Math. Z. (2015) 280:1093–1106
DOI 10.1007/s00209-015-1468-x

Mathematische Zeitschrift

Einstein locally conformal calibrated G_2-structures

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Received: 20 January 2014 / Accepted: 20 April 2015 / Published online: 15 May 2015

Abstract We study locally conformal calibrated G_2-structures whose underlying Riemannian metric is Einstein, showing that in the compact case the scalar curvature cannot be positive. As a consequence, a compact homogeneous 7-manifold cannot admit an invariant Einstein locally conformal calibrated G_2-structure unless the underlying metric is flat. In contrast to the compact case, we provide a non-compact example of homogeneous manifold endowed with a locally conformal calibrated G_2-structure whose associated Riemannian metric is Einstein and non Ricci-flat. The homogeneous Einstein metric is a rank-one extension of a Ricci soliton on the 3-dimensional complex Heisenberg group endowed with a left-invariant coupled SU(3)-structure (ω, Ψ), i.e., such that $d\omega = c\mathrm{Re}(\Psi)$, with $c \in \mathbb{R} - \{0\}$. Nilpotent Lie algebras admitting a coupled SU(3)-structure are also classified.

Mathematics Subject Classification 53C10 · 53C30 · 53C25

1 Introduction

We recall that a seven-dimensional smooth manifold M admits a G_2-structure if the structure group of the frame bundle reduces to the exceptional Lie group G_2. The existence of a G_2-structure is equivalent to the existence of a non-degenerate 3-form φ defined on the whole manifold (see for example [26]) and using this 3-form it is possible to define a Riemannian metric g_φ on M.

Research partially supported by the project PRIN *Varietà reali e complesse: geometria, topologia e analisi armonica*, the project FIRB *Differential Geometry and Geometric functions theory* and by GNSAGA (Indam) of Italy.

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If φ is parallel with respect to the Levi–Civita connection, i.e., $\nabla^{LC}\varphi = 0$, then the holonomy group is contained in G_2, the G_2-structure is called *parallel* and the corresponding manifolds are called G_2*-manifolds*. In this case, the induced metric g_φ is Ricci-flat. The first examples of complete metrics with holonomy G_2 were constructed by Bryant and Salamon [6]. Compact examples of manifolds with holonomy G_2 were obtained first by Joyce [24–26] and then by Kovalev [28] and by Corti, Haskins, Nordström, Pacini [12]. Incomplete Ricci-flat metrics of holonomy G_2 with a 2-step nilpotent isometry group N acting on orbits of codimension 1 were obtained in [9,20]. It turns out that these metrics are locally isometric (modulo a conformal change) to homogeneous metrics on solvable Lie groups, which are obtained as rank one extensions of a six-dimensional nilpotent Lie group endowed with an invariant SU(3)-structure of a special kind, known in the literature as *half-flat* [10].

Examples of compact and non-compact manifolds endowed with non-parallel G_2-structures were given for instance in [1,7–9,14,15,17,19,27,38]. In particular, in [9] conformally parallel G_2-structures on solvmanifolds, i.e., on simply connected solvable Lie groups, were studied. More in general, in [23] it was shown that a seven-dimensional compact Riemannian manifold M admits a locally conformal parallel G_2-structure if and only if it has as covering a Riemannian cone over a compact nearly Kähler 6-manifold such that the covering transformations are homotheties preserving the corresponding parallel G_2-structure.

By [5,11,18], it is evident that the Riemannian scalar curvature of a G_2-structure may be expressed in terms of the 3-form φ and its derivatives. More precisely, in [5] an expression of the Ricci curvature and the scalar curvature in terms of the four intrinsic torsion forms τ_i, $i = 0, \ldots, 3$, and their exterior derivatives was given. Moreover, using this it is possible to show that the scalar curvature has a definite sign for certain classes of G_2-structures.

If $d\varphi = 0$, the G_2-structure is called *calibrated* or *closed*. The geometry of this family of G_2-structures was studied in [11]. Furthermore, Bryant proved in [5] that if the scalar curvature of a closed G_2-structure is non-negative then the G_2-structure is parallel.

