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Finite group actions on 4-manifolds with nonzero Euler characteristic

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Abstract We prove that if X is a compact, oriented, connected 4-dimensional smooth manifold, possibly with boundary, satisfying $\chi(X) \neq 0$, then there exists a natural number C such that any finite group G acting smoothly and effectively on X has an abelian subgroup A generated by two elements which satisfies $[G : A] \leq C$ and $\chi(X^A) = \chi(X)$. Furthermore, if $\chi(X) < 0$ then A is cyclic. This answers positively, for any such X, a question of Étienne Ghys. We also prove an analogous result for manifolds of arbitrary dimension and non-vanishing Euler characteristic, but restricted to pseudofree actions.

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1 Introduction

1.1 Statement of the results

In this paper we prove two results on smooth finite group actions on compact, connected manifolds with non-vanishing Euler characteristic, and possibly with boundary. Our main result is on actions on 4-dimensional manifolds:

Theorem 1.1 *Let X be a compact, orientable, connected 4-dimensional smooth manifold, possibly with boundary, satisfying $\chi(X) \neq 0$. There exists a natural number C such that any finite group G acting smoothly and effectively on X has an abelian subgroup A satisfying $[G : A] \leq C$ and $\chi(X^A) = \chi(X)$. Furthermore, if $\chi(X) > 0$ then A can be generated by 2 elements, and if $\chi(X) < 0$ then A is cyclic.*

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To put Theorem 1.1 in context, recall the following classic theorem of Camille Jordan (see [12] and [4,20] for modern proofs).

Theorem 1.2 (Jordan) *For any natural number n there is some constant* J_n *such that any finite subgroup* $G \subset \mathrm{GL}(n, \mathbb{R})$ *has an abelian subgroup* A *satisfying* $[G : A] \leq \mathrm{J}_n$.

Let us say that a group $\mathcal{G}$ is Jordan if there is some constant C such that any finite subgroup $G \subseteq \mathcal{G}$ has an abelian subgroup A satisfying $[G : A] \leq C$ (this terminology was introduced a few years ago by Popov [24]). It is easy to deduce from Jordan's theorem, Peter–Weyl's theorem, and the existence and uniqueness up to conjugation of maximal compact subgroups, that any finite dimensional Lie group with finitely many connected components is Jordan.

In the mid 1990s, Ghys [7] raised the question of whether the diffeomorphism group of any compact manifold is Jordan. This question appeared in print in [6, Question 13.1].

The first statement of Theorem 1.1 gives a positive answer to Ghys's question for compact connected orientable 4-manifolds with nonzero Euler characteristic. Using the arguments in Subsection 2.3 of [20], one can deduce from Theorem 1.1 that the diffeomorphism groups of compact connected nonorientable 4-manifolds with nonzero Euler characteristic are Jordan (in both cases connectedness is not a crucial property, as long as the manifolds are compact and hence have finitely many connected components).

There are other cases in which Ghys's question is known to have an affirmative answer. In [20] it was proved that if a compact connected n-dimensional manifold X admits one-dimensional integral cohomology classes $\alpha_1, \ldots, \alpha_n$ whose product is nonzero then $\mathrm{Diff}(X)$ is Jordan. This applies for example to tori T^n of arbitrary dimension. Zimmermann [30] proved, using Perelman's proof of Thurston's geometrization conjecture, that if X is a compact 3-manifold then $\mathrm{Diff}(X)$ is Jordan.

In [21] it was proved that $\mathrm{Diff}(S^n)$ and $\mathrm{Diff}(\mathbb{R}^n)$ are Jordan for any n; the paper [21] also proves that if X is compact and has nonzero Euler characteristic then $\mathrm{Diff}(X)$ is Jordan. It should be noted that [21] uses a result of A. Turull and the author [23] which is based on the classification of finite simple groups (CFSG). In contrast, the present paper only uses very basic and standard techniques of finite transformation groups. Note on the other hand that the part of Theorem 1.1 which refers the the fixed point set of the abelian group does not follow from the results in [21].

Roughly one year after the first version of this paper appeared as a preprint [22], Csikós et al. [3] proved that $\mathrm{Diff}(T^2 \times S^2)$ is not Jordan, thus giving the first example of a compact manifold whose diffeomorphism group is not Jordan (previously Popov [25] had given a noncompact 4-dimensional example). It seems to be an interesting question to understand which compact 4-manifolds have Jordan diffeomorphism group (the author does not know any counterexample which is not an S^2-fibration over T^2).

Using more sophisticated methods than the present paper, McCooey has proved in [15,16] very strong restrictions on finite groups acting effectively and homologically trivially on general compact, oriented, connected and closed 4-manifolds satisfying $\chi \neq 0$. In particular, the main theorem in [15] implies that if X is a compact simply connected 4-manifold then $\mathrm{Diff}(X)$ is Jordan. The paper [16] contains results on actions on non simply connected compact 4-manifolds, but these results require, besides the homological triviality of the action, some technical restrictions on the manifold, or on the finite group which acts on it, or on the action, so they do not seem to apply to all actions of finite groups on closed 4-manifolds with nonzero Euler characteristic.

For other results, implying a positive answer to Ghys's question for 4-manifolds with vanishing first homology and $b_2 \leq 2$, see e.g., [9,17,18,29].

The following is an immediate consequence of Theorem 1.1.

Corollary 1.3 *Let X be a compact, oriented, connected 4-dimensional smooth manifold, possibly with boundary, satisfying $\chi(X) \neq 0$. There exist constants C, C' with the following properties.*

(1) *Any finite group acting effectively on X can be generated by C elements.*
(2) *For any action of a finite group G on X there exists some point $x \in X$ whose isotropy group satisfies $[G : G_x] \leq C'$.*

Using group theoretical results based on the CFSG, one can prove the first part of the previous corollary for any compact manifold X. Since to the best of the author's knowledge this has not appeared in the literature, we briefly explain the argument.[1] By the main result in [14], there exists an integer r such that, for any prime p, any elementary p-group acting effectively on X has rank at most r. Suppose that Γ is a p-group acting effectively on X; let Γ_0 be a maximal abelian normal subgroup of Γ. The action by conjugation identifies Γ/Γ_0 with a subgroup of $\mathrm{Aut}(\Gamma_0)$. Since Γ_0 can be generated by at most r elements, the Gorchakov–Hall–Merzlyakov–Roseblade lemma (see e.g., Lemma 5 in [26]) implies that Γ/Γ_0 can be generated by at most $r(5r-1)/2$ elements. Hence Γ can be generated by at most $r(5r+1)/2$ elements. According to a theorem proved independently by Guralnick and Lucchini [8,13], if all Sylow subgroups of a finite group G can be generated by at most k elements, then G itself can be generated by at most $k+1$ elements (both [8] and [13] use the CFSG). Hence any finite group acting effectively on X can be generated by at most $r(5r+1)/2+1$ elements.

Our second result is analogous to the first one. Whereas the class of manifolds to which it applies is much wider, it is limited to pseudofree actions. (Recall that an action of a group G on a manifold X is pseudofree if for any nontrivial $g \in G$ the fixed points of g are isolated.)

Theorem 1.4 *Let X be a compact connected manifold, possibly with boundary, with nonzero Euler characteristic. There exists a natural number C such that, if a finite group G acts pseudofreely, smoothly and effectively on X, then G has an abelian subgroup A satisfying $[G : A] \leq C$ and $\chi(X^A) = \chi(X)$, and A can be generated by* $[\dim X/2]$ *elements.*

This theorem is certainly far from answering Ghys's question for the manifolds to which it applies, since the restriction to pseudofree actions is very strong. A complete proof that these manifolds have Jordan diffeomorphism group appears in [21]. The reason we include this theorem in this paper is that the proof of Theorem 1.4 serves as a toy model for the proof of Theorem 1.1 (note that the proof in [21] uses the CFSG, while the arguments we use to prove Theorem 1.4 are completely elementary; in fact, the proof of Theorem 1.4 is similar to a standard proof of the Hurwitz bound on the size of automorphism groups of Riemann surfaces of genus ≥ 2, see [5, §V.1.3]).

1.2 Conventions, notation, and contents

By a natural number we understand a strictly positive integer. The symbol $\subset$ is reserved for strict inclusion. All manifolds in this paper will implicitly be assumed to be smooth and possibly with boundary, and all group actions on manifolds will be smooth. If G is a group and $S_1, \dots, S_r$ are subsets of G, $\langle S_1, \dots, S_r \rangle$ denotes the subgroup of G generated by the elements of $S_1, \dots, S_r$. When we say that a group G can be generated by d elements we mean that there are elements $g_1, \dots, g_d \in G$, *not necessarily distinct*, which generate G. If

[1] I thank A. Jaikin and E. Khukhro for explaining this argument to me.

a group G acts on a set X we denote the stabiliser of $x \in X$ by G_x, and for any subset $S \subseteq G$ we denote $X^S = \{x \in X \mid S \subseteq G_x\}$. If $g \in G$ we write X^g for $X^{\{g\}}$.

