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Product Hardy spaces associated to operators with heat kernel bounds on spaces of homogeneous type

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Abstract The aim of this article is to develop the theory of product Hardy spaces associated with operators which possess the weak assumption of Davies–Gaffney heat kernel estimates, in the setting of spaces of homogeneous type. We also establish a Calderón–Zygmund decomposition on product spaces, which is of independent and use it to study the interpolation of these product Hardy spaces. We then show that under the assumption of generalized Gaussian estimates, the product Hardy spaces coincide with the Lebesgue spaces, for an appropriate range of p.

Keywords Singular integrals · Hardy spaces · Product space · Atomic decomposition · Calderón–Zygmund decomposition

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1 Introduction

Modern harmonic analysis was introduced in the '50s, with the Calderón–Zygmund theory at the heart of it. This theory established criteria for singular integral operators to be bounded on different scales of function spaces, especially the Lebesgue spaces L^p, $1 < p < \infty$. To achieve this goal, an integrated part of the Calderón–Zygmund theory includes the theory of interpolation and the theory of function spaces, in particular end-point spaces such as the Hardy and BMO spaces. The development of the theory of Hardy spaces in $\mathbb{R}^n$ was initiated by Stein and Weiss [42], and was originally tied to the theory of harmonic functions. Real-variable methods were introduced into this subject by Fefferman and Stein [21]; the evolution of their ideas led eventually to characterizations of Hardy spaces via the atomic or molecular decomposition. See for instance [6,41] and [43]. The advent of the atomic and molecular characterizations enabled the extension of the Hardy spaces on Euclidean spaces to the more general setting of spaces of homogeneous type [14].

While the Calderón–Zygmund theory with one parameter was well established in the four decades of the '50s to '80s, multiparameter Fourier analysis was introduced later in the '70s and studied extensively in the '80s by a number of well known mathematicians, including R. Fefferman, S.-Y. A. Chang, R. Gundy, E.M. Stein, and J.L. Journé (see for instance [8–10,22–27,33]). For recent works, see also [5,7,29,30].

It is now understood that there are important situations in which the standard theory of Hardy spaces is not applicable and there is a need to consider Hardy spaces that are adapted to certain linear operators, similarly to the way that the standard Hardy spaces are adapted to the Laplacian. In this new development, the real-variable techniques of [14,21] and [13] are still of fundamental importance.

Recently, a theory of Hardy spaces associated to operators was introduced and developed by many authors. The following are some previous closely related results in the one-parameter setting.

(i) Auscher et al. [2] introduced the Hardy space $H^1_L(\mathbb{R}^n)$ associated to an operator L, and obtained a molecular decomposition, assuming that L has a bounded holomorphic functional calculus on $L^2(\mathbb{R}^n)$ and the kernel of the heat semigroup e^{-tL} has a pointwise Poisson upper bound.
(ii) Under the same assumptions on L as in (i), Duong and Yan [19,20] introduced the space $\mathrm{BMO}_L(\mathbb{R}^n)$ adapted to L and established the duality of $H^1_L(\mathbb{R}^n)$ and $\mathrm{BMO}_{L^*}(\mathbb{R}^n)$, where L^* denotes the adjoint operator of L on $L^2(\mathbb{R}^n)$. Yan [45] also studied the Hardy space $H^p_L(\mathbb{R}^n)$ and duality associated to an operator L under the same assumptions as (ii) for all $0 < p < 1$.

(iii) Auscher et al. [3] and Hofmann and Mayboroda [32], treated Hardy spaces H_L^p, $p \geq 1$, (and in the latter paper, also BMO spaces) adapted, respectively, to the Hodge Laplacian on a Riemannian manifold with a doubling measure, or to a second order divergence form elliptic operator on $\mathbb{R}^n$ with complex coefficients, in which settings pointwise heat kernel bounds may fail.
(iv) Hofmann et al. [31] developed the theory of $H_L^1(X)$ and $\mathrm{BMO}_L(X)$ spaces adapted to a non-negative, self-adjoint operator L whose heat semigroup satisfies the weak Davies–Gaffney bounds, in the setting of a space of homogeneous type X.
(v) Kunstmann and Uhl [35,44] studied the Hardy spaces $H_L^p(X)$, $1 < p < \infty$, associated to operators L satisfying the same conditions as in (iv) as well as the generalized Gaussian estimates for $p_0 \in [1, 2)$, and proved that $H_L^p(X)$ coincides with $L^p(X)$ for $p_0 < p < p_0'$ where p_0' is the conjugate index of p_0.
(vi) Duong and Li [17] considered the Hardy spaces $H_L^p(X)$, $0 < p \leq 1$, associated to non-self-adjoint operators L that generate an analytic semigroup on $L^2(X)$ satisfying Davies–Gaffney estimates and having a bounded holomorphic functional calculus on $L^2(X)$.

In contrast to the above listed established one-parameter theory, the multiparameter theory is much more complicated and is less advanced. In particular, there has not been much progress in the last decade in the direction of the paper [20] on singular integral operators with non-smooth kernels and the related product function spaces.

Deng et al. [16] introduced the product Hardy space $H_L^1(\mathbb{R} \times \mathbb{R})$ associated with an operator L, assuming that L has a bounded holomorphic functional calculus on $L^2(\mathbb{R})$ and the kernel of the heat semigroup e^{-tL} has a pointwise Poisson upper bound.

Recently, Duong et al. [18] defined the product Hardy space $H^1_{L_1,L_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ associated with non-negative self-adjoint operators L_1 and L_2 satisfying Gaussian heat kernel bounds, and then obtained the atomic decomposition, as well as the $H^1_{L_1,L_2}(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2}) \to L^1(\mathbb{R}^{n_1} \times \mathbb{R}^{n_2})$ boundedness of product singular integrals with non-smooth kernels.

In the study of Hardy spaces H^p associated to operators, $1 \leq p < \infty$, the assumptions on these operators determine the relevant properties of the corresponding Hardy spaces. One would start with the definition of Hardy spaces associated to operators under "weak" conditions on the operators so that the definition is applicable to a large class of operators. However, to obtain useful properties such as the coincidence between the Hardy spaces H^p and the Lebesgue spaces L^p, one would expect stronger conditions on the operators are needed. A natural question is to find a weak condition that is still sufficient for the Hardy spaces and Lebesgue spaces to coincide. We do so in part (γ) below.

This article is devoted to the study of Hardy spaces associated to operators, in the setting of product spaces of homogeneous type. Assume that L_1 and L_2 are two non-negative self-adjoint operators acting on $L^2(X_1)$ and $L^2(X_2)$, respectively, where X_1 and X_2 are spaces of homogeneous type, satisfying Davies–Gaffney estimates (DG) (see Sect. 2.2(**c**)). We note that the Davies–Gaffney estimates are a rather weak assumption, as they are known to be satisfied by quite a large class of operators (see Sect. 2.2 below).

Our main results are the following. In this paper we work in the biparameter setting. However our results, methods and techniques extend to the full k-parameter setting.

(α) We define the product Hardy space $H^1_{L_1,L_2}(X_1 \times X_2)$ associated with L_1 and L_2, in terms of the area function, and then obtain the corresponding atomic decomposition (Theorem 2.9). This is the generalisation of the results in [18] from the product of Euclidean spaces under the stronger assumption of Gaussian estimates (GE) (see Sect. 2.2(**a**)) to the product of spaces of homogeneous type with the weaker assumption of Davies–Gaffney estimates (DG).

This is also the extension of [31] from the one-parameter setting to the multiparameter setting. This part is the content of Sect. 3.

(β) We define the product Hardy space $H^p_{L_1,L_2}(X_1 \times X_2)$ for $1 < p < \infty$ associated with L_1 and L_2, and prove the interpolation result that if an operator T is bounded on $L^2(X_1 \times X_2)$ and is also bounded from $H^1_{L_1,L_2}(X_1 \times X_2)$ to $L^1(X_1 \times X_2)$, then it is bounded from $H^p_{L_1,L_2}(X_1 \times X_2)$ to $L^p(X_1 \times X_2)$ for all p with $1 \le p \le 2$ (Theorem 2.12). The proof of this interpolation result relies on the Calderon–Zygmund decomposition in the product setting, obtained in Theorem 2.11 below, which generalizes the classical result of Chang and Fefferman [9] on $H^1(\mathbb{R} \times \mathbb{R})$. This is done in Sect. 4.

(γ) Next we assume that L_1 and L_2 satisfy generalized Gaussian estimates (see Sect. 2.2(**b**)) for some $p_0 \in [1, 2)$. This assumption implies that L_1 and L_2 are injective operators (see Theorem 5.1) and satisfy the Davies–Gaffney estimates. We prove that our product Hardy spaces $H^p_{L_1,L_2}(X_1 \times X_2)$ coincide with $L^p(X_1 \times X_2)$ for $p_0 < p < p_0'$, where p_0' is the conjugate index of p_0 (Theorem 2.13). This is the extension to the multiparameter setting of the one-parameter result in [44], and is carried out in Sect. 5.

Along this line of research, in [11] we study the boundedness of multivariable spectral multipliers on product Hardy spaces on spaces of homogeneous type.

In the following section we introduce our assumptions on the underlying spaces X_1 and X_2 and the operators L_1 and L_2, give some examples of such operators, and then state our main results. Throughout this article, the symbols "c" and "C" denote constants that are independent of the essential variables.

2 Assumptions, and statements of main results

This section contains background material on spaces of homogeneous type, dyadic cubes, heat kernel bounds, and finite propagation speed of solutions to the wave equation, as well as the statements of our main results.

2.1 Spaces of homogeneous type

Definition 2.1 Consider a set X equipped with a quasi-metric d and a measure μ.

(a) A *quasi-metric* d on a set X is a function $d : X \times X \longrightarrow [0, \infty)$ satisfying (i) $d(x, y) = d(y, x) \ge 0$ for all $x, y \in X$; (ii) $d(x, y) = 0$ if and only if $x = y$; and (iii) the *quasi-triangle inequality*: there is a constant $A_0 \in [1, \infty)$ such that for all $x, y, z \in X$,

$$d(x, y) \le A_0[d(x, z) + d(z, y)].$$

We define the quasi-metric ball by $B(x, r) := \{y \in X : d(x, y) < r\}$ for $x \in X$ and $r > 0$. Note that the quasi-metric, in contrast to a metric, may not be Hölder regular and quasi-metric balls may not be open.

(b) We say that a nonzero measure μ satisfies the *doubling condition* if there is a constant C such that for all $x \in X$ and $r > 0$,

$$\mu(B(x, 2r)) \le C\mu(B(x, r)) < \infty. \tag{2.1}$$

(c) We point out that the doubling condition (2.1) implies that there exist positive constants n and C such that for all $x \in X$, $\lambda \ge 1$ and $r > 0$,

$$\mu(B(x, \lambda r)) \le C\lambda^n \mu(B(x, r)). \tag{2.2}$$

Fix such a constant n; we refer to this n as *the upper dimension of* μ.

(d) We say that (X, d, μ) is a *space of homogeneous type* in the sense of Coifman and Weiss if d is a quasi-metric on X and μ is a nonzero measure on X satisfying the doubling condition.

Throughout the whole paper, we assume that $\mu(X) = +\infty$.

It is shown in [14] that every space of homogeneous type X is *geometrically doubling*, meaning that there is some fixed number T such that each ball B in X can be covered by at most T balls of half the radius of B.

We recall the following construction given by Christ [12], which provides an analogue on spaces of homogeneous type of the grid of Euclidean dyadic cubes. The following formulation is taken from [12].

Lemma 2.2 ([12]) *Let (X, d, μ) be a space of homogeneous type. Then there exist a collection $\{I_\alpha^k \subset X : k \in \mathbb{Z}, \alpha \in \mathcal{I}_k\}$ of open subsets of X, where $\mathcal{I}_k$ is some index set, and constants $C_3 < \infty$, $C_4 > 0$, such that*

(i) $\mu(X \setminus \bigcup_\alpha I_\alpha^k) = 0$ for each fixed k, and $I_\alpha^k \cap I_\beta^k = \emptyset$ if $\alpha \neq \beta$;
(ii) for all α, β, k, l with $l \geq k$, either $I_\beta^l \subset I_\alpha^k$ or $I_\beta^l \cap I_\alpha^k = \emptyset$;
(iii) for each (k, α) and each $l < k$ there is a unique β such that $I_\alpha^k \subset I_\beta^l$;
(iv) $\operatorname{diam}(I_\alpha^k) \leq C_3 2^{-k}$; and
(v) each I_α^k contains some ball $B(z_\alpha^k, C_4 2^{-k})$, where $z_\alpha^k \in X$.

The point z_α^k is called the *centre* of the set I_α^k. Informally, we can think of I_α^k as a dyadic cube with diameter roughly 2^{-k}, centered at z_α^k. We write $\ell(I_\alpha^k) := C_3 2^{-k}$.

Given a constant $\lambda > 0$, we define λI_α^k to be the ball

$$\lambda I_\alpha^k := B(z_\alpha^k, \lambda C_3 2^{-k});$$

if $\lambda > 1$ then $I_\alpha^k \subset \lambda I_\alpha^k$. We refer to the ball λI_α^k as the *cube with the same center as I_α^k and diameter* $\lambda \operatorname{diam}(I_\alpha^k)$, or as the λ*-fold dilate* of the cube I_α^k. Since μ is doubling, we have $\mu(\lambda I_\alpha^k) \leq C\mu(B(z_\alpha^k, C_4 2^{-k})) \leq C\mu(I_\alpha^k)$.

2.2 Generalized Gaussian estimates, Davies–Gaffney estimates, and finite propagation speed

Suppose that L is a non-negative self-adjoint operator on $L^2(X)$, and that the semigroup $\{e^{-tL}\}_{t>0}$ generated by L on $L^2(X)$ has the kernel $p_t(x, y)$.

(a) Gaussian estimates: The kernel $p_t(x, y)$ has *Gaussian upper bounds* (GE) if there are positive constants C and c such that for all $x, y \in X$ and all $t > 0$,

$$|p_t(x, y)| \leq \frac{C}{V(x, t^{1/2})} \exp\left(-\frac{d(x, y)^2}{c\,t}\right). \tag{GE}$$

(b) Generalized Gaussian estimates: We say that $\{e^{-tL}\}_{t>0}$ satisfies the *generalized Gaussian estimates* (GGE_p), for a given $p \in [1, 2]$, if there are positive constants C and c such that for all $x, y \in X$ and all $t > 0$,

$$\|P_{B(x,t^{1/2})} e^{-tL} P_{B(y,t^{1/2})}\|_{L^p(X)\to L^{p'}(X)} \leq C V(x, t^{1/2})^{-(1/p-1/p')} \exp\left(-\frac{d(x, y)^2}{c\,t}\right), \tag{GGE_p}$$

where $1/p + 1/p' = 1$.

(c) Davies–Gaffney estimates: We say that $\{e^{-tL}\}_{t>0}$ satisfies the *Davies–Gaffney condition* (DG) if there are positive constants C and c such that for all open subsets $U_1,\ U_2 \subset X$ and all $t > 0$,

$$|\langle e^{-tL} f_1, f_2\rangle| \le C \exp\left(-\frac{\mathrm{dist}(U_1, U_2)^2}{c\,t}\right) \|f_1\|_{L^2(X)} \|f_2\|_{L^2(X)} \tag{DG}$$

for every $f_i \in L^2(X)$ with $\mathrm{supp}\, f_i \subset U_i$, $i = 1, 2$. Here $\mathrm{dist}(U_1, U_2) := \inf_{x\in U_1, y\in U_2} d(x, y)$.

(d) Finite propagation speed: We say that L satisfies the *finite propagation speed property* (FS) for solutions of the corresponding wave equation if for all open sets $U_i \subset X$ and all $f_i \in L^2(U_i)$, $i = 1, 2$, we have

$$\langle \cos(t\sqrt{L}) f_1, f_2\rangle = 0 \tag{FS}$$

for all $t \in (0, d(U_1, U_2))$.

As the following lemma notes, it is known that the Davies–Gaffney estimates and the finite propagation speed property are equivalent. For the proof, see for example [15, Theorem 3.4].

