Laurent polynomial $\Gamma_{\mathcal{H}}(K)$ in u, v is called the HOMFLYPT polynomial. It was first obtained by a slightly different approach, using the Ocneanu trace on $\mathcal{H}_n$ and a rescaling procedure, see [8] and references therein. We followed the approach in [7, section4.5], where the connections between both approaches are specified. We will carefully detail this connection in the more general context of the Yokonuma–Hecke algebras below.

6.2 $\mathbb{Z}/d\mathbb{Z}$-framed braid group and $\mathbb{Z}/d\mathbb{Z}$-framed links

Roughly speaking, a $\mathbb{Z}/d\mathbb{Z}$-framed braid is a usual braid with an element of $\mathbb{Z}/d\mathbb{Z}$ (the *framing*) attached to each strand. Similarly, a $\mathbb{Z}/d\mathbb{Z}$-framed link is a classical link where each connected component carries a framing in $\mathbb{Z}/d\mathbb{Z}$. The notion of isotopy for framed

links is generalized straightforwardly from the classical setting. We refer to [15,16] for more details on framed braids and framed links.

Let $d \in \mathbb{Z}_{\geq 1}$. The $\mathbb{Z}/d\mathbb{Z}$-framed braid group, denoted by $\mathbb{Z}/d\mathbb{Z} \wr B_n$, is (isomorphic to) the semi-direct product of the abelian group $(\mathbb{Z}/d\mathbb{Z})^n$ by the braid group B_n, where the action of B_n on $(\mathbb{Z}/d\mathbb{Z})^n$ is by permutation. In other words, the group $\mathbb{Z}/d\mathbb{Z} \wr B_n$ is generated by elements $\sigma_1, \sigma_2, \ldots, \sigma_{n-1}, t_1, \ldots, t_n$, and relations:

$$\begin{array}{rll} \sigma_i\sigma_j &= \sigma_j\sigma_i & \text{for all } i, j = 1, \ldots, n-1 \text{ such that } |i-j| > 1, \\ \sigma_i\sigma_{i+1}\sigma_i &= \sigma_{i+1}\sigma_i\sigma_{i+1} & \text{for all } i = 1, \ldots, n-2, \\ t_i t_j &= t_j t_i & \text{for all } i, j = 1, \ldots, n, \\ \sigma_i t_j &= t_{s_i(j)}\sigma_i & \text{for all } i = 1, \ldots, n-1 \text{ and } j = 1, \ldots, n, \\ t_j^d &= 1 & \text{for all } j = 1, \ldots, n. \end{array} \tag{49}$$

The closure of a $\mathbb{Z}/d\mathbb{Z}$-framed braid is naturally a $\mathbb{Z}/d\mathbb{Z}$-framed link (the framing of a connected component is the sum of the framings of the strands forming this component after closure). Given a classical link, from the classical Alexander's theorem, we have a classical braid closing to this link, and it is immediate that by adding a suitable framing on this braid, one can obtain any possible framing on the given link. So the analogue of Alexander's theorem is also true for $\mathbb{Z}/d\mathbb{Z}$-framed braids and links.

Moreover, the Markov's theorem has also been generalized to the $\mathbb{Z}/d\mathbb{Z}$-framed setting (see [16, Lemma1] or [15, Theorem 6]). The necessary and sufficient conditions for two $\mathbb{Z}/d\mathbb{Z}$-framed braids to have the same closure up to isotopy is formally the same as for usual braids; namely, the two braids have to be equivalent under the equivalence relation generated by

$$\tilde{\alpha}\tilde{\beta} \sim \tilde{\beta}\tilde{\alpha} \ \ (\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}/d\mathbb{Z} \wr B_n, \ n \geq 1) \qquad \text{and} \qquad \tilde{\alpha}\sigma_n^{\pm 1} \sim \tilde{\alpha} \ \ (\tilde{\alpha} \in \mathbb{Z}/d\mathbb{Z} \wr B_n, \ n \geq 1). \tag{50}$$

The conditions (M1) and (M2) in (36) for a Markov trace $\{\rho_n\}_{n\geq 1}$ on the Yokonuma–Hecke algebras reflect this equivalence relation, and this will allow to use the Markov traces obtained in the previous section to construct isotopy invariants for $\mathbb{Z}/d\mathbb{Z}$-framed links.

A family of morphisms from the group algebra of $\mathbb{Z}/d\mathbb{Z} \wr B_n$ to $Y_{d,n}$ Let γ be another indeterminate and set $R := \mathbb{C}[u^{\pm 1}, v^{\pm 1}, \gamma^{\pm 1}]$. We define:

$$\delta_{Y,n}^{\gamma} \ : \ \sigma_i \mapsto \big(\gamma + (1-\gamma)e_i\big)g_i \ \ (i = 1, \ldots, n-1), \qquad t_j \mapsto t_j \ \ (j = 1, \ldots, n). \tag{51}$$

Lemma 6.2 *The map $\delta_{Y,n}^{\gamma}$ extends to an algebras homomorphism from $R\big[\mathbb{Z}/d\mathbb{Z} \wr B_n\big]$ to $RY_{d,n}$.*

Proof We have to check that the defining relations (49) are satisfied by the images of the generators, and also that the images of the generators are invertible elements of $RY_{d,n}$. For the latter statement, it is easily checked that

$$\Big(\big(\gamma + (1-\gamma)e_i\big)g_i\Big)^{-1} = \big(\gamma^{-1} + (1-\gamma^{-1})e_i\big)g_i^{-1}, \qquad i = 1, \ldots, n-1. \tag{52}$$

The three last relations in (49) are satisfied since the elements e_i's and t_j's commute. Then, if $|i-j| > 1$, a direct calculation shows that the image of the first relation in (49) is

$$\big(\gamma + (1-\gamma)e_i\big)\big(\gamma + (1-\gamma)e_j\big)g_i g_j = \big(\gamma + (1-\gamma)e_j\big)\big(\gamma + (1-\gamma)e_i\big)g_j g_i,$$

since $g_i e_j = e_j g_i$ and $g_j e_i = e_i g_j$ whenever $|i-j| > 1$. This relation is satisfied in $RY_{d,n}$.

Finally, using again the commutation relations between the generators g_i's and t_j, we calculate the image of the first relation in (49), and obtain

$$(\gamma + (1-\gamma)e_i)(\gamma + (1-\gamma)e_{i,i+2})(\gamma + (1-\gamma)e_{i+1})\big(g_i g_{i+1} g_i - g_{i+1} g_i g_{i+1}\big) = 0,$$

where $e_{i,i+2} := \dfrac{1}{d}\sum_{0\leq s\leq d-1} t_i^s t_{i+2}^{-s}$. This relation is also satisfied in $RY_{d,n}$. □

Remark 6.3 If we specialize the parameter γ to 1, we obtain the natural surjective morphism from the group algebra over $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$ of the group $\mathbb{Z}/d\mathbb{Z} \wr B_n$ to its quotient $Y_{d,n}$ with the presentation (7) of Sect. 2. For other values of γ, this morphism is related with other equivalent presentations of $Y_{d,n}$; see below Sect. 6.5.