We will systematically use this convention: when we say that some quantity is A-bounded we mean that that quantity is bounded above by a function depending only on A; here A can either be a number, a manifold (then the upper bound depends on the diffeomorphism class of A), or a tuple of objects. This will hopefully make the reading lighter, but it will naturally prevent us from keeping track of the precise value of the bounds we obtain. In any case, due to the elementary nature of our arguments, the bounds that can be deduced are very likely far from optimal.

We close this introduction with a description of the contents of the paper. Section 2 contains several unrelated results which will be used in the subsequent sections. Section 3 contains the proof of Theorem 1.4. Section 4 contains the proof of Theorem 1.1. The last two sections contain some auxiliary results which are used in the proof of Theorem 1.1: Sect. 5 gathers some results on finite group actions on surfaces (in particular, Lemma 5.3 is the analogue of Theorem 1.1 for surfaces), and Sect. 6 contains some results on finite abelian groups actions on compact 4-manifolds and on C-rigid actions.

2 Preliminaries

2.1 Linearizing group actions

The following result is well known (see e.g., [21, Lemma 2.1]). It implies that the fixed point set of any (smooth) finite group action on a manifold with boundary is a neat submanifold in the sense of [10, §1.4].

Lemma 2.1 *Let a finite group G act smoothly on a manifold X, and let $x \in X^G$. The tangent space $T_x X$ carries a linear action of G, defined as the derivative at x of the action on X, satisfying the following properties.*

(1) *There exist neighborhoods $U \subset T_x X$ and $V \subset X$, of 0 and x resp., such that:*
   (a) *if $x \notin \partial X$ then there is a G-equivariant diffeomorphism $\phi \colon U \to V$;*
   (b) *if $x \in \partial X$ then there is G-equivariant diffeomorphism $\phi \colon U \cap \{\xi \geq 0\} \to V$, where ξ is a nonzero G-invariant element of $(T_x X)^*$ such that $\operatorname{Ker} \xi = T_x \partial X$.*

(2) *If the action of G is effective and X is connected then the action of G on $T_x X$ is effective, so it induces an inclusion $G \hookrightarrow \mathrm{GL}(T_x X)$.*

Lemma 2.2 *Let a finite group G act smoothly on a connected manifold X, and assume that $X^G \neq \emptyset$. Then G has an abelian subgroup A of X-bounded index.*

Proof Let $x \in X^G$. By (2) in Lemma 2.1 there is an embedding $G \hookrightarrow \mathrm{GL}(T_x X)$. The lemma follows from Theorem 1.2 applied to the image of this embedding. □

Lemma 2.3 *Let a finite group G act smoothly and preserving the orientation on a connected oriented manifold X. For any $\gamma \in G$, any connected component of the fixed point set X^γ is a neat submanifold of even codimension in X.*

Proof Combine Lemma 2.1 and the fact that for any $A \in \mathrm{SO}(n, \mathbb{R})$ the difference $n - \dim \operatorname{Ker}(A - \mathrm{Id})$ is even (note that X^γ is not necessarily connected, so it may have components of different dimensions). □

2.2 Finite group actions and cohomology

In the following two lemmas we denote by $b_j(Y; k)$ the jth Betti number of a space Y with coefficients in a field k.

Lemma 2.4 *Let Γ be a finite cyclic group acting on a compact manifold X and let $\gamma \in \Gamma$ be a generator. We have*

$$\chi(X^\Gamma) = \sum_j (-1)^j \operatorname{Tr}(H^j(\gamma) : H^j(X; \mathbb{Q}) \to H^j(X; \mathbb{Q})). \tag{1}$$

In particular, if the action of Γ on $H^*(X; \mathbb{Q})$ is trivial, then $\chi(X^\Gamma) = \chi(X)$. In general,

$$|\chi(X^\Gamma)| \leq \sum_j b_j(X; \mathbb{Q}). \tag{2}$$

Proof Formula (1) is classic, see Exercise 3 in [28, Chap III, 6.17]. To prove (2) note that, since γ has finite order, all the eigenvalues of $H^j(\gamma) : H^j(X; \mathbb{Q}) \to H^j(X; \mathbb{Q})$ have modulus one, so $|\operatorname{Tr}(H^j(\gamma) : H^j(X; \mathbb{Q}) \to H^j(X; \mathbb{Q}))| \leq b_j(X; \mathbb{Q})$. □

Lemma 2.5 *Let $\Gamma \simeq \mathbb{Z}_p$ act on a manifold X. Then*

$$\sum_j b_j(X^\Gamma; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p).$$

Proof This is [1, Theorem III.4.3]. □

2.3 CT and CTO actions

We say that the action of a group G on a manifold X is cohomologically trivial (CT for short) if the induced action of G on $H^*(X; \mathbb{Z})$ is trivial. If X is orientable, then we say that the action is CTO if it is CT and orientation preserving (this makes sense without having to specify an orientation, because a CT action preserves connected components). Of course, if X is closed and orientable then CT implies CTO, but for manifolds with boundary this is not the case.

Lemma 2.6 *For any compact manifold X and any finite group G acting on X there is a subgroup $G_0 \subseteq G$ such that $[G : G_0]$ is X-bounded and the action of G_0 on X is CTO.*

Proof Since X is compact, it has finitely many components, so any group acting on X has a subgroup of X-bounded index which acts preserving connected components and orientation preservingly. Furthermore, the cohomology of X is finitely generated as an abelian group. Let $T \subseteq H^*(X; \mathbb{Z})$ be the torsion. A classic result of Minkowski states that, given any integer k, the size of any finite subgroup of $\mathrm{GL}(k; \mathbb{Z})$ is k-bounded (see [19,27]). So if G is a finite group acting on X, there is a subgroup $G' \subseteq G$ of X-bounded index whose action on $H^*(X; \mathbb{Z})/T$ is trivial. There is also a subgroup $G'' \subseteq G'$ of index at most $|\operatorname{Aut}(T)|$ which acts trivially on T. Let $F := H^*(X; \mathbb{Z})/T$. In terms of a splitting $H^*(X; \mathbb{Z}) \simeq F \oplus T$, the action of G'' on $H^*(X; \mathbb{Z})$ is through lower triangular matrices with ones in the diagonal, so it factors through the group $\operatorname{Hom}(F, T)$, which is finite; hence, there is a subgroup $G_0 \subseteq G''$ of index at most $|\operatorname{Hom}(F, T)|$ whose action on $H^*(X; \mathbb{Z})$ is trivial. □

3 Pseudofree actions: proof of Theorem 1.4

3.1 The singular set and its projection to the orbit space

Consider an arbitrary action of a finite group G on a compact manifold X. Recall that the singular set of the action of G on X is

$$S_X = \bigcup_{g \in G \setminus \{1\}} X^g = \{x \in X \mid G_x \neq \{1\}\}, \tag{3}$$

Let $\pi : X \to Y := X/G$ denote the projection to the orbit space, and let $S_Y := \pi(S_X)$.

Lemma 3.1 *The cohomologies of the spaces Y, S_X and S_Y are finitely generated abelian groups, so $\chi(Y)$, $\chi(S_X)$ and $\chi(S_Y)$ are well defined. Furthermore, we have*

$$\chi(X) - \chi(S_X) = |G|(\chi(Y) - \chi(S_Y)).$$

Proof Let $(\mathcal{C}, \phi)$ be a G-regular triangulation of X. This means that $\mathcal{C}$ is a a G-regular finite simplicial complex (in the sense of Definition 1.2 of [2, Chapter III]—note that the G-regularity of $\mathcal{C}$ implies that $\mathcal{C}/G$ is a simplicial complex) and $\phi : X \to |\mathcal{C}|$ is a G-equivariant homeomorphism. Regular triangulations always exist: the second barycentric subdivision of an arbitrary equivariant triangulation of X (which exists e.g., by [11]) is automatically regular (see Proposition 1.1 in [2, Chapter III]).

