Math. Z. (2016) 282:1169–1190
DOI 10.1007/s00209-015-1584-7

Mathematische Zeitschrift

Moser–Trudinger inequalities for singular Liouville systems

Luca Battaglia[1]

Received: 15 April 2015 / Accepted: 27 October 2015 / Published online: 23 November 2015

Abstract In this paper we prove a Moser–Trudinger inequality for the Euler–Lagrange functional of general singular Liouville systems on a compact surface. We characterize the values of the parameters which yield coercivity for the functional, hence the existence of energy-minimizing solutions for the system, and we give necessary conditions for boundedness from below. We also provide a sharp inequality under assuming the coefficients of the system to be non-positive outside the diagonal. The proofs use a concentration-compactness alternative, Pohožaev-type identities and blow-up analysis.

Keywords Liouville systems · Moser–Trudinger inequality · Coercivity · Minimizing solutions

Mathematics Subject Classification 35A23 · 35J47 · 35J50 · 58E35

1 Introduction

An essential tool in the study of the embeddings of Sobolev spaces is the Moser–Trudinger inequality, which gives compact embedding in any L^p space for finite $p \geq 1$ and also exponential integrability.

If we consider a 2-dimensional compact Riemannian manifold (Σ, g), due to well-known works from Moser [18] and Fontana [13] we get

$$\log \int_\Sigma e^u \mathrm{d}V_g - \fint_\Sigma u \mathrm{d}V_g \leq \frac{1}{16\pi} \int_\Sigma |\nabla u|^2 \mathrm{d}V_g + C \qquad \forall u \in H^1(\Sigma), \tag{1}$$

The author has been supported by the PRIN project *Variational and perturbative aspects of nonlinear differential problems*.

✉ Luca Battaglia
lbatta@sissa.it

1 Université Catholique de Louvain, Institut de Recherche en Mathématique et Physique, Chemin du Cyclotron 2, 1348 Louvain-la-Neuve, Belgium

where $\nabla = \nabla_g$ is the gradient given by the metric g and $C = C_{\Sigma,g}$ is a constant depending only on Σ and g.

This inequality has fundamental importance in the study of the Liouville equations of the kind

$$-\Delta u = \rho\left(\frac{he^u}{\int_\Sigma he^u \mathrm{d}V_g} - 1\right), \tag{2}$$

where $\Delta = \Delta_g$ is the Laplace-Beltrami operator, ρ a positive real parameter, h a positive smooth function and Σ is supposed, without loss of generality, to have area equal to $|\Sigma| = 1$. In fact, the solutions of (2) are critical points of the functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 \mathrm{d}V_g - \rho\left(\log\int_\Sigma he^u \mathrm{d}V_g - \int_\Sigma u\mathrm{d}V_g\right);$$

using the inequality (1) we can control the last term by the Dirichlet energy, thus showing that I_ρ is bounded from below on $H^1(\Sigma)$ if and only if ρ is smaller or equal to 8π.

Equations like (2) have great importance in different contexts like the Gaussian curvature prescription problem (see for instance [6,7]) and abelian Chern–Simons models in theoretical physics ([21,24]).

An extension of the inequality (1), which takes into consideration power-type weights, was given by Chen [8] and Trojanov [22]. For a given $p \in \Sigma$ and $\alpha \in (-1, 0]$, they showed that

$$(1+\alpha)\left(\log\int_\Sigma d(\cdot, p)^{2\alpha} e^u \mathrm{d}V_g - \int_\Sigma u\mathrm{d}V_g\right) \le \frac{1}{16\pi}\int_\Sigma |\nabla u|^2 \mathrm{d}V_g + C \qquad \forall\, u \in H^1(\Sigma). \tag{3}$$

This inequality allows to treat singularities in the Eq. (2), that is to consider equations like

$$-\Delta u = \rho\left(\frac{he^u}{\int_\Sigma he^u \mathrm{d}V_g} - 1\right) - 4\pi\sum_{m=1}^M \alpha_m(\delta_{p_m} - 1), \tag{4}$$

where we take arbitrary $p_1, \ldots, p_M \in \Sigma$ and $\alpha_m > -1$ for any $m \in \{1, \ldots, M\}$.

This is a natural extension of (2), which allows to consider the same problems in a more general context. For instance, it arises in the Gaussian curvature prescription problem on surfaces with conical singularities and in Chern–Simons vortices theory.

Defining G_p as the Green function of $-\Delta$ on Σ centered at a point p, through the change of variables

$$u \mapsto u + 4\pi\sum_{m=1}^M \alpha_m G_{p_m} \tag{5}$$

Equation (4) becomes

$$-\Delta u = \rho\left(\frac{\widetilde{h}e^u}{\int_\Sigma \widetilde{h}e^u \mathrm{d}V_g} - 1\right)$$

with $\widetilde{h} = he^{-4\pi\sum_{m=1}^M \alpha_m G_{p_m}}$.

Since G_p has the same behavior as $\frac{1}{2\pi}\log\frac{1}{d(\cdot,p)}$ around p, then $\widetilde{h}$ behaves like $d(\cdot, p_m)^{2\alpha_m}$ around each singular point p_m, hence the energy functional

$$I_\rho(u) = \frac{1}{2}\int_\Sigma |\nabla u|^2 \mathrm{d}V_g - \rho\left(\log\int_\Sigma \widetilde{h}e^u \mathrm{d}V_g - \int_\Sigma u \mathrm{d}V_g\right)$$

can be estimated, as in the regular case, using (3).

The purpose of this paper is to extend inequality (3) to singular Liouville systems of the type

$$-\Delta u_i = \sum_{j=1}^N a_{ij}\rho_j\left(\frac{h_j e^{u_j}}{\int_\Sigma h_j e^{u_j}\mathrm{d}V_g} - 1\right) - 4\pi\sum_{m=1}^M \alpha_{im}(\delta_{p_m} - 1), \quad i = 1, \dots, N,$$

where $A = (a_{ij})$ is a $N \times N$ symmetric positive definite matrix and ρ_i, h_i, α_{im} are as before.

Applying, similarly to (5), the change of variables

$$u_i \mapsto u_i + 4\pi\sum_{m=1}^M \alpha_{im} G_{p_m},$$

the system becomes

$$-\Delta u_i = \sum_{j=1}^N a_{ij}\rho_j\left(\frac{\widetilde{h}_j e^{u_j}}{\int_\Sigma \widetilde{h}_j e^{u_j}\mathrm{d}V_g} - 1\right), \quad i = 1, \dots, N, \tag{6}$$

with $\widetilde{h}_j$ having the same behavior around the singular points.

The system has a variational formulation with the energy functional

$$J_\rho(u) := \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g - \sum_{i=1}^N \rho_i\left(\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g - \int_\Sigma u_i \mathrm{d}V_g\right), \tag{7}$$

with a^{ij} indicating the entries of the inverse matrix A^{-1} of A.

A recent paper by the author and Malchiodi ([2]) gives an answer for the particular case of the $SU(3)$ Toda system, that is $N = 2$ and A is the Cartan matrix

$$\begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}.$$

This is a particularly interesting case, due to its application in the description of holomorphic curves in $\mathbb{CP}^N$ in geometry ([3,5,9]) and in the non-abelian Chern–Simons theory in physics ([12,21,24]).

The authors prove a sharp inequality, that is they show that the functional J_ρ is bounded from below if and only if both the parameters ρ_i are less or equal than $4\pi\min\{1, 1 + \min_m \alpha_{im}\}$, thus extending the result in the regular case from [15].

Concerning general regular Liouville systems, Wang [23] gave necessary and sufficient conditions for the boundedness from below of J_ρ, following previous results in [10,11] for the problem on Euclidean domains with Dirichlet boundary conditions. Analogous results were given in [20] for the standard unit sphere $\left(\mathbb{S}^2, g_0\right)$ and in [19] for a similar problem.

In these papers, the authors introduce, for any $\mathcal{I} \subset \{1, \dots, N\}$, the following function of the parameter ρ:

$$\Lambda_{\mathcal{I}}(\rho) = 8\pi\sum_{i\in\mathcal{I}} \rho_i - \sum_{i,j\in\mathcal{I}} a_{ij}\rho_i\rho_j.$$

What they prove is boundedness from below for J_ρ for any $\rho \in \mathbb{R}^N_+$ which satisfies $\Lambda_{\mathcal{I}}(\rho) > 0$ for all the subsets $\mathcal{I}$ of $\{1, \dots, N\}$, whereas $\inf_{H^1(\Sigma)^N} J_\rho = -\infty$ whenever $\Lambda_{\mathcal{I}}(\rho) < 0$ for some $\mathcal{I} \subset \{1, \dots, N\}$.

The first main result of this paper is an extension of the results from [10,11,23] to the case of singularities.

Similarly to Liouville equation, we will have to multiply some quantities by $1 + \alpha_{im}$. Precisely, we have:

Theorem 1.1 *Let J_ρ be the functional defined by (7) and set, for $\rho \in \mathbb{R}^N_{>0}$, $x \in \Sigma$ and $i \in \mathcal{I} \subset \{1, \dots, N\}$:*

$$\alpha_i(x) = \begin{cases} \alpha_{im} & \text{if } x = p_m \\ 0 & \text{otherwise} \end{cases} \qquad \Lambda_{\mathcal{I},x}(\rho) := 8\pi \sum_{i \in \mathcal{I}} (1 + \alpha_i(x))\rho_i - \sum_{i,j \in \mathcal{I}} a_{ij}\rho_i\rho_j$$

$$\Lambda(\rho) := \min_{\mathcal{I} \subset \{1,\dots,N\}, x \in \Sigma} \Lambda_{\mathcal{I},x}(\rho). \tag{8}$$

Then, J_ρ is bounded from below if $\Lambda(\rho) > 0$, whereas J_ρ is unbounded from below if $\Lambda(\rho) < 0$.

Notice that, in the definition of Λ, the minimum makes sense because it is taken in a finite set, since $\alpha_i(x) = 0$ for all points of Σ but a finite number, and for all the former points $\Lambda_{\mathcal{I},x}$ is defined in the same way.

As a consequence of this result, we obtain information about the existence of solutions for the system (6).

Corollary 1.2 *The functional J_ρ is coercive in $\overline{H}^1(\Sigma)$ if and only if $\Lambda(\rho) > 0$.*

Therefore, if this occurs, then J_ρ admits a minimizer u which solves (6).