6.3 Invariants for classical and $\mathbb{Z}/d\mathbb{Z}$-framed links from $Y_{d,n}$

Invariants for $\mathbb{Z}/d\mathbb{Z}$-framed links from $Y_{d,n}$ Let $\{\rho_n\}_{n\geq 1}$ be a Markov trace on the Yokonuma–Hecke algebras $\{Y_{d,n}\}_{n\geq 1}$ (defined by Conditions (M1) and (M2) in (36)), and extend it R-linearly to $\{RY_{d,n}\}_{n\geq 1}$.

Let $\tilde{K}$ be a $\mathbb{Z}/d\mathbb{Z}$-framed link and $\tilde{\beta}_{\tilde{K}} \in \mathbb{Z}/d\mathbb{Z} \wr B_n$ a $\mathbb{Z}/d\mathbb{Z}$-framed braid on n strands having $\tilde{K}$ as its closure. We define a map $\mathrm{F}\Gamma^{\gamma}_{Y,\rho}$ from the set of $\mathbb{Z}/d\mathbb{Z}$-framed links to the ring R by

$$\mathrm{F}\Gamma^{\gamma}_{Y,\rho}(\tilde{K}) = \rho_n \circ \delta^{\gamma}_{Y,n}(\tilde{\beta}_{\tilde{K}}), \tag{53}$$

where the map $\delta^{\gamma}_{Y,n}$ is defined in (51). For $\mu^0 \in \mathrm{Comp}^0_d$, we denote by $\mathrm{F}\Gamma^{\gamma}_{Y,\mu_0}$ the map corresponding to the Markov trace $\{\rho_{\mu^0,n}\}_{n\geq 1}$ considered in (45). The classification of Markov traces of the previous section, together with the construction detailed in this section, lead to the following result.

Theorem 6.4 **1.** *For any Markov trace $\{\rho_n\}_{n\geq 1}$ on $\{Y_{d,n}\}_{n\geq 1}$, the map $\mathrm{F}\Gamma^{\gamma}_{Y,\rho}$ is an isotopy invariant for $\mathbb{Z}/d\mathbb{Z}$-framed links with values in $R = \mathbb{C}[u^{\pm 1}, v^{\pm 1}, \gamma^{\pm 1}]$.*

2. *The set of invariants for $\mathbb{Z}/d\mathbb{Z}$-framed links obtained from the Yokonuma–Hecke algebras via this construction consists of all R-linear combinations of invariants from the set*

$$\left\{ \mathrm{F}\Gamma^{\gamma}_{Y,\mu^0} \;\middle|\; \mu^0 \in \mathrm{Comp}^0_d \right\}. \tag{54}$$

Proof **1.** By the Markov's theorem for $\mathbb{Z}/d\mathbb{Z}$-framed links, we have to check that the map given by $\{\rho_n \circ \delta^{\gamma}_{Y,n}\}_{n\geq 1}$ from the set of $\mathbb{Z}/d\mathbb{Z}$-framed braids to the ring R coincide on equivalent braids, for the equivalence relation generated by the moves in (50).

From the trace condition (M1) in (36) for $\{\rho_n\}_{n\geq 1}$ and the fact that the maps $\delta^{\gamma}_{Y,n}$ $(n \geq 1)$ are algebra morphisms (Lemma 6.2), it follows at once that the map $\rho_n \circ \delta^{\gamma}_{Y,n}$ $(n \geq 1)$ coincides on $\tilde{\alpha}\tilde{\beta}$ and $\tilde{\beta}\tilde{\alpha}$ for any two braids $\tilde{\alpha}, \tilde{\beta} \in \mathbb{Z}/d\mathbb{Z} \wr B_n$.

Next, let $n \geq 1$ and $\tilde{\alpha} \in \mathbb{Z}/d\mathbb{Z} \wr B_n$. We set $x_{\tilde{\alpha}} := \delta^{\gamma}_{Y,n}(\tilde{\alpha}) \in Y_{d,n}$. Note that, seeing $x_{\tilde{\alpha}}$ as an element of $Y_{d,n+1}$, we also have $x_{\tilde{\alpha}} := \delta^{\gamma}_{Y,n+1}(\tilde{\alpha})$, by definition of $\{\delta^{\gamma}_{Y,n}\}_{n\geq 1}$. From (51) and (52), we have

$$\rho_{n+1} \circ \delta^{\gamma}_{Y,n+1}(\tilde{\alpha}\sigma_n^{\pm 1}) = \rho_{n+1}\Big(x_{\tilde{\alpha}}\big(\gamma^{\pm 1} + (1-\gamma^{\pm 1})e_n\big)g_n^{\pm 1}\Big),$$

and we need to prove that it is equal to $\rho_n \circ \delta^{\gamma}_{Y,n}(\tilde{\alpha}) = \rho_n(x_{\tilde{\alpha}})$. This will follow from the Markov condition (M2) in (36) for $\{\rho_n\}_{n\geq 1}$ together with the following fact:

$$\rho_{n+1}(x e_n g_n^{\pm 1}) = \rho_{n+1}(x g_n^{\pm 1}), \qquad \text{for any } n \geq 1 \text{ and } x \in Y_{d,n}. \tag{55}$$

This last relation is true for any linear map κ on $Y_{d,n+1}$ satisfying the trace condition since, for $s \in \{1, \dots, d\}$,

$$\kappa(xt_{n+1}^{-s}t_n^s g_n^{\pm 1}) = \kappa(xt_{n+1}^{-s}g_n^{\pm 1}t_{n+1}^s) = \kappa(t_{n+1}^s xt_{n+1}^{-s}g_n^{\pm 1}) = \kappa(xg_n^{\pm 1}),$$

where we used successively the relation $t_n g_n^{\pm 1} = g_n^{\pm 1} t_{n+1}$, the trace condition and the fact that t_{n+1} commutes with $x \in Y_{d,n}$.