The quotient $\mathcal{C}/G$ is a simplicial complex and the homeomorphism $\phi : X \to |\mathcal{C}|$ descends to a homeomorphism $\phi_Y : Y \to |\mathcal{C}/G|$ (here we use the homeomorphism $|\mathcal{C}|/G \simeq |\mathcal{C}/G|$ described at the end of Section 1 in [2, Chapter III]). Hence $H^*(Y; \mathbb{Z})$ is a finitely generated abelian group, so $\chi(Y)$ is well defined. Let $\mathcal{C}' = \{\sigma \in \mathcal{C} \mid G_\sigma \neq \{1\}\}$. The regularity of $\mathcal{C}$ implies that $\phi(S_X) = |\mathcal{C}'|$ and $\phi_Y(S_Y) = |\mathcal{C}'/G|$, which imply that $\chi(S_X)$ and $\chi(S_Y)$ are well defined. Since Euler characteristics can be computed counting simplices in triangulations, we have

$$\chi(X) - \chi(S_X) = \sum_{\sigma \in \mathcal{C} \setminus \mathcal{C}'} (-1)^{\dim \sigma}, \quad \chi(Y) - \chi(S_Y) = \sum_{[\sigma] \in (\mathcal{C}/G) \setminus (\mathcal{C}'/G)} (-1)^{\dim \sigma}.$$

Since G acts freely on $\mathcal{C} \setminus \mathcal{C}'$ (and, of course, preserving dimensions), we have

$$\sum_{\sigma \in \mathcal{C} \setminus \mathcal{C}'} (-1)^{\dim \sigma} = |G| \left(\sum_{[\sigma] \in (\mathcal{C}/G) \setminus (\mathcal{C}'/G)} (-1)^{\dim \sigma} \right),$$

which proves the lemma. □

3.2 Proof of Theorem 1.4

Consider a pseudofree effective action of a finite group G on a compact connected manifold X with nonzero Euler characteristic. By Lemma 2.6 we may replace G by a subgroup of X-bounded index and assume that G acts trivially on $H^*(X; \mathbb{Z})$. By Lemma 2.4, for any $\gamma \in G \setminus \{1\}$ the set X^γ consists of $\chi(X)$ points. This implies, if $\chi(X) < 0$, that $G = \{1\}$, so Theorem 1.4 is true in this case.

Let us assume for the rest of the proof that $\chi := \chi(X)$ is positive. Denote for convenience $d = |G|$. Since $S_X = \bigcup_{\gamma \in G \setminus \{1\}} X^\gamma$,

$$|S_X| \leq (d - 1)\chi.$$

Let $Y = X/G$. By [28, Chap II, Prop 9.13] we have $H_*(X;\mathbb{Q})^G \simeq H_*(Y;\mathbb{Q})$, so $\chi(Y) = \chi$. Lemma 3.1 gives

$$|S_Y| = \frac{(d-1)\chi + |S_X|}{d} \le \frac{2(d-1)\chi}{d} \le 2\chi.$$

This implies that the number r of G-orbits in S_X is at most 2χ. Let $d/a_1,\dots,d/a_r$ be the number of elements of the G-orbits in S_X, and assume that $a_1 \ge \cdots \ge a_r$. Then $|S_X| = \sum d/a_j$, so Lemma 3.1 implies that

$$\frac{d}{\chi a_1} + \cdots + \frac{d}{\chi a_r} - 1 = \frac{d(r-\chi)}{\chi}.$$

The following lemma implies that $d/(\chi a_1)$ is (χ, r)-bounded, hence X-bounded.

Lemma 3.2 *Suppose that $d, e_1,\dots,e_l, a$ are positive integers satisfying: $e_1 \ge \cdots \ge e_l$, each e_j divides d, and*

$$\frac{d}{e_1} + \cdots + \frac{d}{e_l} - 1 = \frac{dt}{a}. \tag{4}$$

for some integer t. Then d/e_1 is (a,l)-bounded.

Proof Consider for any $(l,a) \in \mathbb{N}^2$ the set $\mathcal{S}(l,a) \subset \mathbb{N}^{l+1} \times \mathbb{Z}$ consisting of tuples $(d, e_1,\dots,e_l,t)$ satisfying (4), $e_1 \ge \cdots \ge e_l$ and $e_j | d$ for each j. Define $D\colon \mathbb{N}\times\mathbb{N}\to\mathbb{N}$ recursively as follows: $D(1,a) := a$ and $D(l,a) := \max\{D(l-1,aj) \mid 1 \le j \le al\}$ for each $l > 1$. In fact $D(l,a) = D(l-1,a^2 l)$.

We prove that for any $(d,e_1,\dots,e_l,t) \in \mathcal{S}(l,a)$ we have $e_1 \ge d/D(l,a)$ using induction on l. For the case $l=1$, suppose that $(d,e_1,t) \in \mathcal{S}(1,a)$ and let $d = e_1 g$, where $g \in \mathbb{N}$. Rearranging (4) we deduce that g divides a, which implies $g \le a$, so $e_1 = d/g \ge d/a = d/D(1,a)$. Now assume that $l > 1$ and that the inequality has been proved for smaller values of l. Let $(d,e_1,\dots,e_l,t) \in \mathcal{S}(l,a)$. Since each e_j divides d, we have $d/e_j \ge 1$ for each j, so the left hand side in (4) is positive. This implies that $t \ge 1$. Using $e_1 \ge \cdots \ge e_l$ we can estimate $d/a \le ld/e_l$, so $1 \le e_l \le al$. Furthermore, (4) implies

$$\frac{d}{e_1} + \cdots + \frac{d}{e_{l-1}} - 1 = \frac{dt}{a} - \frac{d}{e_l} = \frac{d(te_l - a)}{ae_l},$$

so $(d, e_1,\dots,e_{l-1}, te_l - a)$ belongs to $\mathcal{S}(l-1,aj)$ for some $1 \le j \le al$. Using the induction hypothesis we deduce that $e_1 \ge d/D(l-1,aj) \ge d/D(l,a)$. □

Let $x \in S_X$ be one of the points whose G-orbit has d/a_1 elements. Then $[G : G_x] = d/a_1$ is X-bounded. By Lemma 2.2 there is an abelian subgroup $G_a \subseteq G_x$ of X-bounded index. Since G_a is abelian and can be identified with a subgroup of $\mathrm{GL}(\dim X, \mathbb{R})$ (see the proof of Lemma 2.2) it follows that there exists a subgroup $G_b \subseteq G_a$ of $(\dim X)$-bounded index which can be generated by $[\dim X/2]$ elements. Let $\gamma \in G_b$ be any nontrivial element. Since $X^{G_b} \subseteq X^\gamma$ and X^γ consists of χ points, the subgroup $A \subseteq G_b$ fixing each element of X^γ satisfies $[G_b : A] \le \chi!$ and $\gamma \in A$. The latter implies $X^A \subseteq X^\gamma$ so $X^A = X^\gamma$. Since A is a subgroup of an abelian subgroup which can be generated by $[\dim X/2]$ elements, A can also be generated by $[\dim X/2]$ elements. Finally, the index $[G : A]$ is X-bounded, so the proof of Theorem 1.4 is complete.

4 Proof of Theorem 1.1

4.1 Basic idea of the proof: C-rigid actions

Let G be a finite group acting effectively on a compact, connected and oriented 4-manifold X satisfying $\chi(X) \neq 0$. Roughly speaking, the proof of Theorem 1.1 is based, as the proof of Theorem 1.4, on estimating the Euler characteristic of the singular set $S_X = \bigcup_{g\in G\setminus\{1\}} X^g$ and deducing the existence of some point $x \in X$ whose stabilizer has X-bounded index $[G : G_x]$.

However, estimating in a useful way $\chi(S_X)$ for general actions is much more difficult than in the case of pseudofree actions. If the action is trivial in cohomology (which we may assume, replacing G by a subgroup of X-bounded index) then $\chi(X^g) = \chi(X)$ for every $g \in G$, but to compute $\chi(S_X)$ (say, using the inclusion–exclusion principle) one needs to control the numbers $\chi(X^{g_1} \cap \cdots \cap X^{g_k})$ for different $g_1, \ldots, g_k \in G$, and there is no general formula for this quantity.[2]

To circumvent this difficulty we replace the singular set S_X by a set $S'_X \subset X$ whose Euler characteristic is much easier to compute and which is in some sense a *uniform approximation* of S_X; by the latter we mean that there exist X-bounded constants, $1 < C_1 \leq C_2$, independent of G, such that the isotropy group of any point in S'_X (resp. in the complementary of S'_X) has at least C_1 (resp. at most C_2) elements. The actual definition of S'_X uses the notion of C-rigid subgroup of G, which we next explain (see Sect. 6 for more details).

Let C be a natural number. We say that the action of a subgroup $A \subseteq G$ is C-rigid if A is abelian and for any subgroup $A' \subseteq A$ satisfying $[A : A'] \leq C$ we have $X^{A'} = X^A$. Sometimes, abusing terminology, we simply say that A is C-rigid. The following two trivial properties of C-rigidity will be implicitly used in our arguments. First, if $C \leq C'$ and $A \subseteq G$ is a C'-rigid subgroup then A is also C-rigid. Second, if $A \subseteq G$ is C-rigid, and $A_0 \subseteq A$ is a subgroup, then A_0 is C_0-rigid for any C_0 such that $C_0[A : A_0] \leq C$.

In Sect. 6 we prove the following properties for any finite group action G on X:

(a) there exists some X-bounded constant C_χ such that for any C_χ-rigid subgroup $A \subseteq G$ we have $\chi(X^A) = \chi(X) \neq 0$ and each connected component of X^A is even dimensional (Lemma 6.5);
(b) for any C there exists a (C, X)-bounded constant Λ_C such that any abelian subgroup $A \subseteq G$ has a C-rigid subgroup $A_0 \subseteq A$ satisfying $[A : A_0] \leq \Lambda_C$ (Lemma 6.4); more precisely, Λ_C will denote the minimal number with that property, and this implies that Λ_C is a nondecreasing function of C.

