Math. Z. (2016) 282:165–175
DOI 10.1007/s00209-015-1536-2

Mathematische Zeitschrift

Riemannian submersions from compact four manifolds

Xiaoyang Chen[1]

Received: 8 October 2014 / Accepted: 8 July 2015 / Published online: 19 September 2015

Abstract We show that under certain conditions, a nontrivial Riemannian submersion from positively curved four manifolds does *not* exist. This gives a partial answer to a conjecture due to Fred Wilhelm. We also prove a rigidity theorem for Riemannian submersions with totally geodesic fibers from compact four-dimensional Einstein manifolds.

1 Introduction

A smooth map $\pi\colon (M, g) \to (N, h)$ is a Riemannian submersion if π_* is surjective and satisfies the following property:

$$g_p(v, w) = h_{\pi(p)}(\pi_* v, \pi_* w)$$

for any v, w that are tangent vectors in TM_p and perpendicular to the kernel of π_*.

A fundamental problem in Riemannian geometry is to study the interaction between curvature and topology. A lot of important work has been done in this direction. In this paper we study a similar problem for Riemannian submersions:

Problem *Explore the structure of* π *under additional curvature assumptions of* (M, g).

When (M, g) has constant sectional curvature, we have the following classification results ([8,23,24]).

Theorem 1.1 *Let* $\pi\colon (M^m, g) \to (N, h)$ *be a nontrivial Riemannian submersion (i.e.* $0 < dim N < dim M$*) with connected fibers, where* (M^m, g) *is compact and has constant sectional curvature* c.

The author is supported in part by NSF DMS-1209387.

✉ Xiaoyang Chen
xychen100@gmail.com

1 Department of Mathematics, University of Notre Dame, Notre Dame, IN 46637, USA

1. *If $c < 0$, then there is no such Riemannian submersion.*
2. *If $c = 0$, then locally π is the projection of a metric product onto one of its factors.*
3. *If $c > 0$ and M^m is simply connected, then π is metrically congruent to the Hopf fibration, i.e, there exist isometries $f_1 \colon M^m \to \mathbb{S}^m$ and $f_2 \colon N \to \mathbb{P}(\mathbb{K})$ such that $pf_1 = f_2\pi$, where p is the standard projection from $\mathbb{S}^m$ to projective spaces $\mathbb{P}(\mathbb{K})$.*

However, very little is known about the structure of π if (M, g) is not of constant curvature. In this paper we consider two different curvature conditions:

1. (M, g) has positive sectional curvature.
2. (M, g) is an Einstein manifold.

When (M, g) has positive sectional curvature, we have the following important conjecture due to Fred Wilhelm.

Conjecture 1 *Let $\pi \colon (M, g) \to (N, h)$ be a nontrivial Riemannian submersion, where (M, g) is a compact Riemannian manifold with positive sectional curvature. Then $dim(F) < dim(N)$, where F is the fiber of π.*

By Frankel's theorem [7], it is not hard to see that Conjecture 1 is true if at least two fibers of π are totally geodesic. In fact, since any two fibers do not intersect with each other, Frankel's theorem implies that $2\, dim(F) < dim(M)$. Hence $dim(F) < dim(N)$. If all fibers of π are totally geodesic, we have the following stronger result which is due to Florit and Ziller [6]. See also Propositions 2.4, 2.5 in [27].

Proposition 1.2 *Let $\pi \colon (M, g) \to (N, h)$ be a nontrivial Riemannian submersion such that all fibers of π are totally geodesic, where (M, g) is a compact Riemannian manifold with positive sectional curvature. Then $dim(F) < \rho(dim(N)) + 1$, where F is any fiber of π and $\rho(n)$ is the maximal number of linearly independent vector fields on S^{n-1}.*

Notice that we always have $\rho(dim(N)) + 1 \leq dim(N) - 1 + 1 = dim(N)$ and equality holds if and only $dim(N) = 2, 4$ or 8.

Remark 1 It would be very interesting to know whether one can replace $dim(F) < dim(N)$ by $dim(F) < \rho(dim(N)) + 1$ in Conjecture 1. It would be the Riemannian analogue of Toponogov's Conjecture (page 1727 in [19]) and would imply that $dim(N)$ must be even (In fact, if $dim(N)$ is odd, then $\rho(dim(N)) = 0$. Hence $dim(F) < \rho(dim(N)) + 1$ implies $dim(F) = 0$ and hence π is trivial, contradiction). In particular, there would be no Riemannian submersion with one-dimensional fibers from even-dimensional manifolds with positive sectional curvature.

When $dim(M) = 4$, Conjecture 1 is equivalent to the following conjecture.

Conjecture 2 *There is no nontrivial Riemannian submersion from any compact four manifold (M^4, g) with positive sectional curvature.*

In fact, suppose there exists such a Riemannian submersion $\pi \colon (M^4, g) \to (N, h)$. Then Conjecture 1 would imply $dim(N) = 3$. Hence the Euler number of M^4 is zero. On the other hand, since (M^4, g) has positive sectional curvature, $H^1(M^4, \mathbb{R}) = 0$ by Bochner's vanishing theorem ([17], page 208). By Poincaré duality, the Euler number of M^4 is positive. Contradiction.