Theorem 1.1 leaves an open question about what happens when $\Lambda(\rho) = 0$. In this case, as we will see in the following Sections, one encounters blow-up phenomena which are not yet fully known for general systems.

Anyway, we can say something more if we assume in addition $a_{ij} \le 0$ for any $i, j \in \{1, \dots, N\}$ with $i \ne j$. First of all, we notice that in this case

$$\Lambda(\rho) = \min_{i \in \{1,\dots,N\}} \left(8\pi(1 + \widetilde{\alpha}_i)\rho_i - a_{ii}\rho_i^2\right), \qquad \text{where}$$

$$\widetilde{\alpha}_i := \min_{m \in \{1,\dots,M\}, x \in \Sigma} \alpha_i(x) = \min\left\{0, \min_{m \in \{1,\dots,M\}} \alpha_{im}\right\}; \tag{9}$$

hence the sufficient condition in Theorem 1.1 is equivalent to assuming $\rho_i < \frac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}}$ for any i.

With this assumption, studying what happens when $\Lambda_{\mathcal{I}}(\rho) = 0$ is reduced to a single-component local blow-up, which can be treated by using an inequality from [1]. Therefore, we get the following sharp result:

Theorem 1.3 *Let J_ρ be defined by (7), $\widetilde{\alpha}_i$ as in (9) and $\Lambda(\rho)$ as in Theorem 1.1, and suppose $a_{ij} \le 0$ for any $i, j \in \{1, \dots, N\}$ with $i \ne j$.*

Then, J_ρ is bounded from below on $H^1(\Sigma)^N$ if and only if $\Lambda(\rho) \ge 0$, namely if and only if $\rho_i \le \frac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}}$ for any $i \in \{1, \dots, N\}$.

We remark that the assuming A to be positive definite is necessary. If it is not, then J_ρ is unbounded from below for any ρ.

In fact, suppose there exists $v \in \mathbb{R}^N$ such that $\sum_{i,j=1}^N a^{ij} v_i v_j \le -\theta |v|^2$ for some $\theta > 0$. Then, we consider the family of functions $u^\lambda(x) := \lambda v \cdot x$; by Jensen's inequality we get

$$\begin{aligned} J_\rho\left(u^\lambda\right) &\le \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i^\lambda \cdot \nabla u_j^\lambda \mathrm{d}V_g - \sum_{i=1}^N \rho_i \int_\Sigma \log \widetilde{h}_i \mathrm{d}V_g \\ &\le -\frac{\theta}{2}\lambda^2|v|^2 + C \\ &\underset{n\to+\infty}{\longrightarrow} -\infty. \end{aligned}$$

We also notice that, with respect to the scalar case, in Theorem 1.1 and Corollary 1.2 the positive coefficients α_{im}'s may affect the definition of $\Lambda(\rho)$, hence the conditions for coercivity and boundedness from below of J_ρ.

On the other hand, under the assumptions of Theorem 1.3, coercivity and boundedness from below only depend on the negative α_{im}'s, just like for the scalar equation.

The plan of this paper is the following: in Sect. 2 we will introduce some notations and some preliminary results which will be used throughout the rest of the paper. In Sect. 3 we will show a sort of Concentration-compactness theorem, showing the possible non-compactness phenomena for solutions of the system (6). Finally, in Sects. 4 and 5 we will give the proof of the two main theorems.

2 Notations and preliminaries

In this section, we will give some useful notation and some known preliminary results which will be needed to prove the two main theorems.

Given two points $x, y \in \Sigma$, we will indicate the metric distance on Σ between them as $d(x, y)$. We will indicate the open metric ball centered in p having radius r as

$$B_r(x) := \{y \in \Sigma : d(x, y) < r\}.$$

For any subset of a topological space $A \subset X$ we indicate its closure as $\overline{A}$ and its interior part as $\mathring{A}$.

Given a function $u \in L^1(\Sigma)$, the symbol $\overline{u}$ will indicate the average of u on Σ. Since we assume $|\Sigma| = 1$, we can write:

$$\overline{u} = \int_\Sigma u \mathrm{d}V_g = \fint_\Sigma u \mathrm{d}V_g.$$

We will indicate the subset of $H^1(\Sigma)$ which contains the functions with zero average as

$$\overline{H}^1(\Sigma) := \left\{u \in H^1(\Sigma) : \overline{u} = 0\right\}.$$

Since the functional J_ρ defined by (7) is invariant by addition of constants, it will not be restrictive to study it on $\overline{H}^1(\Sigma)^N$ rather than on $H^1(\Sigma)^N$.

We will indicate with the letter C large constants which can vary among different lines and formulas. To underline the dependence of C on some parameter α, we indicate with C_α and so on.

We will denote as $o_\alpha(1)$ quantities which tend to 0 as α tends to 0 or to $+\infty$ and we will similarly indicate bounded quantities as $O_\alpha(1)$, omitting in both cases the subscript(s) when it is evident from the context.

First of all, we need a result from Brezis and Merle [4]. It is a classical estimate about exponential integrability of solutions of some elliptic PDEs.

Lemma 2.1 ([4], Theorem 1) *Take* $r > 0,\ \Omega := B_r(0) \subset \mathbb{R}^2,\ f \in L^1(\Omega)$ *with* $\|f\|_{L^1(\Omega)} < 4\pi$ *and* u *solving*

$$\begin{cases} -\Delta u = f & in\ \Omega \\ u = 0 & on\ \partial\Omega \end{cases}.$$

Then, for any $q \in \left[1, \frac{4\pi}{\|f\|_{L^1(\Omega)}}\right)$ *there exists a constant* $C = C_{q,r}$ *such that* $\int_\Omega e^{q|u(x)|}\mathrm{d}x \le C$.

A crucial role in the proof of both Theorems 1.1 and 1.3 will be played by the concentration values of the sequences of solutions of (6).

For a sequence $u^n = \{u_1^n, \ldots, u_N^n\}_{n\in\mathbb{N}}$ of solutions of (6) with $\rho = \rho^n = \{\rho_1^n, \ldots, \rho_N^n\}$, we define (up to subsequences), for $i \in \{1, \ldots, N\}$, the concentration value of its i^{th} component around a point $x \in \Sigma$ as

$$\sigma_i(x) := \lim_{r\to 0}\lim_{n\to+\infty} \rho_i^n \frac{\int_{B_r(x)} \widetilde{h}_i e^{u_i^n}\mathrm{d}V_g}{\int_\Sigma \widetilde{h}_i e^{u_i^n}\mathrm{d}V_g}. \tag{10}$$

In a recent paper ([16], see also [14] for the regular case) it was proved, by a Pohožaev identity, that the concentration values satisfy the following algebraic relation, which involves the same quantities as in Theorem 1.1:

Proposition 2.2 ([14], Lemma 2.2; [16], Proposition 3.1) *Let* $\{u^n\}_{n\in\mathbb{N}}$ *be a sequence of solutions of* (6), $\alpha_i(x)$ *and* $\Lambda_{\mathcal{I},x}$ *as in* (8) *and* $\sigma(x) = (\sigma_1(x), \ldots, \sigma_N(x))$ *as in* (10). *Then,*

$$\Lambda_{\{1,\ldots,N\},x}(\sigma(x)) = 8\pi\sum_{i=1}^N (1+\alpha_i(x))\sigma_i(x) - \sum_{i,j=1}^N a_{ij}\sigma_i(x)\sigma_j(x) = 0.$$

To study the concentration phenomena of solutions of (6) we will use the following simple but useful calculus Lemma:

Lemma 2.3 ([15], Lemma 4.4) *Let* $\{a^n\}_{n\in\mathbb{N}}$ *and* $\{b^n\}_{n\in\mathbb{N}}$ *two sequences of real numbers satisfying*

$$a^n \underset{n\to+\infty}{\longrightarrow} +\infty \qquad \lim_{n\to+\infty}\frac{b^n}{a^n} \le 0.$$

Then, there exists a smooth function $F : [0, +\infty) \to \mathbb{R}$ *which satisfies, up to subsequences,*

$$0 < F'(t) < 1 \quad \forall t > 0 \qquad F'(t) \underset{t\to+\infty}{\longrightarrow} 0 \qquad F\left(a^n\right) - b^n \underset{n\to+\infty}{\longrightarrow} +\infty.$$

Finally, as anticipated in the introduction, we will need a singular Moser–Trudinger inequality for Euclidean domains by Adimurthi and Sandeep [1], and its straightforward corollary.

Theorem 2.4 ([1], Theorem 2.1) *For any* $r > 0,\ \alpha \in (-1, 0]$ *there exists a constant* $C = C_{\alpha,r}$ *such that if* $\Omega := B_r(0) \subset \mathbb{R}^2$ *and* $u \in H_0^1(\Omega)$, *then*

$$\int_\Omega |\nabla u(x)|^2\mathrm{d}x \le 1 \Rightarrow \int_\Omega |x|^{2\alpha}e^{4\pi(1+\alpha)u(x)^2}\mathrm{d}x \le C$$

Corollary 2.5 *For any $r > 0$, $\alpha \in (-1, 0]$ there exists a constant $C = C_{\alpha,r}$ such that if $\Omega := B_r(0) \subset \mathbb{R}^2$ and $u \in H_0^1(\Omega)$, then*

$$(1+\alpha)\log\int_\Omega |x|^{2\alpha} e^{u(x)}\mathrm{d}x \le \frac{1}{16\pi}\int_\Omega |\nabla u(x)|^2\mathrm{d}x + C$$

Proof By the elementary inequality $u \le \theta u^2 + \frac{1}{4\theta}$ with $\theta = \frac{4\pi(1+\alpha)}{\int_\Omega |\nabla u(y)|^2\mathrm{d}y}$ we get

$$\begin{aligned}(1+\alpha)\log\int_\Omega |x|^{2\alpha}e^{u(x)}\mathrm{d}x &\le (1+\alpha)\log\int_\Omega |x|^{2\alpha}e^{\theta u(x)^2+\frac{1}{4\theta}}\mathrm{d}x \\ &= \frac{1}{16\pi}\int_\Omega |\nabla u(y)|^2\mathrm{d}y + (1+\alpha)\log \\ &\quad\times\int_\Omega |x|^{2\alpha} e^{4\pi(1+\alpha)\left(\frac{u(x)}{\sqrt{\int_\Omega |\nabla u(y)|^2\mathrm{d}y}}\right)^2}\mathrm{d}x \\ &\le \frac{1}{16\pi}\int_\Omega |\nabla u(y)|^2\mathrm{d}y + C.\end{aligned}$$

□

3 A Concentration-compactness theorem

The aim of this section is to prove a result which describes the concentration phenomena for the solutions of (6), extending what was done for the two-dimensional Toda system in [2,17].