We say that a G_2-structure φ is *Einstein* if the underlying Riemannian metric g_φ is Einstein. In [5,11] it was proved, as an analogous of Goldberg conjecture for almost-Kähler manifolds, that on a compact manifold an Einstein (or, more in general, with divergence-free Weyl tensor [11]) calibrated G_2-structure has holonomy contained in G_2. In the non-compact case, Cleyton and Ivanov showed that the same result is true with the additional assumption that the G_2-structure is $*$-Einstein, but it still an open problem to see if there exist (even incomplete) Einstein metrics underlying calibrated G_2-structures. Recently, some negative results were proved in the case of non-compact homogeneous spaces in [16]. In particular, the authors showed that a seven-dimensional solvmanifold cannot admit any left-invariant calibrated G_2-structure inducing an Einstein metric g_φ unless g_φ is flat.

In the present paper, we are mainly interested in the geometry of *locally conformal calibrated* G_2-structures, i.e., G_2-structures whose associated metric is conformally equivalent (at least locally) to the metric induced by a calibrated G_2-structure.

In Sect. 3, we prove that a compact manifold endowed with an Einstein locally conformal calibrated G_2-structure has non-positive scalar curvature (and then has either zero or negative curvature if it is also connected) and we show that a compact homogeneous 7-manifold cannot admit an invariant Einstein locally conformal calibrated G_2-structure unless the underlying metric is flat.

In the last section, we give a non-compact example of a homogeneous manifold endowed with an Einstein locally conformal calibrated G_2-structure. The homogeneous manifold is a solvmanifold, thus this example and the aforementioned result of [16] highlight a different behaviour of calibrated and locally conformal calibrated G_2-structures. Moreover, the homogeneous Einstein metric is a rank-one extension of a Ricci soliton on the complex

Heisenberg group induced by a coupled SU(3)-structure (ω, Ψ) such that $d\omega = -\mathrm{Re}(\Psi)$. Recall that a half-flat SU(3)-structure is said to be *coupled* if $d\omega$ is proportional to $\mathrm{Re}(\Psi)$ at each point (see [37]). Finally, we classify nilpotent Lie groups admitting a left-invariant coupled SU(3)-structure, showing that the complex Heisenberg group is, up to isomorphisms, the only nilpotent Lie group admitting a coupled SU(3)-structure (ω, Ψ) whose associated metric is a Ricci soliton.

2 Preliminaries on G_2 and SU(3)-structures

Let $(e_1, \ldots, e_7)$ be the standard basis of $\mathbb{R}^7$ and $\left(e^1, \ldots, e^7\right)$ be the corresponding dual basis. We set

$$\varphi = e^{123} + e^{145} + e^{167} + e^{246} - e^{257} - e^{347} - e^{356},$$

where for simplicity e^{ijk} stands for the wedge product $e^i \wedge e^j \wedge e^k$ in $\Lambda^3((\mathbb{R}^7)^*)$. The subgroup of $\mathrm{GL}(7, \mathbb{R})$ fixing φ is G_2. The basis $(e^1, \ldots, e^7)$ is an oriented orthonormal basis for the underlying metric and the orientation is determined by the inclusion $G_2 \subset \mathrm{SO}(7)$. The group G_2 also fixes the 4-form

$$*\varphi = e^{4567} + e^{2367} + e^{2345} + e^{1357} - e^{1346} - e^{1256} - e^{1247},$$

where $*$ denotes the Hodge star operator determined by the associated metric and orientation.

We recall that a G_2-structure on a 7-manifold M is characterized by a positive 3-form φ. Indeed, it turns out that there is a $1-1$ correspondence between G_2-structures on a 7-manifold and 3-forms for which the bilinear form B_φ defined by

$$B_\varphi(X, Y) = \frac{1}{6}\, i_X\varphi \wedge i_Y\varphi \wedge \varphi$$

is positive definite, where i_X denotes the contraction by X. A 3-form φ for which B_φ is positive definite defines a unique Riemannian metric g_φ and volume form dV_φ such that for any couple of vectors X and Y on M the following relation holds

$$g_\varphi(X, Y) dV_\varphi = \frac{1}{6}\, i_X\varphi \wedge i_Y\varphi \wedge \varphi.$$

As in [11], we let

$$\varphi = \frac{1}{6}\varphi_{ijk} e^{ijk}$$

and define the $*$-Ricci tensor of the G_2-structure as

$$\rho^*_{sm} := R_{ijkl}\varphi_{ijs}\varphi_{klm}.$$

A G_2-structure is said to be $*$-Einstein if the traceless part of the $*$-Ricci tensor vanishes, i.e., if $\rho^* = \frac{s^*}{7} g$, where s^* is the trace of ρ^*.