Let $\pi \colon (M, g) \to (N, h)$ be a Riemannian submersion. We say that a function f defined on M is basic if f is constant along each fiber. A vector field X on M is basic if it is horizontal and is π-related to a vector field on N. In other words, X is the horizontal lift of some vector field on N. Let H be the mean curvature vector field of the fibers and A be the O'Neill tensor of π. We denote by $\|A\|$ the norm of A, i.e., $\|A\|^2 = \sum_{i,j} \|A_{X_i} X_j\|^2$, where $\{X_i\}$ is a local orthonormal basis of the horizontal distribution of π. The next theorem gives a partial answer to Conjecture 2.

Theorem 1.3 *There is no nontrivial Riemannian submersion from any compact four manifold with positive sectional curvature such that either $\|A\|$ or H is basic.*

We emphasize that in Conjecture 1 the assumption that (M, g) has positive sectional curvature can *not* be replaced by (M, g) has positive sectional curvature *almost* everywhere, namely, (M, g) has nonnegative sectional curvature everywhere and has positive sectional curvature on an open and dense subset of M. Such counterexamples were firstly constructed by M. Kerin in [15]. In fact, he constructed Riemannian metrics on $M^{13} = (S^7 \times S^7)/S^1$ and $N^{11} = (S^7 \times S^7)/S^3$ with positive sectional curvature *almost* everywhere. Equipped with these metrics, there exist Riemannian submersions $M^{13} \to \mathbb{CP}^3$ and $N^{11} \to S^4$ such that in each case the fibre is S^7. Here we provide a new counterexample. Let g be the metric on $S^2 \times S^3$ constructed by Wilking [25] which has positive sectional curvature *almost* everywhere. Then by a theorem of Tapp [20], g can be extended to a nonnegatively curved metric $\tilde{g}$ on $S^2 \times \mathbb{R}^4$ such that $(S^2 \times S^3, g)$ becomes the distance sphere of radius 1 about the soul. By Proposition 5.1 below, we get a Riemannian submersion $\pi\colon (S^2 \times S^3, g) \to (S^2, h)$, where h is the induced metric from $\tilde{g}$ on the soul S^2. This example together with Kerin's examples show that in Conjecture 1 the assumption that (M, g) has positive sectional curvature can *not* be replaced by (M, g) has positive sectional curvature *almost* everywhere.

Riemannian submersions are also important in the study of compact Einstein manifolds, for example, see [3]. Our next theorem gives a complete classification of Riemannian submersions with totally geodesic fibers from compact four-dimensional Einstein manifolds.

Theorem 1.4 *Suppose $\pi\colon (M^4, g) \to (N, h)$ is a Riemannian submersion, where (M^4, g) is a compact four-dimensional Einstein manifold. If all fibers of π are totaly geodesic and have dimension 2, then locally π is the projection of a metric product $B^2(c) \times B^2(c)$ onto one of the factors, where $B^2(c)$ is a two-dimensional compact manifold with constant curvature c.*

If the dimension of the fibers of π is 1 or 3 (all fibers are not necessarily totally geodesic), then the Euler number of M^4 is zero. By a theorem of Berger [2,14], (M^4, g) must be flat. Hence by a theorem of Walschap [23], locally π is the projection of a metric product onto one of the factors.

2 Preliminaries

In this section we recall some definitions and facts on Riemannian submersions which will be used in this paper. We refer to [16] for more details.

Let $\pi\colon (M, g) \to (N, h)$ be a Riemannian submersion. Then π induces an orthogonal splitting $TM = \mathcal{H} \oplus \mathcal{V}$, where $\mathcal{V}$ is tangent to the fibers and $\mathcal{H}$ is the orthogonal complement of $\mathcal{V}$. We write $Z = Z^h + Z^v$ for the corresponding decomposition of $Z \in TM$. The O'Neill tensor A is given by

$$A_X Y = (\nabla_X Y)^v = \frac{1}{2}([X, Y])^v,$$

where $X, Y \in \mathcal{H}$ and are π-related to some vector field on N, respectively.

Fix $X \in \mathcal{H}$, define A_X^* by

$$A_X^*\colon \mathcal{V} \to \mathcal{H}$$
$$V \mapsto -(\nabla_V X)^h.$$

Then A_X^* is the dual of A_X.

Define the mean curvature vector field H of π by

$$H = \sum_i (\nabla_{V_i} V_i)^h,$$

where $\{V_i\}_{i=1}^k$ is any orthonormal basis of $\mathcal{V}$ and $k = dim\mathcal{V}$.

Define the mean curvature form ω of π by

$$\omega(Z) = g(H, Z),$$

where $Z \in TM$. It is clear that $i_V\omega = \omega(V) = 0$ for any $V \in \mathcal{V}$.

We say that a function f defined on M is basic if f is constant along each fiber. A vector field X on M is basic if it is horizontal and is π-related to a vector field on N. In other words, X is the horizontal lift of some vector field on N. A differential form α on M is called to be basic if and only if $i_V\alpha = 0$ and $\mathcal{L}_V\alpha = 0$ for any $V \in \mathcal{V}$, where $\mathcal{L}_V\alpha$ is the Lie derivative of α.