We actually have to normalize such solutions to bypass the issue of the invariance by translation by constants and to have the parameter ρ multiplying only the constant term.

In fact, for any solution u of (6) the functions

$$v_i := u_i - \log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g + \log\rho_i \tag{11}$$

solve

$$\begin{cases} -\Delta v_i = \sum_{j=1}^N a_{ij}\left(\widetilde{h}_j e^{v_j} - \rho_j\right) \\ \int_\Sigma \widetilde{h}_i e^{v_i}\mathrm{d}V_g = \rho_i \end{cases} \qquad i = 1, \ldots, N. \tag{12}$$

Moreover, we can rewrite in a shorter way (10) as

$$\sigma_i(x) = \lim_{r\to 0}\lim_{n\to+\infty}\int_{B_r(x)} \widetilde{h}_i^n e^{v_i^n}\mathrm{d}V_g.$$

For such functions, we get the following concentration-compactness alternative:

Theorem 3.1 *Let $\{u^n\}_{n\in\mathbb{N}}$ be a sequence of solutions of (6) with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho \in \mathbb{R}_+^N$ and $\widetilde{h}_i^n = V_i^n\widetilde{h}_i$ with $V_i^n \underset{n\to+\infty}{\longrightarrow} 1$ in $C^1(\Sigma)^N$, $\{v^n\}_{n\in\mathbb{N}}$ be defined as in (11) and $\mathcal{S}_i$ be defined, for $i \in \{1, \ldots, N\}$, by*

$$\mathcal{S}_i := \left\{x \in \Sigma : \exists x^n \underset{n\to+\infty}{\longrightarrow} x \text{ such that } v_i^n\left(x^n\right) \underset{n\to+\infty}{\longrightarrow} +\infty\right\}. \tag{13}$$

Then, up to subsequences, one of the following occurs:

- *If* $\mathcal{S}_i = \emptyset$ *for any* $i \in \{1, \ldots, N\}$, *then* $v^n \underset{n\to+\infty}{\longrightarrow} v$ *in* $W^{2,q}(\Sigma)^N$ *for some* $q > 1$ *and some* v *which solves* (12).
- *If* $\mathcal{S}_i \neq \emptyset$ *for some* i, *then it is a finite set for all such* i*'s. If this occurs, then there is a subset* $\mathcal{I} \subset \{1, \ldots, N\}$ *such that* $v_j^n \underset{n\to+\infty}{\longrightarrow} -\infty$ *in* $L^\infty_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'}\right)$ *for any* $j \in \mathcal{I}$ *and* $v_j^n \underset{n\to+\infty}{\longrightarrow} v_j$ *in* $W^{2,q}_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'}\right)$ *for some* $q > 1$ *and some suitable* v_j, *for any* $j \in \{1, \ldots, N\} \setminus \mathcal{I}$.

Since $\widetilde{h}_j$ is smooth outside the points p_m's, the estimates in $W^{2,q}(\Sigma)$ are actually in $C^{2,\alpha}_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{m=1}^M p_m\right)$ and the estimates in $W^{2,q}_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{j'=1}^N \mathcal{S}_{j'}\right)$ are actually in $C^{2,\alpha}_{\mathrm{loc}}\left(\Sigma \setminus \left(\bigcup_{j'=1}^N \mathcal{S}_{j'} \cup \bigcup_{m=1}^M p_m\right)\right)$. Anyway, estimates in $W^{2,q}$ will suffice in most of the paper.

To prove Theorem 3.1 we need two preliminary lemmas.

The first is a Harnack-type alternative for sequences of solutions of PDEs. It is inspired by [4,17].

Lemma 3.2 *Let* $\Omega \subset \Sigma$ *be a connected open subset,* $\{f^n\}_{n\in\mathbb{N}}$ *a bounded sequence in* $L^q_{\mathrm{loc}}(\Omega) \cap L^1(\Omega)$ *for some* $q > 1$ *and* $\{w^n\}_{n\in\mathbb{N}}$ *bounded from above and solving* $-\Delta w^n = f^n$ *in* Ω.

Then, up to subsequences, one of the following alternatives holds:

- w^n *is uniformly bounded in* $L^\infty_{\mathrm{loc}}(\Omega)$.
- $w^n \underset{n\to+\infty}{\longrightarrow} -\infty$ *in* $L^\infty_{\mathrm{loc}}(\Omega)$.

Proof Take a compact set $\mathcal{K} \Subset \Omega$ and cover it with balls of radius $\frac{r}{2}$, with r smaller than the injectivity radius of Σ. By compactness, we can write $\mathcal{K} \subset \bigcup_{h=1}^H B_{\frac{r}{2}}(x_h)$. If the second alternative does not occur, then up to relabeling we get $\sup_{B_r(x_1)} w^n \geq -C$.

Then, we consider the solution z^n of

$$\begin{cases} -\Delta z^n = f^n & \text{in } B_r(x_1) \\ z^n = 0 & \text{on } \partial B_r(x_1) \end{cases},$$

which is bounded in $L^\infty(B_r(x_1))$ by elliptic estimates. This means that, for a large constant C, the function $C - w^n + z^n$ is positive, harmonic and bounded from below on $B_r(x_1)$, and moreover its infimum is bounded from above; therefore, applying the Harnack inequality (which is allowed since r is small enough) we get that $C - w^n + z^n$ is uniformly bounded in $L^\infty\left(B_{\frac{r}{2}}(x_1)\right)$, hence w^n is.

At this point, by connectedness, we can relabel the index h in such a way that $B_{\frac{r}{2}}(x_h) \cap B_{\frac{r}{2}}(x_{h+1}) \neq \emptyset$ for any $h \in \{1, \ldots, H-1\}$ and we repeat the argument for $B_{\frac{r}{2}}(x_2)$: since it has nonempty intersection with $B_{\frac{r}{2}}(x_1)$, we have $\sup_{B_r(x_2)} w^n \geq -C$, hence we get boundedness in $L^\infty\left(B_{\frac{r}{2}}(x_2)\right)$. In the same way, we obtain the same result in all the balls $B_{\frac{r}{2}}(x_h)$, whose union contains $\mathcal{K}$, therefore w^n must be uniformly bounded on $\mathcal{K}$ and we get the conclusion. □

The second Lemma basically says that if all the concentration values in a point are under a certain threshold, and in particular if all of them equal zero, then compactness occurs around that point.

On the other hand, if a point belongs to some set $\mathcal{S}_i$, then at least a fixed amount of mass has to accumulate around it; hence, being the total mass uniformly bounded from above, this can occur only for a finite number of points, so we deduce the finiteness of the $\mathcal{S}_i$'s.

Precisely, we have the following, inspired again by [17], Lemma 4.4:

Lemma 3.3 *Let* $\{v^n\}_{n\in\mathbb{N}}$ *and* $\mathcal{S}_i$ *be as in* (13) *and* σ_i *as in* (10)*, and suppose* $\sigma_i(x) < \sigma_i^0$ *for any* $i \in \{1, \ldots, N\}$*, where*

$$\sigma_i^0 := \frac{4\pi \min\left\{1, 1 + \min_{j\in\{1,\ldots,N\}, m\in\{1,\ldots,M\}} \alpha_{jm}\right\}}{\sum_{j=1}^N a_{ij}^+}.$$

Then, $x \notin \mathcal{S}_i$ *for any* $i \in \{1, \ldots, N\}$*.*

Proof First of all we notice that σ_i^0 is well-defined for any i because $a_{ii} > 0$, hence $\sum_{j=1}^N a_{ij}^+ > 0$.

Under the hypotheses of the Lemma, for large n and small r we have

$$\int_{B_r(x)} \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g < \sigma_i^0. \tag{14}$$

Let us consider w_i^n and z_i^n defined by

$$\begin{cases} -\Delta w_i^n = -\sum_{j=1}^N a_{ij}\rho_j^n & \text{in } B_r(x) \\ w_i^n = 0 & \text{on } \partial B_r(x) \end{cases}, \qquad \begin{cases} -\Delta z_i^n = \sum_{j=1}^N a_{ij}^+ \widetilde{h}_j^n e^{v_j^n} & \text{in } B_r(x) \\ z_i^n = 0 & \text{on } \partial B_r(x) \end{cases}. \tag{15}$$

Is it evident that the w_i^n's are uniformly bounded in $L^\infty(B_r(x))$.

As for the z_i^n's, we can suppose to be working on a Euclidean disc, up to applying a perturbation to $\widetilde{h}_i^n$ which is smaller as r is smaller, hence for r small enough we still have the strict estimate (14).

Therefore, we get

$$\left\|-\Delta z_i^n\right\|_{L^1(B_r(x))} = \sum_{j=1}^N a_{ij}^+ \int_{B_r(x)} \widetilde{h}_j^n e^{v_j^n} \mathrm{d}V_g < \sum_{j=1}^N a_{ij}^+ \sigma_j^0 \le 4\pi \min\{1, 1+\alpha_i(x)\}$$

and we can apply Lemma 2.1 to obtain $\int_{B_r(x)} e^{q|z_i^n|}\mathrm{d}V_g \le C$ for some $q > \frac{1}{\min\{1,1+\alpha_i(x)\}}$.

If $\alpha_i(x) \ge 0$, then taking $q \in \left(1, \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_r(x))}}\right)$ we have

$$\int_{B_r(x)} \left(\widetilde{h}_i^n e^{z_i^n}\right)^q \mathrm{d}V_g \le C_r \int_{B_r(x)} e^{q|z_i^n|}\mathrm{d}V_g \le C.$$

On the other hand, if $\alpha_i(x) < 0$, we choose

$$q \in \left(1, \frac{4\pi}{\left\|-\Delta z_i^n\right\|_{L^1(B_r(x))} - 4\pi\alpha_i(x)}\right) \qquad q' \in \left(\frac{4\pi}{4\pi - q\left\|-\Delta z_i^n\right\|_{L^1(B_r(x))}}, \frac{1}{-\alpha_i(x)q}\right)$$

and, applying Hölder's inequality,

$$\begin{aligned} \int_{B_r(x)} \left(\widetilde{h}_i^n e^{z_i^n}\right)^q \mathrm{d}V_g &\le C_r \int_{B_r(x)} d(\cdot, x)^{2q\alpha_i(x)} e^{qz_i^n}\mathrm{d}V_g \\ &\le C\left(\int_{B_r(x)} d(\cdot,x)^{2qq'\alpha_i(x)}\mathrm{d}V_g\right)^{\frac{1}{q'}} \left(\int_{B_r(x)} e^{q\frac{q'}{q'-1}|z_i^n|}\mathrm{d}V_g\right)^{1-\frac{1}{q'}} \\ &\le C, \end{aligned}$$

because $qq'\alpha_i(x) > -1$ and $q\frac{q'}{q'-1}\alpha_i(x) < \frac{4\pi}{\|-\Delta z_i^n\|_{L^1(B_r(x))}}$. Hence $\widetilde{h}_i^n e^{z_i^n}$ is uniformly bounded in $L^q(B_r(x))$ for some $q > 1$.