To define S'_X we take a suitable X-bounded number C and we set $S'_X = \bigcup_A X^A$, where A runs over the set of nontrivial C-rigid subgroups of G. This is an approximation of S_X in the previous sense: property (b) and Jordan's theorem guarantees that if $x \in X\setminus S'_X$ then G_x can not be too big, whereas the definition of rigidity implies that if A is nontrivial and C-rigid then $|A| > C$, from which we deduce that if $x \in S'_X$ then $|G_x| > C$.

The actual definition of C is given in formula (5) below. The reader should think of C as a big but X-bounded number. The choice of C guarantees that each connected component of S'_X has the same Euler characteristic as X. The action of G on X induces an action on the set of connected components of S'_X, and we will prove that the number of G-orbits of connected components of S'_X is X-bounded (Lemma 4.6). From this we will deduce, using the same

[2] However, for some restricted classes of groups acting on X one can study in detail the topology of the singular set; in the case of minimal non-abelian groups, this is done in [15,16], and it is the crucial ingredient of the proofs.

arithmetic arguments as in the proof of Theorem 1.4, that there is some point in X satisfying $[G : G_x] \leq C'$, where C' is X-bounded.

4.2 Details of the proof

The next three paragraphs are devoted to proving some useful properties of rigid group actions. The proof of Theorem 1.1 is in Sect. 4.2.4.

4.2.1 J_4-rigid groups

Here J_4 refers to the constant in Jordan's Theorem 1.2 for $n = 4$.

Lemma 4.1 *Suppose that G is a finite group acting on X and that $A_1, A_2 \subseteq G$ are abelian subgroups satisfying $X^{A_1} \cap X^{A_2} \neq \emptyset$. If A_1 is J_4-rigid then there is a subgroup $A_2' \subseteq A_2$ such that $[A_2 : A_2'] \leq \mathrm{J}_4$ and A_2' preserves X^{A_1}.*

Proof Let $\Gamma = \langle A_1, A_2 \rangle \subseteq G$. Since $X^{\Gamma} = X^{A_1} \cap X^{A_2} \neq \emptyset$, Lemma 2.2 implies that there exists an abelian subgroup $H \subseteq \Gamma$ satisfying $[\Gamma : H] \leq \mathrm{J}_4$. Since $[A_1 : A_1 \cap H] \leq \mathrm{J}_4$ and A_1 is J_4-rigid, we have $X^{A_1} = X^{A_1 \cap H}$. Let $A_2' = A_2 \cap H$. Then $[A_2 : A_2'] \leq \mathrm{J}_4$. Finally, since H is abelian the action of $A_2' \subseteq H$ on X preserves $X^{A_1 \cap H} = X^{A_1}$. □

4.2.2 The constant C

Define, for any compact manifold Y, the following numbers

$$b_+(Y) := \sum_{j \geq 0} \max\{b_j(Y; \mathbb{F}_p) \mid p \text{ prime}\}, \quad b_-(Y) := \sum_{j \geq 0} \min\{b_j(Y; \mathbb{F}_p) \mid p \text{ prime}\}$$

and denote by $\mathcal{S}(X)$ the set of diffeomorphism classes of compact connected surfaces Σ satisfying $b_-(\Sigma) \leq b_+(X)$. Abusing language, we will sometimes say that a surface belongs to $\mathcal{S}(X)$ meaning that its diffeomorphism type belongs to $\mathcal{S}(X)$. Note that $\mathcal{S}(X)$ is never empty, as it always contains S^2.

By Lemma 5.3 (which is the analogue for surfaces of Theorem 1.1), for any compact surface Σ there exists a Σ-bounded natural number $C(\Sigma)$ such that any finite group G acting effectively on Σ has an abelian subgroup A satisfying $[G : A] \leq C(\Sigma)$ and $\chi(\Sigma^A) = \chi(\Sigma)$. The classification theorem of compact connected surfaces implies that

$$C_{\mathrm{surf}} = \max\{C(\Sigma) \mid \Sigma \text{ compact surface}, \Sigma \in \mathcal{S}(X)\}$$

is finite and X-bounded. This is well defined because $\mathcal{S}(X) \neq \emptyset$, and we have $C_{\mathrm{surf}} \geq 1$.

By Lemma 6.2 there is some X-bounded constant C_f such that for any finite abelian subgroup A acting on X the fixed point set X^A has at most C_f connected components. Recall that C_χ denotes an X-bounded constant with the property that if a finite group G acts effectively on X and $A \subseteq G$ is any C_χ-rigid subgroup then $\chi(X^A) = \chi(X)$ and each connected component of X^A is even dimensional (see Lemma 6.5). Let

$$C_\delta = \max\{C_\chi, C_f C_{\mathrm{surf}}\}.$$

The following lemma shows part of the significance of the number C_δ.

Lemma 4.2 *Let G be a finite group acting on X in a CTO way, and let $A_1, A_2 \subseteq G$ be C_δ-rigid subgroups. If the intersection $A_1 \cap A_2$ is nontrivial then $X^{A_1} \cap X^{A_2} \neq \emptyset$.*

Proof Let $a \in A_1 \cap A_2$ be a nontrivial element. Since the action of G is CTO, by Lemma 2.4 we have $\chi(X^a) = \chi(X)$ and by Lemma 2.3 each connected component of X^a is even dimensional. Choose a connected component $Y \subseteq X^a$ satisfying $\chi(Y) \neq 0$. The group A_i preserves X^a, and the subgroup $A_i' \subseteq A_i$ preserving Y satisfies $[A_i : A_i'] \leq C_f$. Since A_i is C_δ-rigid we have $X^{A_i'} = X^{A_i}$. If Y is a point then $\emptyset \neq Y \subseteq X^{A_1'} \cap X^{A_2'}$, hence $X^{A_1} \cap X^{A_2} \neq \emptyset$. If Y is a surface then $Y \in \mathcal{S}(X)$ (Lemma 5.1) so by Lemma 5.3 the group $\Gamma = \langle A_1', A_2' \rangle$ (which acts on Y) has an abelian subgroup $A \subseteq \Gamma$ of index $[\Gamma : A] \leq C_{\text{surf}}$ and satisfying $\chi(Y^A) = \chi(Y) \neq 0$, so $Y^A \neq \emptyset$. We have $X^{A_i' \cap A} = X^{A_i}$ because of C_δ-rigidity so

$$X^{A_1} \cap X^{A_2} \subseteq X^\Gamma \subseteq X^A \subseteq X^{A_1' \cap A} \cap X^{A_2' \cap A}$$

implies $X^{A_1} \cap X^{A_2} = X^A$, and $X^A \neq \emptyset$ because $Y^A \subseteq X^A$. □

Let Λ_δ be the X-bounded number, given by Lemma 6.4, with the property that any abelian finite group A acting on X has a C_δ-rigid subgroup A_0 of index at most Λ_δ. Define the following constant:

$$C := \max\{C_\chi, \mathrm{J}_4\, C_f C_{\text{surf}}, \mathrm{J}_4\, \Lambda_\delta, 2\,\mathrm{J}_4\}. \tag{5}$$

The expression in the right hand side is redundant, since Λ_δ can not be smaller than C_χ; we include the constant C_χ inside the maximum for clarity.

4.2.3 Properties of C-rigid groups

In the following two lemmas we prove that C-rigid subgroups of finite groups acting on X have particularly nice properties.

Lemma 4.3 *Suppose that G is a finite group acting on X and that $A_1, A_2, A_3 \subseteq G$ are C-rigid subgroups satisfying $X^{A_1} \cap X^{A_2} \neq \emptyset \neq X^{A_1} \cap X^{A_3}$. Then $X^{A_2} \cap X^{A_3} \neq \emptyset$.*

Proof By Lemma 4.1 there exist subgroups $A_2' \subseteq A_2$ and $A_3' \subseteq A_3$ satisfying $[A_j : A_j'] \leq \mathrm{J}_4$ such that both A_2' and A_3' preserve X^{A_1}. Let $\Gamma = \langle A_2', A_3' \rangle \subseteq G$. Since A_1 is C_χ-rigid, there is a connected component $Y \subseteq X^{A_1}$ such that $\chi(Y) \neq 0$. Since X^{A_1} has at most C_f connected components, the subgroup $\Gamma' \subseteq \Gamma$ preserving Y satisfies $[\Gamma : \Gamma'] \leq C_f$. The subvariety Y is even dimensional, so it is either a point or an element of $\mathcal{S}(X)$ (Lemma 6.1). In the first case we have $Y \subseteq X^{A_2' \cap \Gamma'} \cap X^{A_3' \cap \Gamma'} = X^{A_2} \cap X^{A_3}$, the second equality following from $[A_j : A_j' \cap \Gamma'] \leq C$ $(j = 2, 3)$ and rigidity. If $Y \in \mathcal{S}(X)$ then by Lemma 5.3 there is a subgroup $\Gamma'' \subseteq \Gamma'$ satisfying $[\Gamma' : \Gamma''] \leq C_{\text{surf}}$ and $\chi(Y^{\Gamma''}) = \chi(Y) \neq 0$, so $Y^{\Gamma''} \neq \emptyset$. So $Y^{\Gamma''} \subseteq X^{A_2' \cap \Gamma''} \cap X^{A_3' \cap \Gamma''} = X^{A_2} \cap X^{A_3}$, again because of $[A_j : A_j' \cap \Gamma''] \leq C$ and rigidity. □