The set of basic forms of M, denoted by $\Omega_b(M)$, constitutes a subcomplex

$$d\colon \Omega_b^r(M) \to \Omega_b^{r+1}(M)$$

of the De Rham complex $\Omega(M)$. The basic cohomology of M, denoted by $H_b^*(M)$, is defined to be the cohomology of $(\Omega_b(M), d)$.

Proposition 2.1 *The inclusion map* $i\colon \Omega_b(M) \to \Omega(M)$ *induces an injective map*

$$H_b^1(M) \to H_{DR}^1(M).$$

Proof See pages 33–34, Proposition 4.1 in [22]. □

3 Proof of Theorem 1.3

Let (M^m, g) be an m-dimensional compact manifold with positive sectional curvature, $m \geq 4$ and (N^2, h) be a 2-dimensional compact Riemannian manifold. Now we are going to prove the following theorem which implies Theorem 1.3.

Theorem 3.1 *There is no Riemannian submersion* $\pi\colon (M^m, g) \to (N^2, h)$ *such that*

1. *the Euler numbers of the fibers are nonzero and*
2. *either* $\|A\|$ *or* H *is basic.*

Remark 2 If Conjecture 1 is true, then there would be no Riemannian submersion $\pi\colon (M^m, g) \to (N^2, h)$, where (M^m, g) has positive sectional curvature and $m \geq 4$.

Before we prove Theorem 3.1, we firstly show how to derive Theorem 1.3. The proof is by contradiction. Suppose there exists a nontrivial Riemannian submersion $\pi\colon (M^4, g) \to (N, h)$ such that either $\|A\|$ or H is basic, where (M^4, g) is a compact four manifold with positive sectional curvature. Since (M^4, g) has positive sectional curvature, $H^1(M^4, \mathbb{R}) = 0$ by Bochner's vanishing theorem ([17], page 208). By Poincaré duality, $\chi(M^4) = 2+b_2(M^4)$ is positive. By a theorem of Hermann [13], π is a locally trivial fibration. Then $\chi(M^4) = \chi(N)\chi(F)$, where F is any fiber of π. It follows that $dim(N) = 2$ and $\chi(F)$ is nonzero (hence all fibers have nonzero Euler numbers), which is a contradiction by Theorem 3.1.

The proof of Theorem 3.1 is again by contradiction. Suppose $\pi\colon (M^m, g) \to (N^2, h)$ is a Riemannian submersion satisfying the conditions in Theorem 3.1. By passing to its oriented double cover, we can assume that N^2 is oriented. The idea of the proof of Theorem 3.1 is to construct a nowhere vanishing vector field (or line field) on some fiber of π, which will imply the Euler numbers of the fibers are zero. Contradiction.

Since (M, g) has positive sectional curvature, by Theorem 1.3 in [23], $\|A\|$ can not be identical to zero on M. Hence there exists $p \in M$ such that $\|A\|(p) \neq 0$.

If $\|A\|$ is basic, then $\|A\| \neq 0$ at any point on F_p, where F_p is the fiber at p. Let X, Y be any orthonormal oriented basic vector fields in some open neighborhood of F_p. Then $\|A_X Y\|^2 = \frac{1}{2}\|A\|^2 \neq 0$ at any point on F_p. Define a map s by

$$s\colon F_p \to TF_p$$
$$x \mapsto \frac{A_X Y}{\|A_X Y\|}(x).$$

Let Z, W be another orthonormal oriented basic vector fields. Then $Z = aX + bY$ and $W = cX + dY$, $ad - bc > 0$. Then

$$A_Z W = (ad - bc)A_X Y.$$

Hence s does not depend on the choice of X, Y. Then s is a nowhere vanishing vector field on F_p. Thus the Euler number of F_p is zero. Contradiction.

If H is basic, the construction of such nowhere vanishing vector field (or line field) is more complicated. Under the assumption that H is basic, we firstly construct a metric $\hat{g}$ on M^m such that $\pi\colon (M^m, \hat{g}) \to (N^2, h)$ is still a Riemannian submersion and all fibers are minimal submanifolds with respect to $\hat{g}$. Of course, in general $\hat{g}$ can *not* have positive sectional curvature everywhere. However, the crucial point is that there exists some fiber F_0 such that $\hat{g}$ has positive sectional curvature at all points on F_0. Pick any fiber F_1 which is close enough to F_0. Then using the Synge's trick, we construct a continuous codimension one distribution on F_1. Thus the Euler number of F_1 is zero. Contradiction.