Now, let us consider $v_i^n - z_i^n - w_i^n$: it is a subharmonic sequence by construction, so for any $y \in B_{\frac{r}{2}}(x)$ we get

$$\begin{aligned} v_i^n(y) - z_i^n(y) - w_i^n(y) &\le \fint_{B_{\frac{r}{2}}(y)} \left(v_i^n - z_i^n - w_i^n\right) \mathrm{d}V_g \\ &\le C \int_{B_{\frac{r}{2}}(y)} (v_i^n - z_i^n - w_i^n)^+ \mathrm{d}V_g \\ &\le C \int_{B_r(x)} \left((v_i^n - z_i^n)^+ + (w_i^n)^-\right) \mathrm{d}V_g \\ &\le C \left(1 + \int_{B_r(x)} \left(v_i^n - z_i^n\right)^+ \mathrm{d}V_g\right). \end{aligned}$$

Moreover, since the maximum principle yields $z_i^n \ge 0$, taking $\theta = \begin{cases} 1 & \text{if } \alpha_i(x) \le 0 \\ \in \left(0, \frac{1}{1+\alpha_i(x)}\right) & \text{if } \alpha_i(x) > 0 \end{cases}$, we get

$$\begin{aligned} \int_{B_r(x)} \left(v_i^n - z_i^n\right)^+ \mathrm{d}V_g &\le \int_{B_r(x)} (v_i^n)^+ \mathrm{d}V_g \\ &\le \frac{1}{e\theta} \int_{B_r(x)} e^{\theta v_i^n} \mathrm{d}V_g \\ &\le C \left\| \left(\widetilde{h}_i^n\right)^{-\theta} \right\|_{L^{\frac{1}{1-\theta}}(B_r(x))} \left(\int_{B_r(x)} \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g\right)^\theta \\ &\le C. \end{aligned}$$

Therefore, we showed that $v_i^n - z_i^n - w_i^n$ is bounded from above in $B_{\frac{r}{2}}(x)$, that is $e^{v_i^n - z_i^n - w_i^n}$ is uniformly bounded in $L^\infty\left(B_{\frac{r}{2}}(x)\right)$. Since the same holds for $e^{w_i^n}$ and $\widetilde{h}_i^n e^{z_i^n}$ is uniformly bounded in $L^q\left(B_{\frac{r}{2}}(x)\right)$ for some $q > 1$, we deduce that also

$$\widetilde{h}_i^n e^{v_i^n} = \widetilde{h}_i^n e^{z_i^n} \, e^{v_i^n - z_i^n - w_i^n} \, e^{w_i^n}$$

is bounded in the same $L^q\left(B_{\frac{r}{2}}(x)\right)$.

Thus, we have an estimate on $\|-\Delta z_i^n\|_{L^q\left(B_{\frac{r}{2}}(x)\right)}$ for any $i \in \{1, \dots, N\}$, hence by standard elliptic estimates we deduce that z_i^n is uniformly bounded in $L^\infty\left(B_{\frac{r}{2}}(x)\right)$. Therefore, we also deduce that

$$v_i^n = \left(v_i^n - z_i^n - w_i^n\right) + z_i^n + w_i^n$$

is bounded from above on $B_{\frac{r}{2}}(x)$, which is equivalent to saying $x \notin \bigcup_{i=1}^N \mathcal{S}_i$. □

From this proof, we notice that, under the assumptions of Theorem 1.3, the same result holds for any single index $i \in \{1, \dots, N\}$. In other words, the upper bound on one σ_i implies that $x \notin \mathcal{S}_i$.

Corollary 3.4 *Suppose $a_{ij} \le 0$ for any $i \ne j$.*
Then, for any given $i \in \{1, \dots, N\}$ the following conditions are equivalent:

- $x \in \mathcal{S}_i$.
- $\sigma_i(x) \neq 0$.
- $\sigma_i(x) \geq \sigma_i' = \frac{4\pi \min\{1, 1+\min_m \alpha_{im}\}}{a_{ii}}$.

Proof The third statement trivially implies the second and the second implies the first, since if v_i^n is bounded from above in $B_r(x)$ then $\widetilde{h}_i^n e^{v_i^n}$ is bounded in $L^q(B_r(x))$. Finally, if $\sigma_i(x) < \sigma_i'$ then the sequence $\widetilde{h}_i^n e^{z_i^n}$ defined by (15) is bounded in L^q for $q > 1$,so one can argue as in Lemma 3.3 to get boundedness from above of v_i^n around x, that is $x \notin \mathcal{S}_i$. □

We can now prove the main theorem of this Section.

Proof of Theorem 3.1 If $\mathcal{S}_i = \emptyset$ for any i, then $e^{v_i^n}$ is bounded in $L^\infty(\Sigma)$, so $-\Delta v_i^n$ is bounded in $L^q(\Sigma)$ for any

$$q \in \left[1, \frac{1}{-\min_{j\in\{1,\dots,N\}, m\in\{1,\dots,M\}} \alpha_{jm}}\right).$$

Therefore, we can apply Lemma 3.2 to v_i^n on Σ, where we must have the first alternative for every i, since otherwise the dominated convergence would give $\int_\Sigma \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} 0$ which is absurd; standard elliptic estimates allow to conclude compactness in $W^{2,q}(\Sigma)$.

Suppose now $\mathcal{S}_i \neq \emptyset$ for some i; from Lemma 3.3 we deduce

$$|\mathcal{S}_i|\sigma_i^0 \leq \sum_{x\in\mathcal{S}_i} \max_j \sigma_j(x) \leq \sum_{j=1}^N \sum_{x\in\mathcal{S}_i} \sigma_j(x) \leq \sum_{j=1}^N \rho_j,$$

hence $\mathcal{S}_i$ is finite.

For any $j \in \{1,\dots,N\}$, we can apply Lemma 3.2 on $\Sigma\setminus\bigcup_{j'=1}^N \mathcal{S}_{j'}$ with $f^n = \sum_{j'=1}^N a_{jj'}\left(\widetilde{h}_{j'}^n e^{v_{j'}^n} - \rho_{j'}^n\right)$, since the last function is bounded in $L^q_{\mathrm{loc}}\left(\Sigma\setminus\bigcup_{j'=1}^N \mathcal{S}_{j'}\right)$.

Therefore, either v_j^n goes to $-\infty$ or it is bounded in L^∞_{loc}, and in the last case we get compactness in $W^{2,q}_{\mathrm{loc}}$ by applying again standard elliptic regularity. □

4 Proof of Theorem 1.1

Here we will prove the theorem which gives sufficient and necessary conditions for the functional J_ρ to be bounded from below.

In other words, setting

$$E := \left\{\rho \in \mathbb{R}_+^N : J_\rho \text{ is bounded from below on } H^1(\Sigma)^N\right\}, \tag{16}$$

we will prove that $\{\Lambda > 0\} \subset E \subset \{\Lambda \geq 0\}$.

As a first thing, we notice that the set E is not empty and it verifies a simple monotonicity condition.

Lemma 4.1 *The set E defined by (16) is nonempty.*

Moreover, for any $\rho \in E$ then $\rho' \in E$ provided $\rho_i' \leq \rho_i$ for any $i \in \{1,\dots,N\}$.

Proof Let $\theta > 0$ be the biggest eigenvalue of the matrix (a_{ij}). Then,

$$J_\rho(u) \geq \sum_{i=1}^N \left(\frac{1}{2\theta}\int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - \rho_i\left(\log\int_\Sigma \widetilde{h}_i e^{u_i}\mathrm{d}V_g - \overline{u_i}\right)\right).$$

Therefore, from scalar Moser–Trudinger inequality (3), we deduce that J_ρ is bounded from below if $\rho_i \le \frac{8\pi(1+\widetilde{\alpha}_i)}{\theta}$, hence $E \neq \emptyset$.

Suppose now $\rho \in E$ and $\rho'_i \le \rho_i$ for any i. Then, through Jensen's inequality, we get

$$\begin{aligned} J_{\rho'}(u) &= J_\rho(u) + \sum_{i=1}^N (\rho_i - \rho'_i) \log \int_\Sigma e^{u_i - \overline{u_i} + \log \widetilde{h}_i} \mathrm{d}V_g \\ &\ge -C + \sum_{i=1}^N (\rho_i - \rho'_i) \int_\Sigma \log \widetilde{h}_i \mathrm{d}V_g \\ &\ge -C \end{aligned}$$

for any $u \in H^1(\Sigma)^N$, hence the claim. □

It is interesting to observe that a similar monotonicity condition is also satisfied by the set $\{\Lambda > 0\}$ (although one can easily see that it is not true if we replace Λ with $\Lambda_{\mathcal{I},x}$).

Lemma 4.2 *Let $\rho, \rho' \in \mathbb{R}_+^N$ be such that $\Lambda(\rho) > 0$ and $\rho'_i \le \rho_i$ for any $i \in \{1, \dots, N\}$. Then, $\Lambda(\rho') > 0$.*

Proof Suppose by contradiction $\Lambda(\rho') \le 0$, that is $\Lambda_{\mathcal{I},x}(\rho') \le 0$ for some $\mathcal{I}, x$.

This cannot occur for $\mathcal{I} = \{i\}$ because it would mean $\rho'_i \ge \frac{8\pi(1+\alpha_i(x))}{a_{ii}}$, so the same inequality would for ρ_i, hence $\Lambda(\rho) \le \Lambda_{\mathcal{I},x}(\rho) \le 0$.