On a 7-manifold endowed with a G_2-structure, the action of G_2 on the tangent spaces induces an action of G_2 on the exterior algebra $\Lambda^p(M)$, for any $p \geq 2$. In [4], it was shown that there are irreducible G_2-module decompositions

$$\Lambda^2((\mathbb{R}^7)^*) = \Lambda^2_7((\mathbb{R}^7)^*) \oplus \Lambda^2_{14}((\mathbb{R}^7)^*),$$
$$\Lambda^3((\mathbb{R}^7)^*) = \Lambda^3_1((\mathbb{R}^7)^*) \oplus \Lambda^3_7((\mathbb{R}^7)^*) \oplus \Lambda^3_{27}((\mathbb{R}^7)^*),$$

where $\Lambda^p_k((\mathbb{R}^7)^*)$ denotes an irreducible G_2-module of dimension k. Using the previous decomposition of p-forms, in [5] a simple expression of $d\varphi$ and $d*\varphi$ was obtained, where $*$ denotes the Hodge operator defined by the metric g_φ and the volume form dV_φ. More precisely, for any G_2-structure φ there exist unique differential forms $\tau_0 \in \Lambda^0(M)$, $\tau_1 \in \Lambda^1(M)$, $\tau_2 \in \Lambda^2_{14}(M)$, $\tau_3 \in \Lambda^3_{27}(M)$, such that

$$\begin{aligned} d\varphi &= \tau_0 * \varphi + 3\tau_1 \wedge \varphi + *\tau_3, \\ d * \varphi &= 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi, \end{aligned}$$

where $\Lambda^p_k(M)$ denotes the space of sections of the bundle $\Lambda^p_k(T^*M)$.

In the case of a closed G_2 structure we have

$$\begin{aligned} d\varphi &= 0, \\ d * \varphi &= \tau_2 \wedge \varphi. \end{aligned}$$

By the results of [5], the scalar curvature is given by

$$\mathrm{Scal}(g_\varphi) = -\frac{1}{2}|\tau_2|^2$$

and from this it is clear that it cannot be positive.

For a locally conformal calibrated G_2-structure φ one has $\tau_0 \equiv 0$ and $\tau_3 \equiv 0$, so

$$\begin{aligned} d\varphi &= 3\tau_1 \wedge \varphi, \\ d * \varphi &= 4\tau_1 \wedge *\varphi + \tau_2 \wedge \varphi, \end{aligned}$$

and taking the exterior derivative of the former it is easy to show that τ_1 is a closed 1-form. Moreover, in this case the scalar curvature has not a definite sign as one can check from its expression

$$\mathrm{Scal}(g_\varphi) = 12\delta\tau_1 + 30|\tau_1|^2 - \frac{1}{2}|\tau_2|^2,$$

where δ denotes the adjoint of the exterior derivative d with respect to the metric g_φ.

If the only nonzero intrinsic torsion form is τ_1, we have the so called *locally conformal parallel* G_2-structures. They are named in this way since a conformal change of the metric g_φ associated to a G_2-structure of this kind gives (at least locally) the metric induced by a parallel G_2-structure. In this case

$$\begin{aligned} d\varphi &= 3\tau_1 \wedge \varphi, \\ d * \varphi &= 4\tau_1 \wedge *\varphi. \end{aligned}$$

We will give an example of such a structure at the end of Sect. 4.

We recall that a six-dimensional smooth manifold admits an SU(3)-structure if the structure group of the frame bundle can be reduced to SU(3). It is possible to show that the existence of an SU(3)-structure is equivalent to the existence of an almost Hermitian structure (h, J, ω) and a unit $(3, 0)$-form Ψ.