Now we are going to explain the proof of Theorem 3.1 in details. We firstly need the following lemmas:

Lemma 3.2 *Suppose ω is the mean curvature form of a Riemannian submersion from compact Riemannian manifolds. If ω is a basic form, then it is a closed form.*

Proof See page 82 in [22] for a proof. □

Lemma 3.3 *Suppose $\pi\colon (M^m, g) \to (N, h)$ is a Riemannian submersion such that H is basic, where (M^m, g) is a compact Riemannian manifold with positive sectional curvature. Then there exists a metric $\hat{g}$ on M^m such that $\pi\colon (M^m, \hat{g}) \to (N, h)$ is still a Riemannian submersion and all fibers are minimal submanifolds with respect to $\hat{g}$. Furthermore, there exists some fiber F_0 such that $\hat{g}$ has positive sectional curvature at all points on F_0.*

Proof The idea is to use partial conformal change of metrics along the fibers, see also page 82 in [22]. Let ω be the mean curvature form of π. Since H is basic, ω is a basic form. Then ω is closed by Lemma 3.2. So $[\omega]$ defines a cohomological class in $H_b^1(M^m)$. Because

(M^m, g) has positive sectional curvature, $H^1_{DR}(M^m) = 0$ by Bochner's vanishing theorem ([17], page 208). By Proposition 2.1, we see that $H^1_b(M^m) = 0$. Then there exists a basic function f globally defined on M^m such that $\omega = df$. Define $\hat{f} = f - max_{p\in M^m} f(p)$. Then $max_{p\in M^m} \hat{f}(p) = 0$ and $\omega = d\hat{f}$. Let $\lambda = e^{\hat{f}}$ and define

$$\hat{g} = (\lambda^{\frac{2}{k}} g_v) \oplus g_h,$$

where $k = dim(M^m) - dim(N)$, g_v/g_h are the vertical/horizontal components of g, respectively.

Since the horizontal components of g remains unchanged, $\pi\colon (M^m, \hat{g}) \to (N, h)$ is still a Riemannian submersion. Now we compute the mean curvature form $\hat{\omega}$ associated to $\hat{g}$. Let $\{V_i\}_{i=1}^k$ be vertical vector fields satisfying $g(V_i, V_j) = \delta_i^j$. With respect to $\hat{g}$, the mean curvature vector field are given by $\hat{H} = (\sum_{i=1}^k \hat{\nabla}_{\hat{V}_i} \hat{V}_i)^h$, where $\hat{V}_i = \lambda^{-\frac{1}{k}} V_i$ and $\hat{\nabla}$ is the Levi-Civita connection associated to $\hat{g}$. For any basic vector field X, we have

$$\hat{\omega}(X) = \hat{g}(\hat{H}, X) = \hat{g}\left(\sum_{i=1}^k \hat{\nabla}_{\hat{V}_i} \hat{V}_i, X\right).$$

By the $Koszul's formula$, we get

$$\begin{aligned}
2\hat{\omega}(X) &= \hat{V}_i \hat{g}(\hat{V}_i, X) + \hat{V}_i \hat{g}(X, \hat{V}_i) - X\hat{g}(\hat{V}_i, \hat{V}_i) \\
&\quad + \hat{g}([\hat{V}_i, \hat{V}_i], X) - \hat{g}([\hat{V}_i, X], \hat{V}_i) - \hat{g}([\hat{V}_i, X], \hat{V}_i) \\
&= -X\hat{g}(\hat{V}_i, \hat{V}_i)) - 2\hat{g}([\hat{V}_i, X], \hat{V}_i) \\
&= -Xg(V_i, V_i) - 2\lambda^{\frac{2}{k}} g\left(\left[\lambda^{-\frac{1}{k}} V_i, X\right], \lambda^{-\frac{1}{k}} V_i\right) \\
&= -Xg(V_i, V_i) - 2g([V_i, X], V_i) + 2\lambda^{\frac{1}{k}} X(\lambda^{-\frac{1}{k}}) g(V_i, V_i) \\
&= -Xg(V_i, V_i) - 2g([V_i, X], V_i) - 2\, dlog\lambda(X).
\end{aligned}$$

On the other hand, by the $Koszul's formula$ again, we get

$$2\omega(X) = 2g(H, X) = -Xg(V_i, V_i) - 2g([V_i, X], V_i).$$

So we get

$$2\hat{\omega}(X) = 2\omega(X) - 2\, dlog\lambda(X).$$

Hence

$$\hat{\omega} = \omega - dlog\lambda = \omega - d\hat{f} = 0.$$

It follows that all fibers of π are minimal submanifolds with respect to $\hat{g}$.

Let $e^{2\phi}(p) = \lambda^{\frac{2}{k}}(p)$, $p \in M^m$. Then

$$\hat{g} = e^{2\phi} g_v \oplus g_h.$$

Note for any $p \in M^m$, $0 < e^{2\phi}(p) \le 1$. Moreover, we have $max_{p\in M^m} e^{2\phi}(p) = 1$. Let $p_0 \in M^m$ such that $e^{2\phi}(p_0) = 1$ and F_0 be the fiber of π passing through p_0. Since f is a basic function on M^m, $e^{2\phi}$ is also basic. Then $e^{2\phi} \equiv 1$ on F_0, which will play a crucial role for our purpose. Of course, in general $\hat{g}$ can *not* have positive sectional curvature everywhere. However, we will see that $\hat{g}$ still has positive sectional curvature at all points on F_0. (The reader should compare it to the following fact: Let $\hat{h} = e^{2f} h$ be a conformal change of h, where h is a Riemannian metric on M with positive sectional curvature. Then $\hat{h}$ still has positive sectional curvature at those points where f attains its maximum value.) This can be seen by the results of Chapter 2 in [10], in particular 2.1.23–2.1.25 in page 52. We provide some details here. Let ∇, $R/\hat{\nabla}$, $\hat{R}$ be the Levi-Civita connection and curvature tensor with respect to $g/\hat{g}$. Given a nonzero vertical vector V and horizontal vector X, by 2.1.19 in page 51 or 2.1.24 in page 52 in [10], we have