Therefore, there must be some $\mathcal{I}, x$ such that $\Lambda_{\mathcal{I},x}(\rho') \le 0$ and $\Lambda_{\mathcal{I}\setminus\{i\},x}(\rho') > 0$ for any $i \in \mathcal{I}$; this implies

$$\begin{aligned} 0 &< \Lambda_{\mathcal{I}\setminus\{i\},x}(\rho') - \Lambda_{\mathcal{I},x}(\rho') \\ &= 2\sum_{j\in\mathcal{I}} a_{ij}\rho'_i\rho'_j - a_{ii}{\rho'_i}^2 - 8\pi(1+\alpha_i(x))\rho'_i \\ &< \rho'_i \left(2\sum_{j\in\mathcal{I}} a_{ij}\rho'_j - 8\pi(1+\alpha_i(x)) \right). \end{aligned} \tag{17}$$

It will be not restrictive to suppose, from now on, $\rho'_1 \le \rho_1$ and $\rho'_i = \rho_i$ for any $i \ge 2$, since the general case can be treated by exchanging the indices and iterating.

Assuming this, we must have $1 \in \mathcal{I}$, therefore we obtain:

$$\begin{aligned} 0 &< \Lambda_{\mathcal{I},x}(\rho) - \Lambda_{\mathcal{I},x}(\rho') \\ &= 8\pi(1+\alpha_1(x))(\rho_1 - \rho'_1) - a_{11}\left({\rho'_1}^2 - \rho_1^2\right) - 2\sum_{j\in\mathcal{I}\setminus\{1\}} a_{1j}(\rho'_1 - \rho_1)\rho_j \\ &= (\rho_1 - \rho'_1)\left(8\pi(1+\alpha_1(x)) - a_{11}(\rho'_1 + \rho_1) - 2\sum_{j\in\mathcal{I}\setminus\{1\}} a_{1j}\rho_j \right) \\ &< (\rho_1 - \rho'_1)\left(8\pi(1+\alpha_1(x)) - 2\sum_{j\in\mathcal{I}} a_{1j}\rho'_j \right), \end{aligned}$$

which is negative by (17). We found a contradiction. □

We will now show that if the parameter ρ lies in the interior of E then not only the functional is bounded from below but it is coercive in the space of zero-average functions. In particular, this fact allows to deduce the "if" part in Corollary 1.2 from Theorem 1.1.

On the other hand, if ρ belongs to the boundary of E, then the scenario is quite different.

Lemma 4.3 *Suppose $\rho \in \mathring{E}$. Then, there exists a constant $C = C_\rho$ such that*

$$J_\rho(u) \geq \frac{1}{C}\sum_{i=1}^N \int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - C.$$

Moreover, J_ρ admits a minimizer which solves (6).

Proof Choosing $\delta \in \left(0, \frac{d(\rho,\partial E)}{\sqrt{N}|\rho|}\right)$ one has $(1+\delta)\rho \in E$, so

$$\begin{aligned} J_\rho(u) &= \frac{\delta}{2(1+\delta)}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g + \frac{1}{1+\delta}J_{(1+\delta)\rho}(u) \\ &\geq \frac{\delta}{2\theta(1+\delta)}\sum_{i=1}^N \int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - C, \end{aligned}$$

hence we get the former claim.

To get the latter, we notice that, due to invariance by translation, any minimizer can be supposed to be in $\overline{H}^1(\Sigma)^N$; therefore, we can restrict J_ρ to this subspace. Here, the above inequality implies coercivity, and it is immediate to see that J_ρ is also lower semi-continuous, hence the existence of minimizers follows from direct methods of calculus of variations. □

Lemma 4.4 *Suppose $\rho \in \partial E$. Then, there exists a sequence $\{u^n\}_{n\in\mathbb{N}} \subset H^1(\Sigma)^N$ such that*

$$\sum_{i=1}^N \int_\Sigma \left|\nabla u_i^n\right|^2 \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} +\infty \qquad \lim_{n\to+\infty}\frac{J_\rho(u^n)}{\sum_{i=1}^N \int_\Sigma \left|\nabla u_i^n\right|^2 \mathrm{d}V_g} \leq 0$$

Proof We first notice that $(1-\delta)\rho \in E$ for any $\delta \in (0,1)$. In fact, otherwise, from Lemma 4.1 we would get $\rho' \notin E$ as soon as $\rho_i' \geq (1-\delta)\rho_i$ for some i, hence $\rho \notin \partial E$.

Now, suppose by contradiction that for any sequence u^n one gets

$$\sum_{i=1}^N \int_\Sigma \left|\nabla u_i^n\right|^2 \mathrm{d}V_g \underset{n\to+\infty}{\longrightarrow} +\infty \qquad \Rightarrow \qquad \frac{J_\rho(u^n)}{\sum_{i=1}^N \int_\Sigma \left|\nabla u_i^n\right|^2 \mathrm{d}V_g} \geq \varepsilon > 0.$$

Therefore, we would have

$$J_\rho(u) \geq \frac{\varepsilon}{2}\sum_{i=1}^N \int_\Sigma |\nabla u_i|^2 \mathrm{d}V_g - C;$$

hence, indicating as θ' the smallest eigenvalue of the matrix A, for small δ we would get

$$\begin{aligned} J_\rho(u) &= (1+\delta)J_{(1+\delta)\rho}(u) - \frac{\delta}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g \\ &\geq \left((1+\delta)\frac{\varepsilon}{2} - \frac{\delta}{2\theta'}\right)\sum_{i=1}^N \int_\Sigma |\nabla u_i|^2 - C \\ &\geq -C. \end{aligned}$$

So we obtain $(1+\delta)\rho \in E$; being also $(1-\delta)\rho \in E$ (by Lemma 4.1), we get a contradiction with $\rho \in \partial E$. □

To see what happens when $\rho \in \partial E$, we build an auxiliary functional using Lemma 2.3.

Lemma 4.5 *Fix $\rho' \in \partial E$ and define:*

$$a^n_{\rho'} := \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u^n_i \cdot \nabla u^n_j \mathrm{d}V_g \qquad b^n_{\rho'} := J_{\rho'}\left(u^n\right)$$

$$J'_{\rho',\rho}(u) = J_\rho(u) - F_{\rho'}\left(\frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g\right),$$

where u^n is given by Lemma 4.4 and $F_{\rho'}$ by Lemma 2.3.

If $\rho \in \mathring{E}$, then $J'_{\rho',\rho}$ is bounded from below on $H^1(\Sigma)^N$ and its infimum is achieved by a solution of

$$-\Delta\left(u_i - \sum_{i,j=1}^N a^{ij} f u_j\right) = \sum_{j=1}^N a_{ij}\rho_j\left(\frac{\widetilde{h}_j e^{u_j}}{\int_\Sigma \widetilde{h}_j e^{u_j}\mathrm{d}V_g} - 1\right), \qquad i = 1, \ldots, N,$$

with $f = \left(F_{\rho'}\right)'\left(\frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i \cdot \nabla u_j \mathrm{d}V_g\right)$.

On the other hand, $J'_{\rho',\rho'}$ is unbounded from below.

Proof For $\rho \in \mathring{E}$, we can argue as in Lemma 4.3, since the continuity follows from the regularity of F and the coercivity from the behavior of F' at the infinity.

For $\rho = \rho'$, if we take u^n as in Lemma 4.4 we get

$$J'_{\rho',\rho'}\left(u^n\right) = b^n_{\rho'} - F_{\rho'}\left(a^n_{\rho'}\right) \underset{n\to+\infty}{\longrightarrow} -\infty.$$

□

Now we can prove the first half of Theorem 1.1, that is J_ρ is bounded from below if $\Lambda(\rho) > 0$.

Proof of $\{\Lambda > 0\} \subset E$ Suppose by contradiction there is some $\rho' \in \partial E$ with $\Lambda(\rho) > 0$ and take a sequence $\rho^n \in E$ with $\rho^n \underset{n\to+\infty}{\longrightarrow} \rho'$.

Then, by Lemma 4.5, the auxiliary functional J_{ρ',ρ^n} admits a minimizer u^n, so the functions v^n_i defined as in (11) solve

$$\begin{cases} -\Delta v^n_i = \sum_{j,j'=1}^N a_{ij} b^{jj',n}\left(\widetilde{h}_j e^{v^n_j} - \rho^n_j\right) & i = 1, \ldots, N \\ \int_\Sigma \widetilde{h}^n_i e^{v^n_i}\mathrm{d}V_g = \rho^n_i \end{cases}$$

where $b^{ij,n}$ is the inverse matrix of $b^n_{ij} := \delta_{ij} - a^{ij} f^n$, hence $b^{ij,n} \underset{n\to+\infty}{\longrightarrow} \delta_{ij}$.

We can then apply Theorem 3.1. The first alternative is excluded, since otherwise we would get, for any $u \in H^1(\Sigma)^N$,

$$J'_{\rho',\rho'}(u) = \lim_{n\to+\infty} J'_{\rho',\rho^n}(u) \geq \lim_{n\to+\infty} J'_{\rho',\rho^n}\left(v^n\right) = J'_{\rho',\rho'}(v) > -\infty,$$

thus contradicting Lemma 4.5.

Therefore, blow up must occur; this means, by Lemma 3.3, that $\sigma_i(p) \neq 0$ for some $i \in \{1, \ldots, N\}$ and some $p \in \Sigma$.