2. This is simply a reformulation of the classification of Markov traces $\{\rho_n\}_{n\geq 1}$ on $\{Y_{d,n}\}_{n\geq 1}$ given by Theorem 5.3 leading to the basis $\left\{ \{\rho_{\mu^0,n}\}_{n\geq 1} \mid \mu^0 \in \mathrm{Comp}_d^0 \right\}$ in (46). □

Invariants for classical links from $Y_{d,n}$ The classical braid group B_n is naturally a subgroup of the $\mathbb{Z}/d\mathbb{Z}$-framed braid group $\mathbb{Z}/d\mathbb{Z} \wr B_n$ (a classical braid is seen as a $\mathbb{Z}/d\mathbb{Z}$-framed braid with all framings equal to 0). Therefore, one can restrict the maps $\mathrm{F}\Gamma_{Y,\rho}^{\gamma}$ in (53) to classical links, and obtain invariants for classical links since the Markov's theorem is formally the same for classical and $\mathbb{Z}/d\mathbb{Z}$-framed links; compare (48) and (50).

Explicitly, let K be a link and $\beta_K \in B_n$ a braid on n strands having K as its closure. We now see β_K as an element of the $\mathbb{Z}/d\mathbb{Z}$-framed braid group $\mathbb{Z}/d\mathbb{Z} \wr B_n$ and we set

$$\Gamma_{Y,\rho}^{\gamma}(K) = \rho_n \circ \delta_{Y,n}^{\gamma}(\beta_K). \tag{56}$$

For $\mu^0 \in \mathrm{Comp}_d^0$, we denote by $\Gamma_{Y,\mu^0}^{\gamma}$ the map corresponding to the Markov trace $\{\rho_{\mu^0,n}\}_{n\geq 1}$ considered in (45). According to the above discussion, the following corollary is immediately deduced from Theorem 6.4

Corollary 6.5 **1.** *For any Markov trace $\{\rho_n\}_{n\geq 1}$ on $\{Y_{d,n}\}_{n\geq 1}$, the map $\Gamma_{Y,\rho}^{\gamma}$ is an isotopy invariant for classical links with values in $R = \mathbb{C}[u^{\pm 1}, v^{\pm 1}, \gamma^{\pm 1}]$.*

2. *The set of invariants for classical links obtained from the Yokonuma–Hecke algebras via this construction consists of all R-linear combinations of invariants from the set*

$$\left\{ \Gamma_{Y,\mu^0}^{\gamma} \mid \mu^0 \in \mathrm{Comp}_d^0 \right\}. \tag{57}$$

Note that, in the definition of the invariant $\Gamma_{Y,\rho}^{\gamma}(K)$ in (56), even though the word β_K only contains generators σ_i (and no t_j), the image $\delta_{Y,n}^{\gamma}(\beta_K)$ in the algebra $Y_{d,n}$ does involve in general the generators t_j (more precisely, it involves the elements e_i). Indeed, first, the image of σ_i by the map $\delta_{Y,n}^{\gamma}$ contains the idempotent e_i. Besides, even if γ is specialized to 1, as soon as one σ_i^2 for example appears in β_K, then the last relation of (7) is used to calculate $\rho_n \circ \delta_{Y,n}^{1}(\beta_K)$, and this last relation involves the idempotent e_i.

Example 6.6 Let $d = 2$. We will explicitly give $\Gamma_{Y,\mu^0}^{\gamma}(K)$ for $\mu^0 \in \mathrm{Comp}_2^0$ and some classical links K. Using the notations of [19], let $K_1 =$ L10a46 and $K_2 =$ L10a110. For each of these two links, one can find in [19] a braid on 4 strands closing to the link. Namely, the braid $\beta_1 = \sigma_1^2\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^3\sigma_2^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1$ admits K_1 as its closure, while the braid $\beta_1 = \sigma_1^{-1}\sigma_2^3\sigma_1^{-1}\sigma_3^{-1}\sigma_2^3\sigma_3^{-1}$ admits K_2 as its closure. Thus we can use the calculations made in Example 3.6.

- We first consider $\mu^0 = (1, 0)$ or $\mu^0 = (0, 1)$. Then we have, by definition, $\Gamma_{Y,\mu^0}^{\gamma}(K_i) = \tau_4 \circ \Psi_\mu \circ \delta_{Y,4}^{\gamma}(\beta_i)$ $(i = 1, 2)$, where μ is the composition $(4, 0)$ or $(0, 4)$, and τ_4 comes from the unique Markov trace $\{\tau_n\}_{n\geq 1}$ on the Iwahori–Hecke algebras. It is straightforward to see that in this situation, from the formulas in Example 3.6, we have $\delta_{Y,4}^{\gamma}(\beta_i) = \delta_{H,4}(\beta_i)$,

and in turn that we have $\Gamma^{\gamma}_{Y,\mu^0}(K_i) = \Gamma_{\mathcal{H}}(K_i)$ (the HOMFLYPT polynomial). This is a general property of the invariants $\Gamma^{\gamma}_{Y,\mu^0}$ when $|\mu^0| = 1$; see Proposition 6.12 below.

- Then we consider $\mu^0 = (1, 1)$. By definition, we have $(i = 1, 2)$

$$\Gamma^{\gamma}_{Y,\mu^0}(K_i) = \Big((\tau_3\otimes\tau_1)\circ \mathrm{Tr}\circ\Psi_{(3,1)} + (\tau_1\otimes\tau_3)\circ\mathrm{Tr}\circ\Psi_{(1,3)} + (\tau_2\otimes\tau_2)\circ\mathrm{Tr}\circ\Psi_{(2,2)}\Big)\circ\delta^{\gamma}_{Y,4}(\beta_i), \tag{58}$$

where Tr is the usual trace of a matrix. So we need first to calculate $\Psi_\mu \circ \delta^{\gamma}_{Y,4}(\beta_i)$ for $\mu = (3, 1), (1, 3), (2, 2)$. Take for example $i = 1$ and $\mu = (3, 1)$. According to Example 3.6, the generators g_1, g_2 and g_3 map under $\Psi_{(3,1)} \circ \delta^{\gamma}_{Y,4}$ respectively to

$$\begin{pmatrix} \cdot & u\gamma & \cdot & \cdot \\ u\gamma & \cdot & \cdot & \cdot \\ \cdot & \cdot & T_1 & \cdot \\ \cdot & \cdot & \cdot & T_1 \end{pmatrix}, \quad \begin{pmatrix} T_1 & \cdot & \cdot & \cdot \\ \cdot & \cdot & u\gamma & \cdot \\ \cdot & u\gamma & \cdot & \cdot \\ \cdot & \cdot & \cdot & T_2 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} T_2 & \cdot & \cdot & \cdot \\ \cdot & T_2 & \cdot & \cdot \\ \cdot & \cdot & \cdot & u\gamma \\ \cdot & \cdot & u\gamma & \cdot \end{pmatrix}.$$