$$\begin{aligned} e^{-2\phi}\hat{g}(\hat{R}(X,V,V),X) &= g(R(X,V,V),X) - (1-e^{2\phi})g(A_X^*V, A_X^*V) \\ &+ 2g(\nabla\phi, X)g(X, B(V,V)) - (Hess\phi(X,X) + g(\nabla\phi, X)^2)g(V,V), \end{aligned}$$

where B is the second fundamental form of the fibers and $Hess\phi$ is the Hessian of ϕ with respect to g. Since $e^{2\phi}$ attains its maximum value 1 at all points on F_0, we see $\nabla\phi \equiv 0$ and $Hess\phi(X,X) \le 0$ on F_0. Then we get

$$e^{-2\phi}\hat{g}(\hat{R}(X,V,V),X) \ge g(R(X,V,V),X) > 0.$$

Adapting the above argument to other tangent planes, one can check that $\hat{g}$ has positive sectional curvature at all points on F_0. □

Now we can give a proof of Theorem 3.1 under the assumption that H is basic. We prove it by contradiction. Let $\pi\colon (M^m, g) \to (N^2, h)$ be a Riemannian submersion such that H is basic and the fibers have nonzero Euler numbers, where (M^m, g) has positive sectional curvature and $m \ge 4$. By Lemma 3.3, there exists a metric $\hat{g}$ on M^m such that $\pi\colon (M^m, \hat{g}) \to (N^2, h)$ is still a Riemannian submersion and all fibers of π are minimal submanifolds with respect to $\hat{g}$. Furthermore, there exists some fiber F_0 such that $\hat{g}$ has positive sectional curvature at all points in F_0. Let r be a fixed positive number such that the normal exponential map of F_0 is a diffeomorphism when restricted to the tubular neighborhood of F_0 with radius r. By continuity of sectional curvature, there exists ϵ, $0 < \epsilon < r$ such that $\hat{g}$ has positive sectional curvature at the ϵ neighborhood of F_0. Choose another fiber F_1 such that $0 < \hat{d}(F_0, F_1) < \epsilon$, where $\hat{d}(F_0, F_1)$ is the distance between F_0 and F_1 with respect to $\hat{g}$. Since $\pi\colon (M^m, \hat{g}) \to (N^2, h)$ is a Riemannian submersion, F_0 and F_1 are equidistant. On the other hand, since $0 < \hat{d}(F_0, F_1) < \epsilon$, then for any point $q \in F_1$, there is a *unique* point $p \in F_0$ such that $\hat{d}(p,q) = \hat{d}(F_0, F_1)$. Let $L = \hat{d}(p,q)$ and $\gamma\colon [0, L] \to M^m$, $\gamma(0) = p$, $\gamma(L) = q$ be the *unique* minimal geodesic with unit speed realizing the distance between p and q. Let $V \subseteq T_q(M^m)$ be the subspace of vectors $v = X(L)$ where X is a parallel field along γ such that $X(0) \in T_p(F_0)$. Then

$$\begin{aligned} dim(V \cap T_q(F_1)) &= dim(V) + dim(T_q(F_1)) - dim(V + T_q(F_1)) \\ &\ge (m-2) + (m-2) - (m-1) = m-3. \end{aligned}$$

We claim that $dim(V \cap T_q(F_1)) = m-3$. If not, then $dim(V \cap T_q(F_1)) = m-2$. Let X_i, $i = 1, \cdots m-2$, be orthonormal parallel fields along γ such that $X_i(0) \in T_p(F_0)$, $X_i(L) \in T_q(F_1)$. For each i, choose a variation $f_i(s,t)$ of γ such that $f_i(s,0) \in F_0$, $f_i(s,L) \in F_1$ for small s and $\frac{\partial f_i(0,t)}{\partial s} = X_i(t)$. By construction, $\dot{X}_i(t) = \hat{\nabla}_{\dot{\gamma}} X_i(t) = 0$ for all t, where $\hat{\nabla}$ is the Levi-Civita connection with respect to $\hat{g}$. By the second variation formula, for $i = 1, \cdots m-2$, we have

$$\begin{aligned} \frac{1}{2}\frac{d^2 E_i(s)}{ds^2}_{|s=0} &= \int_0^L (\hat{g}(\dot{X}_i, \dot{X}_i) - \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i))dt \\ &+ \hat{g}(\hat{B}_1(X_i, X_i), \dot{\gamma})(L) - \hat{g}(\hat{B}_0(X_i, X_i), \dot{\gamma})(0) \\ &= -\int_0^L \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i)dt + \hat{g}(\hat{B}_1(X_i, X_i), \dot{\gamma})(L) - \hat{g}(\hat{B}_0(X_i, X_i), \dot{\gamma})(0), \end{aligned}$$

where $E_i(s) = \int_0^L \hat{g}(\frac{\partial f_i(s,t)}{\partial t}, \frac{\partial f_i(s,t)}{\partial t})dt$, $\hat{R}$ is the curvature tensor of $\hat{g}$ and $\hat{B}_j$ is the second fundamental form of F_j with respect to $\hat{g}$, $j = 0, 1$.