Lemma 4.4 *Suppose that G is a finite group acting on X and that $A_1, \ldots, A_r \subseteq G$ are C-rigid subgroups (with r arbitrary) satisfying $X^{A_1} \cap X^{A_j} \neq \emptyset$ for every j. Let $Z := X^{A_1} \cap \cdots \cap X^{A_r}$. Then there is a C_δ-rigid subgroup $A \subseteq G$ such that $Z = X^A$. In particular (since $C_\delta \geq C_\chi$), $\chi(Z) = \chi(X)$ (so $Z \neq \emptyset$), every connected component of Z is even dimensional, and Z has at most C_f connected components.*

Proof We first prove that $Z \neq \emptyset$. Assume that $r \geq 2$, otherwise the claim is trivial. The proof of the claim is very similar to that of the previous lemma. For every $j \geq 2$ there exists a subgroup $A_j' \subseteq A_j$ satisfying $[A_j : A_j'] \leq \mathrm{J}_4$ such that A_j' preserves X^{A_1}. Let

$\Gamma = \langle A_2', \dots, A_r' \rangle \subseteq G$. Let $Y \subseteq X^{A_1}$ be a connected component such that $\chi(Y) \neq 0$. The subgroup $\Gamma' \subseteq \Gamma$ preserving Y satisfies $[\Gamma : \Gamma'] \leq C_f$. If Y is a point then setting $A_j'' := A_j' \cap \Gamma'$ we have $Y \subseteq X^{A_2''} \cap \dots \cap X^{A_r''}$ and $[A_j : A_j''] \leq C$, so by rigidity $X^{A_j''} = X^{A_j}$ for every j, which implies that $Z = X^{A_1''} \cap \dots \cap X^{A_r''}$, and we are done. If $Y \in \mathcal{S}(X)$ then there is a subgroup $\Gamma'' \subseteq \Gamma'$ satisfying $[\Gamma' : \Gamma''] \leq C_{\text{surf}}$ and $\chi(Y^{\Gamma''}) = \chi(Y) \neq 0$, so $Y^{\Gamma''} \neq \emptyset$. Setting $A_j'' := A_j' \cap \Gamma''$ we have $Y^{\Gamma''} \subseteq X^{A_2''} \cap \dots \cap X^{A_r''}$ and $[A_j : A_j''] \leq C$, and the proof is finished as in the case where Y is a point.

Let $T = \langle A_1, \dots, A_r \rangle \subseteq G$. We have $X^T = Z \neq \emptyset$, so by Lemma 2.2 there is an abelian subgroup $H \subseteq T$ of index at most J_4. By Lemma 6.4 there is a C_χ-rigid subgroup $H_\chi \subseteq H$ satisfying $[H : H_\chi] \leq \Lambda_\chi$. Let $A_j' = A_j \cap H_\chi$. Then $[A_j : A_j'] \leq C$, so $X^{A_j} = X^{A_j'}$ for every j because A_j is C-rigid. Thus

$$Z = X^\Gamma \subseteq X^{H_\chi} \subseteq X^{A_1'} \cap \dots \cap X^{A_r'} = Z,$$

which implies $Z = X^{H_\chi}$. Hence $A := H_\chi$ has the desired property. □

4.2.4 The sets $\mathcal{F}$ and $\mathcal{H}$ and the proof of Theorem 1.1

Recall our assumptions: G is a finite group acting effectively on a compact, oriented and connected 4-manifold X satisfying $\chi(X) \neq 0$. By Lemma 2.6 we may (and do) replace G by a subgroup of X-bounded index whose action on X is CTO. Let C be the constant defined in (5) above. Define the following collection of (not necessarily connected) submanifolds of X:

$$\mathcal{F} = \{X^A \mid A \subseteq G,\ A \text{ is nontrivial and } C\text{-rigid}\}.$$

The action of G on X induces an action on $\mathcal{F}$, since $g\,X^A = X^{gAg^{-1}}$ for any $g \in G$ and $A \subseteq G$ is C-rigid if and only if gAg^{-1} is. Let $\approx$ be the relation between elements of $\mathcal{F}$ which identifies $F, F' \in \mathcal{F}$ whenever $F \cap F' \neq \emptyset$. By Lemma 4.3 this is an equivalence relation. Let

$$\mathcal{H} := \mathcal{F}/\approx\,.$$

The action of G on $\mathcal{F}$ preserves the relation $\approx$, so it descends to an action on $\mathcal{H}$.

We are going to prove in Lemma 4.6 below that $|\mathcal{H}/G|$ is X-bounded. Before that, we introduce some notation and a preliminary result (Lemma 4.5). For any $H \in \mathcal{H}$ define

$$X_H := \bigcap_{F \in \mathcal{F},\, [F]=H} F.$$

Then $\{X_H \mid H \in \mathcal{H}\}$ is a collection of disjoint (not necessarily connected) submanifolds of X. By Lemma 4.4 we have $\chi(X_H) = \chi(X) \neq 0$, all connected components of X^H are even dimensional, and X_H has at most C_f connected components (recall that C_f is defined a few lines before Lemma 4.2).

We denote by $G_H \subseteq G$ the isotropy group of each element $H \in \mathcal{H}$. The action of G_H on X preserves X_H. Let

$$G(H) = \{g \in G \mid \exists Y \subset X \text{ such that } Y \text{ is a connected component of } X_H \text{ and of } X^g\}.$$

Note that unlike G_H the subset $G(H)$ is not a subgroup of G.

Lemma 4.5 *There exist an X-bounded number C_1 such that for any $H \in \mathcal{H}$ satisfying $|G_H| > C_1$ we have $|G(H)| \geq |G_H|/(2C_1)$.*

Proof Choose a connected component $Y \subseteq X_H$ satisfying $\chi(Y) \neq \emptyset$. Since X_H has at most C_f connected components there is a subgroup $G'_H \subseteq G_H$ whose action on X_H preserves Y and such that $[G_H : G'_H] \leq C_f$. Recall that Y is even dimensional.

Suppose that Y consists of a unique point $y \in X$. Then $G'_H \subseteq G_y$, so $|G_y| \geq |G'_H| \geq |G_H|/C_f$. By Jordan's Theorem 1.2 there is an abelian subgroup $B \subseteq G_y$ satisfying $[G_y : B] \leq \mathrm{J}_4$. Since B acts linearly on T_yX there is a splitting $T_yX = L_1 \oplus L_2$ where L_1, L_2 are 2-dimensional linear subspaces of T_yX preserved by the action of B. Let $B_j \subseteq B$ be the subgroup fixing all vectors of L_j. We prove that $|B_1| \leq |B|/3$. If this were not true, then the subgroup $G_1 \subseteq G_y$ fixing all vectors of L_1 would satisfy $[G_y : G_1] \geq 2\,\mathrm{J}_4$. For any C-rigid subgroup $A \subseteq G_y \subseteq G$ we would have $X^A = X^{A \cap G_1}$ since $[A : A \cap G_1] \leq [G_y : G_1] \leq 2\,\mathrm{J}_4 \leq C$, which would imply by Lemma 2.1 that T_yX^A contains L_1. But since y is an isolated point of X_H, there must exist some nontrivial C-rigid subgroup $A \subseteq G$ such that $L_1 \nsubseteq T_yX^A$, a contradiction. Hence $|B_1| \leq |B|/3$, and for the same reason $|B_2| \leq |B|/3$. Now $B \setminus (B_1 \cup B_2) \subseteq G(H)$ so $|G(H)| \geq |B \setminus (B_1 \cup B_2)| \geq |B|/3 \geq |G_y|/(3\,\mathrm{J}_4) \geq |G_H|/(3\,\mathrm{J}_4\,C_f)$.

Now suppose that $\dim Y = 2$ so that $Y \in \mathcal{S}(X)$. By Lemma 5.3 there is a subgroup $G''_H \subseteq G'_H$ such that $\chi(Y^{G''_H}) = \chi(Y) \neq 0$ and such that $[G'_H : G''_H] \leq C_{\mathrm{surf}}$. We have $X^{G''_H} \neq \emptyset$, so by Lemma 2.2 there is an abelian subgroup $A \subseteq G''_H$ such that $[G''_H : A] \leq \mathrm{J}_4$. By Lemma 6.4 there exists an X-bounded number Λ with the property that A has a C-rigid subgroup $A_0 \subseteq A$ satisfying $[A : A_0] \leq \Lambda$. Let

$$C'_1 := C_f C_{\mathrm{surf}}\,\mathrm{J}_4\,\Lambda.$$

Then $[G_H : A_0] \leq C'_1$. If $|G_H| > C'_1$ then A_0 is nontrivial, so X^{A_0} is an element of $\mathcal{F}$. Since $\emptyset \neq Y^{G''_H} \subseteq X^{A_0}$, the $\approx$-class of X^{A_0} is H, hence $X_H \subseteq X^{A_0}$. Since Y is 2-dimensional and all connected components of X^{A_0} are even dimensional, it follows that $A_0 \setminus \{1\} \subseteq G(H)$. Since $A_0 \neq \{1\}$ we have $|G(H)| \geq |A_0 \setminus \{1\}| \geq |A_0|/2 \geq |G_H|/(2C'_1)$.