Lemma 2.3 *Let L be a non-negative self-adjoint operator acting on $L^2(X)$. Then the finite propagation speed property (FS) and the Davies–Gaffney estimates (DG) are equivalent.*

Remark 2.4 Note that when $p = 2$, it is shown in [15, Lemma3.1] that the generalized Gaussian estimates are the same as the Davies–Gaffney estimates (DG). Also, when $p = 1$, the generalized Gaussian estimates (GGE_p) are equivalent to the Gaussian estimates (GE) (see [4, Proposition 2.9]). By Hölder's inequality, we see that if an operator satisfies the generalized Gaussian estimates for some p with $1 < p < 2$, then it also satisfies the generalized Gaussian estimates (GGE_q) for all q with $p < q \le 2$. In particular,

$$(GE) \iff (\mathrm{GGE}_p) \text{ with } p = 1 \implies (\mathrm{GGE}_p) \text{ with } p \in (1, 2] \implies (DG) \iff (FS).$$

We also note that if the generalized Gaussian estimates (GGE_p) hold for some $p \in [1, 2)$, then the operator L is injective on $L^2(X)$ (see Theorem 5.1).

Suppose L is a non-negative self-adjoint operator acting on $L^2(X)$, and satisfying the Davies–Gaffney estimates (DG). Then the following result holds.

Lemma 2.5 (Lemma 3.5, [31]) *Let $\varphi \in C_0^\infty(\mathbb{R})$ be an even function with $\mathrm{supp}\,\varphi \subset (-1, 1)$. Let Φ denote the Fourier transform of φ. Then for every $\kappa = 0, 1, 2, \dots,$ and for every $t > 0$, the kernel $K_{(t^2L)^\kappa \Phi(t\sqrt{L})}(x, y)$ of the operator $(t^2L)^\kappa \Phi(t\sqrt{L})$, which is defined via spectral theory, satisfies*

$$\mathrm{supp}\ K_{(t^2L)^\kappa \Phi(t\sqrt{L})}(x, y) \subseteq \Big\{(x, y) \in X \times X : d(x, y) \le t\Big\}. \tag{2.3}$$

Examples. We now describe some operators where property (FS) and the estimates (GGE_p) hold for some p with $1 \le p < 2$.

Let $V \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ be a non-negative function. The Schrödinger operator with potential V is defined by $L = -\Delta + V$ on $\mathbb{R}^n$, where $n \ge 3$. From the well-known Trotter–Kato product formula, it follows that the semigroup e^{-tL} has a kernel $p_t(x, y)$ satisfying

$$0 \le p_t(x, y) \le (4\pi t)^{-\frac{n}{2}} \exp\left(-\frac{|x - y|^2}{4t}\right) \quad \text{forall } t > 0,\ \ x, y \in \mathbb{R}^n. \tag{2.4}$$

See [39, p. 195]. It follows that property (FS) and the estimates (GGE$_p$) hold with $p = 1$.

Next we consider inverse square potentials, that is $V(x) = c/|x|^2$. Fix $n \geq 3$ and assume that $c > -(n-2)^2/4$. Define $L := -\Delta + V$ to be the standard quadratic form on $L^2(\mathbb{R}^n, dx)$. The classical Hardy inequality

$$-\Delta \geq \frac{(n-2)^2}{4}|x|^{-2}, \tag{2.5}$$

shows that for all $c > -(n-2)^2/4$, the self-adjoint operator L is non-negative. Set $p_c^* := n/\sigma$, and $\sigma := \max\{(n-2)/2 - \sqrt{(n-2)^2/4 + c}, 0\}$. If $c \geq 0$ then the semigroup $\exp(-tL)$ is pointwise bounded by the Gaussian semigroup and hence acts on all L^p spaces with $1 \leq p \leq \infty$. If $c < 0$, then $\exp(-tL)$ acts as a uniformly bounded semigroup on $L^p(\mathbb{R}^n)$ for $p \in ((p_c^*)', p_c^*)$ and the range $((p_c^*)', p_c^*)$ is optimal (see for example [37]). It is known (see for instance [15]) that L satisfies property (FS) and the estimates (GGE$_p$) for all $p \in ((p_c^*)', 2n/(n+2)]$. If $c \geq 0$, then $p = (p_c^*)' = 1$ is included.

It is also known (see [36]) that the estimates (GGE$_p$) hold for some p with $1 \leq p < 2$ (and hence the property (FS) also holds) when L is the second order Maxwell operator with measurable coefficient matrices, or the Stokes operator with Hodge boundary conditions on bounded Lipschitz domains in $\mathbb{R}^3$, or the time-dependent Lamé system equipped with homogeneous Dirichlet boundary conditions.

2.3 Main results: product Hardy spaces associated with operators

We begin this section by defining the Hardy space $H^2(X_1 \times X_2)$. Next we introduce the area function Sf, and use it to define the Hardy space $H^1_{L_1,L_2}(X_1 \times X_2)$ associated to L_1 and L_2 (Definition 2.6). We define $(H^1_{L_1,L_2}, 2, N)$-atoms $a(x_1, x_2)$ (Definition 2.7) and use them to define the atomic Hardy space $H^1_{L_1,L_2,at,N}(X_1 \times X_2)$ (Definition 2.8). We show that these two definitions of this Hardy space coincide (Theorem 2.9). We also define the Hardy space $H^p_{L_1,L_2}(X_1 \times X_2)$ associated to L_1 and L_2, via a modified area function (Definition 2.10). We present the Calderón–Zygmund decomposition of the Hardy spaces $H^p_{L_1,L_2}(X_1 \times X_2)$ (Theorem 2.11). We use this decomposition to establish two interpolation results and to show that $H^p_{L_1,L_2}(X_1 \times X_2)$ coincides with $L^p(X_1 \times X_2)$ for an appropriate range of p (Theorems 2.12 and 2.13).

We work with the product of spaces of homogeneous type $(X_1, d_1, \mu_1) \times (X_2, d_2, \mu_2)$. Here, for $i = 1, 2$, (X_i, d_i, μ_i) is a space of homogeneous type with upper dimension n_i, as in Definition 2.1, and $\mu_i(X_i) = \infty$.

Following [3], one can define the $L^2(X_1 \times X_2)$-*adapted Hardy space*

$$H^2(X_1 \times X_2) := \overline{R(L_1 \otimes L_2)}, \tag{2.6}$$

that is, the closure of the range of $L_1 \otimes L_2$ in $L^2(X_1 \times X_2)$. Then $L^2(X_1 \times X_2)$ is the orthogonal sum of $H^2(X_1 \times X_2)$ and the null space $N(L_1 \otimes L_2) = \{f \in L^2(X_1 \times X_2) : (L_1 \otimes L_2)f = 0\}$.

We shall work with the domain $(X_1 \times \mathbb{R}_+) \times (X_2 \times \mathbb{R}_+)$ and its distinguished boundary $X_1 \times X_2$. For $x = (x_1, x_2) \in X_1 \times X_2$, denote by $\Gamma(x)$ the product cone $\Gamma(x) := \Gamma_1(x_1) \times \Gamma_2(x_2)$, where $\Gamma_i(x_i) := \{(y_i, t_i) \in X_i \times \mathbb{R}_+ : d_i(x_i, y_i) < t_i\}$ for $i = 1, 2$.

Our first definition of the product Hardy space $H^1_{L_1,L_2}(X_1 \times X_2)$ associated to operators L_1 and L_2 is via an appropriate area function. For $i = 1, 2$, suppose that L_i is a non-negative self-adjoint operator on X_i such that the corresponding heat semigroup e^{-tL_i} satisfies the

Davies–Gaffney estimates (DG). Given a function f on $L^2(X_1 \times X_2)$, the *area function* Sf associated with the operators L_1 and L_2 is defined by

$$Sf(x) := \left(\iint_{\Gamma(x)} \left| \left(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2} \right) f(y) \right|^2 \frac{d\mu_1(y_1) dt_1 d\mu_2(y_2) dt_2}{t_1 V(x_1, t_1) t_2 V(x_2, t_2)} \right)^{1/2}. \tag{2.7}$$

Since L_1 and L_2 are non-negative self-adjoint operators, it is known from H_∞ functional calculus [38] that there exist constants C_1 and C_2 with $0 < C_1 \le C_2 < \infty$ such that

$$\|Sf\|_2 \le C_2 \|f\|_2$$

for all $f \in L^2(X_1 \times X_2)$, and (by duality)

$$C_1 \|f\|_2 \le \|Sf\|_2$$

for all $f \in H^2(X_1 \times X_2)$.

Definition 2.6 For $i = 1, 2$, let L_i be a non-negative self-adjoint operator on $L^2(X_i)$ such that the corresponding heat semigroup e^{-tL_i} satisfies the Davies–Gaffney estimates (DG). The *Hardy space* $H^1_{L_1, L_2}(X_1 \times X_2)$ *associated to* L_1 *and* L_2 is defined as the completion of the set

$$\{f \in H^2(X_1 \times X_2) : \|Sf\|_{L^1(X_1 \times X_2)} < \infty\}$$

with respect to the norm

$$\|f\|_{H^1_{L_1, L_2}(X_1 \times X_2)} := \|Sf\|_{L^1(X_1 \times X_2)}.$$

We now introduce the notion of $(H^1_{L_1, L_2}, 2, N)$-atoms associated to operators.

Definition 2.7 Let N be a positive integer. A function $a(x_1, x_2) \in L^2(X_1 \times X_2)$ is called an $(H^1_{L_1, L_2}, 2, N)$-*atom* if it satisfies the following conditions:

(1) there is an open set Ω in $X_1 \times X_2$ with finite measure such that $\operatorname{supp} a \subset \Omega$; and
(2) a can be further decomposed as

$$a = \sum_{R \in m(\Omega)} a_R,$$

where $m(\Omega)$ is the set of all maximal dyadic rectangles contained in Ω, and for each $R \in m(\Omega)$ there exists a function b_R such that for all $\sigma_1, \sigma_2 \in \{0, 1, \ldots, N\}$, b_R belongs to the range of $L_1^{\sigma_1} \otimes L_2^{\sigma_2}$ in $L^2(X_1 \times X_2)$ and

(i) $a_R = \left(L_1^N \otimes L_2^N\right) b_R$;
(ii) $\operatorname{supp} \left(L_1^{\sigma_1} \otimes L_2^{\sigma_2}\right) b_R \subset \overline{C} R$;
(iii) $||a||_{L^2(X_1 \times X_2)} \le \mu(\Omega)^{-1/2}$ and

$$\sum_{R = I \times J \in m(\Omega)} \ell(I)^{-4N} \ell(J)^{-4N} \left\| \left(\ell(I)^2 L_1\right)^{\sigma_1} \otimes \left(\ell(J)^2 L_2\right)^{\sigma_2} b_R \right\|^2_{L^2(X_1 \times X_2)} \le \mu(\Omega)^{-1}.$$

Here $R = I \times J$, $\overline{C}$ is a fixed constant, and $\overline{C}R$ denotes the product $\overline{C}I \times \overline{C}J$ of the balls which are the $\overline{C}$-fold dilates of I and J respectively, as defined in Section 3.

We can now define an atomic $H^1_{L_1,L_2,at,N}$ space, which we shall show is equivalent to the space $H^1_{L_1,L_2}$ defined above via area functions.

Definition 2.8 Let N be a positive integer with $N > \max\{n_1, n_2\}/4$, where n_i is the upper doubling dimension of X_i, $i = 1, 2$. We say that $f = \sum \lambda_j a_j$ is an *atomic* $(H^1_{L_1,L_2}, 2, N)$-*representation of* f if $\{\lambda_j\}_{j=0}^{\infty} \in \ell^1$, each a_j is an $(H^1_{L_1,L_2}, 2, N)$-atom, and the sum converges in $L^2(X_1 \times X_2)$. Set

$$\mathbb{H}^1_{L_1,L_2,at,N}(X_1 \times X_2) := \big\{f : f \text{ has an atomic } (H^1_{L_1,L_2}, 2, N)\text{-representation}\big\},$$

with the norm given by

$$\|f\|_{\mathbb{H}^1_{L_1,L_2,at,N}(X_1\times X_2)} := \inf\Big\{\sum_{j=0}^{\infty}|\lambda_j| : f = \sum_j \lambda_j a_j \text{ is an atomic } (H^1_{L_1,L_2}, 2, N)\text{-representation}\Big\}. \tag{2.8}$$

The *Hardy space* $H^1_{L_1,L_2,at,N}(X_1 \times X_2)$ is then defined as the completion of $\mathbb{H}^1_{L_1,L_2,at,N}(X_1 \times X_2)$ with respect to this norm.

Our first result is that the "area function" and "atomic" H^1 spaces coincide, with equivalent norms, if the parameter $N > \max\{n_1, n_2\}/4$.

Theorem 2.9 *Let (X_i, d_i, μ_i) be spaces of homogeneous type with upper dimension n_i, for $i = 1, 2$. Suppose $N > \max\{n_1, n_2\}/4$. Then*

$$H^1_{L_1,L_2}(X_1 \times X_2) = H^1_{L_1,L_2,at,N}(X_1 \times X_2).$$

Moreover,

$$\|f\|_{H^1_{L_1,L_2}(X_1\times X_2)} \sim \|f\|_{H^1_{L_1,L_2,at,N}(X_1\times X_2)},$$

where the implicit constants depend only on N, n_1 and n_2.

It follows that Definition 2.8 always yields the same Hardy space $H^1_{L_1,L_2,at,N}(X_1 \times X_2)$, independent of the particular choice of $N > \max\{n_1, n_2\}/4$.

The proof of Theorem 2.9 will be given in Section 3.

We turn from the case of $p = 1$ to the Hardy spaces $H^p_{L_1,L_2}(X_1 \times X_2)$ associated to L_1 and L_2, for $1 < p < \infty$.

Definition 2.10 Let L_1 and L_2 be two non-negative, self-adjoint operators acting on $L^2(X_1)$ and $L^2(X_1)$ respectively, satisfying the Davies–Gaffney condition (DG).

(i) For each p with $1 < p \leq 2$, the *Hardy space* $H^p_{L_1,L_2}(X_1 \times X_2)$ *associated to* L_1 *and* L_2 is the completion of the space $\Big\{f \in H^2(X_1 \times X_2) : Sf \in L^p(X_1 \times X_2)\Big\}$ in the norm

$$\|f\|_{H^p_{L_1,L_2}(X_1,X_2)} = \|Sf\|_{L^p(X_1,X_2)}.$$

(ii) For each p with $2 < p < \infty$, the *Hardy space* $H^p_{L_1,L_2}(X_1, X_2)$ *associated to* L_1 *and* L_2 is the completion of the space $D_{K_0,p}$ in the norm

$$\|f\|_{H^p_{L_1,L_2}(X_1,X_2)} := \|S_{K_0}f\|_{L^p(X_1,X_2)}, \quad \text{with} \quad K_0 := \max\left\{\left[\frac{n_1}{4}\right], \left[\frac{n_2}{4}\right]\right\} + 1,$$

where

$$S_K f(x) := \left(\int_{\Gamma(x)} |(t_1^2L_1)^K e^{-t_1^2L_1} \otimes (t_2^2L_2)^K e^{-t_2^2L_2} f(y)|^2 \frac{d\mu_1(y_1)}{V(x_1,t_1)}\frac{dt_1}{t_1}\frac{d\mu_2(y_2)}{V(x_2,t_2)}\frac{dt_2}{t_2}\right)^{1/2}, \tag{2.9}$$

and

$$D_{K,p} := \left\{f \in H^2(X_1 \times X_2) : \ S_K f \in L^p(X_1 \times X_2)\right\}.$$

Next we develop the Calderón–Zygmund decomposition of the Hardy spaces $H^p_{L_1,L_2}(X_1 \times X_2)$, which is a generalization of the result of Chang and Fefferman [9].