Since F_0 and F_1 are minimal submanifolds in $(M^m, \hat{g})$, we have

$$\sum_{i=1}^{m-2} \hat{B}_j(X_i, X_i) = 0,\ j = 0, 1.$$

Then

$$\frac{1}{2}\sum_{i=1}^{m-2} \frac{d^2E_i(s)}{ds^2}_{|s=0} = -\sum_{i=1}^{m-2}\int_0^L \hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i)dt.$$

Since $\hat{g}$ has positive sectional curvature at the ϵ neighborhood of F_0 and $0 < \hat{d}(F_0, F_1) < \epsilon$, we see that $\hat{R}(X_i, \dot{\gamma}, \dot{\gamma}, X_i) < 0$. Hence

$$\frac{1}{2}\sum_{i=1}^{m-2} \frac{d^2E_i(s)}{ds^2}_{|s=0} < 0.$$

Then there exists some i_0 such that $\frac{d^2E_{i_0}(s)}{ds^2}_{|s=0} < 0$, which contradicts that γ is a minimal geodesic realizing the distance between F_0 and F_1. So $dim(V \cap T_q(F_1)) = m - 3$. Since $dim(T_q(F_1)) = m-2$, then $V \cap T_q(F_1)$ is a codimension one subspace of $T_q(F_1)$. Since q is arbitrary on F_1, by doing the same construction as above for any q, then we get a continuous codimension one distribution on F_1. Thus the Euler number of F_1 is zero. Contradiction.

4 Proof of Theorem 1.4

In this section we prove Theorem 1.4. Suppose $\pi\colon (M^4, g) \to (N^2, h)$ is a Riemannian submersion with totally geodesic fibers, where (M^4, g) is a compact four-dimensional Einstein manifold. We are going to show that the A tensor of π vanishes and then locally π is the projection of a metric product onto one of the factors. We firstly need the following lemmas:

Lemma 4.1 *Let π be a Riemannian submersion with totally geodesic fibers from compact Riemannian manifolds, then all fibers are isometric to each other.*

Proof See [13]. □

Lemma 4.2 *Suppose $\pi\colon (M^4, g) \to (N^2, h)$ is a Riemannian submersion with totally geodesic fibers, where (M^4, g) is a compact four-dimensional Einstein manifold. Let c_1, c_2 be the sectional curvature of $(F^2, g_{|F^2})$ and (N^2, h), respectively, where $g_{|F^2}$ is the restriction of g to the fibers F^2. Let $Ric(g) = \lambda g$ for some λ. Then*

(i) $2c_1 + \|A\|^2 = 2\lambda$;
(ii) $2c_2 \circ \pi - 2\|A\|^2 = 2\lambda$;
(iii) $\|A\|^2 = \frac{2}{3}(c_2 \circ \pi - c_1)$,

where $\|A\|^2 = \|A_X^*U\|^2 + \|A_X^*V\|^2 + \|A_Y^*U\|^2 + \|A_Y^*V\|^2$. Here $X, Y/U, V$ is an orthonormal basis of $\mathcal{H}/\mathcal{V}$, respectively.

Proof See page 250, Corollary 9.62 in [3]. For completeness, we give a proof here.

Let $U, V/X, Y$ are orthonormal basis of $\mathcal{V}/\mathcal{H}$, respectively. Then by O'Neill's formula ([16]) , we have

$$\begin{aligned}
\lambda &= Ric(U, U) = c_1 + \|A_X^* U\|^2 + \|A_Y^* U\|^2;\\
\lambda &= Ric(V, V) = c_1 + \|A_X^* V\|^2 + \|A_Y^* V\|^2;\\
\lambda &= Ric(X, X) = c_2 \circ \pi - 3\|A_X Y\|^2 + \|A_X^* U\|^2 + \|A_X^* V\|^2;\\
\lambda &= Ric(Y, Y) = c_2 \circ \pi - 3\|A_X Y\|^2 + \|A_Y^* U\|^2 + \|A_Y^* V\|^2.
\end{aligned}$$

On the other hand, by direct calculation, we see that $2\|A_X Y\|^2 = \|A\|^2$. Hence

$$\begin{aligned}
&2c_1 + \|A\|^2 = 2\lambda;\\
&2c_2 \circ \pi - 2\|A\|^2 = 2\lambda;\\
&\|A\|^2 = \frac{2}{3}(c_2 \circ \pi - c_1).
\end{aligned}$$

□

By Lemmas 4.1 and 4.2, we see that c_1, $\|A\|$ are constants on M^4 and c_2 is a constant on N^2.