By Proposition 2.2 follows $\Lambda(\sigma) \leq 0$. On the other hand, since $\sigma_i \leq \rho'_i$ for any i, Lemma 4.2 yields $\Lambda(\rho') \leq 0$, which contradicts our assumptions. □

To prove the unboundedness from below of J_ρ in the case $\Lambda(\rho) < 0$ we will use suitable test functions, whose properties are described by the following:

Lemma 4.6 *Define, for* $x \in \Sigma$ *and* $\lambda > 0$, $\varphi = \varphi^{\lambda,x}$ *as*

$$\varphi_i := -2(1+\alpha_i(x)) \log \max\{1, \lambda d(\cdot, x)\}.$$

Then, as $\lambda \to +\infty$, *one has*

$$\int_\Sigma \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}V_g = 8\pi(1+\alpha_i(x))(1+\alpha_j(x)) \log\lambda + O(1)$$
$$\overline{\varphi_i} = -2(1+\alpha_i(x))\log\lambda + O(1)$$
$$\int_\Sigma \widetilde{h}_i e^{\sum_{j=1}^N \theta_j\varphi_j} \mathrm{d}V_g \geq C\lambda^{-2(1+\alpha_i(x))} \qquad \text{if} \quad \sum_{i=1}^N \theta_j(1+\alpha_j(x)) > 1+\alpha_i(x).$$

Proof It holds

$$\nabla\varphi_i = \begin{cases} 0 & \text{if } d(\cdot,x) < \frac{1}{\lambda} \\ -2(1+\alpha_i(x))\frac{\nabla d(\cdot,x)}{d(\cdot,x)} & \text{if } d(\cdot,x) > \frac{1}{\lambda} \end{cases}.$$

Therefore, being $|\nabla d(\cdot, x)| = 1$ almost everywhere on Σ:

$$\begin{aligned} &\int_\Sigma \nabla\varphi_i \cdot \nabla\varphi_j \mathrm{d}V_g \\ &= 4(1+\alpha_i(x))(1+\alpha_j(x)) \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(x)} \frac{\mathrm{d}V_g}{d(\cdot,x)^2} \\ &= 8\pi(1+\alpha_i(x))(1+\alpha_j(x))\log\lambda + O(1). \end{aligned}$$

For the average of φ_i, we get

$$\begin{aligned} \int_\Sigma \varphi_i \mathrm{d}V_g &= -2(1+\alpha_i(x)) \int_{\Sigma\setminus B_{\frac{1}{\lambda}}(x)} (\log\lambda + \log d(\cdot,x))\mathrm{d}V_g + O(1) \\ &= -2(1+\alpha_i(x))\log\lambda + O(1). \end{aligned}$$

For the last estimate, choose $r > 0$ such that $\overline{B_\delta(x)}$ does not contain any of the points p_m for $m = 1, \ldots, M$, except possibly x.

Then, outside such a ball, $e^{\sum_{j=1}^N \theta_j\varphi_j} \leq C\lambda^{-2\sum_{j=1}^N \theta_j(1+\alpha_j(x))}$.

Therefore, under the assumptions of the Lemma,

$$\int_{\Sigma\setminus B_\delta(x)} \widetilde{h}_i e^{\sum_{i=1}^N \theta_j\varphi_j} \mathrm{d}V_g = o\left(\lambda^{-2(1+\alpha_i(x))}\right),$$

hence

$$\begin{aligned} \int_\Sigma \widetilde{h}_i e^{\sum_{i=1}^N \theta_j\varphi_j} \mathrm{d}V_g &\geq \int_{B_\delta(x)} \widetilde{h}_i e^{\sum_{i=1}^N \theta_j\varphi_j} \mathrm{d}V_g \\ &\geq C\left(\int_{B_{\frac{1}{\lambda}}(x)} d(\cdot,x)^{2\alpha_i(x)} \mathrm{d}V_g + \frac{1}{\lambda^{2\sum_{j=1}^N \theta_j(1+\alpha_j(x))}}\right. \end{aligned}$$

$$\left.\int_{A_{\frac{1}{\lambda},\delta}(x)} d(\cdot,x)^{2\alpha_i(x)-2\sum_{i=1}^N \theta_j(1+\alpha_j(x))}\mathrm{d}V_g\right)$$
$$\geq C\lambda^{-2(1+\alpha_i(x))},$$

which concludes the proof. □

Proof of $E \subset \{\Lambda \geq 0\}$ Take $\rho, \mathcal{I}, x$ such that $\Lambda_{\mathcal{I},x}(\rho) < 0$ and $\Lambda_{\mathcal{I}\setminus\{i\},x}(\rho) \geq 0$ for any $i \in \mathcal{I}$, and consider the family of functions $\left\{u^\lambda\right\}_{\lambda>0}$ defined by

$$u_i^\lambda := \sum_{j\in\mathcal{I}} \frac{a_{ij}\rho_j}{4\pi(1+\alpha_i(x))}\varphi_j^{\lambda,x}.$$

By Jensen's inequality we get

$$\begin{aligned} J_\rho\left(u^\lambda\right) &\leq \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla u_i^\lambda\cdot\nabla u_j^\lambda \mathrm{d}V_g + \sum_{i\in\mathcal{I}}\rho_i\left(\overline{u_i^\lambda} - \log\int_\Sigma \widetilde{h}_i e^{u_i^\lambda}\mathrm{d}V_g\right) + C \\ &= \frac{1}{2}\sum_{i,j\in\mathcal{I}} \frac{a_{ij}\rho_i\rho_j}{16\pi^2(1+\alpha_i(x))(1+\alpha_j(x))}\int_\Sigma \nabla\varphi_i\cdot\nabla\varphi_j \mathrm{d}V_g \\ &\quad + \sum_{i,j\in\mathcal{I}} \frac{a_{ij}\rho_i\rho_j}{4\pi(1+\alpha_j(x))}\overline{\varphi_j} - \sum_{i\in\mathcal{I}}\rho_i\log\int_\Sigma \widetilde{h}_i e^{\sum_{j\in\mathcal{I}}\frac{a_{ij}\rho_j}{4\pi(1+\alpha_j(x))}\varphi_j}\mathrm{d}V_g + C. \end{aligned}$$

At this point, we would like to apply Lemma 4.6 to estimate $J_\rho\left(u^\lambda\right)$. To be able to do this, we have to verify that

$$\frac{1}{4\pi}\sum_{j\in\mathcal{I}} a_{ij}\rho_j > 1+\alpha_i(x) \qquad \forall i \in \mathcal{I}.$$

If $\mathcal{I} = \{i\}$, then $\rho_i > \frac{8\pi(1+\alpha_i(x))}{a_{ii}}$, so it follows immediately. For the other cases, it follows from (17).

So we can apply Lemma 4.6 and we get from the previous estimates:

$$\begin{aligned} J_\rho\left(u^\lambda\right) &\leq \left(\frac{1}{4\pi}\sum_{i,j\in\mathcal{I}} a_{ij}\rho_i\rho_j - \frac{1}{2\pi}\sum_{i,j\in\mathcal{I}} a_{ij}\rho_i\rho_j + 2\sum_{i\in\mathcal{I}}\rho_i(1+\alpha_i(x))\right)\log\lambda + C \\ &= -\frac{\Lambda_{\mathcal{I},x}(\rho)}{4\pi}\log\lambda + C \underset{n\to+\infty}{\longrightarrow} -\infty. \end{aligned}$$

□

Proof of Corollary 1.2 The coercivity in the case $\Lambda < 0$, hence the existence of minimizing solutions for (6) follows from Theorem 1.1 and Lemma 4.3.

If instead $\Lambda(\rho) \geq 0$, then one can find out the lack of coercivity by arguing as before with the sequence u^λ, which verifies

$$\sum_{i=1}^N \int_\Sigma \left|\nabla u_i^\lambda\right|^2 \mathrm{d}V_g \underset{\lambda\to+\infty}{\longrightarrow} +\infty \qquad J_\rho\left(u^\lambda\right) \leq -\frac{\Lambda_{\mathcal{I},x}(\rho)}{4\pi}\log\lambda + C \leq C.$$

□

5 Proof of Theorem 1.3

Here we will finally prove a sharp inequality in the case when the matrix a_{ij} has non-positive entries outside its main diagonal.

As already pointed out in the introduction, the function $\Lambda(\rho)$ can be written in a much shorter form under these assumptions, so the condition $\Lambda(\rho) \geq 0$ is equivalent to $\rho_i \leq \frac{8\pi(1+\widetilde{\alpha}_i)}{a_{ii}}$ for any $i \in \{1, \ldots, N\}$.

Moreover, thanks to Lemma 4.1, in order to prove Theorem 1.3 for all such ρ's it will suffice to consider

$$\rho^0 := \left(\frac{8\pi(1+\widetilde{\alpha}_1)}{a_{11}}, \ldots, \frac{8\pi(1+\widetilde{\alpha}_N)}{a_{NN}}\right). \tag{18}$$

By what we proved in the previous Section, for any sequence $\rho^n \underset{n\to+\infty}{\nearrow} \rho^0$ one has

$$\inf_{H^1(\Sigma)^N} J_{\rho^n} = J_{\rho^n}(u^n) \geq -C_{\rho^n},$$

so Theorem 1.3 will follow by showing that, for a given sequence $\{\rho^n\}_{n\in\mathbb{N}}$, the constant $C_n = C_{\rho^n}$ can be chosen independently of n.

As a first thing, we provide a Lemma which shows the possible blow-up scenarios for such a sequence u^n.

Here, the assumption on a_{ij} is crucial since it reduces largely the possible cases.

Lemma 5.1 *Let ρ^0 be as in (18), $\{\rho^n\}_{n\in\mathbb{N}}$ such that $\rho^n \nearrow \rho^0$, u^n a minimizer of J_{ρ^n} and v^n as in (11). Then, up to subsequences, there exists a set $\mathcal{I} \subset \{1, \ldots, N\}$ such that:*

- *If $i \in \mathcal{I}$, then $\mathcal{S}_i = \{x_i\}$ for some $x_i \in \Sigma$ which satisfy $\widetilde{\alpha}_i = \alpha_i(x_i)$ and $\sigma_i(x_i) = \rho_i^0$, and $v_i^n \underset{n\to+\infty}{\longrightarrow} -\infty$ in $L^\infty_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{j\in\mathcal{I}}\{x_j\}\right)$.*
- *If $i \notin \mathcal{I}$, then $\mathcal{S}_i = \emptyset$ and $v_i^n \underset{n\to+\infty}{\longrightarrow} v_i$ in $W^{2,q}_{\mathrm{loc}}\left(\Sigma \setminus \bigcup_{j\in\mathcal{I}}\{x_j\}\right)$ for some $q > 1$ and some suitable v_i.*

Moreover, if $a_{ij} < 0$ then $x_i \neq x_j$.

Proof From Theorem 3.1 we get a $\mathcal{I} \subset \{1, \ldots, N\}$ such that $\mathcal{S}_i \neq \emptyset$ for $i \in \mathcal{I}$.

If $\mathcal{S}_i \neq \emptyset$, then by Corollary 3.4 one gets

$$0 < \sigma_i(x) \leq \rho_i^0 \leq \frac{8\pi(1+\alpha_i(x))}{a_{ii}}$$

for all $x \in \mathcal{S}_i$, hence

$$\begin{aligned} 0 &= \Lambda_{\{1,\ldots,N\},x}(\sigma(x)) \\ &\geq \sum_{j=1}^N \left(8\pi(1+\alpha_j(x))\sigma_j(x) - a_{jj}\sigma_j(x)^2\right) \\ &\geq 8\pi(1+\alpha_i(x))\sigma_i(x) - a_{ii}\sigma_i(x)^2 \\ &\geq 0. \end{aligned} \tag{19}$$

Therefore, all these inequalities must actually be equalities.