Hence setting $C_1 := \max\{C'_1, 3\,\mathrm{J}_4\,C_f\}$ the lemma holds true. □

Lemma 4.6 $|\mathcal{H}/G| \leq C_1 + 2C_1C_f$.

Proof Choose for each $H \in \mathcal{H}$ a C_δ-rigid subgroup $A(H) \subseteq G$ such that $X_H = X^{A(H)}$. If $H \neq H'$ are elements of $\mathcal{H}$, then since $X^{A(H)} \cap X^{A(H')} = \emptyset$ we have $A(H) \cap A(H') = \{1\}$ by Lemma 4.2. Consequently, $|\mathcal{H}| = s \leq |G|$. Let

$$\mathcal{H}_{\mathrm{small}} = \{H \in \mathcal{H} \mid |G_H| \leq C_1\}, \quad \mathcal{H}_{\mathrm{big}} = \{H \in \mathcal{H} \mid |G_H| > C_1\}.$$

Both subsets $\mathcal{H}_{\mathrm{small}}, \mathcal{H}_{\mathrm{big}} \subseteq \mathcal{H}$ are G-invariant. Each G-orbit in $\mathcal{H}_{\mathrm{small}}$ has at least $|G|/C_1$ elements, so the bound $|\mathcal{H}_{\mathrm{small}}| \leq |\mathcal{H}| \leq |G|$ implies that $\mathcal{H}_{\mathrm{small}}$ contains at most C_1 orbits, i.e., $|\mathcal{H}_{\mathrm{small}}/G| \leq C_1$. To estimate the number of orbits in $\mathcal{H}_{\mathrm{big}}$ we use the following:

$$|G| \cdot |\mathcal{H}_{\mathrm{big}}/G| = \sum_{H \in \mathcal{H}_{\mathrm{big}}} |G_H| \leq 2C_1 \sum_{H \in \mathcal{H}_{\mathrm{big}}} |G(H)| \leq 2C_1C_f|G|.$$

The equality follows from a simple counting argument, the first inequality follows from Lemma 4.5, and the second inequality follows from the fact that the submanifolds $\{X_H\}$ are disjoint and that for any $g \in G$ the number of connected components of X^g is at most C_f. Dividing both extremes by $|G|$ we deduce $|\mathcal{H}_{\mathrm{big}}/G| \leq 2C_1C_f$ which combined with the estimate on $|\mathcal{H}_{\mathrm{small}}/G|$ proves the lemma. □

For any $H \in \mathcal{H}$ let

$$Y_H = \bigcup_{F \in \mathcal{F},\,[F]=H} H.$$

By Lemma 4.4 and the inclusion–exclusion principle, we have $\chi(Y_H) = \chi(X)$. Let $H_1, \ldots, H_r \in \mathcal{H}$ be representatives of the orbits of the action of G on $\mathcal{H}$. Let

$$d = |G|$$

and let $e_j = |G_{H_j}|$. Since for different $H, H' \in \mathcal{H}$ we have $Y_H \cap Y_{H'} = \emptyset$, we have

$$\chi\left(\bigcup_{F\in\mathcal{F}} F\right) = \left(\frac{d}{e_1}\chi(Y_{H_1}) + \cdots + \frac{d}{e_r}\chi(Y_{H_r})\right) = \chi(X)\left(\frac{d}{e_1} + \cdots + \frac{d}{e_r}\right). \tag{6}$$

Lemma 4.7 *The difference $\chi(X) - \chi\left(\bigcup_{F\in\mathcal{F}} F\right)$ is divisible by d/a, where a is an X-bounded divisor of d.*

Proof Let Λ be the same X-bounded number as in the proof of Lemma 4.5. Let $x \in X$. If $|G_x| > \mathrm{J}_4\,\Lambda$ then, by Jordan's Theorem 1.2 and Lemma 6.4, there is a nontrivial C-rigid subgroup $A \subseteq G_x$, so $x \in X^A \subseteq \bigcup_{F\in\mathcal{F}} F$. So the isotropy group of any point in $X \setminus \bigcup_{F\in\mathcal{F}} F$ has at most $\mathrm{J}_4\,\Lambda$ elements. Now take a G-regular triangulation of X (see the proof of Lemma 3.1). The regularity of the triangulation implies that the isotropy group of any simplex is contained in the isotropy group of any of its points. Hence each orbit of simplexes in $X \setminus \bigcup_{F\in\mathcal{F}} F$ has size d/e, where e is a divisor of d and $e \leq \mathrm{J}_4\,\Lambda$, so e divides $a := \mathrm{GCD}(d, (\mathrm{J}_4\,\Lambda)!)$. Now $\chi(X) - \chi\left(\bigcup_{F\in\mathcal{F}} F\right)$ can be computed as the alternate sum of numbers of simplexes in each dimension which are not contained in $\bigcup_{F\in\mathcal{F}} F$. Grouping the simplexes in G-orbits, the result follows immediately. □

Combining the previous lemma with (6) we obtain the following equality:

$$\frac{d}{e_1} + \cdots + \frac{d}{e_r} - 1 = \frac{dt}{a},$$

where t is an integer and a is an X-bounded divisor of d. By Lemma 4.6, r is also X-bounded. We can assuming (reordering if necessary) that $e_1 \geq \cdots \geq e_r$. Lemma 3.2 gives $|G_{H_1}| = e_1 \geq d/K$ for some constant K depending only on r and a, so K is X-bounded. It follows that $[G : G_{H_1}] = d/e_1 \leq K$ is X-bounded.

By the arguments in the proof of Lemma 4.5, there is a subgroup $G_2 \subseteq G_{H_1}$ of X-bounded index such that $X_{H_1}^{G_2} \neq \emptyset$. By Lemma 2.2 and Lemmas 6.4 and 6.5 there exists an abelian subgroup $A \subseteq G_2$ of X-bounded index such that $\chi(X^A) = \chi(X)$, and A can be generated by 2 elements. If $\chi(X) < 0$ then, since $\chi(X^A) < 0$, there is at least one connected component of X^A which is a surface. If $\Sigma \subseteq X^A$ is one such component and $x \in \Sigma$, then the linearization of the action of A near x gives an embedding $A \hookrightarrow \mathrm{GL}(T_xX/T_x\Sigma)$ preserving the orientation and a metric (see Lemma 2.1), so we may identify A with a subgroup of $\mathrm{SO}(2, \mathbb{R})$; hence A is cyclic.

Since $[G : A]$ is X-bounded, the proof of Theorem 1.1 is now complete.

5 Finite groups acting on surfaces

In this section we consider finite group actions on surfaces. The main result is Lemma 5.3, which is the analogue in two dimensions of Theorem 1.1.

Lemma 5.1 *Let Σ be a compact connected surface. For any finite abelian group A acting on Σ the number of connected components of Σ^A is Σ-bounded.*

Proof It clearly suffices to consider nontrivial actions. So let A be a finite abelian group acting on Σ and assume that there is an element $a \in A$ acting nontrivially on Σ. We distinguish two possibilities.

If all connected components of Σ^a are zero dimensional then, by (2) in Lemma 2.4 we have $|\Sigma^a| = \chi(\Sigma^a) \leq b(\Sigma; \mathbb{Q}) := \sum_{j=0}^{2} b_j(\Sigma; \mathbb{Q})$. Since $\Sigma^A \subseteq \Sigma^a$, the result follows.

Now assume that Σ^a contains some one-dimensional component. Any such component is (diffeomorphic to) either a circle or a closed interval. For $j = 0, 1$ let $\Sigma^a_j \subseteq \Sigma^a$ denote the union of the connected components whose Euler characteristic is j. By (2) in Lemma 2.4 we have $|\pi_0(\Sigma^a_1)| \leq b(\Sigma; \mathbb{Q})$. Let us now bound $|\pi_0(\Sigma^a_0)|$, which is equal to the number of circles in Σ^a. The fact that Σ^a has a codimension one connected component implies, by (1) in Lemma 2.1, that a has order 2. Let $\langle a \rangle = \{1, a\}$. Then $\Sigma' := \Sigma/\langle a \rangle$ is a surface with corners, so it is homeomorphic to a surface with boundary. We may bound

$$\chi(\Sigma') = (\chi(\Sigma) + |\pi_0\left(\Sigma^a_1\right)|)/2 \geq \chi(\Sigma)/2$$

using an A-regular triangulation on Σ (see the proof of Lemma 3.1) and computing Euler characteristics in terms of counting simplices. As a topological surface, Σ' is the complementary in a compact connected surface S of finitely many disjoint open discs; $\chi(\Sigma')$ is equal to $\chi(S)$ minus the number of discs, and the latter can be identified with $|\pi_0(\partial\Sigma')|$. By the classification of compact connected surfaces we have $\chi(\Sigma) \leq 2$; this gives $\chi(\Sigma') \leq 2 - |\pi_0(\partial\Sigma')|$ or, equivalently,

$$|\pi_0(\partial\Sigma')| \leq 2 - \chi(\Sigma')$$

Each connected component of Σ^a_0 contributes to a connected component of $\partial\Sigma'$. We deduce that $|\pi_0(\Sigma^a_0)| \leq 2 - \chi(\Sigma)/2$.