Fix $p \in M^4$. Locally we can always choose basic vector fields X, Y such that X, Y is an orthonormal basis of the horizontal distribution. At point p, since the image of A_X^* is perpendicular to X and $dim\mathcal{V} = dim\mathcal{H} = 2$, A_X^* must have nontrivial kernel. Then there exists some $v \in \mathcal{V}$ such that $\|v\| = 1$ and $A_X^*(v) = 0$. Extend v to be a local unit vertical vector field V and choose U such that U, V is a local orthonormal basis of $\mathcal{V}$.

Lemma 4.3

$$\begin{aligned}
A_X^* V(p) = 0;\\
A_Y^* V(p) = 0.
\end{aligned}$$

Proof We already see $A_X^* V(p) = A_{X,p}^*(v) = 0$. Since A^* is the dual of A, at point p, we have

$$\begin{aligned}
A_Y^* V &= g(A_Y^* V, X)X = g(V, A_Y X)X\\
&= -g(V, A_X Y)X = -g(A_X^* V, Y)X = 0.
\end{aligned}$$

□

Since all fibers of π are totally geodesic, by O'Neill's formula ([16]), we see that $K(X, U) = \|A_X^* U\|^2$. Because (M^4, g) is Einstein, at point p, we have

$$\begin{aligned}
\lambda &= Ric(U, U) = c_1 + \|A_X^* U\|^2 + \|A_Y^* U\|^2;\\
\lambda &= Ric(V, V) = c_1 + \|A_X^* V\|^2 + \|A_Y^* V\|^2;
\end{aligned}$$

Combined with Lemma 4.3, we see that $\lambda = c_1$ and $\|A_X^* U\|^2(p) = 0$, $\|A_Y^* U\|^2(p) = 0$. Then $\|A\|^2(p) = 0$. Hence $\|A\|^2 \equiv 0$ on M^4 and $c_1 = c_2$. Let $c = c_1 = c_2$. Then locally π is the projection of a metric product $B^2(c) \times B^2(c)$ onto one of the factors, where $B^2(c)$ is a two-dimensional compact manifold with constant curvature c.

5 Conjecture 1 and the Weak Hopf Conjecture

In this section we point out several interesting corollaries of Conjecture 1.

Suppose (E, g) is a complete, open Riemannian manifold with nonnegative sectional curvature. By a well known theorem of Cheeger and Gromoll [4], E contains a compact totally geodesic submanifold Σ, called the soul, such that E is diffeomorphic to the normal bundle of Σ. Let Σ_r be the distance sphere to Σ of radius r. Then for small $r > 0$, the induced metric on Σ_r has nonnegative sectional curvature by a theorem of Guijarro and Walschap [11]. In [9], Gromoll and Tapp proposed the following conjecture:

Weak Hopf Conjecture *Let $k \geq 3$. Then for any complete metric with nonnegative sectional curvature on $S^n \times \mathbb{R}^k$, the induced metric on the boundary of a small metric tube about the soul can not have positive sectional curvature.*

The case $n = 2, k = 3$ is of particular interest since the metric tube of the soul is diffeomorphic to $S^2 \times S^2$.

Recall that a map between metric spaces $f : X \to Y$ is a submetry if for all $x \in X$ and $r \in [0, r(x)]$ we have that $f(B(x, r)) = B(f(x), r)$, where $B(p, r)$ denotes the open metric ball centered at p of radius x and $r(x)$ is some positive continuous function. If both X and Y are Riemannian manifolds, then f is a Riemannian submersion of class $C^{1,1}$ by a theorem of Berestovskii and Guijarro [1].

Proposition 5.1 *Suppose Σ is a soul of (E, g), where (E, g) is a complete, open Riemannian manifold with nonnegative sectional curvature. If the induced metric on Σ_r has positive sectional curvature at some point for some $r > 0$, then there is a Riemannian submersion from Σ_r to Σ with fibers S^{l-1}, where $l = dim(E) - dim(\Sigma)$.*

Proof In fact, by a theorem of Guijarro and Walschap in [12], if Σ_r has positive sectional curvature at some point, the normal holonomy group of Σ acts transitively on Σ_r. By Corollary 5 in [26], we get a submetry $\pi : (E, g) \to \Sigma \times [0, +\infty)$ with fibers S^{l-1}, where $\Sigma \times [0, +\infty)$ is endowed with the product metric. Then $\pi : (\pi^{-1}(\Sigma \times (0, +\infty)), g) \to \Sigma \times (0, +\infty)$ is also a submetry. By a theorem of Berestovskii and Guijarro in [1], π is a $C^{1,1}$ Riemannian submersion. Then $\Sigma_r = \pi^{-1}(\Sigma \times \{r\})$ and $\pi : \Sigma_r \to \Sigma$ is also a $C^{1,1}$ Riemannian submersion with fibers S^{l-1}, where Σ_r is endowed with the induced metric from (E, g). □

Proposition 5.2 *When $k > n$, Conjecture 1 implies Weak Hopf Conjecture.*

Proof Suppose for some complete metric g on $S^n \times \mathbb{R}^k$ with nonnegative sectional curvature, the induced metric on Σ_r has positive sectional curvature for some $r > 0$, where Σ is a soul. Since $S^n \times \mathbb{R}^k$ is diffeomorphic to the normal bundle of Σ, we see that Σ is a homotopy sphere and $dim(\Sigma) = n$. By Proposition 5.1, we get a Riemannian submersion from Σ_r to Σ with fibers S^{k-1}, where Σ_r is endowed with the induced metric from g and hence has positive sectional curvature. Since $k > n$, we see $k - 1 \geq n$, which is impossible if Conjecture 1 is true for $C^{1,1}$ Riemannian submersions. □

Remark 3 If the question in Remark 1 after Proposition 1.2 has a positive answer, then by Proposition 5.1 again, any small metric tube about the soul can *not* have positive sectional curvature when the soul is odd-dimensional. This would give a solution to a question asked by Tapp in [21].