From the last, we have $\sigma_i(x) = \rho_i^0 = \frac{8\pi(1+\alpha_i(x))}{a_{ii}}$, hence $\alpha_i(x) = \widetilde{\alpha}_i$. On the other hand, since $\sum_{x\in\mathcal{S}_i} \sigma_i(x) \leq \rho_i^0$, it must be $\sigma_i(x) = 0$ for all but one $x_i \in \mathcal{S}_i$, so Corollary 3.4 yields $\mathcal{S}_i = \{x_i\}$.

Let us now show that $v_i^n \underset{n\to+\infty}{\longrightarrow} -\infty$ in L^∞_{loc}.

Otherwise, Theorem 3.1 would imply $v_i^n \underset{n\to+\infty}{\longrightarrow} v_i$ almost everywhere, therefore by Fatou's Lemma we would get the following contradiction:

$$\sigma_i(x_i) < \int_\Sigma \widetilde{h}_i e^{v_i} \mathrm{d}V_g + \sigma_i(x_i) \le \int_\Sigma \widetilde{h}_i^n e^{v_i^n} \mathrm{d}V_g = \rho_i^n \le \rho_i = \sigma_i(x_i).$$

Since also inequality (19) has to be an equality, we get $a_{ij}\sigma_i(x_i)\sigma_j(x_i)$ for any $i, j \in \mathcal{I}$, so whenever $a_{ij} < 0$ there must be $\sigma_j(x_i) = 0$, so $x_i \neq x_j$.

Finally, if $\mathcal{S}_i = \emptyset$, the convergence in $W^{2,q}_{\mathrm{loc}}$ follows from what we just proved and Theorem 3.1. □

We basically showed that if a component of the sequence v^n blows up, then all its mass concentrates at a single point which has the lowest singularity coefficient.

The next Lemma gives some more important information about the convergence or the blow-up of the components of v^n.

Lemma 5.2 *Let v_i^n, v_i, ρ^0, $\mathcal{I}$ and x_i as in Lemma 5.1.*
Then,

- *If $i \in \mathcal{I}$, then the sequence $v_i^n - \overline{v_i^n}$ converges to some G_i in $W^{2,q}_{\mathrm{loc}}\left(\Sigma\setminus\bigcup_{j\in\mathcal{I}}\{x_j\}\right)$ for some $q > 1$ and weakly in $W^{1,q'}(\Sigma)$ for any $q' \in (1,2)$, and G_i solves:*

$$\begin{cases} -\Delta G_i = \sum_{j\in\mathcal{I}} a_{ij}\rho_j^0\left(\delta_{x_j} - 1\right) + \sum_{j\notin\mathcal{I}} a_{ij}\left(\widetilde{h}_j e^{v_j} - \rho_j^0\right) \\ \overline{G_i} = 0 \end{cases}.$$

- *If $i \notin \mathcal{I}$, then $v_i^n \underset{n\to+\infty}{\longrightarrow} v_i$ in the same space, and v_i solves:*

$$\begin{cases} -\Delta v_i = \sum_{j\in\mathcal{I}} a_{ij}\rho_j^0\left(\delta_{x_j} - 1\right) + \sum_{j\notin\mathcal{I}} a_{ij}\left(\widetilde{h}_j e^{v_j} - \rho_j^0\right) \\ \int_\Sigma \widetilde{h}_i e^{v_i}\mathrm{d}V_g = \rho_i^0 \end{cases}. \tag{20}$$

Proof From Lemma 5.1 follows that, for $i \in \mathcal{I}$, $\widetilde{h}_i^n e^{v_i^n} \underset{n\to\infty}{\rightharpoonup} \rho_i^0\delta_{x_i}$ in the sense of measures; in fact, for any $\phi \in C(\Sigma)$

$$\begin{aligned}\left|\int_\Sigma \widetilde{h}_i^n e^{v_i^n}\phi \mathrm{d}V_g - \rho_i^0\phi(x_i)\right| &\le \int_\Sigma \widetilde{h}_i^n e^{v_i^n}|\phi - \phi(x_i)|\mathrm{d}V_g + \left|\rho_i^n - \rho_i^0\right||\phi(x_i)| \\ &\le \varepsilon\int_{B_\delta(x_i)} \widetilde{h}_i^n e^{v_i^n}\mathrm{d}V_g + 2\|\phi\|_{L^\infty(\Sigma)}\int_{\Sigma\setminus B_\delta(x_i)} \widetilde{h}_i^n e^{v_i^n}\mathrm{d}V_g \\ &\quad + \left|\rho_i^n - \rho_i^0\right|\|\phi\|_{L^\infty(\Sigma)} \\ &\le \varepsilon\rho_i^n + 2\|\phi\|_{L^\infty(\Sigma)}o(1) + o(1)\|\phi\|_{L^\infty(\Sigma)},\end{aligned}$$

which is, choosing properly ε, arbitrarily small. Therefore, v_i solves (20).

On the other hand, if $q' \in (1,2)$, then $\frac{q'}{q'-1} > 2$, so any function $\phi \in W^{1,\frac{q'}{q'-1}}(\Sigma)$ is actually continuous, hence

$$\begin{aligned}&\left|\int_\Sigma \nabla\left(v_i^n - \overline{v_i^n} - G_i\right)\cdot\nabla\phi \mathrm{d}V_g\right| \\ &= \left|\int_\Sigma \left(-\Delta v_i^n + \Delta G_i\right)\phi dV_h\right|\end{aligned}$$

$$\leq \sum_{j\in\mathcal{I}} a_{ij}\left|\int_\Sigma \widetilde{h}_j e^{v_j^n}\phi \mathrm{d}V_g - \rho_j^0\phi(p)\right|$$
$$+\sum_{j\notin\mathcal{I}} a_{ij}\left|\int_\Sigma \widetilde{h}_j\left(e^{v_j^n}-e^{v_j}\right)\phi \mathrm{d}V_g\right| \underset{n\to+\infty}{\longrightarrow} 0.$$

Therefore, we get weak convergence in $W^{1,q'}(\Sigma)$ for any $q'\in(1,2)$; standard elliptic estimates yield convergence in $W^{2,q}_{\mathrm{loc}}\left(\Sigma\setminus\bigcup_{j\in\mathcal{I}}\{x_j\}\right)$.

In the same way we prove the same convergence of v_i^n to v_i. □

From these information about the blow-up profile of v^n we deduce an important fact which will be used to prove the main Theorem:

Corollary 5.3 *Let v^n and x_i be as in Lemmas 5.1 and 5.2 and w^n be defined by $w_i^n = \sum_{j=1}^N a^{ij}v_j^n$ for $i\in\{1,\dots,N\}$.*

Then, $w_i^n - \overline{w_i^n}$ is uniformly bounded in $W^{2,q}_{\mathrm{loc}}(\Sigma\setminus\{x_i\})$ for some $q>1$ if $i\in\mathcal{I}$, whereas if $i\notin\mathcal{I}$ it is bounded in $W^{2,q}(\Sigma)$.

Proof Since $-\Delta w_i^n = \widetilde{h}_i^n e^{v_i^n} - \rho_i^n$, the claim follows from the boundedness of $e^{v_i^n}$ in $L^\infty_{\mathrm{loc}}(\Sigma\setminus\{x_i\})$ and from standard elliptic estimates. □

The last Lemma we need is a localized scalar Moser–Trudinger inequality for the blowing-up sequence.

Lemma 5.4 *Let w_i^n be as in Corollary 5.3 and x_i as in the previous Lemmas. Then, for any $i\in\mathcal{I}$ and any small $r>0$ one has*

$$\frac{a_{ii}}{2}\int_{B_r(x_i)}\left|\nabla w_i^n\right|^2 \mathrm{d}V_g - \rho_i^n\left(\log\int_{B_r(x_i)}\widetilde{h}_i e^{a_{ii}w_i^n}\mathrm{d}V_g - a_{ii}\overline{w_i^n}\right)\geq -C_r.$$

Proof Since Σ is locally conformally flat, we can choose r small enough so that we can apply Corollary 2.5 up to modifying $\widetilde{h}_i^n$. We also take r so small that $\overline{B_r(x_i)}$ contains neither any x_j for $x_j\neq x_i$ nor any p_m for $m=1,\dots,M$ (except possibly x_i).

Let z^n be the solution of

$$\begin{cases} -\Delta z_i^n = \widetilde{h}_i^n e^{v_i^n} - \rho_i^n & \text{in } B_r(x_i)\\ z_i^n = 0 & \text{on } \partial B_r(x_i)\end{cases}.$$

Then, $w_i^n - \overline{w_i^n} - z_i^n$ is harmonic and it has the same value as $w_i^n - \overline{w_i^n}$ on $\partial B_r(x_i)$, so from standard estimates

$$\left\|w_i^n - \overline{w_i^n} - z_i^n\right\|_{C^1(B_r(x_i))}\leq C\left\|w_i^n - \overline{w_i^n}\right\|_{C^1(\partial B_r(x_i))}\leq C.$$

From Lemma 5.2 we get

$$\left|\int_{B_r(x_i)}\left|\nabla w_i^n\right|^2\mathrm{d}V_g - \int_{B_r(x_i)}\left|\nabla z_i^n\right|^2\mathrm{d}V_g\right| = \left|\int_{B_r(x_i)}\left|\nabla\left(w_i^n - z_i^n\right)\right|^2\mathrm{d}V_g\right.$$
$$\left. + 2\int_{B_r(x_i)}\nabla w_i^n\cdot\nabla\left(w_i^n - z_i^n\right)\mathrm{d}V_g\right|$$
$$\leq \int_{B_r(x_i)}\left|\nabla\left(w_i^n - z_i^n\right)\right|^2\mathrm{d}V_g$$

$$+2\left\|\nabla w_i^n\right\|_{L^1(\Sigma)}\left\|\nabla\left(w_i^n-z_i^n\right)\right\|_{L^\infty(B_r(x_i))}$$
$$\le C_r.$$