Theorem 2.11 *Fix p with* $1 < p < 2$. *Take* $\alpha > 0$ *and* $f \in H^p_{L_1,L_2}(X_1 \times X_2)$. *Then we may write* $f = g + b$, *where* $g \in H^2_{L_1,L_2}(X_1 \times X_2)$ *and* $b \in H^1_{L_1,L_2}(X_1 \times X_2)$, *such that*

$$\|g\|^2_{H^2_{L_1,L_2}(X_1\times X_2)} \leq C\alpha^{2-p}\|f\|^p_{H^p_{L_1,L_2}(X_1\times X_2)}$$

and

$$\|b\|_{H^1_{L_1,L_2}(X_1\times X_2)} \leq C\alpha^{1-p}\|f\|^p_{H^p_{L_1,L_2}(X_1\times X_2)}.$$

Here C is an absolute constant.

As a consequence of the above Calderón–Zygmund decomposition, we obtain the following interpolation result.

Theorem 2.12 *Suppose that* L_1 *and* L_2 *are non-negative self-adjoint operators such that the corresponding heat semigroups* e^{-tL_1} *and* e^{-tL_2} *satisfy the Davies–Gaffney estimates (DG). Let T be a sublinear operator which is bounded on* $L^2(X_1 \times X_2)$ *and bounded from* $H^1_{L_1,L_2}(X_1\times X_2)$ *to* $L^1(X_1\times X_2)$. *Then T is bounded from* $H^p_{L_1,L_2}(X_1\times X_2)$ *to* $L^p(X_1\times X_2)$ *for all* $1 < p < 2$.

The proofs of Theorems 2.11 and 2.12 will be given in Sect. 4.

Next, we establish the relationship between the Hardy spaces $H^p_{L_1,L_2}(X_1 \times X_2)$ and the Lebesgue spaces $L^p(X_1 \times X_2)$ for a certain range of p.

First note that under the assumption of Gaussian upper bounds (GE), following the approaches used in [31] in the one-parameter setting, we can obtain that $H^p_{L_1,L_2}(X_1 \times X_2) = L^p(X_1 \times X_2)$ for all $1 < p < \infty$. Second, if one assumes only the Davies–Gaffney estimates on the heat semigroups of L_1 and L_2, then for $1 < p < \infty$ and $p \neq 2$, $H^p_{L_1,L_2}(X_1 \times X_2)$ may or may not coincide with the space $L^p(X_1 \times X_2)$. An example where the classical Hardy space can be different from the Hardy space associated to an operator L is when L is the elliptic divergence form operator with complex, bounded measurable coefficients on $\mathbb{R}^n$; see [32]. However, it can be verified by spectral analysis that $H^2_{L_1,L_2}(X_1 \times X_2) = H^2(X_1 \times X_2)$. Here the $L^2(X_1 \times X_2)$-adapted Hardy space $H^2(X_1 \times X_2)$ is as defined in (2.6) above.

Theorem 2.13 *Suppose that L_1 and L_2 are non-negative self-adjoint operators on $L^2(X_1)$ and $L^2(X_2)$, respectively. Suppose that there exists some $p_0 \in [1, 2)$ such that L_1 and L_2 satisfy the generalized Gaussian estimates* ($\mathrm{GGE}_{\mathrm{p}_0}$). *Let p_0' satisfy $1/p_0 + 1/p_0' = 1$.*

(i) We have $H^p_{L_1,L_2}(X_1 \times X_2) = L^p(X_1 \times X_2)$ for all p such that $p_0 < p < p_0'$, with equivalent norms $\|\cdot\|_{H^p_{L_1,L_2}}$ and $\|\cdot\|_{L^p}$.

(ii) Let T be a sublinear operator which is bounded on $L^2(X_1 \times X_2)$ and bounded from $H^1_{L_1,L_2}(X_1 \times X_2)$ to $L^1(X_1 \times X_2)$. Then T is bounded on $L^p(X_1 \times X_2)$ for all p such that $p_0 < p < p_0'$.

The proof of Theorem 2.13 will be given in Section 5.

3 Characterization of the Hardy space $H^1_{L_1,L_2}(X_1 \times X_2)$ in terms of atoms

The goal of this section is to provide the proof of Theorem 2.9.

Our strategy is as follows: by density, it is enough to show that when $N > \max\{n_1, n_2\}/4$, we have

$$\mathbb{H}^1_{L_1,L_2,at,N}(X_1 \times X_2) = H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$$

with equivalent norms. The proof of this fact proceeds in two steps.

Step 1. $\mathbb{H}^1_{L_1,L_2,at,N}(X_1 \times X_2) \subseteq H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$, for $N > \max\{n_1, n_2\}/4$. This step relies on the fact that the area function S is bounded on $L^2(X_1 \times X_2)$ and that $\|Sa\|_{L^1(X_1 \times X_2)}$ is uniformly bounded for every atom a.
Step 2. $H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2) \subseteq \mathbb{H}^1_{L_1,L_2,at,N}(X_1 \times X_2)$, for all $N \in \mathbb{N}$. In the proof of this step we use the tent space approach to construct the atoms in the Hardy spaces associated to operators in the product setting.

We take these in order.

Proof of Step 1 The conclusion of Step 1 is an immediate consequence of the following pair of Lemmata.

Lemma 3.1 *Fix $N \in \mathbb{N}$. Assume that T is a linear operator, or a non-negative sublinear operator, satisfying the weak-type* (2,2) *bound*

$$\left|\{x \in X_1 \times X_2 : |Tf(x)| > \eta\}\right| \le C\eta^{-2}\|f\|^2_{L^2(X_1 \times X_2)}, \text{ for all } \eta > 0,$$

and that for every $(H^1_{L_1,L_2}, 2, N)$-atom a, we have

$$\|Ta\|_{L^1(X_1 \times X_2)} \le C \tag{3.1}$$

with constant C independent of a. Then T is bounded from $\mathbb{H}^1_{L_1,L_2,at,N}(X_1 \times X_2)$ to $L^1(X_1 \times X_2)$, and

$$\|Tf\|_{L^1(X_1 \times X_2)} \le C\|f\|_{\mathbb{H}^1_{L_1,L_2,at,N}(X)}.$$

Therefore, by density, T extends to a bounded operator from $H^1_{L_1,L_2,at,N}(X_1 \times X_2)$ to $L^1(X_1 \times X_2)$.

The proof of Lemma 3.1 follows directly from that of the one-parameter version: Lemma 4.3 in [31]. The proof given there is independent of the number of parameters. We omit the details here.

Lemma 3.2 *Let a be an $(H^1_{L_1,L_2}, 2, N)$-atom with $N > \max\{n_1, n_2\}/4$. Let S denote the area function defined in (2.7). Then*

$$\|Sa\|_1 \le C, \tag{3.2}$$

where C is a positive constant independent of a.

Given Lemma 3.2, we may apply Lemma 3.1 with $T = S$ to obtain

$$\|f\|_{H^1_{L_1,L_2}(X_1\times X_2)} := \|Sf\|_{L^1(X_1\times X_2)} \le C\|f\|_{\mathbb{H}^1_{L_1,L_2,at,N}(X_1\times X_2)},$$

from which Step 1 follows.

To finish Step 1, it therefore suffices to verify estimate (3.2) in Lemma 3.2. To do so, we apply Journé's covering lemma.

We recall from [28] the formulation of Journé's Lemma [34,40] in the setting of spaces of homogeneous type. Let (X_i, d_i, μ_i), $i = 1, 2$, be spaces of homogeneous type and let $\{I^{k_i}_{\alpha_i} \subset X_i\}$, $i = 1, 2$, be open cubes as in Lemma 2.2. Let $\mu = \mu_1 \times \mu_2$ denote the product measure on $X_1 \times X_2$. The open set $I^{k_1}_{\alpha_1} \times I^{k_2}_{\alpha_2}$ for $k_1, k_2 \in \mathbb{Z}$, $\alpha_1 \in I_{k_1}$ and $\alpha_2 \in I_{k_2}$, is called a *dyadic rectangle in* $X_1 \times X_2$. Let $\Omega \subset X_1 \times X_2$ be an open set of finite measure. Denote by $m(\Omega)$ the maximal dyadic rectangles contained in Ω, and by $m_i(\Omega)$ the family of dyadic rectangles $R \subset \Omega$ which are maximal in the x_i-direction, for $i = 1, 2$.

In what follows, we let $R = I \times J$ denote any dyadic rectangle in $X_1 \times X_2$. Given $R = I \times J \in m_1(\Omega)$, let $\widehat{J}$ be the largest dyadic cube containing J such that

$$\mu\big((I \times \widehat{J}) \cap \Omega\big) > \frac{1}{2}\,\mu(I \times \widehat{J}).$$

Similarly, given $R = I \times J \in m_2(\Omega)$, let $\widehat{I}$ be the largest dyadic cube containing I such that

$$\mu\big((\widehat{I} \times J) \cap \Omega\big) > \frac{1}{2}\,\mu(\widehat{I} \times J).$$

Also, let $w(x)$ be any increasing function such that $\sum_{j=0}^{\infty} j\,w(c2^{-j}) < \infty$, where c is a fixed positive constant. In particular, we may take $w(x) = x^\delta$ for any $\delta > 0$.

Lemma 3.3 ([28]) *Let $\Omega \subset X_1 \times X_2$ be an open set with finite measure. Then*

$$\sum_{R=I\times J\in m_1(\Omega)} \mu(R)\,w\left(\frac{\ell(J)}{\ell(\widehat{J})}\right) \le C\mu(\Omega) \tag{3.3}$$

and

$$\sum_{R=I\times J\in m_2(\Omega)} \mu(R)\,w\left(\frac{\ell(I)}{\ell(\widehat{I})}\right) \le C\mu(\Omega), \tag{3.4}$$

for some constant C independent of Ω.

Proof of Lemma 3.2 Given an $(H^1_{L_1,L_2}, 2, N)$-atom a, let Ω be an open set of finite measure in $X_1 \times X_2$ as in Definition 2.7 such that $a = \sum_{R\in m(\Omega)} a_R$ is supported in Ω.

For each rectangle $R = I \times J \subset \Omega$, let I^* be the largest dyadic cube in X_1 containing I such that $I^* \times J \subset \widetilde{\Omega}$, where $\widetilde{\Omega} := \{x \in X_1 \times X_2 : M_s(\chi_\Omega)(x) > 1/2\}$ and M_s denotes the strong maximal function. Next, let J^* be the largest dyadic cube in X_2 containing J such that $I^* \times J^* \subset \widetilde{\widetilde{\Omega}}$, where $\widetilde{\widetilde{\Omega}} := \{x \in X_1 \times X_2 : M_s(\chi_{\widetilde{\Omega}})(x) > 1/2\}$.

Now let R^* be the 100-fold dilate of $I^* \times J^*$ concentric with $I^* \times J^*$. That is, $R^* = 100I^* \times 100J^*$ is the product of the balls $100I^*$ and $100J^*$ centered at the centers of I^* and J^* respectively, as defined in Sect. 2. An application of the strong maximal function theorem shows that $\mu\big(\cup_{R\subset\Omega} R^*\big) \le C\mu(\widetilde{\widetilde{\Omega}}) \le C\mu(\widetilde{\Omega}) \le C\mu(\Omega)$.

Then we write

$$\|Sa\|_{L^1(X_1\times X_2)} = \|Sa\|_{L^1(\cup R^*)} + \|Sa\|_{L^1((\cup R^*)^c)}.$$

Thus, by Hölder's inequality and the property (iii) of the $(H^1_{L_1,L_2,}, 2, N)$-atom, we see that the first term on the right-hand side is bounded by

$$\|Sa\|_{L^1(\cup R^*)} \le \mu(\cup R^*)^{1/2}\|Sa\|_{L^2(X_1\times X_2)} \le C\mu(\Omega)^{1/2}\|a\|_{L^2(X_1\times X_2)} \le C.$$

Now it suffices to prove that

$$\int_{(\bigcup R^*)^c} |Sa(x_1, x_2)|\, d\mu_1(x_1)\, d\mu_2(x_2) \le C. \tag{3.5}$$

From the definition of a, we see that the left-hand side of (3.5) is controlled by

$$\begin{aligned}
&\sum_{R\in m(\Omega)} \int_{(R^*)^c} |Sa_R(x_1, x_2)|\, d\mu_1(x_1)\, d\mu_2(x_2)\\
&\le \sum_{R\in m(\Omega)} \int_{(100I^*)^c\times X_2} |Sa_R(x_1, x_2)|\, d\mu_1(x_1)\, d\mu_2(x_2)\\
&\quad + \sum_{R\in m(\Omega)} \int_{X_1\times(100J^*)^c} |Sa_R(x_1, x_2)|\, d\mu_1(x_1)\, d\mu_2(x_2)\\
&=: D + E.
\end{aligned} \tag{3.6}$$

It suffices to verify that the term D is bounded by a positive constant C independent of the atom a, since the estimate for E follows symmetrically. For the term D, by splitting the region of integration $(100I^*)^c \times X_2$ into $(100I^*)^c \times 100J$ and $(100I^*)^c \times (100J)^c$, we write D as $D^{(a)} + D^{(b)}$.

Let us first estimate the term $D^{(a)}$. Using Hölder's inequality, we have

$$D^{(a)} \le C \sum_{R\in m(\Omega)} \mu_2(J)^{1/2} \int_{(100I^*)^c} \left(\int_{100J} |Sa_R(x_1, x_2)|^2\, d\mu_2(x_2)\right)^{1/2} d\mu_1(x_1). \tag{3.7}$$

Next, we claim that

$$\begin{aligned}
&\int_{(100I^*)^c} \left(\int_{100J} |Sa_R(x_1, x_2)|^2\, d\mu_2(x_2)\right)^{1/2} d\mu_1(x_1)\\
&\le C\left(\frac{\ell(I)}{\ell(I^*)}\right)^{\epsilon_1} \mu_1(I)^{1/2}\left(\ell(I)^{-4N}\ell(J)^{-4N}\|(\mathbb{1}_1 \otimes (\ell(J)^2 L)^N)b_R\|^2_{L^2(X_1\times X_2)}\right)^{1/2}
\end{aligned} \tag{3.8}$$

for some $\epsilon_1 > 0$. Assuming this claim holds, then by using Hölder's inequality, Journé's Lemma and property (2)(iii) of Definition 2.7, we have

$$\begin{aligned} D^{(a)} &\le C \left(\sum_{R\in m(\Omega)} \mu(R) \left(\frac{\ell(I)}{\ell(I^*)} \right)^{2\epsilon_1} \right)^{\frac{1}{2}} \\ &\quad \times \left(\sum_{R\in m(\Omega)} \ell(I)^{-4N} \ell(J)^{-4N} \|(\mathbb{1}_1 \otimes (\ell(J)^2 L)^N) b_R\|^2_{L^2(X_1\times X_2)} \right)^{\frac{1}{2}} \\ &\le C\mu(\Omega)^{\frac{1}{2}} \mu(\Omega)^{-\frac{1}{2}} \\ &\le C. \end{aligned}$$

It remains to verify the claim (3.8). Set $a_{R,2} = (\mathbb{1}_1 \otimes L_2^N) b_R$; then $a_R = (L_1^N \otimes \mathbb{1}_2) a_{R,2}$. Then, from the definition of the area
function, we have

$$\begin{aligned} &\int_{100J} |Sa_R(x_1, x_2)|^2 \, d\mu_2(x_2) \\ &= \int_{100J} \int_{\Gamma_1(x_1)} \int_{\Gamma_2(x_2)} \left| \left((t_1^2 L_1)^{N+1} e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2} \right) (a_{R,2})(y_1, y_2) \right|^2 \\ &\quad \times \frac{d\mu_2(y_2) dt_2}{t_2 V(x_2, t_2)} \frac{d\mu_1(y_1) dt_1}{t_1^{1+4N} V(x_1, t_1)} \, d\mu_2(x_2) \\ &= \int_{\Gamma_1(x_1)} \Bigg[\int_{100J} \int_{\Gamma_2(x_2)} \\ &\quad \times \left| t_2^2 L_2 e^{-t_2^2 L_2} \left((t_1^2 L_1)^{N+1} e^{-t_1^2 L_1} a_{R,2}(y_1, \cdot) \right) (y_2) \right|^2 \frac{d\mu_2(y_2) dt_2}{t_2 V(x_2, t_2)} \, d\mu_2(x_2) \Bigg] \frac{d\mu_1(y_1) dt_1}{t_1^{1+4N} V(x_1, t_1)} \\ &\le C \int_{\Gamma_1(x_1)} \int_{X_2} \left| (t_1^2 L_1)^{N+1} e^{-t_1^2 L_1} a_{R,2}(y_1, x_2) \right|^2 d\mu_2(x_2) \frac{d\mu_1(y_1) dt_1}{t_1^{1+4N} V(x_1, t_1)}, \end{aligned} \tag{3.9}$$

where the last inequality follows from the L^2 estimate of the area function on X_2.