Since SU(3) is the stabilizer of the transitive action of G_2 on the 6-sphere S^6, it follows that a G_2-structure on a 7-manifold induces an SU(3)-structure on any oriented hypersurface. If the G_2-structure is parallel, then the SU(3)-structure is half-flat [10]. In terms of the forms (ω, Ψ) this means $d(\omega \wedge \omega) = 0$, $d(\mathrm{Re}(\Psi)) = 0$.

In our computations we will use another characterization of SU(3)-structures which follows from the results of [22,36]. We describe it here. Consider a six-dimensional oriented real vector space V, a k-form on V is said to be *stable* if its GL(V)-orbit is open. Let

$A : \Lambda^5(V^*) \to V \otimes \Lambda^6(V^*)$ denote the canonical isomorphism given by $A(\gamma) = w \otimes \Omega$, where $i_w\Omega = \gamma$, and define for a fixed 3-form $\sigma \in \Lambda^3(V^*)$

$$K_\sigma : V \to V \otimes \Lambda^6(V^*), \quad K_\sigma(w) = A((i_w\sigma) \wedge \sigma)$$

and

$$\lambda : \Lambda^3(V^*) \to (\Lambda^6(V^*))^{\otimes 2}, \quad \lambda(\sigma) = \frac{1}{6}\mathrm{tr} K_\sigma^2.$$

A 3-form σ is stable if and only if $\lambda(\sigma) \neq 0$ and whenever this happens it is possible to define a volume form by $\sqrt{|\lambda(\sigma)|} \in \Lambda^6(V^*)$, where the positively oriented root is chosen, and an endomorphism

$$J_\sigma = \frac{1}{\sqrt{|\lambda(\sigma)|}} K_\sigma,$$

which is a complex structure when $\lambda(\sigma) < 0$.

A pair of stable forms $(\omega, \sigma) \in \Lambda^2(V^*) \times \Lambda^3(V^*)$ is called *compatible* if $\omega \wedge \sigma = 0$ and *normalized* if $J_\sigma^*\sigma \wedge \sigma = \frac{2}{3}\omega^3$ (the latter identity is non-zero since a 2-form ω is stable if and only if $\omega^3 \neq 0$). Such a pair defines a (pseudo) Euclidean metric $h(\cdot, \cdot) = \omega(J_\sigma \cdot, \cdot)$. As a consequence, on a six-dimensional smooth manifold N there is a one to one correspondence between SU(3)-structures and pairs $(\omega, \sigma) \in \Lambda^2(N) \times \Lambda^3(N)$ such that for each point $p \in N$ the pair of forms defined on T_pN (ω_p, σ_p) is stable, compatible, normalized, has $\lambda(\sigma_p) < 0$ and induces a Riemannian metric $h_p(\cdot, \cdot) = \omega_p(J_{\sigma_p} \cdot, \cdot)$. In this case we have $\Psi = \sigma + \mathrm{i}J_\sigma^*\sigma$ and, then, $\sigma = \mathrm{Re}(\Psi)$. We refer to h as the *associated Riemannian metric* to the SU(3)-structure (ω, σ).

An SU(3)-structure (ω, σ) on a 6-manifold N is called *coupled* if $d\omega = c\sigma$, with c a non-zero real number. Note that in particular a coupled SU(3)-structure is half-flat since $d(\omega^2) = 0$ and $d\sigma = 0$ and its intrinsic torsion belongs to the space $\mathcal{W}_1^- \oplus \mathcal{W}_2^-$, where $\mathcal{W}_1^- \cong \mathbb{R}$ and $\mathcal{W}_2^- \cong \mathfrak{su}(3)$ (see [10]).

It is interesting to notice that the product manifold $N \times \mathbb{R}$, where N is a 6-manifold endowed with a coupled SU(3)-structure (ω, σ), has a natural locally conformal calibrated G_2-structure defined by

$$\varphi = \omega \wedge dt + \sigma.$$

Indeed,

$$d\varphi = c\sigma \wedge dt = c\varphi \wedge dt,$$

since in local coordinates the components of σ are functions defined on N and thus they do not depend on t. Then, $\tau_0 \equiv 0$, $\tau_3 \equiv 0$ and $\tau_1 = \left(-\frac{1}{3}c\right) dt$.