To complete the argument in this case, note that $\Sigma^A \subseteq \Sigma^a$. This implies that Σ^A contains at most as many one-dimensional connected components as Σ^a, so we only need to bound the number of zero dimensional connected components (i.e., the isolated points) of Σ^A. Each isolated point in Σ^A is either an isolated point in Σ^a or belongs to a one-dimensional connected component of Σ^a. Since we have a bound on $|\pi_0(\Sigma^a)|$, it suffices to bound uniformly the number of isolated points in Σ^A which can belong to a given one-dimensional component of Σ^a. If $Y \subseteq \Sigma^a$ is one such component and Y contains an isolated point of Σ^A, then the action of A on Σ^a preserves Y and we can identify $\Sigma^A \cap Y$ with the fixed point set of the action of A on Y. To finish the proof it suffices to check that Y^A contains at most 2 points. Let $g \in A$ be an element acting nontrivially on Y. Then Y^g is a finite set of points, and $|Y^g| \leq b_0(Y; \mathbb{Q}) + b_1(Y; \mathbb{Q}) \leq 2$ by (2) in Lemma 2.4. Since $Y^A \subseteq Y^g$, the result follows. □

Lemma 5.2 *For any compact connected surface Σ and any finite abelian group A acting on Σ there is an abelian subgroup $A_0 \subseteq A$ such that $[A : A_0]$ is Σ-bounded and $\chi(\Sigma^{A_0}) = \chi(\Sigma)$.*

Proof Let Σ be a compact connected surface, and let an abelian group A act on Σ. By Lemma 2.6 there exists a subgroup $A' \subseteq A$ whose action on Σ is CT and $[A : A']$ is Σ-bounded. If the action of A' on Σ is trivial, then we set $A_0 := A'$ and we are done. Otherwise, there exists some $a \in A'$ acting nontrivially on Σ. By Lemma 2.4, $\chi(\Sigma^a) = \chi(\Sigma)$. By Lemma 5.1 the number of connected components of Σ^a is Σ-bounded. It follows that there exists a subgroup $A_0 \subseteq A$ of Σ-bounded index whose action on Σ^a preserves each connected component and is orientation preserving on each component of Σ^a. We claim that $\chi(\Sigma^{A_0}) = \chi(\Sigma^a)$. To prove this, it suffices to check that for any connected component

$Y \subseteq \Sigma^a$ we have $\chi(Y) = \chi(Y^{A_0})$. But each such Y is a closed manifold of dimension at most 1, so $\chi(Y) = \chi(Y^{A_0})$ follows from Lemma 2.4 and the fact that A_0 acts on Y preserving the orientation. □

Lemma 5.3 *For any compact connected surface Σ and any finite group G acting effectively on Σ then there is an abelian subgroup $A \subseteq G$ whose index $[G : A]$ is Σ-bounded and which satisfies $\chi(\Sigma^A) = \chi(\Sigma)$.*

Proof In view of Lemma 5.2 it suffices to check that, for any compact connected surface Σ, any finite group acting effectively on Σ has an abelian subgroup of Σ-bounded index. To prove this, suppose first that $\partial\Sigma$ is empty. If Σ is orientable, then the lemma is Theorem 1.3 in [20] (if furthermore $\chi(\Sigma) \neq 0$ then it also follows from Theorem 1.4 and Lemma 2.3 of the present paper). If Σ is not orientable, then the arguments of Section 2.3 in [20] allow to deduce the lemma from the orientable case. Now suppose that $\partial\Sigma$ is nonemtpy, say with k connected components. Let a finite group G act on Σ. Replacing G by a subgroup of index at most k, we can assume that G fixes one connected component $Y \subset \partial\Sigma$. Considering the restriction of the action to Y we get a morphism of groups $G \to \mathrm{Diff}(Y)$ which we claim to be injective. This follows from the fact that a finite order diffeomorphism of Σ which is the identity on Y is automatically the identity on the whole Σ, which in turn is a consequence of (1.b) in Lemma 2.1. So to finish the proof we need to prove that a finite subgroup of $\mathrm{Diff}(S^1)$ has an abelian subgroup of uniformly bounded index. This the simplest case of Theorem 1.4 in [20], but it can also be proved directly observing that, since all metrics in S^1 are isometric up to rescaling, choosing an invariant metric on S^1 gives an embedding of the group in a dihedral group. □

6 *C*-rigid group actions on 4-manifolds

In this section we prove some facts on finite group actions on compact 4-manifolds and on rigidity that were used in Sect. 4 when proving Theorem 1.1.

6.1 Bounding the number of components of fixed point sets

The following notation, which is recalled for convenience, was defined in Sect. 4.2.2. For any space Y with finitely generated homology we set $b_+(Y) := \sum_{j\geq 0} \max\{b_j(Y; \mathbb{F}_p) \mid p \text{ prime}\}$ and $b_-(Y) := \sum_{j\geq 0} \min\{b_j(Y; \mathbb{F}_p) \mid p \text{ prime}\}$. For any 4-dimensional oriented manifold X we denote by $\mathcal{S}(X)$ the set of diffeomorphism classes of compact connected surfaces Σ such that $b_-(\Sigma) \leq b_+(X)$.

Lemma 6.1 *Let X be a 4-dimensional compact connected oriented manifold X, and let H be a group acting nontrivially on X preserving the orientation. The connected components of X^H are neat submanifolds of dimensions 0, 1 or 2. Any two-dimensional connected component of X^H is diffeomorphic to an element of $\mathcal{S}(X)$.*

Proof That X^H is a (not necessarily connected) neat submanifold of X follows from (1.b) in Lemma 2.1. By Lemma 2.3, for any $h \in H$ the connected components of X^h are zero or two-dimensional; hence, the dimension of any connected component of X^H is at most two. To prove the last statement, suppose that $Y \subset X^H$ is a two-dimensional connected component. Let $h \in H$ be an element acting nontrivially; replacing h by a power h^r we may assume that the diffeomorphism of X induced by the action of h has primer order. Since the connected components of X^h have dimension at most 2, the inclusion $X^H \subset X^h$ implies that Y is a connected component of X^h. Then, by Lemma 2.5, $b_-(Y) \leq b_+(X)$, so Y is diffeomorphic to an element of $\mathcal{S}(X)$. □

Lemma 6.2 *For any compact 4-dimensional oriented manifold X and any finite abelian group A acting on X the number of connected components of X^A is X-bounded.*

Proof Let X be a 4-dimensional oriented manifold. Let A be a finite abelian group acting on X. If the action of A is trivial then there is nothing to prove. Otherwise, let $a \in A$ be an element acting nontrivially on X through a diffeomorphism of order p, where p is any prime. By Lemma 2.5 we have

$$\sum_j b_j(X^a; \mathbb{F}_p) \leq \sum_j b_j(X; \mathbb{F}_p) \leq b_+(X),$$

so X^a has at most $b_+(X)$ connected components, and each connected component $Y \subseteq X^a$ satisfies $b_-(Y) \leq b_+(X)$. Since $X^A \subseteq X^a$, it suffices to prove that for connected component of X^a contains an X-bounded amount of connected components of X^A. By Lemma 2.3 the connected components of X^a are either points or surfaces. Of course each isolated point in X^a contains at most one connected component of X^A. Now suppose that $Y \subseteq X^a$ is a surface. Then Y is diffeomorphic to some element of $\mathcal{S}(X)$. If $Y \cap X^a = \emptyset$, then there is nothing to prove. Otherwise, the action of A on X^a leaves Y fixed. By Lemma 5.1, the number of connected components of Y^A i Y-bounded. Since Y is diffeomorphic to an element of $\mathcal{S}(X)$, the argument is finished using the classification of compact surfaces, which implies that for every N the set of diffeomorphism types of compact surfaces Σ satisfying $b_-(\Sigma) \leq N$ is finite. □

Lemma 6.3 *For any compact 4-dimensional oriented manifold X, and any chain of inclusions $\emptyset \neq Y_1 \subsetneq Y_2 \subsetneq \cdots \subsetneq Y_r$ of neat*[3] *submanifolds of X satisfying $|\pi_0(Y_j)| \leq k$ for each j, we have $r \leq \binom{5+k}{5}$.*

Proof This is a particular case of Lemma 7.1 in [21]. □

6.2 Definition and basic results on C-rigid abelian group actions

Let A be a finite group acting on a compact 4-manifold X and let C be a natural number. Recall (see Sect. 4.1) that (the action of) A is said to be C-rigid if A is abelian and for any subgroup $A_0 \subseteq A$ satisfying $[A : A_0] \leq C$ we have $X^{A_0} = X^A$.