Define $U_j(I) = 2^j I \backslash 2^{j-1} I$ for $j \ge 1$. Then we see that $(100I^*)^c \subset \cup_{j>4} U_j(I)$. Moreover, we have that $\mu_1(U_j(I)) \le C 2^{jn_1} \mu_1(I)$. Then, by Hölder's inequality and the estimate in (3.9), we get

$$\begin{aligned} &\int_{(100I^*)^c} \left(\int_{100J} |Sa_R(x_1, x_2)|^2 \, d\mu_2(x_2) \right)^{\frac{1}{2}} d\mu_1(x_1) \\ &\le C \sum_{j>4} \mu_1(U_j(I))^{1/2} \mu_1(I)^{\frac{1}{2}} \Bigg(\int_{(100I^*)^c \bigcap U_j(I)} \int_0^\infty \int_{d_1(x_1,y_1)<t_1} \int_{X_2} \\ &\quad \times \left| (t_1^2 L_1)^{N+1} e^{-t_1^2 L_1} a_{R,2}(y_1, x_2) \right|^2 d\mu_2(x_2) \frac{d\mu_1(y_1) dt_1}{t_1^{1+4N} V(x_1, t_1)} \, d\mu_1(x_1) \Bigg)^{\frac{1}{2}}. \end{aligned}$$

Next, we split the integral area $(0, \infty)$ for t_1 into three parts: $(0, \ell(I))$, $(\ell(I), d_1(x_1, x_I)/4)$ and $(d_1(x_1, x_I)/4, \infty)$. Then the right-hand side of the above inequality is bounded by the sum of the following three terms

$$D_1^{(a)} + D_2^{(a)} + D_3^{(a)},$$

where

$$D_1^{(a)} := C\sum_{j>4} 2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\left\{\int_{X_2}\int_{(100I^*)^c\bigcap U_j(I)}\int_0^{\ell(I)}\int_{d_1(x_1,y_1)<t_1}\right.$$
$$\left.\times\left|(t_1^2L_1)^{N+1}e^{-t_1^2L_1}a_{R,2}(y_1,x_2)\right|^2\frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}V(x_1,t_1)}\,d\mu_1(x_1)\,d\mu_2(x_2)\right\}^{\frac{1}{2}},$$

and $D_2^{(a)}$ and $D_3^{(a)}$ are the same as $D_1^{(a)}$ with the integral $\int_0^{\ell(I)}$ replaced by $\int_{\ell(I)}^{d_1(x_1,x_I)/4}$ and $\int_{d_1(x_1,x_I)/4}^{\infty}$, respectively. Here we use x_I to denote the center of the dyadic cube I.

We first consider the term $D_1^{(a)}$. We define $E_j(I) := \{y_1 : d_1(x_1,y_1) < \ell(I)$ for some $x_1 \in (100I^*)^c \cap U_j(I)\}$. Then we can see that $\mathrm{dist}(E_j(I), I) > 2^{j-2}\ell(I) + \ell(I^*)$. Now we have

$$\begin{aligned}D_1^{(a)} &\le C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\\
&\quad\times\left\{\int_{X_2}\int_0^{\ell(I)}\int_{E_j(I)}\left|(t_1^2L_1)^{N+1}e^{-t_1^2L_1}\alpha_{R,2}(y_1,x_2)\right|^2\frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}}\,d\mu_2(x_2)\right\}^{\frac{1}{2}}\\
&\le C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\left\{\int_0^{\ell(I)}e^{-(2^{j-2}\ell(I)+\ell(I^*))^2/(ct_1^2)}\frac{dt_1}{t_1^{1+4N}}\,\|a_{R,2}\|^2_{L^2(X_1\times X_2)}\right\}^{\frac{1}{2}}\\
&\le C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\left\{\frac{\ell(I)^\beta}{(2^{j-2}\ell(I)+\ell(I^*))^\beta}\,\ell(I)^{-4N}\,\|a_{R,2}\|^2_{L^2(X_1\times X_2)}\right\}^{\frac{1}{2}},\end{aligned}$$

where the second inequality follows from the Davies–Gaffney estimates, and the third inequality follows from the fact that $e^{-x} \le x^{-\beta}$ for all $x > 0$ and $\beta > 0$ and that we choose β satisfying $\beta > 4N$.

Moreover, noting that

$$\sum_{j>4}2^{jn_1/2}\frac{\ell(I)^{\beta/2}}{(2^{j-2}\ell(I)+\ell(I^*))^{\beta/2}} \le \left(\frac{\ell(I)}{\ell(I^*)}\right)^{n_1/2-\beta/2}, \tag{3.10}$$

we obtain that $D_1^{(a)}$ is bounded by the right-hand side of (3.8) for $\epsilon_1 := \beta/2 - n_1/2$.

Next we consider the term $D_2^{(a)}$. Similarly, we set

$$F_j(I) := \{y_1 : d_1(x_1,y_1) < d_1(x_1,x_I)/4 \text{ for some } x_1 \in (100I^*)^c \cap U_j(I)\}.$$

We see that $\mathrm{dist}(F_j(I), I) > 2^{j-3}\ell(I) + \ell(I^*)$. Now we have

$$\begin{aligned}D_2^{(a)} &\le C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\\
&\quad\times\left\{\int_{X_2}\int_{\ell(I)}^{\infty}\int_{F_j(I)}\left|(t_1^2L_1)^{N+1}e^{-t_1^2L_1}a_{R,2}(y_1,x_2)\right|^2\frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}}\,d\mu_2(x_2)\right\}^{\frac{1}{2}}\end{aligned}$$

$$\leq C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\left\{\int_{\ell(I)}^{\infty}e^{-(2^{j-3}\ell(I)+\ell(I^*))^2/(ct_1^2)}\frac{dt_1}{t_1^{1+4N}}\ \|a_{R,2}\|^2_{L^2(X_1\times X_2)}\right\}^{\frac{1}{2}}$$

$$\leq C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\left\{\frac{\ell(I)^{\beta}}{(2^{j-3}\ell(I)+\ell(I^*))^{\beta}}\ \ell(I)^{-4N}\ \|a_{R,2}\|^2_{L^2(X_1\times X_2)}\right\}^{\frac{1}{2}},$$

where the second inequality follows from the Davies–Gaffney estimates, and β is chosen to satisfy $n_1<\beta<4N$. Now using (3.10), we obtain that $D_2^{(a)}$ is bounded by the right-hand side of (3.8) for $\epsilon_1:=\beta/2-n_1/2$.

Now we turn to the term $D_3^{(a)}$. Since $x_1\in(100I^*)^c\cap U_j(I)$, we can see that $d(x_1,x_I)>2^{j-1}\ell(I)+\ell(I^*)$. Thus, the Davies–Gaffney estimates imply that

$$D_3^{(a)}\leq C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}$$

$$\times\left\{\int_{X_2}\int_{2^{j-1}\ell(I)+\ell(I^*)}^{\infty}\int_{X_1}\left|(t_1^2L_1)^{N+1}e^{-t_1^2L_1}a_{R,2}(y_1,x_2)\right|^2\frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}}\,d\mu_2(x_2)\right\}^{\frac{1}{2}}$$

$$\leq C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\left\{\int_{2^{j-1}\ell(I)+\ell(I^*)}^{\infty}\frac{dt_1}{t_1^{1+4N}}\ \|a_{R,2}\|^2_{L^2(X_1\times X_2)}\right\}^{\frac{1}{2}}$$

$$\leq C\sum_{j>4}2^{jn_1/2}\mu_1(I)^{\frac{1}{2}}\left\{\frac{\ell(I)^{4N}}{(2^{j-1}\ell(I)+\ell(I^*))^{4N}}\ \ell(I)^{-4N}\ \|a_{R,2}\|^2_{L^2(X_1\times X_2)}\right\}^{\frac{1}{2}},$$

Now using (3.10), we obtain that $D_3^{(a)}$ is bounded by the right-hand side of (3.8) for $\epsilon_1:=2N-n_1/2$.

Combining the estimates of $D_1^{(a)}$, $D_2^{(a)}$ and $D_3^{(a)}$, we obtain that the claim (3.8) holds for $\epsilon_1:=\beta/2-n_1/2$, and hence $D^{(a)}$ is uniformly bounded.

We now consider the term $D^{(b)}$. Similar to the estimates for the term $D^{(a)}$, we set $U_{j_1}(I)=2^{j_1}I\backslash 2^{j_1-1}I$ for $j_1\geq 1$ and $U_{j_2}(J)=2^{j_2}J\backslash 2^{j_2-1}J$ for $j_2\geq 1$. Then we have $(100I^*)^c\subset\cup_{j_1>4}U_{j_1}(I)$ and $(100J)^c\subset\cup_{j_2>4}U_{j_2}(J)$. Moreover, we have the following measure estimate for the annuli: $\mu_1(U_{j_1}(I))\leq C2^{j_1n_1}\mu_1(I)$ and $\mu_2(U_{j_2}(J))\leq C2^{j_2n_2}\mu_2(J)$. Now we have

$$D^{(b)}=\sum_{R\in m(\Omega)}\int_{(100I^*)^c}\int_{(100J)^c}|Sa_R(x_1,x_2)|\,d\mu_1(x_1)d\mu_2(x_2)$$

$$\leq\sum_{R\in m(\Omega)}\sum_{j_1>4}\sum_{j_2>4}\int_{(100I^*)^c\cap U_{j_1}(I)}\int_{(100S)^c\cap U_{j_2}(J)}|Sa_R(x_1,x_2)|\,d\mu_1(x_1)d\mu_2(x_2)$$

$$\leq C\sum_{R\in m(\Omega)}\mu(R)^{1/2}\sum_{j_1>4}\sum_{j_2>4}2^{j_1n_1/2}2^{j_2n_2/2}$$

$$\times\left(\int_{(100I^*)^c\cap U_{j_1}(I)}\int_{(100J)^c\cap U_{j_2}(J)}|Sa_R(x_1,x_2)|^2\,d\mu_1(x_1)d\mu_2(x_2)\right)^{\frac{1}{2}},\quad(3.11)$$

where the second inequality follows from Hölder's inequality.

We claim that

$$\sum_{j_1>4}\sum_{j_2>4} 2^{j_1 n_1/2} 2^{j_2 n_2/2}\left(\int_{(100I^*)^c\bigcap U_{j_1}(I)}\int_{(100J)^c\bigcap U_{j_2}(J)} |Sa_R(x_1,x_2)|^2\, d\mu_1(x_1)d\mu_2(x_2)\right)^{\frac{1}{2}}$$
$$\leq C\left(\frac{\ell(I)}{\ell(I^*)}\right)^{\epsilon_1}\left(\ell(I)^{-4N}\ell(J)^{-4N}\|b_R\|^2_{L^2(X_1\times X_2)}\right)^{1/2} \tag{3.12}$$

for some $\epsilon_1 > 0$, which, together with (3.11), implies that

$$\begin{aligned}
D^{(b)} &\leq C\sum_{R\in m(\Omega)}\mu(R)^{1/2}\left(\frac{\ell(I)}{\ell(I^*)}\right)^{\epsilon_1}\left(\ell(I)^{-4N}\ell(J)^{-4N}\|b_R\|_{L^2(X_1\times X_2)^2}\right)^{1/2}\\
&\leq C\left(\sum_{R\in m(\Omega)}\mu(R)\left(\frac{\ell(I)}{\ell(I^*)}\right)^{2\epsilon_1}\right)^{1/2}\left(\sum_{R\in m(\Omega)}\ell(I)^{-4N}\ell(J)^{-4N}\|b_R\|_{L^2(X_1\times X_2)^2}\right)^{1/2}\\
&\leq C\mu(\Omega)^{1/2}\mu(\Omega)^{-1/2}\\
&\leq C.
\end{aligned}$$

From the definitions of the area function Sf and the $(H^1_{L_1,L_2}, 2, N)$-atom a_R, we have

$$\begin{aligned}
&|Sa_R(x)|^2\\
&=\int_0^\infty\int_{d_1(x_1,y_1)<t_1}\int_0^\infty\int_{d_2(x_2,y_2)<t_2}\left|(t_1^2L_1)^{N+1}e^{-t_1^2L_1}\otimes(t_2^2L_2)^{N+1}e^{-t_2^2L_2}(b_R)(y_1,y_2)\right|^2\\
&\quad\times\frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}V(x_1,t_1)}\frac{d\mu_2(y_2)dt_2}{t_2^{1+4N}V(x_2,t_2)}.
\end{aligned}$$

Similarly to the estimate for the term $D^{(a)}$, we split the region of integration $(0,\infty)$ for t_1 into three parts $(0,\ell(I))$, $(\ell(I), d_1(x_1,x_I)/4)$ and $(d_1(x_1,x_I)/4,\infty)$, and the region of integration $(0,\infty)$ for t_2 into three parts $(0,\ell(J))$, $(\ell(J), d_2(x_2,x_J)/4)$ and $(d_2(x_2,x_J)/4,\infty)$. Hence $|Sa_R(x)|^2$ is decomposed into

$$\begin{aligned}
&|Sa_R(x)|^2\\
&=\Bigg(\int_0^{\ell(I)}\int_0^{\ell(J)}+\int_0^{\ell(I)}\int_{\ell(J)}^{\frac{d_2(x_2,x_J)}{4}}+\int_0^{\ell(I)}\int_{\frac{d_2(x_2,x_J)}{4}}^\infty+\int_{\ell(I)}^{\frac{d_1(x_1,x_I)}{4}}\int_0^{\ell(J)}+\int_{\ell(I)}^{\frac{d_1(x_1,x_I)}{4}}\int_{\ell(J)}^{\frac{d_2(x_2,x_J)}{4}}\\
&\quad+\int_{\ell(I)}^{\frac{d_1(x_1,x_I)}{4}}\int_{\frac{d_2(x_2,x_J)}{4}}^\infty+\int_{\frac{d_1(x_1,x_I)}{4}}^\infty\int_0^{\ell(J)}+\int_{\frac{d_1(x_1,x_I)}{4}}^\infty\int_{\ell(J)}^{\frac{d_2(x_2,x_J)}{4}}+\int_{\frac{d_1(x_1,x_I)}{4}}^\infty\int_{\frac{d_2(x_2,x_J)}{4}}^\infty\Bigg)\\
&\quad\times\int_{d_1(x_1,y_1)<t_1}\int_{d_2(x_2,y_2)<t_2}\left|(t_1^2L_1)^{N+1}e^{-t_1^2L_1}\otimes(t_2^2L_2)^{N+1}e^{-t_2^2L_2}(b_R)(y_1,y_2)\right|^2\\
&\quad\times\frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}V(x_1,t_1)}\frac{d\mu_2(y_2)dt_2}{t_2^{1+4N}V(x_2,t_2)}\\
&=: \mathbf{d}_{1,1}(x_1,x_2)+\mathbf{d}_{1,2}(x_1,x_2)+\mathbf{d}_{1,3}(x_1,x_2)+\mathbf{d}_{2,1}(x_1,x_2)+\mathbf{d}_{2,2}(x_1,x_2)\\
&\quad+\mathbf{d}_{2,3}(x_1,x_2)+\mathbf{d}_{3,1}(x_1,x_2)+\mathbf{d}_{3,2}(x_1,x_2)+\mathbf{d}_{3,3}(x_1,x_2)\\
&=\sum_{\iota=1}^3\sum_{\kappa=1}^3\mathbf{d}_{\iota,\kappa}(x_1,x_2).
\end{aligned}$$

Now for $\iota = 1, 2, 3$ and $\kappa = 1, 2, 3$ we set

$$D^{(b)}_{\iota,\kappa} := C \sum_{j_1>4} \sum_{j_2>4} 2^{\frac{j_1 n_1}{2}} 2^{\frac{j_2 n_2}{2}} \times \left(\int_{(100I^*)^c \bigcap U_{j_1}(I)} \int_{(100J)^c \bigcap U_{j_2}(J)} \mathbf{d}_{\iota,\kappa}(x_1, x_2)\, d\mu_1(x_1) d\mu_2(x_2) \right)^{\frac{1}{2}}.$$