3 Einstein locally conformal calibrated G_2-structures on compact manifolds

We will show now that a seven-dimensional, compact, smooth manifold M endowed with an Einstein locally conformal calibrated G_2-structure φ has $\mathrm{Scal}(g_\varphi) \leq 0$. It is worth observing here that, up to now, there are no known examples of smooth manifolds endowed with a locally conformal calibrated G_2-structure whose associated metric is Ricci-flat (and then has zero scalar curvature).

First of all recall that given a Riemannian manifold (M, g) of dimension $n \geq 3$ it is possible to define the so called *conformal Yamabe constant* $Q(M, g)$ in the following way: set $a_n := \frac{4(n-1)}{n-2}$, $p_n := \frac{2n}{n-2}$ and let $C_c^\infty(M)$ denote the set of compactly supported smooth real valued functions on M. Then

$$Q(M, g) := \inf_{u \in C_c^\infty(M), u \not\equiv 0} \left\{ \frac{\int_M (a_n |du|_g^2 + u^2 \mathrm{Scal}(g)) dV_g}{(\int_M |u|^{p_n} dV_g)^{\frac{2}{p_n}}} \right\}.$$

The sign of $Q(M, g)$ is a conformal invariant, in particular the following characterization holds:

Proposition 3.1 *If (M, g) is a compact Riemannian manifold of dimension $n \geq 3$, then $Q(M, g)$ is negative/zero/positive if and only if g is conformal to a Riemannian metric of negative/zero/positive scalar curvature.*

Using the conformal Yamabe constant it is possible to prove the following

Theorem 3.2 *Let M be a seven-dimensional, compact, smooth manifold endowed with an Einstein locally conformal calibrated G_2-structure φ. Then* $\mathrm{Scal}(g_\varphi) \leq 0$. *Moreover, if M is connected,* $\mathrm{Scal}(g_\varphi)$ *is either zero or negative.*

Proof Suppose that $\mathrm{Scal}(g_\varphi) > 0$, then the 1-form τ_1 is exact. Indeed, since $d\tau_1 = 0$, we can consider the de Rham class $[\tau_1] \in H^1_{\mathrm{dR}}(M)$ and take the harmonic 1-form ξ representing $[\tau_1]$, that is, $\tau_1 = \xi + df$, where $\Delta\xi = 0$ and $f \in C^\infty(M)$. ξ has to vanish everywhere on M since it is compact, oriented and has positive Ricci curvature. Then $\tau_1 = df$. Let us consider $\tilde{\varphi} := e^{-3f}\varphi$, it is clear that $\tilde{\varphi}$ is a G_2-structure defined on M. Moreover

$$\begin{aligned} d\tilde{\varphi} &= d(e^{-3f}\varphi) \\ &= -3e^{-3f} df \wedge \varphi + e^{-3f} d\varphi \\ &= -3e^{-3f} \tau_1 \wedge \varphi + e^{-3f}(3\tau_1 \wedge \varphi) \\ &= 0, \end{aligned}$$

so $\tilde{\varphi}$ is a closed G_2-structure and $\mathrm{Scal}(g_{\tilde{\varphi}}) \leq 0$ by [5]. We have $g_{\tilde{\varphi}} = e^{-2f} g_\varphi$, that is, $g_{\tilde{\varphi}}$ is conformal to the Riemannian metric g_φ of positive scalar curvature, then the conformal Yamabe constant $Q(M, g_{\tilde{\varphi}})$ is positive by the previous characterization.