Performing the matrix multiplication corresponding to the braid β_1 given above, we obtain

$$\Psi_{(3,1)} \circ \delta^{\gamma}_{Y,4}(\beta_1) = \begin{pmatrix} \cdot & \cdot & (u\gamma)^2\, T_1^2T_2^{-1}T_1^{-1}T_2T_1^{-1} & \cdot \\ \cdot & \dfrac{T_1^2T_2^{-1}T_1^3T_2T_1}{(u\gamma)^4} & \cdot & \cdot \\ \cdot & \cdot & \cdot & u\gamma\, T_2^{-1}T_1^3T_2^{-1} \\ (u\gamma)^5\, T_1^{-1}T_2^{-1}T_1^{-1}T_2^{-1}T_1 & \cdot & \cdot & \cdot \end{pmatrix}$$

This gives a contribution (i.e. a term in the sum (58)) to $\Gamma^{\gamma}_{Y,\mu^0}(K_1)$ equal to $\dfrac{1}{(u\gamma)^4}\tau_3$ $(T_1^2T_2^{-1}T_1^3T_2T_1)$. It is easy to see that the composition $(1, 3)$ gives the same contribution to $\Gamma^{\gamma}_{Y,\mu^0}(K_1)$. A similar calculation shows that the composition $(2, 2)$ gives a contribution equal to 0 (this can also be deduced without calculation from the fact that the underlying permutation of β_1 is $(1, 2, 4)$ and this cycle structure makes impossible for $\Psi_{(2,2)} \circ \delta^{\gamma}_{Y,4}(\beta_1)$ to have a non-zero diagonal term).

A similar procedure for β_2 shows that the compositions $(3, 1)$ and $(1, 3)$ give both a contribution to $\Gamma^{\gamma}_{Y,\mu^0}(K_2)$ equal to 0, while the composition $(2, 2)$ gives a contribution equal to $\frac{2}{(u\gamma)^4}\tau_2(T_1^3)^2$.

Quite remarkably, even though the two calculations involve different compositions and different elements of Iwahori–Hecke algebras, these two calculations lead finally to

$$\Gamma^{\gamma}_{Y,\mu^0}(K_1) = \Gamma^{\gamma}_{Y,\mu^0}(K_2) = \frac{2}{(u\gamma)^4}(2u^2 - u^4 + v^2)^2.$$

From [19], we note that these two links K_1, K_2 are topologically different and are however not distinguished by the HOMFLYPT polynomial. We just checked that they are not distinguished neither by the invariants $\Gamma^{\gamma}_{Y,\mu^0}$ (when $d = 2$) coming from the Yokonuma–Hecke algebras.

In fact, we will prove below in Proposition 6.13 that (for any d), the set of invariants $\{\Gamma^{\gamma}_{Y,\mu^0}\}$ is topologically equivalent to the HOMFLYPT polynomial when restricted to classical knots. We note that the question remains open for classical links which are not knots. However, computational data seem to indicate that the invariants are topologically equivalent as well for all classical links (we checked this by direct calculations, as in the example here, for $d = 2$ and links up to 10 crossings; see also [2]).

6.4 Comparison of invariants for classical links

As already explained, when calculating the invariant for a classical link using the Yokonuma–Hecke algebras, the additional generators t_j play a non-trivial role, and therefore these invariants are *a priori* different from the HOMFLYPT polynomial. In this part, we will compare the set of invariants for classical links obtained from the Yokonuma–Hecke algebras with the HOMFLYPT polynomial. The main question is whether or not they are topologically equivalent.

According to Corollary 6.5, we can express any invariant for classical links obtained from the algebras $Y_{d,n}$ via the above construction as a linear combination of the invariants denoted $\Gamma^{\gamma}_{Y,\mu^0}$, where $\mu^0 \in \mathrm{Comp}^0_d$. The main question is whether we have, for any two classical links K_1, K_2,

$$\Big(\forall \mu^0 \in \mathrm{Comp}^0_d,\ \ \Gamma^{\gamma}_{Y,\mu^0}(K_1) = \Gamma^{\gamma}_{Y,\mu^0}(K_2)\Big) \quad \overset{?}{\Longleftrightarrow} \quad \Gamma_{\mathcal{H}}(K_1) = \Gamma_{\mathcal{H}}(K_2).$$

In this part, we will show first that the set $\{\Gamma^{\gamma}_{Y,\mu^0} \mid \mu^0 \in \mathrm{Comp}^0_d\}$ contains $\Gamma_{\mathcal{H}}$, so that only one half of the equivalence is not trivial. Secondly, we will show that this equivalence is true whenever we restrict our attention to classical knots. We refer to [1,2] for similar results about Juyumaya–Lambropoulou invariants. Note that these invariants are shown in Sect. 6.5 below to be certain linear combinations of the set $\{\Gamma^{\gamma}_{Y,\mu^0} \mid \mu^0 \in \mathrm{Comp}^0_d\}$ for some specific value of γ.

The HOMFLYPT polynomial from $Y_{d,n}$ Among the basic invariants $\Gamma^{\gamma}_{Y,\mu^0}$, we consider in this paragraph the ones for which $|\mu^0| = 1$.

Proposition 6.7 *Let $\mu^0 \in \mathrm{Comp}_d(1)$. We have, for any classical link K,*

$$\Gamma^{\gamma}_{Y,\mu^0}(K) = \Gamma_{\mathcal{H}}(K).$$

In particular, the set of invariants for classical links obtained from $Y_{d,n}$ contains the HOMFLYPT polynomial.

Proof Let $\mu^0 \in \mathrm{Comp}_d(1)$ (so that μ^0 automatically belongs to Comp^0_d). So there exists $a \in \{1, \dots, d\}$ such that $\mu^0 = (0, \dots, 0, 1, 0, \dots, 0)$ with 1 in a-th position. Note that a composition μ with d parts satisfies $[\mu] = \mu^0$ if and only if $\mu = (0, \dots, 0, n, 0, \dots, 0)$ with n in a-th position for some $n \geq 1$.

So let $n \geq 1$ and $\mu = (0, \dots, 0, n, 0, \dots, 0)$ with n in a-th position. In this situation, we have $\mathcal{H}^{\mu} \cong \mathcal{H}_n$ and $m_{\mu} = 1$. According to Formula (45), the linear function $\rho_{\mu^0,n}$ is then given by

$$\rho_{\mu^0,n}(x) = \tau_n \circ \Psi_{\mu}(E_{\mu} x), \qquad \text{for any } x \in Y_{d,n}. \tag{59}$$

The defining formula (22) for the isomorphism Ψ_{μ} becomes simply $\Psi_{\mu}(E_{\mu} g_w) = T_w$ for $w \in \mathfrak{S}_n$, and in particular, we have

$$\Psi_{\mu}(E_{\mu} g_i) = T_i, \qquad \text{for any } i = 1, \dots, n-1. \tag{60}$$