We first consider $D^{(b)}_{1,1}$. Similar to the estimate in $D^{(a)}_1$, we define $E_{j_1}(I) := \{y_1 : d_1(x_1, y_1) < \ell(I)$ for some $x_1 \in (100I^*)^c \bigcap U_{j_1}(I)\}$ and $E_{j_2}(J) := \{y_2 : d_2(x_2, y_2) < \ell(J)$ for some $x_2 \in (100J)^c \bigcap U_{j_2}(J)\}$. Then we get $\mathrm{dist}(E_{j_1}(I), I) > 2^{j_1-2}\ell(I) + \ell(I^*)$ and $\mathrm{dist}(E_{j_2}(J), J) > 2^{j_1-2}\ell(J)$. Now we have

$$\begin{aligned}
&\int_{(100I^*)^c \bigcap U_{j_1}(I)} \int_{(100J)^c \bigcap U_{j_2}(J)} \mathbf{d}_{1,1}(x_1, x_2)\, d\mu_1(x_1) d\mu_2(x_2) \\
&= \int_0^{\ell(I)} \int_{E_{j_1}(I)} \int_0^{\ell(J)} \int_{E_{j_2}(J)} \left| (t_1^2 L_1)^{N+1} e^{-t_1^2 L_1} \otimes (t_2^2 L_2)^{N+1} e^{-t_2^2 L_2} (b_R)(y_1, y_2) \right|^2 \\
&\quad \times \frac{d\mu_1(y_1) dt_1}{t_1^{1+4N}} \frac{d\mu_2(y_2) dt_2}{t_2^{1+4N}} \\
&\le C \int_0^{\ell(I)} e^{-(2^{j_1-2}\ell(I)+\ell(I^*))^2/(ct_1^2)} \frac{dt_1}{t_1^{1+4N}} \int_0^{\ell(J)} e^{-(2^{j_2-2}\ell(J))^2/(ct_2^2)} \frac{dt_2}{t_2^{1+4N}} \, \|b_R\|^2_{L^2(X_1 \times X_2)} \\
&\le C \frac{\ell(I)^\beta}{(2^{j_1-2}\ell(I) + \ell(I^*))^\beta} \, \ell(I)^{-4N} 2^{-j_2\beta} \ell(J)^{-4N} \|b_R\|^2_{L^2(X_1 \times X_2)},
\end{aligned}$$

where the second inequality follows from the Davies–Gaffney estimates, and the third inequality follows from the fact that $e^{-x} \le x^{-\beta}$ for all $x > 0$ and $\beta > 0$ and that we choose β satisfying $\beta > 4N$.

Thus,

$$\begin{aligned}
D^{(b)}_{1,1} &\le C \sum_{j_1>4} 2^{\frac{j_1 n_1}{2}} \frac{\ell(I)^{\frac{\beta}{2}}}{(2^{j_1-2}\ell(I) + \ell(I^*))^{\frac{\beta}{2}}} \\
&\qquad \times \sum_{j_2>4} 2^{\frac{j_2 n_2}{2}} 2^{\frac{-j_2\beta}{2}} \left(\ell(I)^{-4N} \ell(J)^{-4N} \|b_R\|_{L^2(X_1 \times X_2)^2} \right)^{1/2} \\
&\le C \left(\frac{\ell(I)}{\ell(I^*)} \right)^{\epsilon_1} \left(\ell(I)^{-4N} \ell(J)^{-4N} \|b_R\|^2_{L^2(X_1 \times X_2)} \right)^{1/2},
\end{aligned}$$

where the second inequality follows from (3.10) with $\epsilon_1 := \beta/2 - n_1/2$. Note that $\beta > \max\{n_1, n_2\}$ follows from the fact that $N > \max\{n_1/4, n_2/4\}$.

As for $D^{(b)}_{1,2}$, similar to the term $D^{(a)}_2$, set $F_{j_2}(J) := \{y_2 : d_2(x_2, y_2) < d_2(x_2, x_J)/4$ for some $x_2 \in (100J)^c \bigcap U_{j_2}(J)\}$. Then we can see that $\mathrm{dist}(F_{j_2}(J), J) > 2^{j_2-3}\ell(J)$. Now we have

$$\begin{aligned}
&\int_{(100I^*)^c \bigcap U_{j_1}(I)} \int_{(100J)^c \bigcap U_{j_2}(J)} \mathbf{d}_{1,2}(x_1, x_2)\, d\mu_1(x_1) d\mu_2(x_2) \\
&= \int_0^{\ell(I)} \int_{E_{j_1}(I)} \int_{\ell(J)}^{\frac{d_2(x_2, x_J)}{4}} \int_{F_{j_2}(J)} \left| (t_1^2 L_1)^{N+1} e^{-t_1^2 L_1} \otimes (t_2^2 L_2)^{N+1} e^{-t_2^2 L_2} (b_R)(y_1, y_2) \right|^2
\end{aligned}$$

$$\times \frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}} \frac{d\mu_2(y_2)dt_2}{t_2^{1+4N}}$$

$$\leq C \int_0^{\ell(I)} e^{-(2^{j_1-2}\ell(I)+\ell(I^*))^2/(ct_1^2)} \frac{dt_1}{t_1^{1+4N}} \int_{\ell(J)}^{\infty} e^{-(2^{j_2-2}\ell(J))^2/(ct_2^2)} \frac{dt_2}{t_2^{1+4N}} \|b_R\|^2_{L^2(X_1\times X_2)}$$

$$\leq C \frac{\ell(I)^{\beta_1}}{(2^{j_1-2}\ell(I)+\ell(I^*))^{\beta_1}} \ell(I)^{-4N} 2^{-j_2\beta_2} \ell(J)^{-4N} \|b_R\|^2_{L^2(X_1\times X_2)},$$

where the second inequality follows from the Davies–Gaffney estimates, and the third inequality follows from the fact that $e^{-x} \leq x^{-\beta}$ for all $x > 0$ and $\beta > 0$, and that we choose β_1 satisfying $\beta_1 > 4N$ and β_2 satisfying $n_2 < \beta_2 < 4N$. Hence, similar to the estimate of the term $D^{(b)}_{1,1}$,

$$D^{(b)}_{1,2} \leq C \left(\frac{\ell(I)}{\ell(I^*)}\right)^{\epsilon_1} \left(\ell(I)^{-4N} \ell(J)^{-4N} \|b_R\|^2_{L^2(X_1\times X_2)}\right)^{1/2}$$

with $\epsilon_1 := \beta_1/2 - n_1/2$. Note that $\beta_1 > n_1$ follows from the fact that $N > n_1/4$.

As for $D^{(b)}_{1,3}$, since $x_2 \in (100J)^c \cap U_{j_2}(J)$, we see that $d_2(x_2, x_J) > 2^{j_2-1}\ell(J)$. Thus, the Davies–Gaffney estimates imply that

$$\int_{(100I^*)^c \bigcap U_{j_1}(I)} \int_{(100J)^c \bigcap U_{j_2}(J)} \mathbf{d}_{1,3}(x_1, x_2)\, d\mu_1(x_1) d\mu_2(x_2)$$

$$= \int_0^{\ell(I)} \int_{E_{j_1}(I)} \int_{2^{j_2-1}\ell(J)}^{\infty} \int_{X_2} \left|(t_1^2 L_1)^{N+1} e^{-t_1^2 L_1} \otimes (t_2^2 L_2)^{N+1} e^{-t_2^2 L_2}(b_R)(y_1, y_2)\right|^2$$

$$\times \frac{d\mu_1(y_1)dt_1}{t_1^{1+4N}} \frac{d\mu_2(y_2)dt_2}{t_2^{1+4N}}$$

$$\leq C \int_0^{\ell(I)} e^{-(2^{j_1-2}\ell(I)+\ell(I^*))^2/(ct_1^2)} \frac{dt_1}{t_1^{1+4N}} \int_{2^{j_2-1}\ell(J)}^{\infty} \frac{dt_2}{t_2^{1+4N}} \|b_R\|^2_{L^2(X_1\times X_2)}$$

$$\leq C \frac{\ell(I)^{\beta_1}}{(2^{j_1-2}\ell(I)+\ell(I^*))^{\beta_1}} \ell(I)^{-4N} 2^{-4Nj_2} \ell(J)^{-4N} \|b_R\|^2_{L^2(X_1\times X_2)},$$

in which we choose $\beta_1 > 4N$. Hence, we have

$$D^{(b)}_{1,3} \leq C \left(\frac{\ell(I)}{\ell(I^*)}\right)^{\epsilon_1} \left(\ell(I)^{-4N} \ell(J)^{-4N} \|b_R\|^2_{L^2(X_1\times X_2)}\right)^{1/2}$$

with $\epsilon_1 := \beta_1/2 - n_1/2$. Note that $\beta_1 > n_1$ follows from the fact that $N > n_1/4$.

For the remaining terms $D^{(b)}_{\iota,\kappa}$ for $\iota = 2, 3$ and $\kappa = 1, 2, 3$, we estimate the integral with respect to the first variable t_1 in a way similar to that for $D^{(a)}_{\iota}$ above, while for the integral with respect to t_2, we use an estimate similar to that used for the t_2 integral in $D^{(b)}_{1,\kappa}$ above. This completes the estimate of $D^{(b)}$, and hence that of D.

The estimate for the term E is symmetric to that of D.

Combining the estimates of D and E, we obtain (3.5), which, together with the fact that $\|Sa\|_{L^1(\cup R^*)} \leq C$, yields the estimate (3.2). Thus Lemma 3.2 is proved. □

This completes the proof of Step 1. □

Proof of Step 2 Our goal is to show that every function $f \in H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$ has an $(H^1_{L_1,L_2}, 2, M)$-atom representation, with appropriate quantitative control of the coefficients. To this end, we follow the standard tent space approach, and we are now ready to establish the atomic decomposition of $H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$.

Proposition 3.4 *Suppose $M \geq 1$. If $f \in H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$, then there exist a family of $(H^1_{L_1,L_2}, 2, M)$-atoms $\{a_j\}_{j=0}^{\infty}$ and a sequence of numbers $\{\lambda_j\}_{j=0}^{\infty} \in \ell^1$ such that f can be represented in the form $f = \sum \lambda_j a_j$, with the sum converging in $L^2(X_1 \times X_2)$, and*

$$\|f\|_{\mathbb{H}^1_{L_1,L_2,at,N}(X_1\times X_2)} \leq C \sum_{j=0}^{\infty} |\lambda_j| \leq C\|f\|_{H_{L_1,L_2}(X_1\times X_2)},$$

where C is independent of f. In particular,

$$H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2) \subset \mathbb{H}^1_{L_1,L_2,at,M}(X_1 \times X_2).$$

Proof Let $f \in H^1_{L_1,L_2}(X_1 \times X_2) \cap L^2(X_1 \times X_2)$. For each $\ell \in \mathbb{Z}$, define

$$\Omega_\ell := \{(x_1, x_2) \in X_1 \times X_2 : Sf > 2^\ell\},$$
$$B_\ell := \left\{R = I^{k_1}_{\alpha_1} \times I^{k_2}_{\alpha_2} : \mu(R \cap \Omega_\ell) > \frac{1}{2A_0}\mu(R),\ \mu(R \cap \Omega_{\ell+1}) \leq \frac{1}{2A_0}\mu(R)\right\}, \text{ and}$$
$$\widetilde{\Omega}_\ell := \left\{(x_1, x_2) \in X_1 \times X_2 : \mathcal{M}_s(\chi_{\Omega_\ell}) > \frac{1}{2A_0}\right\},$$

where $\mathcal{M}_s$ is the strong maximal function on $X_1 \times X_2$.

For each rectangle $R = I^{k_1}_{\alpha_1} \times I^{k_2}_{\alpha_2}$ in $X_1 \times X_2$, the *tent* $T(R)$ is defined as

$$T(R) := \left\{(y_1, y_2, t_1, t_2) :\ (y_1, y_2) \in R, t_1 \in (2^{-k_1}, 2^{-k_1+1}], t_2 \in (2^{-k_2}, 2^{-k_2+1}]\right\}.$$

For brevity, in what follows we will write $\chi_{T(R)}$ for $\chi_{T(R)}(y_1, y_2, t_1, t_2)$.

Using the reproducing formula, we can write

$$\begin{aligned}
f(x_1, x_2) &= \int_0^\infty \int_0^\infty \psi(t_1\sqrt{L_1})\psi(t_2\sqrt{L_2})(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(x_1, x_2)\frac{dt_1 dt_2}{t_1 t_2}\\
&= \int_0^\infty \int_0^\infty \int_{X_1}\int_{X_2} K_{\psi(t_1\sqrt{L_1})}(x_1, y_1)K_{\psi(t_2\sqrt{L_2})}(x_2, y_2)\\
&\quad \times (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2)d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1 dt_2}{t_1 t_2}\\
&= \sum_{\ell\in\mathbb{Z}}\sum_{R\in B_\ell}\int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1, y_1)K_{\psi(t_2\sqrt{L_2})}(x_2, y_2)\\
&\quad \times (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2)d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1 dt_2}{t_1 t_2}\\
&=: \sum_{\ell\in\mathbb{Z}} \lambda_\ell a_\ell(x_1, x_2).
\end{aligned} \tag{3.13}$$

Here the coefficients λ_ℓ are defined by

$$\lambda_\ell := C\left\|\left(\sum_{R\in B_\ell}\int_0^\infty\int_0^\infty \left|(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2)\right|^2 \chi_{T(R)}\frac{dt_1 dt_2}{t_1 t_2}\right)^{1/2}\right\|_{L^2} \mu(\widetilde{\Omega}_\ell)^{1/2},$$

Also the functions $a_\ell(x_1, x_2)$ are defined by

$$a_\ell(x_1, x_2) := \frac{1}{\lambda_\ell} \sum_{R\in B_\ell} \int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1, y_1) K_{\psi(t_2\sqrt{L_2})}(x_2, y_2) \\ \times (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2) \frac{dt_1 dt_2}{t_1 t_2}.$$

First, it is easy to verify property (1) in Definition 2.7, since from Lemma 2.5 and the definition of the sets B_ℓ and $\widetilde{\Omega}_\ell$, we obtain that $a_\ell(x_1, x_2)$ is supported in $\widetilde{\Omega}_\ell$.

Next, we can further write

$$a_\ell(x_1, x_2) = \sum_{\overline{R}\in m(\widetilde{\Omega}_\ell)} a_{\overline{R}}(x_1, x_2),$$

where

$$a_{\overline{R}} := \sum_{R\in B_\ell, R\subset \overline{R}} \frac{1}{\lambda_\ell} \int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1, y_1) K_{\psi(t_2\sqrt{L_2})}(x_2, y_2) \\ \times (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2)\, d\mu_1(y_1) d\mu_2(y_2) \frac{dt_1 dt_2}{t_1 t_2}.$$

Then property (i) of (2) in Definition 2.7 holds, since $a_{\overline{R}}$ can be further written as

$$a_{\overline{R}} = (L_1^N \otimes L_2^N) b_{\overline{\mathrm{R}}},$$

where

$$b_{\overline{R}} := \sum_{R\in B_\ell, R\subset \overline{R}} \frac{1}{\lambda_\ell} \int_{T(R)} t_1^{2N} t_2^{2N} K_{\phi(t_1\sqrt{L_1})}(x_1, y_1) K_{\phi(t_2\sqrt{L_2})}(x_2, y_2) \\ \times (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2)\, d\mu_1(y_1) d\mu_2(y_2) \frac{dt_1 dt_2}{t_1 t_2}.$$

Next, from Lemma 2.5, we obtain that property (ii) of (2) in Definition 2.7 holds.