Since M is compact, it has finite volume and is complete as a consequence of the well known Hopf–Rinow Theorem. Then, by [34, Corollary 2.2] we have that $Q(M, g_{\tilde{\varphi}}) \leq 0$, which is in contrast with the previous result. □

As a consequence of the previous proposition we have the

Corollary 3.3 *A seven-dimensional, compact, homogeneous, smooth manifold M cannot admit an invariant locally conformal calibrated Einstein G_2-structure φ, unless the underlying metric g_φ is flat.*

Proof Recall that a homogeneous Einstein manifold with negative scalar curvature is not compact [3]. Thus, every seven-dimensional, compact, homogeneous, smooth manifold M with an invariant G_2-structure φ whose associated metric is Einstein has $\mathrm{Scal}(g_\varphi) \geq 0$. Combining this result with the previous proposition we have $\mathrm{Scal}(g_\varphi) = 0$ and, in particular, g_φ is Ricci-flat. The statement then follows recalling that in the homogeneous case Ricci flatness implies flatness [2]. □

4 Noncompact homogeneous examples and coupled SU(3)-structures

In this section, after recalling some facts about noncompact homogeneous Einstein manifolds, we first study the classification of coupled SU(3)-structures on nilmanifolds and then we construct an example of a locally conformal calibrated G_2-structure φ inducing an Einstein (non Ricci-flat) metric on a noncompact homogeneous manifold.

All the known examples of noncompact homogeneous Einstein manifolds are solvmanifolds, i.e., simply connected solvable Lie groups S endowed with a left-invariant metric (see for instance the recent survey [32]). D. Alekseevskii conjectured that these might exhaust the class of non-compact homogeneous Einstein manifolds (see [3, 7.57]).

Lauret in [33] showed that every Einstein solvmanifold is *standard*, i.e., it is a solvable Lie group S endowed with a left-invariant metric such that the orthogonal complement $\mathfrak{a} = [\mathfrak{s}, \mathfrak{s}]^{\perp}$, where $\mathfrak{s}$ is the Lie algebra of S, is abelian. We recall that given a metric nilpotent Lie algebra $\mathfrak{n}$ with an inner product $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$, a metric solvable Lie algebra $(\mathfrak{s} = \mathfrak{n} \oplus \mathfrak{a}, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ is called a *metric solvable extension* of $(\mathfrak{n}, \langle \cdot, \cdot \rangle_{\mathfrak{n}})$ if $[\mathfrak{s}, \mathfrak{s}] = \mathfrak{n}$ and the restrictions to $\mathfrak{n}$ of the Lie bracket of $\mathfrak{s}$ and of the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ coincide with the Lie bracket of $\mathfrak{n}$ and with $\langle \cdot, \cdot \rangle_{\mathfrak{n}}$, respectively. The dimension of $\mathfrak{a}$ is called the *algebraic rank* of $\mathfrak{s}$.

In [21, 4.18], it was proved that the study of standard Einstein metric solvable Lie algebras reduces to the rank-one metric solvable extension of a nilpotent Lie algebra (i.e., those for which $\dim(\mathfrak{a}) = 1$). Indeed, by [21] the metric Lie algebra of any $(n+1)$-dimensional rank-one solvmanifold can be modelled on $(\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}H, \langle \cdot, \cdot \rangle_{\mathfrak{s}})$ for some nilpotent Lie algebra $\mathfrak{n}$, with the inner product $\langle \cdot, \cdot \rangle_{\mathfrak{s}}$ such that $\langle H, \mathfrak{n} \rangle_{\mathfrak{s}} = 0$, $\langle H, H \rangle_{\mathfrak{s}} = 1$ and the Lie bracket on $\mathfrak{s}$ given by

$$[H, X]_{\mathfrak{s}} = DX, \quad [X, Y]_{\mathfrak{s}} = [X, Y]_{\mathfrak{n}},$$

where $[\cdot, \cdot]_{\mathfrak{n}}$ denotes the Lie bracket on $\mathfrak{n}$ and D is some derivation of $\mathfrak{n}$. By [30], a left-invariant metric h on a nilpotent Lie group N is a Ricci soliton if and only if the Ricci operator satisfies $\mathrm{Ric}(h) = \mu I + D$, for some $\mu \in \mathbb{R}$ and some derivation D of $\mathfrak{n}$, when h is identified with an inner product on $\mathfrak{n}$ or, equivalently, if and only if (N, h) admits a metric standard extension whose corresponding standard solvmanifold is Einstein. The inner product h is also called *nilsoliton*.

Using the results of [29,31], in [39] all the seven-dimensional rank-one Einstein solvmanifolds were determined, proving that each one of the 34 nilpotent Lie algebras $\mathfrak{n}$ of dimension 6 admits a rank-one solvable extension which can be endowed with an Einstein inner product.