Note that $E_{\mu} e_i = E_{\mu}$, for any $i = 1, \dots, n-1$, since, for the considered μ, we have $E_{\mu} t_i = E_{\mu} t_{i+1}$ (both are equals to $\xi_a E_{\mu}$). Therefore

$$E_{\mu} \delta^{\gamma}_{Y,n}(\sigma_i) = E_{\mu}\big(\gamma + (1-\gamma) e_i\big) g_i = E_{\mu} g_i, \qquad \text{for any } i = 1, \dots, n-1. \tag{61}$$

To conclude, let $\beta \in B_n$ be a classical braid. Equations (60)-(61), together with the fact that Ψ_μ is a morphism, yields

$$\Psi_\mu\big(E_\mu \delta^\gamma_{Y,n}(\beta)\big) = \delta_{\mathcal{H},n}(\beta),$$

which gives in turn, using (59), that $\rho_{\mu^0,n} \circ \delta^\gamma_{Y,n}(\beta) = \tau_n \circ \delta_{\mathcal{H},n}(\beta)$. This is the desired result. □

Equivalence of invariants for classical knots Let $\beta \in B_n$, for some $n \geq 1$, be a classical braid. From the presentation (47) of B_n, there is a surjective morphism from B_n to the symmetric group $\mathfrak{S}_n$ given by $\sigma_i \mapsto s_i = (i, i+1)$ for $i = 1, \ldots, n-1$. We will denote $\bar{\beta} \in \mathfrak{S}_n$ the image of β and refer to it as the underlying permutation of β.

Now, the necessary and sufficient condition for the closure of β to be a knot (that is, a link with only one connected component) is that the underlying permutation $\bar{\beta}$ leaves no non-trivial subset of $\{1, \ldots, n\}$ invariant. In other words, the closure of β is a knot if and only if $\bar{\beta}$ is a cycle of length n.

Proposition 6.8 *For any classical knot K and any $\mu^0 \in \mathrm{Comp}_d^0$,*

$$\Gamma^\gamma_{Y,\mu^0}(K) = \begin{cases} \Gamma_{\mathcal{H}}(K) & \text{if } |\mu^0| = 1, \\ 0 & \text{otherwise.} \end{cases}$$

In particular, for classical knots, the invariants obtained from $Y_{d,n}$ are topologically equivalent to the HOMFLYPT polynomial.

Proof Let K be a classical knot and $\beta \in B_n$, for some $n \geq 1$, a classical braid closing to K. To save space during the proof, we set $x_\beta := \delta^\gamma_{Y,n}(\beta) \in Y_{d,n}$.

Let $\mu^0 \in \mathrm{Comp}_d^0$ with $|\mu^0| > 1$. According to Proposition 6.7, we only have to prove that $\Gamma^\gamma_{Y,\mu^0}(K) = 0$ which is equivalent to $\rho_{\mu^0,n}(x_\beta) = 0$. We will actually prove the following stronger statement:

$$\mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu\big(E_{\chi_k^\mu} x_\beta\big) = 0, \quad \text{for any } \mu \in \mathrm{Comp}_d(n) \text{ such that } [\mu] = \mu^0, \\ \text{and any } k \in \{1, \ldots, m_\mu\}.$$

The required assertion will then follow from (45) and the fact that $E_\mu x_\beta = \sum_{1 \leq k \leq m_\mu} E_{\chi_k^\mu} x_\beta$.

We first note that, in the framed braid group $\mathbb{Z}/d\mathbb{Z} \wr B_n$, we have $\beta t_j = t_{\bar{\beta}(j)} \beta$, for $j = 1, \ldots, n$, due to the fourth relation in (49). Therefore, in $Y_{d,n}$, we have $x_\beta t_j = t_{\bar{\beta}(j)} x_\beta$, for $j = 1, \ldots, n$, and in turn $x_\beta E_\chi = E_{\bar{\beta}(\chi)} x_\beta$ for any character χ of $(\mathbb{Z}/d\mathbb{Z})^n$.

Then let $\mu \in \mathrm{Comp}_d(n)$ such that $[\mu] = \mu^0$ and $k \in \{1, \ldots, m_\mu\}$. We recall that $\pi \in \mathfrak{S}_n$ satisfies $\pi(\chi_k^\mu) = \chi_k^\mu$ if and only if π belongs to a subgroup of $\mathfrak{S}_n$ conjugated to $\mathfrak{S}^\mu$ (namely, to $\pi_{k,\mu} \mathfrak{S}^\mu \pi_{k,\mu}^{-1}$ with the notations of Sect. 2). By the assumption on μ, we have at least two integers $a, b \in \{1, \ldots, d\}$ such that $\mu_a, \mu_b \geq 1$, and thus the subgroup $\mathfrak{S}^\mu = \mathfrak{S}_{\mu_1} \times \cdots \times \mathfrak{S}_{\mu_d}$ contains no cycle of length n. This means in particular that $\bar{\beta}^{-1}(\chi_k^\mu) \neq \chi_k^\mu$ since $\bar{\beta}$ is a cycle of length n as K is a knot.

Finally, we write $E_{\chi_k^\mu} x_\beta = E^2_{\chi_k^\mu} x_\beta = E_{\chi_k^\mu} x_\beta E_{\bar{\beta}^{-1}(\chi_k^\mu)}$ and we conclude the proof with the following calculation

$$\mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(E_{\chi_k^\mu} x_\beta E_{\bar{\beta}^{-1}(\chi_k^\mu)}) = \mathrm{Tr}_{\mathrm{Mat}_{m_\mu}} \circ \Psi_\mu(E_{\bar{\beta}^{-1}(\chi_k^\mu)} E_{\chi_k^\mu} x_\beta) = 0,$$

where we used that $E_\chi E_{\chi'} = 0$ if $\chi \neq \chi'$. □

Remark 6.9 Note that in general, for an arbitrary Markov trace $\{\rho_n\}_{n\geq 1}$, the invariant $\Gamma^{\gamma}_{Y,\rho}(K)$ is an element of the ring $R = \mathbb{C}[u^{\pm 1}, v^{\pm 1}, \gamma^{\pm 1}]$. For a classical knot K, Proposition 6.8 asserts in particular that every invariant $\Gamma^{\gamma}_{Y,\mu^0}(K)$ ($\mu^0 \in \mathrm{Comp}^0_d$) belongs actually to the subring $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$. Further, for a classical link K, Proposition 6.7 asserts in particular that, when $\mu^0 \in \mathrm{Comp}_d(1)$, the invariant $\Gamma^{\gamma}_{Y,\mu^0}(K)$ belongs as well to the subring $\mathbb{C}[u^{\pm 1}, v^{\pm 1}]$. So for classical links, the parameter γ in fact starts to play a non-trivial role when K is not a knot and $|\mu^0| > 1$ (see Example 6.6).