Lemma 6.4 *Let X be a compact connected 4-manifold. For any natural number C there exists a (C, X)-bounded constant Λ such that any finite abelian group A acting on X has a subgroup of index at most Λ whose action on X is C-rigid.*

Proof By Lemma 6.2 there is an X-bounded constant C_f such that for any finite abelian group A acting on X the fixed point set X^A has at most C_f connected components. Let $C' := \binom{5+C_f}{5}$. We prove that $\Lambda := C^{C'-1}$ has the stated property. Let A be a finite abelian group acting on X in a CTO way and assume by contradiction that no subgroup of A of index at most Λ is C-rigid. Then we may construct recursively a sequence of subgroups

[3] See [10, §1.4] for the definition of neat submanifold.

$A =: A_0 \supset A_1 \supset \cdots \supset A_{C'}$ satisfying $[A_i : A_{i+1}] \leq C$ and $X^{A_i} \subset X^{A_{i+1}}$ for each i; indeed, once $A_0, A_1, \ldots, A_i, i < C'$, have been constructed we have $[A : A_i] \leq C^i \leq C^{C'-1}$ so by our initial assumption on A the group A_i is not C-rigid; hence, we may pick a subgroup $A_{i+1} \subset A_i$ such that $[A_i : A_{i+1}] \leq C$ and $X^{A_i} \subset X^{A_{i+1}}$. By Lemma 6.2, each X^{A_i} has at most C_f connected components, so we obtain a contradiction with Lemma 6.3. ◻

Lemma 6.5 *Let X be a compact connected 4-manifold. There exists an X-bounded constant C_χ such that any finite abelian group A acting on X in a C_χ-rigid way satisfies $\chi(X^A) = \chi(X)$ and each connected component of X^A is even dimensional.*

Proof It suffices to prove that any finite abelian group A acting on X has a subgroup A' of X-bounded index such that $\chi(X) = \chi(X^{A'})$ and each connected component of $X^{A'}$ is even dimensional. So suppose that A is a finite abelian group acting on X. By Lemma 2.6 we may take a subgroup $A_1 \subseteq A$ of X-bounded index whose action on X is CTO. If A_1 acts trivially on X then we set $A' := A_1$ and we are done. Otherwise there exists some $a \in A_1$ whose action on X is nontrivial. By Lemma 2.4 $\chi(X^a) = \chi(X)$ and by Lemma 6.2 the number of connected components of X^a is X-bounded. Hence the subgroup $A_2 \subseteq A_1$ preserving each connected component of X^a and whose action on each connected component of X^a is orientation preserving has X-bounded index $[A_1 : A_2]$. By Lemma 5.2, Lemma 6.1, and the classification of compact surfaces, there is a subgroup $A_3 \subseteq A_2$ of X-bounded index such that for every two-dimensional connected component Y of X^a we have $\chi(Y^{A_3}) = \chi(Y)$. We may clearly assume that $a \in A_3$. Since the action of A_3 on each two-dimensional connected component $Y \subseteq X^a$ is orientation preserving, Y^{A_3} is even dimensional. For every zero-dimensional connected component $Y \subseteq X^a$ we obviously have $\chi(Y^{A_3}) = \chi(Y)$. Since a acts on X preserving the orientation, each connected component of X^a has dimension 0 or 2, by Lemma 2.3. It then follows, as in the proof of Lemma 5.2, that $\chi(X^{A_3}) = \chi(X^a) = \chi(X)$ and that each connected component of X^A is even dimensional. ◻

References

1. Borel, A.: Seminar on Transformation Groups, Annals of Mathematics Studies, vol. 46. Princeton University Press, Princeton (1960)
2. Bredon, G.E.: Introduction to Compact Transformation Groups, Pure and Applied Mathematics, vol. 46. Academic Press, New York (1972)
3. Csikós, B., Pyber, L., Szabó, E.: Diffeomorphism groups of compact 4-manifolds are not always Jordan. Preprint. arXiv:1411.7524
4. Curtis, C.W., Reiner, I.: Representation Theory of Finite Groups and Associative Algebras. AMS Chelsea Publishing, Providence (2006) **(Reprint of the 1962 original.)**
5. Farkas, H.M., Kra, I.: Riemann Surfaces, Graduate Texts in Mathematics, vol. 71. Springer, New York (1980)
6. Fisher, D.: Groups Acting on Manifolds: Around the Zimmer Program, Geometry, Rigidity, and Group Actions, pp. 72–157. Chicago Lectures in Mathematics University of Chicago Press, Chicago (2011). arXiv:0809.4849
7. Ghys, É.: The following talks: Groups of diffeomorphisms, Colóquio brasileiro de matemáticas, Rio de Janeiro (Brasil), July 1997; The structure of groups acting on manifolds, Annual meeting of the Royal Mathematical Society, Southampton (UK), March 1999; Some open problems concerning group actions, Groups acting on low dimensional manifolds, Les Diablerets (Switzerland), March 2002; Some open problems in foliation theory, Foliations 2006, Tokyo (Japan), September 2006
8. Guralnick, R.M.: On the number of generators of a finite group. Arch. Math. (Basel) **53**, 521–523 (1989)
9. Hambleton, I., Lee, R.: Finite group actions on $\mathbb{P}_2(\mathbb{C})$. J. Algebra **116**(1), 227–242 (1988)
10. Hirsch, M.W.: Differential Topology, Graduate Texts in Mathematics. Springer, New York (1976)
11. Illman, S.: Smooth equivariant triangulations of G-manifolds for G a finite group. Math. Ann. **233**, 199–220 (1978)

12. Jordan, C.: Mémoire sur les équations différentielles linéaires à intégrale algébrique. J. Reine Angew. Math. **84**, 89–215 (1878)
13. Lucchini, A.: A bound on the number of generators of a finite group. Arch. Math. (Basel) **53**, 313–317 (1989)
14. Mann, L.N., Su, J.C.: Actions of elementary p-groups on manifolds. Trans. Am. Math. Soc. **106**, 115–126 (1963)
15. McCooey, M.P.: Symmetry groups of four-manifolds. Topology **41**, 835–851 (2002)
16. McCooey, M.P.: Symmetry groups of non-simply-connected four-manifolds. Preprint. arXiv:0707.3834
17. Mecchia, M., Zimmermann, B.: On finite simple and nonsolvable groups acting on closed 4-manifolds. Pac. J. Math. **243**(2), 357–374 (2009)
18. Mecchia, M., Zimmermann, B.: On finite groups acting on homology 4-spheres and finite subgroups of SO(5). Topol. Appl. **158**(6), 741–747 (2011)
19. Minkowski, H.: Zur Theorie der positiven quadratischen Formen. J. Reine Angew. Math. **101**, 196–202 (1887) **(See also collected works. I, 212–218, Chelsea Publishing Company, 1967.)**
20. Mundet i Riera, I.: Jordan's theorem for the diffeomorphism group of some manifolds. Proc. Am. Math. Soc. **138**, 2253–2262 (2010)
21. Mundet i Riera, I.: Finite group actions on homology spheres and manifolds with nonzero Euler characteristic. Preprint. arXiv:1403.0383v3
22. Mundet i Riera, I.: Finite group actions on 4-manifolds with nonzero Euler characteristic. Preprint. arXiv:1312.3149v2
23. Mundet i Riera, I., Turull, A.: Boosting an analogue of Jordan's theorem for finite groups. Adv. Math. **272**, 820–836 (2015)
24. Popov, V.L.: On the Makar-Limanov, Derksen invariants, and finite automorphism groups of algebraic varieties. In: Peter Russells Festschrift, Proceedings of the Conference on Affine Algebraic Geometry Held in Professor Russells Honour, 15 June 2009, McGill University, Montreal, Volume 54 of Centre de Recherches Mathématiques CRM Proceedings and Lecture Notes, pp. 289-311 (2011)
25. Popov, V.L.: Finite subgroups of diffeomorphism groups. Preprint. arXiv:1310.6548
26. Roseblade, J.E.: On groups in which every subgroup is subnormal. J. Algebra **2**, 402–412 (1965)
27. Serre, J.-P.: Bounds for the Orders of the Finite Subgroups of $G(k)$, Group Representation Theory. EPFL Press, Lausanne (2007)
28. tom Dieck, T.: Transformation Groups, de Gruyter Studies in Mathematics, vol. 8. Walter de Gruyter & Co., Berlin (1987)
29. Wilczyński, D.: Group actions on the complex projective plane. Trans. Am. Math. Soc. **303**(2), 707–731 (1987)
30. Zimmermann, B.: On Jordan type bounds for finite groups acting on compact 3-manifolds. Arch. Math. **103**, 195–200 (2014)