Moreover,

$$\int_{B_r(x_i)}\widetilde{h}_i e^{a_{ii}\left(w_i^n-\overline{w_i^n}\right)}\mathrm{d}V_g \le e^{a_{ii}\left\|w_i^n-\overline{w_i^n}-z_i^n\right\|_{L^\infty(B_r(x_i))}}\int_{B_r(x_i)}\widetilde{h}_i e^{a_{ii}z_i^n}\mathrm{d}V_g$$
$$\le C_r\int_{B_r(x_i)}d(\cdot,x_i)^{2\widetilde{\alpha}_i}e^{a_{ii}z_i^n}\mathrm{d}V_g.$$

Therefore, since $\widetilde{\alpha}_i\le 0$ and $a_{ii}\rho_i^n\le 8\pi(1+\widetilde{\alpha}_i)$, we can apply Corollary 2.5 to get the claim:

$$\frac{a_{ii}}{2}\int_{B_r(x_i)}\left|\nabla w_i^n\right|^2\mathrm{d}V_g-\rho_i^n\log\int_{B_r(x_i)}\widetilde{h}_i e^{a_{ii}\left(w_i^n-\overline{w_i^n}\right)}\mathrm{d}V_g$$
$$\ge\frac{1}{2a_{ii}}\int_{B_r(x_i)}\left|\nabla\left(a_{ii}z_i^n\right)\right|^2\mathrm{d}V_g$$
$$-\rho_i^n\log\int_{B_r(x_i)}d(\cdot,x_i)^{2\widetilde{\alpha}_i}e^{a_{ii}z_i^n}\mathrm{d}V_g-C_r$$
$$\ge -C_r$$

□

Proof of Theorem 1.3 As noticed before, it suffices to prove the boundedness from below of $J_{\rho^n}(u^n)$ for a sequence $\rho^n \underset{n\to+\infty}{\nearrow}\rho^0$ and a sequence of minimizers u^n for J_{ρ^n}. Moreover, due to invariance by addition of constants, one can consider v^n in place of u^n.

Let us start by estimating the term involving the gradients.

From Corollary 5.3 we deduce that the integral of $|\nabla w_i^n|^2$ outside a neighborhood of x_i is uniformly bounded for any $i\in\mathcal{I}$, and the integral on the whole Σ is bounded if $i\notin\mathcal{I}$.

For the same reason, the integral of $a_{ij}\nabla w_i^n\cdot\nabla w_j^n$ on the whole surface is uniformly bounded. In fact, if $a_{ij}\neq 0$, then $x_i\neq x_j$, then

$$\left|\int_\Sigma\nabla w_i^n\cdot\nabla w_j^n\mathrm{d}V_g\right|\le\int_{\Sigma\setminus B_r(x_j)}\left|\nabla w_i^n\cdot\nabla w_j^n\right|\mathrm{d}V_g+\int_{\Sigma\setminus B_r(x_i)}\left|\nabla w_i^n\cdot\nabla w_j^n\right|\mathrm{d}V_g$$
$$\le\left\|\nabla w_i^n\right\|_{L^{q'}(\Sigma)}\left\|\nabla w_j^n\right\|_{L^{q''}(\Sigma\setminus B_r\{x_j\})}$$
$$+\left\|\nabla w_i^n\right\|_{L^{q''}(\Sigma\setminus B_r\{x_i\})}\left\|\nabla w_j^n\right\|_{L^{q'}(\Sigma)}$$
$$\le C_r,$$

with q as in Corollary 5.3, $q'=\begin{cases}\frac{2q}{3q-2}<2 & \text{if } q<2\\ 1 & \text{if } q\ge 2\end{cases}$ and $q''=\begin{cases}\frac{2q}{2-q} & \text{if } q<2\\ \infty & \text{if } q\ge 2\end{cases}$.

Therefore, we can write

$$\sum_{i,j=1}^N a^{ij}\int_\Sigma\nabla v_i^n\cdot\nabla v_j^n\mathrm{d}V_g=\sum_{i,j=1}^N a_{ij}\int_\Sigma\nabla w_i^n\cdot\nabla w_j^n\mathrm{d}V_g$$
$$\ge\sum_{i\in\mathcal{I}}a_{ii}\int_{B_r(x_i)}\left|\nabla w_i^n\right|^2\mathrm{d}V_g-C_r.$$

To deal with the other term in the functional, we use the boundedness of w_i^n away from x_i: choosing r as in Lemma 5.4, we get

$$\begin{aligned}\int_\Sigma \widetilde{h}_i^n e^{v_i^n-\overline{v_i^n}}\mathrm{d}V_g &\le 2\int_{B_r(x_i)} \widetilde{h}_i^n e^{v_i^n-\overline{v_i^n}}\mathrm{d}V_g\\ &= 2\int_{B_r(x_i)} \widetilde{h}_i e^{\sum_{j=1}^N a_{ij}\left(w_j^n-\overline{w_j^n}\right)}\mathrm{d}V_g\\ &\le C_r\int_{B_r(x_i)} \widetilde{h}_i e^{a_{ii}\left(w_i^n-\overline{w_i^n}\right)}\mathrm{d}V_g.\end{aligned}$$

Therefore, using Lemma 5.4 we obtain

$$\begin{aligned}J_{\rho^n}\left(v^n\right) &= \frac{1}{2}\sum_{i,j=1}^N a^{ij}\int_\Sigma \nabla v_i^n\cdot\nabla v_j^n \mathrm{d}V_g - \sum_{i=1}^N \rho_i^n\left(\log\int_\Sigma \widetilde{h}_i^n e^{v_i^n}\mathrm{d}V_g - \overline{v_i^n}\right)\\ &\ge \sum_{i\in\mathcal{I}}\left(\frac{a_{ii}}{2}\int_{B_r(x_i)}\left|\nabla w_i^n\right|^2\mathrm{d}V_g - \rho_i^n\left(\log\int_{B_r(x_i)}\widetilde{h}_i e^{a_{ii}w_i^n}\mathrm{d}V_g - a_{ii}\overline{w_i^n}\right)\right) - C_r\\ &\ge -C_r\end{aligned}$$

Since the choice of r does not depend on n, the proof is complete. □

Acknowledgments The author would like to express his gratitude Professor Andrea Malchiodi for the support and for the discussions about this topic.

References

1. Adimurthi, Sandeep, K.: A singular Moser–Trudinger embedding and its applications. Nonlinear Differ. Equ. Appl. (NoDEA) **13**(5–6), 585–603 (2007)
2. Battaglia, L., Malchiodi, A.: A Moser–Trudinger inequality for the singular Toda system. Bull. Inst. Math. Acad. Sin. (N.S.) **9**(1), 1–23 (2014)
3. Bolton, J., Woodward, L.M.: Some geometrical aspects of the 2-dimensional Toda equations. In: Geometry, topology and physics (Campinas, 1996), pp. 69–81. de Gruyter, Berlin (1997)
4. Brezis, H., Merle, F.: Uniform estimates and blow-up behavior for solutions of $-\Delta u = V(x)e^u$ in two dimensions. Commun. Partial Differ. Equ. **16**(8–9), 1223–1253 (1991)
5. Calabi, E.: Isometric imbedding of complex manifolds. Ann. Math. **2**(58), 1–23 (1953)
6. Chang, S.-Y.A., Yang, P.C.: Prescribing Gaussian curvature on S^2. Acta Math. **159**(3–4), 215–259 (1987)
7. Chang, S.-Y.A., Yang, P.C.: Conformal deformation of metrics on S^2. J. Differ. Geom. **27**(2), 259–296 (1988)
8. Chen, W.X.: A Trüdinger inequality on surfaces with conical singularities. Proc. Am. Math. Soc. **108**(3), 821–832 (1990)
9. Chern, S.S., Wolfson, J.G.: Harmonic maps of the two-sphere into a complex Grassmann manifold. II. Ann. Math. **125**(2), 301–335 (1987)
10. Chipot, M., Shafrir, I., Wolansky, G.: On the solutions of Liouville systems. J. Differ. Equ. **140**(1), 59–105 (1997)
11. Chipot, M., Shafrir, I., Wolansky, G.: Erratum: "On the solutions of Liouville systems" [J. Differential Equations 140 (1997), no. 1, 59–105; MR1473855 (98j:35053)]. J. Differ. Equ. **178**(2), 630 (2002)
12. Dunne, G.: Self-dual Chern–Simons Theories. Lecture Notes in Physics. New series m: Monographs. Springer, (1995)
13. Fontana, L.: Sharp borderline Sobolev inequalities on compact Riemannian manifolds. Comment. Math. Helv. **68**(3), 415–454 (1993)
14. Jost, J., Lin, C., Wang, G.: Analytic aspects of the Toda system. II. Bubbling behavior and existence of solutions. Commun. Pure Appl. Math. **59**(4), 526–558 (2006)
15. Jost, J., Wang, G.: Analytic aspects of the Toda system—I: a Moser–Trudinger inequality. Commun. Pure Appl. Math. **54**(11), 1289–1319 (2001)

16. Lin, C.-S., Wei, J-c, Zhang, L.: Classification of blowup limits for SU(3) singular Toda systems. Anal. PDE **8**(4), 807–837 (2015)
17. Lucia, M., Nolasco, M.: $SU(3)$ Chern–Simons vortex theory and Toda systems. J. Differ. Equ. **184**(2), 443–474 (2002)
18. Moser, J.: A sharp form of an inequality by N. Trudinger. Indiana Univ. Math. J. **20**, 1077–1092 (1970/1971)
19. Shafrir, I., Wolansky, G.: The logarithmic HLS inequality for systems on compact manifolds. J. Funct. Anal. **227**(1), 200–226 (2005)
20. Shafrir, I., Wolansky, G.: Moser–Trudinger and logarithmic HLS inequalities for systems. J. Eur. Math. Soc. (JEMS) **7**(4), 413–448 (2005)
21. Tarantello, G.: Selfdual gauge field vortices: an analytical approach. In: Progress in Nonlinear Differential Equations and their Applications, vol. 72. Birkhäuser Boston Inc., Boston, MA (2008)
22. Troyanov, M.: Prescribing curvature on compact surfaces with conical singularities. Trans. Am. Math. Soc. **324**(2), 793–821 (1991)
23. Wang, G.: Moser–Trudinger inequalities and Liouville systems. C. R. Acad. Sci. Paris Sér. I Math. **328**(10), 895–900 (1999)
24. Yang, Y.: Solitons in Field Theory and Nonlinear Analysis. Springer Monographs in Mathematics. Springer, New York (2001)