6.5 Connections with the approach of Juyumaya–Lambropoulou

Analogue of the Ocneanu trace and invariants Let q be an indeterminate. In [2,10,12–15], the Yokonuma–Hecke algebra is presented as a certain quotient of the group algebra over $\mathbb{C}[q, q^{-1}]$ of the $\mathbb{Z}/d\mathbb{Z}$-framed braid group $\mathbb{Z}/d\mathbb{Z} \wr B_n$. Namely, there are generators $G_1, G_2, \ldots, G_{n-1}$ and $t_1, \ldots, t_n$, satisfying the same relations as in (7) (with g_i replaced by G_i) except the last one, which is replaced by

$$G_i^2 = 1 + (q-1)e_i + (q-1)e_i G_i, \qquad i = 1, \ldots, n-1\,.$$

To avoid confusion, we will denote this algebra by $\widetilde{Y}_{d,n}$, and we will give an explicit isomorphism between $\widetilde{Y}_{d,n}$ and $Y_{d,n}$ later.

Let z be another indeterminate. For convenience, we set $k := \mathbb{C}(\sqrt{q}, z)$. Let $c_1, \ldots, c_{d-1}$ be arbitrary elements of k and set $c_0 := 1$. In [10], it is proved that there is a unique k-linear function tr on the chain, in n, of algebras $k\widetilde{Y}_{d,n}$ with values in k satisfying:

$$\begin{array}{ll}
\text{(C0)}\ \mathrm{tr}(1) = 1, & \\
\text{(C1)}\ \mathrm{tr}(xy) = \mathrm{tr}(yx), & \text{for any } n \geq 1 \text{ and } x, y \in \widetilde{Y}_{d,n};\\
\text{(C2)}\ \mathrm{tr}(xG_n) = z\,\mathrm{tr}(x), & \text{for any } n \geq 1 \text{ and } x \in \widetilde{Y}_{d,n}.\\
\text{(C3)}\ \mathrm{tr}(xt_{n+1}^b) = c_b\,\mathrm{tr}(x), & \text{for any } n \geq 0, x \in \widetilde{Y}_{d,n} \text{ and } b \in \{0, \ldots, d-1\}.
\end{array} \tag{62}$$

Note that here, $\widetilde{Y}_{d,n}$ is identified with a subalgebra of $\widetilde{Y}_{d,n+1}$ for any $n \geq 1$.

In [15], it is explained how to obtain isotopy invariants for classical and framed links from the linear function tr (see also [2,14] for classical and framed links and [13] for singular links). This is done as follows.

First we take the parameters $\{c_0, c_1, \ldots, c_{d-1}\}$ to be solutions of the so-called E-system [14, Appendix]. To do so, we fix a non-empty subset $S \subset \{1, \ldots, d\}$, and we set

$$c_b := \frac{1}{|S|}\sum_{a\in S} \xi_a^b, \qquad \text{for } b = 0, 1, \ldots, d-1, \tag{63}$$

where we recall that $\{\xi_1, \ldots, \xi_d\}$ is the set of d-th roots of unity. We denote tr_S the unique linear function satisfying (62) for the values (63) of the parameters $c_0, c_1, \ldots, c_{d-1}$, and we set:

$$E_S := \frac{1}{|S|}, \qquad \lambda_S := \frac{z + (1-q)E_S}{qz} \qquad \text{and} \qquad D_S := \frac{1}{\sqrt{\lambda_S} z}, \tag{64}$$

where the field k is extended by an element $\sqrt{\lambda_S}$. We denote by k_S this new field.

We let $\widetilde{\delta}^{(S)}_{Y,n}$ be the surjective morphism from $k_S\big[\mathbb{Z}/d\mathbb{Z} \wr B_n\big]$ to $k_S\widetilde{Y}_{d,n}$, defined by

$$\widetilde{\delta}^{(S)}_{Y,n} : \quad \sigma_i \mapsto \sqrt{\lambda_S} G_i \ \ (i = 1, \ldots, n-1), \qquad t_j \mapsto t_j \ \ (j = 1, \ldots, n)\,. \tag{65}$$

The fact that $\widetilde{\delta}^{(S)}_{Y,n}$ defines indeed an algebra morphism follows from the homogeneity of the relations (49) in the braid generators σ_i.

Finally, let $\tilde{K}$ be a $\mathbb{Z}/d\mathbb{Z}$-framed link and $\tilde{\beta}_{\tilde{K}} \in \mathbb{Z}/d\mathbb{Z} \wr B_n$ a $\mathbb{Z}/d\mathbb{Z}$-framed braid on n strands having $\tilde{K}$ as its closure. Then we define the map $\mathrm{F}\Delta_{Y,S}$ from the set of $\mathbb{Z}/d\mathbb{Z}$-framed links to the field k_S by

$$\mathrm{F}\Delta_{Y,S}(\tilde{K}) = D_S^{n-1} \cdot \mathrm{tr}_S \circ \widetilde{\delta}_{Y,n}^{(S)}(\tilde{\beta}_{\tilde{K}}). \tag{66}$$

Theorem 6.10 (Juyumaya–Lambropoulou [15]) *For any $S \subset \{1, \ldots, d\}$, the map $\mathrm{F}\Delta_{Y,S}$ is an isotopy invariant for $\mathbb{Z}/d\mathbb{Z}$-framed links.*

Remark 6.11 As in the previous section, the invariants $\mathrm{F}\Delta_{Y,S}$ can be restricted to give invariants for classical links [14]. We denote $\Delta_{Y,S}$ the corresponding invariants for classical links.

Comparison with invariants $\mathrm{F}\Gamma^{\gamma}_{Y,\rho}$ We keep S a fixed non-empty subset of $\{1, \ldots, d\}$, and $c_0, c_1, \ldots, c_{d-1}$ the associated solution (63) of the E-system. In order to relate the invariant $\mathrm{F}\Delta_{Y,S}$ (respectively, $\Delta_{Y,S}$) to the invariants of the form $\mathrm{F}\Gamma^{\gamma}_{Y,\rho}$ (respectively, $\Gamma^{\gamma}_{Y,\rho}$) obtained in Sect. 6.3, we denote

$$\widetilde{\rho}_{S,n} \ : \ k_S\widetilde{Y}_{d,n} \to k_S, \quad \widetilde{\rho}_{S,n}(x) := D_S^{n-1} \cdot \mathrm{tr}_S(x),$$

and define new generators by

$$g_i := \sqrt{\lambda_S}\big(\sqrt{q} + (1 - \sqrt{q})e_i\big)G_i, \qquad i = 1, \ldots, n-1. \tag{67}$$