We now verify property (iii) of (2). To do so, we write

$$\|a_\ell\|_{L^2(X_1\times X_2)} = \sup_{h:\|h\|_{L^2(X_1\times X_2)}=1} |\langle a_\ell, h\rangle|.$$

Then from the definition of a_ℓ, we have

$$|\langle a_\ell, h\rangle| \\ = \bigg| \int_{X_1\times X_2} \frac{1}{\lambda_\ell} \sum_{R\in B_\ell} \int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1, y_1) K_{\psi(t_2\sqrt{L_2})}(x_2, y_2) \\ \times (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2) d\mu_1(y_1) d\mu_2(y_2) \frac{dt_1 dt_2}{t_1 t_2}\, h(x_1, x_2) d\mu_1(x_1) d\mu_2(x_2) \bigg| \\ \le \frac{1}{\lambda_\ell} \sum_{R\in B_\ell} \int_{T(R)} |\psi(t_1\sqrt{L_1})\psi(t_2\sqrt{L_2})(h)(y_1, y_2)| \\ \times \big| (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1, y_2) \big| d\mu_1(y_1) d\mu_2(y_2) \frac{dt_1 dt_2}{t_1 t_2} \\ \le \frac{1}{\lambda_\ell} \int_{X_1\times X_2} \bigg(\sum_{R\in B_\ell} \int_0^\infty \int_0^\infty |\psi(t_1\sqrt{L_1})\psi(t_2\sqrt{L_2})(h)(y_1, y_2)|^2 \chi_{T(R)} \frac{dt_1 dt_2}{t_1 t_2} \bigg)^{1/2}$$

$$\times\Big(\sum_{R\in B_\ell}\int_0^\infty\int_0^\infty\big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|^2\chi_{T(R)}\frac{dt_1dt_2}{t_1t_2}\Big)^{1/2}d\mu_1(y_1)d\mu_2(y_2)$$
$$\le\frac{C}{\lambda_\ell}\|h\|_{L^2}\Big\|\Big(\sum_{R\in B_\ell}\int_0^\infty\int_0^\infty\big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|^2\chi_{T(R)}\frac{dt_1dt_2}{t_1t_2}\Big)^{1/2}\Big\|_{L^2}$$
$$\le\mu(\widetilde\Omega_\ell)^{-1/2}.$$

In the last inequality, we have used the definition of λ_ℓ.

Similarly, from the definition of the function $b_{\overline{R}}$, we have for each $\sigma_1,\sigma_2\in\{0,1,\dots,N\}$ that

$$\ell(\overline{I})^{-2N}\ell(\overline{J})^{-2N}\|(\ell(\overline{I})^2L_1)^{\sigma_1}\otimes(\ell(\overline{J})^2L_2)^{\sigma_2}b_{\overline{R}}\|_{L^2}$$
$$=\sup_{h:\|h\|_{L^2}=1}\big|\langle\ell(\overline{I})^{-2N}\ell(\overline{J})^{-2N}(\ell(\overline{I})^2L_1)^{\sigma_1}\otimes(\ell(\overline{J})^2L_2)^{\sigma_2}b_{\overline{R}},h\rangle\big|$$
$$\le\sup_{h:\|h\|_{L^2}=1}\frac{C}{\lambda_\ell}\sum_{R\in B_\ell,R\subset\overline{R}}\int_{T(R)}|(\ell(\overline{I})^2L_1)^{\sigma_1}\phi(t_1\sqrt{L_1})\otimes(\ell(\overline{J})^2L_2)^{\sigma_2}\phi(t_2\sqrt{L_2})(h)(y_1,y_2)|$$
$$\times\big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|\,d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}.$$

As a consequence, using the same approach as in the above estimates for a_ℓ, we have

$$\sum_{\overline{R}\in m(\widetilde\Omega_\ell)}\ell(\overline{I})^{-4N}\ell(\overline{J})^{-4N}\|(\ell(\overline{I})^2L_1)^{\sigma_1}\otimes(\ell(\overline{J})^2L_2)^{\sigma_2}b_{\overline{R}}\|_{L^2}^2$$
$$\le\sup_{h:\|h\|_{L^2}=1}\frac{C}{\lambda_\ell^2}\sum_{\overline{R}\in m(\widetilde\Omega_\ell)}\Bigg(\sum_{R\in B_\ell,R\subset\overline{R}}\int_{T(R)}$$
$$\times\ |(\ell(\overline{I})^2L_1)^{\sigma_1}\phi(t_1\sqrt{L_1})\otimes(\ell(\overline{J})^2L_2)^{\sigma_2}\phi(t_2\sqrt{L_2})(h)(y_1,y_2)|$$
$$\times\ \big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|\,d\mu_1(y_1)d\mu_2(y_2)\ \frac{dt_1dt_2}{t_1t_2}\Bigg)^2$$
$$\le\frac{C}{\lambda_\ell^2}\Bigg\|\Bigg(\sum_{R\in B_\ell}\int_0^\infty\int_0^\infty\big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|^2\chi_{T(R)}\ \frac{dt_1dt_2}{t_1t_2}\Bigg)^{1/2}\Bigg\|_{L^2}^2$$
$$\le\mu(\widetilde\Omega_\ell)^{-1}.$$

The last inequality follows from the definition of λ_ℓ.

Combining the above estimate and the estimate for a_ℓ, we have established property (iii) of (2) in Definition 2.7. Thus, each a_ℓ is an $(H^1_{L_1,L_2},2,N)$-atom.

To see that the atomic decomposition $\sum_\ell\lambda_\ell a_\ell$ converges to f in the $L^2(X_1\times X_2)$ norm, we only need to show that $\|\sum_{|\ell|>G}\lambda_\ell a_\ell\|_{L^2(X_1\times X_2)}\to0$ as G tends to infinity. To see this, first note that

$$\Big\|\sum_{|\ell|>G}\lambda_\ell a_\ell\Big\|_{L^2(X_1\times X_2)}=\sup_{h:\,\|h\|_{L^2(X_1\times X_2)=1}}\Big|\Big\langle\sum_{|\ell|>G}\lambda_\ell a_\ell,h\Big\rangle\Big|.$$

Next, we have

$$
\begin{aligned}
&\Big|\langle \sum_{|\ell|>G} \lambda_\ell a_\ell, h\rangle\Big| \\
&= \Big| \int_{X_1\times X_2} \sum_{|\ell|>G} \sum_{R\in B_\ell} \int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1,y_1) K_{\psi(t_2\sqrt{L_2})}(x_2,y_2) \\
&\quad \times (t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1,y_2)\, d\mu_1(y_1) d\mu_2(y_2) \frac{dt_1 dt_2}{t_1 t_2}\, h(x_1,x_2) d\mu_1(x_1) d\mu_2(x_2)\Big| \\
&\le \int_{X_1\times X_2} \Bigg(\sum_{|\ell|>G} \sum_{R\in B_\ell} \int_0^\infty \int_0^\infty |\psi(t_1\sqrt{L_1})\psi(t_2\sqrt{L_2})(h)(y_1,y_2)|^2 \chi_{T(R)} \frac{dt_1 dt_2}{t_1 t_2} \Bigg)^{\frac12} \\
&\quad \times \Bigg(\sum_{|\ell|>G} \sum_{R\in B_\ell} \int_0^\infty \int_0^\infty \big|(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1,y_2)\big|^2 \chi_{T(R)} \frac{dt_1 dt_2}{t_1 t_2} \Bigg)^{\frac12} d\mu_1(y_1) d\mu_2(y_2) \\
&\le C\|h\|_{L^2} \Bigg\| \Bigg(\sum_{|\ell|>G} \sum_{R\in B_\ell} \int_0^\infty \int_0^\infty \big|(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1,y_2)\big|^2 \chi_{T(R)} \frac{dt_1 dt_2}{t_1 t_2} \Bigg)^{\frac12} \Bigg\|_{L^2} \\
&\to 0
\end{aligned}
$$

as G tends to ∞, since $\|Sf\|_2 < \infty$.

This implies that $f = \sum_\ell \lambda_\ell a_\ell$ in the sense of $L^2(X_1 \times X_2)$.

Next, we verify the estimate for the series $\sum_\ell |\lambda_\ell|$. To deal with this, we claim that for each $\ell \in \mathbb{Z}$,

$$
\begin{aligned}
&\sum_{R\in B_\ell} \int_{T(R)} \big|(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2})(f)(y_1,y_2)\big|^2 d\mu_1(y_1) d\mu_2(y_2) \frac{dt_1 dt_2}{t_1 t_2} \\
&\quad \le C 2^{2(\ell+1)} \mu(\widetilde{\Omega}_\ell).
\end{aligned}
$$

First we note that

$$
\int_{\widetilde{\Omega}_\ell \backslash \Omega_{\ell+1}} (Sf)^2(x_1,x_2)\, d\mu_1(x_1) d\mu_2(x_2) \le 2^{2(\ell+1)} \mu(\widetilde{\Omega}_\ell).
$$

Also we point out that

$$
\begin{aligned}
&\int_{\widetilde{\Omega}_\ell \backslash \Omega_{\ell+1}} (Sf)^2(x_1,x_2)\, d\mu_1(x_1) d\mu_2(x_2) \\
&= \int_{\widetilde{\Omega}_\ell \backslash \Omega_{\ell+1}} \int_{\Gamma_1(x_1)} \int_{\Gamma_2(x_2)} \\
&\quad \times \big|\big(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2}\big) f(y_1,y_2)\big|^2 \frac{d\mu_1(y_1) d\mu_2(y_2)\, dt_1 dt_2}{t_1 V(x_1,t_1) t_2 V(x_2,t_2)}\, d\mu_1(x_1) d\mu_2(x_2) \\
&= \int_0^\infty \int_0^\infty \int_{X_1\times X_2} \big|\big(t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2}\big)(f)(y_1,y_2)\big|^2 \\
&\quad \times \mu(\{(x_1,x_2) \in \widetilde{\Omega}_\ell \backslash \Omega_{\ell+1} : d_1(x_1,y_1) < t_1, d_2(x_2,y_2) < t_2\}) \frac{d\mu_1(y_1) d\mu_2(y_2)\, dt_1 dt_2}{t_1 V(x_1,t_1) t_2 V(x_2,t_2)}
\end{aligned}
$$

$$\geq \sum_{R\in B_\ell}\int_{T(R)} \left|\left(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2}\right)(f)(y_1,y_2)\right|^2$$

$$\times\mu(\{(x_1,x_2)\in\widetilde{\Omega}_\ell\backslash\Omega_{\ell+1}: d_1(x_1,y_1)<t_1, d_2(x_2,y_2)<t_2\})\frac{d\mu_1(y_1)d\mu_2(y_2)\,dt_1dt_2}{t_1V(x_1,t_1)t_2V(x_2,t_2)}$$

$$\geq C\sum_{R\in B_\ell}\int_{T(R)} \left|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\right|^2 d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2},$$

where the last inequality follows from the definition of B_ℓ. This shows that the claim holds.

As a consequence, we have

$$\begin{aligned}
&\sum_\ell|\lambda_\ell|\\
&\leq C\sum_\ell\left\|\left(\sum_{R\in B_\ell}\int_0^\infty\int_0^\infty |(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)|^2\chi_{T(R)}\frac{dt_1dt_2}{t_1t_2}\right)^{1/2}\right\|_{L^2}\mu(\widetilde{\Omega}_\ell)^{1/2}\\
&\leq C\sum_\ell\left(\sum_{R\in B_\ell}\int_{T(R)}|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)|^2 d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}\right)^{1/2}\mu(\widetilde{\Omega}_\ell)^{1/2}\\
&\leq C\sum_\ell 2^{\ell+1}\mu(\widetilde{\Omega}_\ell)\leq C\sum_\ell 2^\ell\mu(\Omega_\ell)\\
&\leq C\|Sf\|_{L^1(X_1\times X_2)}\\
&= C\|f\|_{H^1_{L_1,L_2}(X_1\times X_2)}.
\end{aligned}$$

Therefore,

$$\|f\|_{\mathbb{H}^1_{L_1,L_2,at,N}(X_1\times X_2)}\leq C\|f\|_{H^1_{L_1,L_2}(X_1\times X_2)},$$

which completes the proof of Proposition 3.4. □

Step 2 is now complete. This concludes the proof of Theorem 2.9. □

4 Calderón–Zygmund decomposition and interpolation on $H^p_{L_1,L_2}(X_1\times X_2)$

In this section, we provide the proofs of the Calderón–Zygmund decomposition (Theorem 2.11) and the interpolation theorem (Theorem 2.12) on the Hardy spaces $H^p_{L_1\times L_2}(X_1\times X_2)$.

Proof of Theorem 2.11 By density, we may assume that $f\in H^p_{L_1,L_2}(X_1\times X_2)\cap H^2(X_1\times X_2)$. Let $\alpha>0$ and set $\Omega_\ell:=\{(x_1,x_2)\in X_1\times X_2: Sf(x_1,x_2)>\alpha 2^\ell\}$, $\ell\geq 0$. Set

$$B_0:=\left\{R=I^{k_1}_{\alpha_1}\times I^{k_1}_{\alpha_1}: \mu(R\cap\Omega_0)<\frac{1}{2A_0}\mu(R)\right\}$$

and

$$B_\ell:=\left\{R=I^{k_1}_{\alpha_1}\times I^{k_1}_{\alpha_1}: \mu(R\cap\Omega_{\ell-1})\geq\frac{1}{2A_0}\mu(R), \mu(R\cap\Omega_\ell)<\frac{1}{2A_0}\mu(R)\right\}$$

for $\ell\geq 1$.

By using the reproducing formula and the decomposition (3.13) as in the proof of Proposition 3.4, we have

$$
\begin{aligned}
f(x_1,x_2) &= \sum_{\ell\in\mathbb{Z}}\sum_{R\in B_\ell}\int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1,y_1)K_{\psi(t_2\sqrt{L_2})}(x_2,y_2)\\
&\quad\times(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\,d\mu_1(y_1)d\mu_2(y_2)\,\frac{dt_1dt_2}{t_1t_2}\\
&= g(x_1,x_2)+b(x_1,x_2),
\end{aligned}
$$

where

$$
\begin{aligned}
g(x_1,x_2) := \sum_{R\in B_0}\int_{T(R)} &K_{\psi(t_1\sqrt{L_1})}(x_1,y_1)K_{\psi(t_2\sqrt{L_2})}(x_2,y_2)\\
&\times(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}
\end{aligned}
$$

and

$$
\begin{aligned}
b(x_1,x_2) := \sum_{\ell>1}\sum_{R\in B_\ell}\int_{T(R)} &K_{\psi(t_1\sqrt{L_1})}(x_1,y_1)K_{\psi(t_2\sqrt{L_2})}(x_2,y_2)\\
&\times(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}.
\end{aligned}
$$

As for g, by writing $\|g\|_{L^2(X_1\times X_2)}=\sup_{h:\,\|h\|_{L^2}=1}|\langle g,h\rangle|$, and noting that

$$
\begin{aligned}
|\langle g,h\rangle| &= \Big|\sum_{R\in B_0}\int_{T(R)}\psi(t_1\sqrt{L_1})\psi(t_2\sqrt{L_2})(h)(y_1,y_2)\\
&\quad\times(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}\Big|\\
&\le C\|h\|_{L^2}\Big(\sum_{R\in B_0}\int_{T(R)}\big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|^2d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}\Big)^{1/2},
\end{aligned}
$$

we have

$$
\|g\|_{L^2}\le C\Big(\sum_{R\in B_0}\int_{T(R)}\big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|^2d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}\Big)^{1/2}.
$$

Also note that

$$
\begin{aligned}
&\int_{Sf(x_1,x_2)\le\alpha} Sf(x_1,x_2)^2d\mu_1(x_1)d\mu_2(x_2)\\
&= \int_{\Omega_0^c}\int_{\Gamma_1(x_1)}\int_{\Gamma_2(x_2)}\big|\big(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2}\big)(f)(y_1,y_2)\big|^2\\
&\qquad\times\frac{d\mu_1(y_1)d\mu_2(y_2)\,dt_1dt_2}{t_1V(x_1,t_1)t_2V(x_2,t_2)}d\mu_1(x_1)d\mu_2(x_2)\\
&= \int_0^\infty\int_0^\infty\int_{X_1\times X_2}\big|\big(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2}\big)(f)(y_1,y_2)\big|^2\\
&\qquad\times\mu(\{(x_1,x_2)\in\Omega_0^c:\,d_1(x_1,y_1)<t_1,d_2(x_2,y_2)<t_2\})\,\frac{d\mu_1(y_1)d\mu_2(y_2)dt_1dt_2}{t_1V(x_1,t_1)t_2V(x_2,t_2)}\\
&\ge C\sum_{R\in B_\ell}\int_{T(R)}\big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|^2\,d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}.
\end{aligned}
$$