Acknowledgments This paper is a part of my Ph.D thesis at University of Notre Dame [5]. The author would like to express gratitude to his advisor, Professor Karsten Grove, for many helpful discussions. He also thanks Professor Anton Petrunin for discussing the proof of Theorem 3.1. The author benefits a lot from his "Exercises in orthodox geometry" [18]. He also would like to thank the referee for his helpful suggestions concerning the presentation of the paper.

References

1. Berestovskii, V.N., Guijarro, L.: A metric characterization of Riemannian submersions. Ann. Glob. Anal. Geom. **18**(6), 577–588 (2000)
2. Berger, M.: Sur les variéétés d'Einstein compactes, C. R. IIP Réunion Math. Expression Latine, Namur, pp. 35-55 (1965)
3. Besse, A.L.: Einstein Manifolds. Reprint of the 1987 edition. Classics in Mathematics. Springer, Berlin (2008)
4. Cheeger, J., Gromoll, D.: On the structure of complete manifolds of nonnegative curvature. Ann. Math. **96**(3), 413–443 (1972)
5. Chen, X.: Curvature and Riemannian Submersions. Ph.D thesis, University of Notre Dame, (2014)
6. Florit, L., Ziller, W.: Topological obstructions to fatness. Geom. Topol. **15**(2), 891–925 (2011)
7. Frankel, T.: Manifolds with positive curvature. Pac. J. Math. **11**, 165–174 (1961)
8. Gromoll, D., Grove, K.: The low-dimensional metric foliations of Euclidean spheres. J. Differ. Geom. **28**(1), 143–156 (1988)
9. Gromoll, D., Tapp, K.: Nonnegatively curved metrics on $S^2 \times \mathbb{R}^2$. Geom. Dedic. **99**, 127–136 (2003)
10. Gromoll, D., Walschap, G.: Metric Foliations and Curvature, vol. 268. Springer, Birkhäuser Verlag AG, Basel, Boston, Berlin (2009)
11. Guijarro, L., Walschap, G.: The metric projection onto the soul. Trans. Am. Math. Soc. **352**(1), 55–69 (2000)
12. Guijarro, L., Walschap, G.: The dual foliation in open manifolds with nonnegative sectional curvature. Proc. Am. Math. Soc. **136**(4), 1419–1425 (2008)
13. Hermann, R.: A sufficient condition that a mapping of Riemannian manifolds be a fibre bundle. Proc. Am. Math. Soc. **11**, 236–242 (1960)
14. Hitchin, N.: Compact four-dimensional Einstein manifolds. J. Differ. Geom. **9**, 435–441 (1974)
15. Kerin, M.: Some new examples with almost positive curvature. Geom. Topol. **15**, 217–260 (2011)
16. O'Neill, B.: The fundamental equations of a submersion. Mich. Math. J. **13**, 459–469 (1966)
17. Petersen, P.: Riemannian Geometry, Second edition. Graduate Texts in Mathematics, vol. 171. Springer, New York (2006)
18. Petrunin, A.: Exercises in Orthodox Geometry. arXiv:0906.0290v6 [math.HO] (2013)
19. Rovenski, V.: Foliations, submanifolds, and mixed curvature. J. Math. Sci. (New York) **99**(6), 1699–1787 (2000)
20. Tapp, K.: Metrics with nonnegative curvature on $S^2 \times \mathbb{R}^4$. Ann. Glob. Anal. Geom. **42**(1), 61–77 (2012)
21. Tapp, K.: Rigidity for odd-dimensional souls. Geom. Topol. **16**(2), 957–962 (2012)
22. Tondeur, P.: Geometry of Foliations. Monographs in Mathematics, vol. 90. Birkhäuser Verlag, Basel (1997)
23. Walschap, G.: Metric foliations and curvature. J. Geom. Anal. **2**(4), 373–381 (1992)
24. Wilking, B.: Index parity of closed geodesics and rigidity of Hopf fibrations. Invent. Math. **144**(2), 281–295 (2001)
25. Wilking, B.: Manifolds with positive sectional curvature almost everywhere. Invent. Math. **148**(1), 117–141 (2002)
26. Wilking, B.: A duality theorem for Riemannian foliations in nonnegative sectional curvature. Geom. Funct. Anal. **17**(4), 1297–1320 (2007)
27. Ziller, W.: Fatness Revisited. http://www.math.upenn.edu/~wziller/papers/Fat-09