Six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure were classified in [13]. For coupled SU(3)-structures we can show the following

Theorem 4.1 *Let $\mathfrak{n}$ be a six-dimensional, non-abelian, nilpotent Lie algebra admitting a coupled* SU(3)*-structure. Then $\mathfrak{n}$ is isomorphic to one of the following*

$$\mathfrak{n}_9 = (0, 0, 0, e^{12}, e^{14} - e^{23}, e^{15} + e^{34}), \quad \mathfrak{n}_{28} = (0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23}),$$

where for instance $\mathfrak{n}_9 = (0, 0, 0, e^{12}, e^{14} - e^{23}, e^{15} + e^{34})$ means that there exists a basis $(e^1, \dots, e^6)$ of $\mathfrak{n}_9^*$ such that

$$de^j = 0, j = 1, 2, 3, \quad de^4 = e^{12}, \quad de^5 = e^{14} - e^{23}, \quad de^6 = e^{15} + e^{34}.$$

Moreover, the only nilpotent Lie algebra admitting a coupled SU(3)*-structure inducing a nilsoliton is $\mathfrak{n}_{28}$.*

Table 1 Expression of $\lambda(\sigma)$ for the six-dimensional nilpotent Lie algebras admitting a half-flat SU(3)-structure

$\mathfrak{n}$	$(de^1, de^2, de^3, de^4, de^5, de^6)$	$\lambda(\sigma)$	Sign of $\lambda(\sigma)$
$\mathfrak{n}_4$	$(0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{24} + e^{15})$	$4c^4 b_{15}^2(-b_{15}(b_{12} + b_{13}) + b_{14}^2)$	?
$\mathfrak{n}_6$	$(0, 0, e^{12}, e^{13}, e^{23}, e^{14})$	$c^4 b_{15}^4$	≥ 0
$\mathfrak{n}_7$	$(0, 0, e^{12}, e^{13}, e^{23}, e^{14} - e^{25})$	$c^4(b_{14}^2 - b_{15}^2)^2$	≥ 0
$\mathfrak{n}_8$	$(0, 0, e^{12}, e^{13}, e^{23}, e^{14} + e^{25})$	$c^4(b_{14}^2 - b_{15}^2)^2$	≥ 0
$\mathfrak{n}_9$	$(0, 0, 0, e^{12}, e^{14} - e^{23}, e^{15} + e^{34})$	$4c^4 b_{15}^2(-b_{15}(b_9 + b_{13}) + b_{14}^2)$	?
$\mathfrak{n}_{10}$	$(0, 0, 0, e^{12}, e^{14}, e^{15} + e^{23})$	$c^4 b_{15}^4$	≥ 0
$\mathfrak{n}_{11}$	$(0, 0, 0, e^{12}, e^{14}, e^{15} + e^{23} + e^{24})$	$c^4 b_{15}^4$	≥ 0
$\mathfrak{n}_{12}$	$(0, 0, 0, e^{12}, e^{14}, e^{15} + e^{24})$	0	0
$\mathfrak{n}_{13}$	$(0, 0, 0, e^{12}, e^{14}, e^{15})$	0	0
$\mathfrak{n}_{14}$	$(0, 0, 0, e^{12}, e^{13}, e^{14} + e^{35})$	$c^4 b_{14}^4$	≥ 0
$\mathfrak{n}_{15}$	$(0, 0, 0, e^{12}, e^{23}, e^{14} + e^{35})$	$c^4(b_{14}^2 - b_{15}^2)^2$	≥ 0
$\mathfrak{n}_{16}$	$(0, 0, 0, e^{12}, e^{23}, e^{14} - e^{35})$	$c^4(b_{14}^2 + b_{15}^2)^2$	≥ 0
$\mathfrak{n}_{21}$	$(0, 0, 0, e^{12}, e^{13}, e^{14} + e^{23})$	0	0
$\mathfrak{n}_{22}$	$(0, 0, 0, e^{12}, e^{13}, e^{24})$	$c^4 b_{15}^4$	≥ 0
$\mathfrak{n}_{24}$	$(0, 0, 0, e^{12}, e^{13}, e^{23})$	0	0
$\mathfrak{n}_{25}$	$(0, 0, 0, 0, e^{12}, e^{15} + e^{34})$	$c^4 b_{15}^4$	≥ 0
$\mathfrak{n}_{27}$	$(0, 0, 0, 0, e^{12}, e^{14} + e^{25})$	0	0
$\mathfrak{n}_{28}$	$(0, 0, 0, 0, e^{13} - e^{24}, e^{14} + e^{23})$	$-4c^4 b_{15}^4$	≤ 0
$\mathfrak{n}_{29}$	$(0, 0, 0, 0, e^{12}, e^{14} + e^{23})$	0	0
$\mathfrak{n}_{30}$	$(0, 0, 0, 0, e^{12}, e^{34})$	$c^4 b_{15}^4$	≥ 0
$\mathfrak{n}_{31}$	$(0, 0, 0, 0, e^{12}, e^{13})$	0	0
$\mathfrak{n}_{32}$	$(0, 0, 0, 0, 0, e^{12} + e^{34})$	0	0
$\mathfrak{n}_{33}$	$(0, 0, 0, 0, 0, e^{12})$	0	0
$\mathfrak{n}_{34}$	$(0, 0, 0, 0, 0, 0)$	0	0