As a consequence, we have

$$\|g\|_{L^2}^2 \le C \int_{Sf(x_1,x_2)\le\alpha} Sf(x_1,x_2)^2 d\mu_1(x_1)d\mu_2(x_2).$$

It remains to estimate $\|b\|_{H^1_{L_1,L_2}(X_1\times X_2)}$. From the definition of the function $b(x_1,x_2)$, we have

$$\begin{aligned}
&\|b\|_{H^1_{L_1,L_2}(X_1\times X_2)}\\
&\le \sum_{\ell\ge 1}\Big\| \sum_{R\in B_\ell}\int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1,y_1)K_{\psi(t_2\sqrt{L_2})}(x_2,y_2)\\
&\qquad \times(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\,d\mu_1(y_1)d\mu_2(y_2)\,\frac{dt_1dt_2}{t_1t_2}\Big\|_{H^1_{L_1,L_2}(X_1\times X_2)}.
\end{aligned}$$

From the proof of Proposition 3.4, we see that, for $\ell\ge 1$,

$$\begin{aligned}
&\frac{1}{\lambda_\ell}\sum_{R\in B_\ell}\int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1,y_1)K_{\psi(t_2\sqrt{L_2})}(x_2,y_2)\\
&\qquad\times(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}
\end{aligned}$$

is an $(H^1_{L_1,L_2},2,N)$-atom, which we denote it by a_ℓ, where λ_ℓ is the coefficient of a_ℓ defined by

$$\lambda_\ell := C\Big\|\Big(\sum_{R\in B_\ell}\int_0^\infty\int_0^\infty \big|(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)\big|^2\chi_{T(R)}\frac{dt_1dt_2}{t_1t_2}\Big)^{1/2}\Big\|_{L^2}\mu(\widetilde{\Omega}_\ell)^{1/2}.$$

Here we point out that the support of a_ℓ is $\widetilde{\Omega} := \{(x_1,x_2)\in X_1\times X_2 : \mathcal{M}_s(\chi_\Omega)(x_1,x_2) > 1/(2A_0)\}$, where $\Omega_\ell = \{(x_1,x_2)\in X_1\times X_2 : Sf(x_1,x_2) > \alpha 2^\ell\}$. Hence, following the same argument in the proof of Proposition 3.4, we obtain that

$$|\lambda_\ell| \le C\alpha 2^\ell \mu(\Omega_\ell).$$

Moreover, Lemma 3.2 implies that $\|a_\ell\|_{H^1_{L_1,L_2}(X_1\times X_2)} \le C$, where C is a positive constant independent of a_ℓ.

As a consequence, we have

$$\begin{aligned}
&\|b\|_{H^1_{L_1,L_2}(X_1\times X_2)}\\
&\le \sum_{\ell\ge 1}|\lambda_\ell|\Big\|\frac{1}{\lambda_\ell}\sum_{R\in B_\ell}\int_{T(R)} K_{\psi(t_1\sqrt{L_1})}(x_1,y_1)K_{\psi(t_2\sqrt{L_2})}(x_2,y_2)\\
&\qquad\times(t_1^2L_1e^{-t_1^2L_1}\otimes t_2^2L_2e^{-t_2^2L_2})(f)(y_1,y_2)d\mu_1(y_1)d\mu_2(y_2)\frac{dt_1dt_2}{t_1t_2}\Big\|_{H^1_{L_1,L_2}(X_1,X_2)}\\
&\le C\sum_{\ell\ge 1}\alpha 2^\ell\mu(\Omega_\ell)\\
&\le C\int_{Sf(x_1,x_2)>\alpha} Sf(x_1,x_2)d\mu_1(x_1)d\mu_2(x_2)
\end{aligned}$$

$$\leq C\alpha^{1-p}\int_{Sf(x_1,x_2)>\alpha} Sf(x_1,x_2)^p d\mu_1(x_1)d\mu_2(x_2)$$
$$\leq C\alpha^{1-p}\|f\|_{H^p_{L_1,L_2}(X_1,X_2)}.$$

□

We are now ready to prove Theorem 2.12.

Proof of Theorem 2.12 Suppose that T is bounded from $H^1_{L_1,L_2}(X_1\times X_2)$ to $L^1(X_1\times X_2)$ and from $H^2_{L_1,L_2}(X_1\times X_2)$ to $L^2(X_1\times X_2)$. For any given $\lambda>0$ and $f\in H^p_{L_1,L_2}(X_1\times X_2)$, by the Calderón–Zygmund decomposition,

$$f(x_1,x_2)=g(x_1,x_2)+b(x_1,x_2)$$

with

$$\|g\|^2_{H^2_{L_1,L_2}(X_1\times X_2)}\leq C\lambda^{2-p}\|f\|^p_{H^p_{L_1,L_2}(X_1\times X_2)} \quad \text{and}$$
$$\|b\|_{H^1_{L_1,L_2}(X_1\times X_2)}\leq C\lambda^{1-p}\|f\|^p_{H^p_{L_1,L_2}(X_1\times X_2)}.$$

Moreover, we have already proved the estimates

$$\|g\|^2_{H^2_{L_1,L_2}(X_1\times X_2)}\leq C\int_{Sf(x_1,x_2)\leq\alpha} Sf(x_1,x_2)^2\, d\mu_1(x_1)d\mu_2(x_2)$$

and

$$\|b\|^1_{H^1_{L_1,L_2}(X_1\times X_2)}\leq C\int_{Sf(x_1,x_2)>\alpha} Sf(x_1,x_2)\, d\mu_1(x_1)d\mu_2(x_2),$$

which imply that

$$\begin{aligned}\|Tf\|^p_{L^p(X_1\times X_2)} &= p\int_0^\infty \alpha^{p-1}\mu(\{(x_1,x_2):|Tf(x_1,x_2)|>\alpha\})d\alpha\\
&\leq p\int_0^\infty \alpha^{p-1}\mu(\{(x_1,x_2):|Tg(x_1,x_2)|>\alpha/2\})d\alpha\\
&\quad + p\int_0^\infty \alpha^{p-1}\mu(\{(x_1,x_2):|Tb(x_1,x_2)|>\alpha/2\})d\alpha\\
&\leq p\int_0^\infty \alpha^{p-2-1}\int_{Sf(x_1,x_2)\leq\alpha} Sf(x_1,x_2)^2\, d\mu_1(x_1)d\mu_2(x_2)d\alpha\\
&\quad + p\int_0^\infty \alpha^{p-1-1}\int_{Sf(x_1,x_2)>\alpha} Sf(x_1,x_2)\, d\mu_1(x_1)d\mu_2(x_2)d\alpha\\
&\leq C\|f\|^p_{H^p_{L_1,L_2}(X_1\times X_2)}\end{aligned}$$

for any $1<p<2$. Hence, T is bounded from $H^p_{L_1,L_2}(X_1\times X_2)$ to $L^p(X_1\times X_2)$. □

5 The relationship between $H^p_{L_1,L_2}(X_1\times X_2)$ and $L^p(X_1\times X_2)$

Before proving our main result Theorem 2.13, we point out that Theorem 2.13 is an extension of Theorem 4.19 in Uhl's PhD thesis [44, Section 4.4]. In Theorem 4.19 ([44, Section 4.4]), to obtain the coincidence of the Hardy space and the Lebesgue space, Uhl assumed that L is an injective operator on $L^2(X)$. Here we note that if L satisfies the *generalized Gaussian estimates* (GGE_{p_0}) for some $1 \le p_0 < 2$, then L is injective. This result seems new and leads to the fact that $H^2(X_1 \times X_2) = L^2(X_1 \times X_2)$ (see the proof of Theorem 2.13 in this section).

Theorem 5.1 *If L satisfies the generalized Gaussian estimates* (GGE_{p_0}) *for some p_0 with $1 \le p_0 < 2$, then the operator L is injective on $L^2(X)$.*

Proof Take $\phi \in L^2(X)$ with $L\phi = 0$. From the functional calculus,

$$e^{-tL} - I = \int_0^t \frac{\partial}{\partial s} e^{-sL} ds = -\int_0^t Le^{-sL} ds.$$

Then we have

$$(e^{-tL} - I)(\phi) = -\int_0^t Le^{-sL} ds(\phi) = 0,$$

which implies that

$$\phi = e^{-tL}\phi \tag{5.1}$$

holds for all $t > 0$. Note that (5.1) is proved in [31, p. 9].

Next, as shown in Lemma 2.6 of [44], the *generalized Gaussian estimates* (GGE_{p_0}) imply the following $L^2 \to L^{p_0'}$ off-diagonal estimates:

$$\|P_{B(x,\sqrt{t})} e^{-tL} P_{C_j(x,\sqrt{t})}\|_{2\to p_0'} \le CV(x, \sqrt{t})^{-(1/2-1/p_0')} e^{-c4^j}, \tag{5.2}$$

where $C_j(x,r) := B(x, 2^j r)\backslash B(x, 2^{j-1}r)$ for $j \ge 1$ and $C_0(x,r) = B(x,r)$.

As a consequence of Fatou's lemma, (5.1) and (5.2), we have that

$$\begin{aligned}
\|\phi\|_{p_0'} &\le \lim_{t\to\infty} \|P_{B(x,\sqrt{t})}\phi\|_{p_0'} = \lim_{t\to\infty} \|P_{B(x,\sqrt{t})} e^{-tL}\phi\|_{p_0'} \\
&\le \lim_{t\to\infty} \sum_{j=0}^{\infty} \|P_{B(x,\sqrt{t})} e^{-tL} P_{C_j(x,\sqrt{t})}\phi\|_{p_0'} \\
&\le \lim_{t\to\infty} \sum_{j=0}^{\infty} CV(x,\sqrt{t})^{-(1/2-1/p_0')} e^{-c4^j} \|\phi\|_2 \\
&\le \lim_{t\to\infty} CV(x,\sqrt{t})^{1/p_0'-1/2} \|\phi\|_2 \\
&= 0.
\end{aligned}$$

Here in the final step we have used the fact that $\mu(X) = \infty$. Thus, we obtain that $\phi = 0$ a.e. This completes the proof of Theorem 5.1. □

Next, we give a vector-valued version of a theorem about the area function associated with an operator L in the one-parameter setting.

Suppose L is a non-negative self-adjoint operator defined on $L^2(X; H)$, where H is a Hilbert space with a norm $|\cdot|_H$. Moreover, assume that L satisfies the *generalized Gaussian estimates* (GGE_{p_0}) for some p_0 with $1 \le p_0 < 2$.

We now define an area function $S_H : L^2(X; H) \to L^2(X)$ associated with L by

$$S_H f(x) := \left(\int_{\Gamma(x)} \left|\left(t^2 L e^{-t^2 L}\right) f(y)\right|_H^2 \frac{d\mu(y)\, dt}{t V(x,t)} \right)^{1/2}.$$

Then we prove the following boundedness result for S_H.

Theorem 5.2 *Suppose that L is a non-negative self-adjoint operator defined on $L^2(X; H)$ satisfying the generalized Gaussian estimates* (GGE$_{p_0}$) *for some $p_0 \in [1, 2)$. Then there exists a positive constant C such that*

$$\|S_H f\|_{L^p(X)} \le C \|f\|_{L^p(X;H)} \tag{5.3}$$

for all $p \in (p_0, p_0')$ and all $f \in L^p(X; H) \cap L^2(X; H)$.

Proof This boundedness result (5.3) is a vector-valued version of the result (4.15) in Uhl's PhD thesis [44, Section4.4]. We restate Uhl's proof in our vector-valued setting.

Step I. We first prove that $\|S_H f\|_{L^p(X)} \le C\|f\|_{L^p(X;H)}$ for $p_0 < p \le 2$.

To see this, we define

$$g^*_{\lambda,H} f(x) := \left(\int_0^\infty \int_X \left(\frac{t}{d(x,y)+t}\right)^{n\lambda} \left|\left(t^2 L e^{-t^2 L}\right) f(y)\right|_H^2 \frac{d\mu(y) dt}{t V(x,t)} \right)^{1/2},$$

where n is the upper dimension of the doubling measure μ. Then it is easy to see that $\|S_H f\|_{L^p(X)} \le C \|g^*_{\lambda,H} f\|_{L^p(X)}$ for each $\lambda > 1$. Thus, it suffices to prove that for each for p with $p_0 < p \le 2$, there exists a positive constant C such that $\|g^*_{\lambda,H} f\|_{L^p(X)} \le C\|f\|_{L^p(X;H)}$ for all $f \in L^p(X; H)$. We do so by interpolation.

We first show the L^2 boundedness of $g^*_{\lambda,H} f$. To see this, we point out that by Fubini's Theorem,

$$\int_F |g^*_{\lambda,H} f|^2 d\mu(x) = \int_0^\infty \int_X J_{\lambda,F}(y,t) \left| \left(t^2 L e^{-t^2 L}\right) f(y)\right|_H^2 \frac{d\mu(y) dt}{t},$$

with

$$J_{\lambda,F}(y,t) = \int_F \left(\frac{t}{d(x,y)+t}\right)^{D\lambda} \frac{d\mu(x)}{V(x,t)},$$

which holds for any closed set $F \subset X$.

Then we have the estimate

$$J_{\lambda,F}(y,s) \le C_\lambda,$$

where C_λ is a constant depending only on λ and n but not on F, y or s. This estimate follows directly from the inequality (4.16) in Uhl's PhD thesis [44, Section4.4].

As a consequence, we obtain that

$$\begin{aligned}\|g^*_{\lambda,H} f\|_2^2 &\le C_\lambda \int_0^\infty \int_X \left|\left(t^2 L e^{-t^2 L}\right) f(y)\right|_H^2 \frac{d\mu(y) dt}{t} \le C_\lambda \int_0^\infty t^4 e^{-2t^2} \frac{dt}{t} \|f\|^2_{L^2(X;H)} \\ &\le C\|f\|^2_{L^2(X;H)}.\end{aligned}$$

Next we point out that $g^*_{\lambda,H}$ is weak-type (p_0, p_0). All the calculations and ingredients of Uhl's proof in [44, pp.63–74] of this fact for $L^{p_0}(X)$, namely the use of the Calderón–Zygmund decomposition, the L^2-integral, duality in the sense of L^2, the Hardy–Littlewood maximal operator, and the $L^{p_0} \to L^2$ estimate, go through in our vector-valued setting $L^{p_0}(X; H)$. Thus we need only apply the rest of Uhl's proof, replacing the absolute value $|\cdot|$ used there by our norm $|\cdot|_H$.

Step II. We now prove that $\|S_H f\|_{L^p(X)} \le C\|f\|_{L^p(X;H)}$ for $2 \le p < p_0'$.

To see this, we consider the Littlewood–Paley g-function defined by

$$G_H f(x) := \left(\int_0^\infty |t^2 L e^{-t^2 L} f(x)|_H^2 \frac{dt}{t}\right)^{1/2}.$$

We claim that

$$\|G_H f\|_{L^p(X)} \le C\|f\|_{L^p(H)}. \tag{5.4}$$

The proof of (5.4) is exactly the same as that of the proof for the Euclidean, non-vector-valued case in Auscher's paper [1, Section 7.1]. The key ingredient of Auscher's proof is Theorem 2.2 of [1]. It is noted in [1, Remark 7, after Theorem 2.2] that Theorem 2.2 also holds in the vector-valued case. Further, the proof of Theorem 2.2 in Auscher's paper goes through in the case of spaces of homogeneous type.

Auscher's proof of (5.4) requires the Davies–Gaffney estimates and (5.2). The Davies–Gaffney estimates are one of our hypotheses. The estimate (5.2) follows from the generalized Gaussian estimates (GGE$_{p_0}$), as is shown in Lemma 2.6 of [44]. Thus inequality (5.4) holds.