Straightforward calculations show first that this change of generators is invertible since

$$G_i = \sqrt{\lambda_S}^{-1}\big(\sqrt{q}^{-1} + (1 - \sqrt{q}^{-1})e_i\big)g_i, \qquad i = 1, \ldots, n-1, \tag{68}$$

and moreover, that these new generators $g_1, \ldots, g_{n-1}$ satisfy all the defining relation in (7) of $Y_{d,n}$, where

$$u := \sqrt{q\lambda_S} \qquad \text{and} \qquad v := (q-1)\sqrt{\lambda_S}, \tag{69}$$

Thus, Formulas (67) and (68) provide mutually inverse isomorphisms between $k_S\widetilde{Y}_{d,n}$ and $k_SY_{d,n}$, and in turn, the linear maps $\widetilde{\rho}_{S,n}$ ($n \geq 1$) can be seen, via this isomorphism, as linear maps on $k_SY_{d,n}$. We note the following formula, which is derived directly from (69) and (64):

$$v^{-1}(1 - u^2) = \frac{D_S}{|S|}. \tag{70}$$

Proposition 6.12 (i) *The family of linear maps $\{\widetilde{\rho}_{S,n}\}_{n\geq 1}$ satisfies Conditions (M1) and (M2) in (36), and is thus a Markov trace on $\{k_SY_{d,n}\}_{n\geq 1}$.*

(ii) *Moreover, we have*

$$\mathrm{F}\Delta_{Y,S} = \mathrm{F}\Gamma^{\sqrt{q}^{-1}}_{Y,\tilde{\rho}_S}. \tag{71}$$

Proof (i) The trace condition (M1) is obviously satisfied by the linear maps $\widetilde{\rho}_{S,n}$. From Theorem 6.10, it follows that the family of linear maps $\{\widetilde{\rho}_{S,n}\}_{n\geq 1}$ satisfies

$$\widetilde{\rho}_{S,n+1}(xG_n) = \frac{\widetilde{\rho}_{S,n}(x)}{\sqrt{\lambda_S}} \ \text{ and } \ \widetilde{\rho}_{S,n+1}(xG_n^{-1}) = \sqrt{\lambda_S}\,\widetilde{\rho}_{S,n}(x), \ \text{ for any } n \geq 1 \text{ and } x \in Y_{d,n}\,.$$

It follows from Formula (67) and a short calculation that $g_i^{-1} = \sqrt{\lambda_S}^{-1}\big(\sqrt{q}^{-1} + (1 - \sqrt{q}^{-1})e_i\big)G_i^{-1}$. According to this and to Formula (67), the Markov condition (M2) will be satisfied if

$$\widetilde{\rho}_{S,n+1}(xe_nG_n^{\pm 1}) = \widetilde{\rho}_{S,n+1}(xG_n^{\pm 1}), \qquad \text{for any } n \geq 1 \text{ and } x \in Y_{d,n}$$

The end of the proof of Theorem 6.4 item **1**, from Relation (55), can be repeated here.

(ii) This is immediate in view of (66) and (68), taking into account the definition (65) of $\widetilde{\delta}^{(S)}_{Y,n}$. □

At this point, we proved that the invariants $\mathrm{F}\Delta_{Y,S}$ (and thus $\Delta_{Y,S}$ as well) are included in the sets of invariants constructed in this paper. For a given S, to identify precisely to which invariant $\mathrm{F}\Delta_{Y,S}$ corresponds, in view of (71), it remains to determine the Markov trace $\{\widetilde{\rho}_{S,n}\}_{n\geq 1}$ in terms of the classification given in Theorem 5.3.

Proposition 6.13 *Using notations as in Theorem 5.3, the Markov trace* $\{\widetilde{\rho}_{S,n}\}_{n\geq 1}$ *on* $\{\tilde{k}Y_{d,n}\}_{n\geq 1}$ *is given by the following choice of parameters:*

$$\alpha_{\mu^0} = \begin{cases} 0 & \text{if } \mu^0_a > 0 \text{ for some } a \notin S, \\ \dfrac{D_S^{|\mu^0|-1}}{|S|^{|\mu^0|}} & \text{otherwise.} \end{cases} \tag{72}$$

Proof Let $\{\alpha_{\mu^0},\ \mu^0 \in \mathrm{Comp}_d^0\}$ be the set of parameters, which is to be determined, corresponding to $\{\widetilde{\rho}_{S,n}\}_{n\geq 1}$. We recall that, from the classification result, the associated traces $\widetilde{\rho}^{\mu}_S$ are of the form

$$\widetilde{\rho}^{\mu}_S = \alpha_{[\mu]} \cdot \tau_{\mu_1} \otimes \cdots \otimes \tau_{\mu_d}, \qquad \text{for any } \mu \in \mathrm{Comp}_d(n).$$

For $a \in \{1, \ldots, d\}$, we denote by α_a (respectively, χ_a) the parameter (respectively, the character) corresponding to the composition $(0, \ldots, 0, 1, 0, \ldots, 0)$ with 1 in a-th position.

First, Condition (C3) in (62) for $n = 0$ gives

$$\widetilde{\rho}_{S,1}(t_1^b) = c_b, \qquad b = 0, \ldots, d-1.$$

On the other hand, we write $t_1^b = \sum_{1\leq a\leq d} E_{\chi_a}\xi_a^b$, and we obtain

$$\widetilde{\rho}_{S,1}(t_1^b) = \sum_{1\leq a\leq d} \xi_a^b \alpha_a = c_b, \qquad b = 0, \ldots, d-1.$$

Inverting the Vandermonde matrix of size d with coefficients ξ_j^{i-1} in row i and column j, this yields:

$$\alpha_a = \frac{1}{d} \sum_{0\leq b\leq d-1} \xi_a^{-b} c_b, \qquad a = 1, \ldots, d. \tag{73}$$

Taking into account now the values of c_b in (63) corresponding to S, we obtain Formula (72) when $|\mu^0| = 1$.