Then, following the duality argument in Uhl's proof [44, pp.74–75], we obtain that for all $\phi \in L^{(p/2)'}(X)$,

$$|\langle (S_H f)^2, \phi\rangle| \le |\langle (G_H f)^2, \mathcal{M}(|\phi|)\rangle|.$$

Therefore $\|S_H f\|_{L^p(X)} \le \|G_H f\|_{L^p(X)} \le C\|f\|_{L^p(H)}$, as required. □

Remark 5.3 We point out that in Step II in the proof above, we can obtain the following result as well: $\|S_{H,\psi} f\|_{L^p(X)} \le \|G_{H,\psi} f\|_{L^p(X)} \le C\|f\|_{L^p(X;H)}$, where ψ appears in the reproducing formula in (3.13), and

$$S_{H,\psi} f(x) := \left(\int_{\Gamma(x)} \left|\left(\psi(t\sqrt{L})\right) f(y)\right|_H^2 \frac{dy\, dt}{tV(x,t)}\right)^{1/2},$$

and

$$G_{H,\psi} f(x) = \left(\int_0^\infty |\psi(t\sqrt{L}) f(x)|_H^2 \frac{dt}{t}\right)^{1/2}.$$

Now we can prove Theorem 2.13.

Proof of Theorem 2.13 Note that Part (ii) is a consequence of Part (i) and Theorem 2.12. It suffices to prove Part (i).

By Theorem 5.1 we obtain that L_1 and L_2 are injective operators on $L^2(X_1)$ and $L^2(X_2)$, respectively. As a consequence, the null space $N(L_1 \otimes L_2) = \{0\}$, which yields that $H^2(X_1 \times X_2) = L^2(X_1 \times X_2)$ since $L^2(X_1 \times X_2) = H^2(X_1 \times X_2) \oplus N(L_1 \otimes L_2)$. Thus, to prove $H^p_{L_1,L_2}(X_1 \times X_2) = L^p(X_1 \times X_2)$ for $p_0 < p \le 2$, it suffices to prove that for all $f \in L^2(X_1 \times X_2) \cap L^p(X_1 \times X_2)$,

$$\|f\|_{L^p(X_1\times X_2)} \le C\|Sf\|_{L^p(X_1\times X_2)} \le C\|f\|_{L^p(X_1\times X_2)}. \tag{5.5}$$

And then the result $H^p_{L_1,L_2}(X_1 \times X_2) = L^p(X_1 \times X_2)$ for $2 < p < p_0'$ follows from the duality argument. This implies that Part (i) holds.

We now verify (5.5). First, write the area function as

$$\left(\int_{\Gamma(x)} |t_1^2 L_1 e^{-t_1^2 L_1} \otimes t_2^2 L_2 e^{-t_2^2 L_2} f(y)|^2 \frac{d\mu_1(y_1)}{V(x_1,t_1)}\frac{dt_1}{t_1}\frac{d\mu_2(y_2)}{V(x_2,t_2)}\frac{dt_2}{t_2}\right)^{1/2}$$
$$= \left(\int_{\Gamma(x_1)}\left[\int_{\Gamma(x_2)} |\big(t_1^2 L_1 e^{-t_1^2 L_1} F_{t_2,y_2}\big)(y_1)|^2 \frac{d\mu_2(y_2)}{V(x_2,t_2)}\frac{dt_2}{t_2}\right]\frac{d\mu_1(y_1)}{V(x_1,t_1)}\frac{dt_1}{t_1}\right)^{1/2}$$

where $F_{t_2,y_2}(\cdot) = \big(t_2^2 L_2 e^{-t_2^2 L_2} f\big)(\cdot, y_2)$.

For each $x_2 \in X_2$, we define the Hilbert-valued function space $L^2(X_1; H_{x_2})$ via the following H_{x_2} norm:

$$|G_{t_2,y_2}(y_1)|_{H_{x_2}} := \left[\int_{\Gamma(x_2)} |G_{t_2,y_2}(y_1)|^2 \frac{d\mu_2(y_2)}{V(x_2,t_2)}\frac{dt_2}{t_2}\right]^{1/2}.$$

Then L_1 can be extended to act on $L^2(X_1; H_{x_2})$ in a natural way. Also the generalized Gaussian estimates can be extended to the semigroup e^{tL} acting on $L^2(X_1; H_{x_2})$. That is, by Minkowski's inequality

$$\begin{aligned}
&\|P_{B(x_1,t^{1/2})} e^{-tL_1} P_{B(y_1,t^{1/2})} G_{t_2,y_2}(\cdot)\|_{L^{p_0'}(X;H)}\\
&= \Big\| |P_{B(x_1,t^{1/2})} e^{-tL_1} P_{B(y_1,t^{1/2})} G_{t_2,y_2}(\cdot)|_H \Big\|_{L^{p_0'}(X_1)}\\
&\le \Big| \|P_{B(x_1,t^{1/2})} e^{-tL_1} P_{B(y_1,t^{1/2})} G_{t_2,y_2}(\cdot)\|_{L^{p_0'}(X_1)} \Big|_H\\
&\le C V(x_1,t^{1/2})^{-(1/p_0 - 1/p_0')} \exp\Big(-b\frac{d(x_1,y_1)^2}{t}\Big) \big| \|G_{t_2,y_2}\|_{L^p(X_1)} \big|_H\\
&\le C V(x_1,t^{1/2})^{-(1/p_0 - 1/p_0')} \exp\Big(-b\frac{d(x_1,y_1)^2}{t}\Big) \big\| |G_{t_2,y_2}|_H \big\|_{L^p(X_1)}\\
&= C V(x_1,t^{1/2})^{-(1/p_0 - 1/p_0')} \exp\Big(-b\frac{d(x_1,y_1)^2}{t}\Big) \|G_{t_2,y_2}\|_{L^p(X;H)}.
\end{aligned}$$

Define the area function $S_{H_{x_2}}$ from $L^2(H_{x_2})$ to $L^2(X_1)$ by

$$S_{H_{x_2}} G_{t_2,y_2}(x_1) := \left(\int_{\Gamma(x_1)} \big|\big(t^2 L_1 e^{-t^2 L_1}\big) G_{t_2,y_2}(y_1)\big|_H^2 \frac{d\mu_1(y_1)\, dt}{t V(x_1,t)}\right)^{1/2}.$$

Recall that $F_{t_2,y_2}(\cdot) = \big(t_2^2 L_2 e^{-t_2^2 L_2} f\big)(\cdot, y_2)$. So by Theorem 5.2, we have for all $p \in (p_0, p_0')$ that

$$\begin{aligned}
\|Sf\|_{L^p(X_1\times X_2)} &= \|S_{H_{x_2}} F_{t_2,y_2}(x_1)\|_{L^p(X_1\times X_2)}\\
&\le \|\|F_{t_2,y_2}\|_{L^p(H_{x_2})}\|_{L^p(X_2)}\\
&= \|\||F_{t_2,y_2}|_{H_{x_2}}\|_{L^p(X_2)}\|_{L^p(X_1)}\\
&= \left\|\left\|\left[\int_{\Gamma(x_2)} |F_{t_2,y_2}(y_1)|^2 \frac{d\mu_2(y_2)}{V(x_2,t_2)}\frac{dt_2}{t_2}\right]^{1/2}\right\|_{L^p(X_2)}\right\|_{L^p(X_1)}\\
&= \left\|\left\|\left[\int_{\Gamma(x_2)} |\big((t_2^2 L_2) e^{-t_2^2 L_2} f\big)(y_1,y_2)|^2 \frac{d\mu_2(y_2)}{V(x_2,t_2)}\frac{dt_2}{t_2}\right]^{1/2}\right\|_{L^p(X_2)}\right\|_{L^p(X_1)}\\
&\le C\|f\|_{L^p(X_1\times X_2)}.
\end{aligned}$$

We can obtain the other direction, that is, $\|f\|_{L^p(X_1\times X_2)} \le C\|Sf\|_{L^p(X_1\times X_2)}$, by using the reproducing formula and then the standard duality argument and the L^p-boundedness of the area function for $2 \le p < p_0'$. This completes the proof of Theorem 2.12. □

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References

1. Auscher, P.: On necessary and sufficient conditions for L^p and related estimates. Mem. Am. Math. Soc. **186**, no. 871 (2007)
2. Auscher, P., Duong, X.T., and McIntosh, A.: Boundedness of Banach space valued singular integral operators and Hardy spaces, unpublished preprint (2005)
3. Auscher, P., McIntosh, A., Russ, E.: Hardy spaces of differential forms on Riemannian manifolds. J. Geom. Anal. **18**, 192–248 (2008)
4. Blunck, S., Kunstmann, P.C.: Weighted norm estimates and maximal regularity. Adv. Differ. Equ. **7**, 1513–1532 (2002)
5. Bownik, M., Li, B.D., Yang, D.C., Zhou, Y.: Weighted anisotropic product Hardy spaces and boundedness of sublinear operators. Math. Nachr. **283**(3), 392–442 (2010)
6. Carleson, L.: A counterexample for measures bounded on H^p for the bi-disc. Mittag-Leffler Report No. 7 (1974)
7. Chang, D.C., Yang, D.C., Zhou, Y.: Boundedness of sublinear operators on product Hardy spaces and its application. J. Math. Soc. Jpn. **62**(1), 321–353 (2010)
8. Chang, S.-Y.A., Fefferman, R.: A continuous version of the duality of H^1 with BMO on the bi-disc. Ann. Math. **112**, 179–201 (1980)
9. Chang, S.-Y.A., Fefferman, R.: The Calderón–Zygmund decomposition on product domains. Am. J. Math. **104**, 445–468 (1982)
10. Chang, S.-Y.A., Fefferman, R.: Some recent developments in Fourier analysis and H^p-theory on product domains. Bull. Am. Math. Soc. **12**, 1–43 (1985)
11. Chen, P., Duong, X.T., Li, J., Ward, L.A., and Yan, L.X.: Marcinkiewicz-type spectral multipliers on Hardy and Lebesgue spaces on product spaces of homogeneous type (2013) (**submitted**)
12. Christ, M.: A T(b) theorem with remarks on analytic capacity and the Cauchy integral. Colloq. Math. **60**(61), 601–628 (1990)
13. Coifman, R.R., Meyer, Y., Stein, E.M.: Some new function spaces and their applications to harmonic analysis. J. Funct. Anal. **62**, 304–335 (1985)
14. Coifman, R.R., Weiss, G.: Analyse harmonique non-commutative sur certains espaces homogènes. Étude de certaines intégrales singulières. Lecture Notes in Mathamatics, vol. 242. Springer, Berlin (1971)
15. Coulhon, T., Sikora, A.: Gaussian heat kernel upper bounds via Phragmén–Lindelöf theorem. Proc. Lond. Math. Soc. **96**, 507–544 (2008)
16. Deng, D.G., Song, L., Tan, C.Q., Yan, L.X.: Duality of Hardy and BMO spaces associated with operators with heat kernel bounds on product domains. J. Geom. Anal. **17**(3), 455–483 (2007)
17. Duong, X.T., Li, J.: Hardy spaces associated to operators satisfying Davies–Gaffney estimates and bounded holomorphic functional calculus. J. Funct. Anal. **264**(6), 1409–1437 (2013)
18. Duong, X.T., Li, J., Yan, L.X.: Endpoint estimates for singular integrals with non-smooth kernels on product spaces. arXiv:1509.07548
19. Duong, X.T., Yan, L.X.: New function spaces of BMO type, John–Nirenberg inequality, interpolation and applications. Commun. Pure Appl. Math. **58**(10), 1375–1420 (2005)
20. Duong, X.T., Yan, L.X.: Duality of Hardy and BMO spaces associated with operators with heat kernel bounds. J. Am. Math. Soc. **18**, 943–973 (2005)
21. Fefferman, C., Stein, E.M.: H^p spaces of several variables. Acta Math. **129**, 137–195 (1972)
22. Fefferman, R.: Calderón–Zygmund theory for product domains: H^p spaces. Proc. Natl. Acad. Sci. USA **83**(4), 840–843 (1986)

23. Fefferman, R.: Multiparameter Fourier analysis. In: Stein, E.M. (ed.)Beijing Lectures in Harmonic Analysis (Beijing, 1984): Annals of Math Studies, vol. 112, pp. 47-130. Princeton University Press, Princeton (1986)
24. Fefferman, R.: Harmonic analysis on product spaces. Ann. Math. **126**, 109–130 (1987)
25. Fefferman, R.: Multiparameter Calderón–Zygmund theory. In: Christ, M., Kenig, C., Sadosky, C. (eds.) Harmonic Analysis and Partial Differential Equations (Chicago, IL, 1996), pp. 207–221. The University of Chicago Press, Chicago (1999)
26. Fefferman, R., Stein, E.M.: Singular integrals on product spaces. Adv. Math. **45**, 117–143 (1982)
27. Gundy, R., Stein, E.M.: H^p theory for the poly-disc. Proc. Natl. Acad. Sci. USA **76**, 1026–1029 (1972)
28. Han, Y.S., Li, J., Lin, C.C.: Criterions of the L^2 boundedness and sharp endpoint estimates for singular integral operators on product spaces of homogeneous type. To appear in Ann. Scuola Norm. Sup. Pisa Cl. Sci
29. Han, Y.S., Li, J., Lu, G.: Multiparameter Hardy space theory on Carnot–Caratheodory spaces and product spaces of homogeneous type. Trans. Am. Math. Soc. **365**(1), 319–360 (2013)
30. Han, Y.S., Li, J., Lu, G.: Duality of multiparameter Hardy spaces H^p on spaces of homogeneous type. Ann. Scuola Norm. Sup. Pisa Cl. Sci. **IX**(5), 645–685 (2010)
31. Hofmann, S., Lu, G.Z., Mitrea, D., Mitrea, M., Yan, L.X.: Hardy spaces associated to non-negative self-adjoint operators satisfying Davies-Gaffney estimates. Mem. Am. Math. Soc. **214**, no. 1007 (2011)
32. Hofmann, S., Mayboroda, S.: Hardy and BMO spaces associated to divergence form elliptic operators. Math. Ann. **344**(1), 37–116 (2009)
33. Journé, J.-L.: Calderón–Zygmund operators on product spaces. Rev. Mat. Iberoam. **1**, 55–91 (1985)
34. Journé, J.-L.: A covering lemma for product spaces. Proc. Am. Math. Soc. **96**, 593–598 (1986)
35. Kunstmann, P.C., Uhl, M.: Spectral multiplier theorems of Hörmander type on Hardy and Lebesgue spaces. J. Oper. Theory **73**, 27–69 (2015)
36. Kunstmann, P.C., Uhl, M.: L^p-spectral multipliers for some elliptic systems. Proc. Edinb. Math. Soc. **58**, 231–253 (2015)
37. Liskevich, V., Sobol, Z., Vogt, H.: On the L^p theory of C^0-semigroups associated with second-order elliptic operators II. J. Funct. Anal. **193**, 55–76 (2002)
38. McIntosh, A.: Operators which have an H_∞-calculus, Miniconference on operator theory and partial differential equations. In: Proceedings of the Centre for Mathematics and its Applications. vol. 14, pp. 210-231. ANU, Canberra (1986)
39. Ouhabaz, E.M.: Analysis of heat equations on domains. In: London Mathematical Society Monographs, vol. 31. Princeton University Press, Princeton (2004)
40. Pipher, J.: Journé's covering lemma and its extension to higher dimensions. Duke Math. J. **53**, 683–690 (1986)
41. Stein, E.M.: Singular Integrals and Differentiability Properties of Functions. Princeton University Press, Princeton (1970)
42. Stein, E.M., Weiss, G.: On the theory of harmonic functions of several variables. I. The theory of H^p-spaces. Acta Math. **103**, 25–62 (1960)
43. Taibleson, M.W., Weiss, G.: The molecular characterization of certain Hardy spaces. Astérisque **77**, 68–149 (1980)
44. Uhl, M.: Spectral multiplier theorems of Hörmander type via generalized Gaussian estimates, Ph.D. thesis, Karlsruhe Institute of Technology (Jun 2011)
45. Yan, L.X.: Classes of Hardy spaces associated with operators, duality theorem and applications. Trans. Am. Math. Soc. **360**(8), 4383–4408 (2008)