Let $n > 0$. Condition (C3) in (62) now gives

$$\widetilde{\rho}_{S,n+1}(xt_{n+1}^b) = D_S c_b \widetilde{\rho}_{S,n}(x), \qquad x \in Y_{d,n},\ \ b = 0, \ldots, d-1.$$

Let $\mu \in \mathrm{Comp}_d(n)$ and let χ_1^{μ} be the character of $(\mathbb{Z}/d\mathbb{Z})^n$ defined in (19). We then have, by construction,

$$\widetilde{\rho}_{S,n}(E_{\chi_1^{\mu}}) = \widetilde{\rho}^{\mu}_S(1),$$

while, on the other hand, writing $E_{\chi_1^{\mu}} t_{n+1}^b = \sum_{1\leq a\leq d} \xi_a^b E_{\chi_1^{\mu[a]}}$, we have

$$\widetilde{\rho}_{S,n+1}(E_{\chi_1^{\mu}} t_{n+1}^b) = \sum_{1\leq a\leq d} \xi_a^b \widetilde{\rho}^{\mu[a]}_S(1), \qquad b = 0, \ldots, d-1.$$

We conclude that, for any $\mu \in \mathrm{Comp}_d(n)$ and $b = 0, \ldots, d-1$, we have

$$\sum_{1 \le a \le d} \xi_a^b \alpha_{[\mu^{[a]}]} \cdot \tau_{\mu_1}(1) \ldots \tau_{\mu_a+1}(1) \ldots \tau_{\mu_d}(1) = D_S\, c_b\, \alpha_{[\mu]} \cdot \tau_{\mu_1}(1) \ldots \tau_{\mu_a}(1) \ldots \tau_{\mu_d}(1).$$

Inverting the same matrix as above, and using the already obtained formula (73), we conclude that

$$\begin{aligned} &\alpha_{[\mu^{[a]}]} \cdot \tau_{\mu_1}(1) \ldots \tau_{\mu_a+1}(1) \ldots \tau_{\mu_d}(1) \\ &\quad = D_S \alpha_a \alpha_{[\mu]} \cdot \tau_{\mu_1}(1) \ldots \tau_{\mu_a}(1) \ldots \tau_{\mu_d}(1), \qquad a = 1, \ldots, d\,. \end{aligned}$$

Now when $\mu_a = 0$, this yields $\alpha_{[\mu^{[a]}]} = D_S \alpha_a \alpha_{[\mu]}$, which is what is needed to conclude the proof. □

Remark 6.14 Following Remark 5.6 after the proof of Theorem 5.3, we notice that the associated traces corresponding to $\{\widetilde{\rho}_{S,n}\}_{n\ge 1}$ are given, for $\mu \in \mathrm{Comp}_d(n)$, by

$$\widetilde{\rho}_S^{\mu} = \begin{cases} 0 & \text{if } \mu_a > 0 \text{ for some } a \notin S, \\ \dfrac{1}{|S|} \cdot \tau_n & \text{otherwise.} \end{cases}$$

where τ_n acts on $\mathcal{H}^{\mu}$ by restriction from $\mathcal{H}_n$. This follows directly from Proposition 6.13 and (70).

Remark 6.15 Proposition 6.13 gives the explicit decomposition of the Markov trace $\{\widetilde{\rho}_{S,n}\}_{n\ge 1}$ in the basis $\left\{ \{\rho_{\mu^0,n}\}_{n\ge 1} \mid \mu^0 \in \mathrm{Comp}_d^0 \right\}$ and in turn, together with Proposition 6.12, relates explicitly the invariant $\mathrm{F}\Delta_{Y,S}$ with the invariants obtained in this paper. Concretely, we have:

$$\mathrm{F}\Delta_{Y,S} = \sum_{\mu^0 \in \mathrm{Comp}_d^0} \alpha_{\mu^0} \mathrm{F}\Gamma_{Y,\mu^0}^{\sqrt{q}^{-1}},$$

where the coefficients α_{μ^0} are given by (72), and the variables u and v are expressed in terms of variables q and λ_S according to (69).

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References

1. Chlouveraki, M., Juyumaya, J., Karvounis, K., Lambropoulou, S.: Identifying the invariants for classical knots and links from the Yokonuma-Hecke algebras. Preprint (2015). arXiv:1505.06666
2. Chlouveraki, M., Lambropoulou, S.: The Yokonuma–Hecke algebras and the HOMFLYPT polynomial. J. Knot Theory Ramif. **22**(14), 1350080 (2013)
3. Chlouveraki, M., Poulain d'Andecy, L.: Representation theory of the Yokonuma–Hecke algebra. Adv. Math. **259**, 134–172 (2014)
4. Dipper, R., Mathas, A.: Morita equivalences of Ariki–Koike algebras. Math. Z. **240**(3), 579–610 (2002)
5. Du, J., Rui, H.: Ariki–Koike algebras with semisimple bottoms. Math. Z. **234**(4), 807–830 (2000)
6. Geck M., Jacon, N.: Representations of Hecke algebras at roots of unity. In: Algebra and Applications, vol. 15. Springer, London (2011)
7. Geck M., Pfeiffer, G.: Characters of finite Coxeter groups and Iwahori–Hecke algebras. In: London Mathematical Society Monographs. New Series, 21. The Clarendon Press, Oxford University Press, New York (2000)

8. Jones, V.F.R.: Hecke algebra representations of braid groups and links polynomials. Ann. Math. **126**, 335–388 (1987)
9. Juyumaya, J.: Sur les nouveaux générateurs de l'algèbre de Hecke H(G, U,1). J. Algebra **204**, 49–68 (1998)
10. Juyumaya, J.: Markov trace on the Yokonuma–Hecke algebra. J. Knot Theory Ramif. **13**, 25–39 (2004)
11. Juyumaya, J., Kannan, S.: Braid relations in the Yokonuma–Hecke algebra. J. Algebra **239**, 272–297 (2001)
12. Juyumaya, J., Lambropoulou, S.: p-adic framed braids. Topol. Appl. **154**, 1804–1826 (2007)
13. Juyumaya, J., Lambropoulou, S.: An invariant for singular knots. J. Knot Theory Ramif. **18**, 825–840 (2009)
14. Juyumaya, J., Lambropoulou, S.: An adelic extension of the Jones polynomial. In: Banagl, M., Vogel, D. (eds.) The Mathematics of Knots, Contributions in the Mathematical and Computational Sciences, vol. 1. Springer, Berlin (2011)
15. Juyumaya, J., Lambropoulou, S.: p-adic framed braids II. Adv. Math. **234**, 149–191 (2013)
16. Ko, K.H., Smolinsky, L.: The framed braid group and 3-manifolds. Proc. Am. Math. Soc. **115**(2), 541–551 (1992)
17. Lam, T.Y.: Lectures on Modules and Rings, vol. 189. Springer, Berlin (1999)
18. Lusztig, G.: Character sheaves on disconnected groups, VII. Represent. Theory **9**, 209–266 (2005)
19. The Knot Atlas: The Thistlethwaite Link Table. Retrieved January 22, 2015 from http://katlas.org/wiki/The_Thistlethwaite_Link_Table
20. Thiem, N.: Unipotent Hecke algebras of $\mathrm{GL}_n(\mathbb{F}_q)$. J. Algebra **284**, 559–577 (2005)
21. Thiem, N.: A skein-like multiplication algorithm for unipotent Hecke algebras. Trans. Am. Math. Soc. **359**(4), 1685–1724 (2007)
22. Yokonuma, T.: Sur la structure des anneaux de Hecke d'un groupe de Chevalley fini. C. R. Acad. Sci. Paris Ser. I Math. **264**, 344–347 (1967)