Proof By the results in [13], the generic nilpotent Lie algebra $\mathfrak{n}$ admitting a half-flat SU(3)-structure is isomorphic to one of the 24 Lie algebras described in Table 1. Consider on $\mathfrak{n}$ a generic 2-form

$$\omega = b_1 e^{12} + b_2 e^{13} + b_3 e^{14} + b_4 e^{15} + b_5 e^{16} + b_6 e^{23} + b_7 e^{24} + b_8 e^{25} + b_9 e^{26} + b_{10} e^{34} + b_{11} e^{35} + b_{12} e^{36} + b_{13} e^{45} + b_{14} e^{46} + b_{15} e^{56},$$

where $b_i \in \mathbb{R}$, $i = 1, \ldots, 15$, and the 3-form

$$\sigma = c(d\omega), \quad c \in \mathbb{R} - \{0\}.$$

The expression of $\lambda(\sigma)$ for each nilpotent Lie algebra considered is given in Table 1.

We observe that among the 24 nilpotent Lie algebras admitting a half-flat SU(3)-structure we have:

- 1 case ($\mathfrak{n}_{28}$) for which $\lambda(\sigma) < 0$ if $b_{15} \neq 0$,
- 2 cases ($\mathfrak{n}_4$ and $\mathfrak{n}_9$) for which the sign of $\lambda(\sigma)$ depends on ω,
- 21 cases for which $\lambda(\sigma)$ cannot be negative.

Therefore, the 21 algebras having $\lambda(\sigma) \geq 0$ do not admit any coupled SU(3)-structure. Consider $\mathfrak{n}_4$, it has structure equations

$$(0, 0, e^{12}, e^{13}, e^{14} + e^{23}, e^{24} + e^{15}).$$

First of all, observe that if $b_{15} = 0$ then $\lambda(\sigma) = 0$. So if we want to find an SU(3)-structure we have to look for ω with $b_{15} \neq 0$. Moreover, σ induces an almost complex structure if and only if $\lambda(\sigma)$ is negative, then we have to suppose in addition that $b_{15}(b_{12} + b_{13}) > b_{14}^2$. Since we want ω to be the 2-form associated to an SU(3)-structure, it must be a form of type (1, 1) and this happens if and only if $\omega(\cdot, \cdot) = \omega(J\cdot, J\cdot)$, where $J = J_\sigma$. Computing the previous identity with respect to the considered frame, we have that the following equations have to be satisfied by the components of ω:

$$\omega_{ab} = \sum_{k,m=1}^{6} J_a^k J_b^m \omega_{km}, \quad 1 \leq a < b \leq 6$$

(observe that $\omega_{12} = b_1, \omega_{13} = b_2$ and so on). Using these equations it is possible to write four of the b_i in terms of the remaining and obtain a new expression for ω. We can now compute the matrix associated to $h(\cdot, \cdot) = \omega(J\cdot, \cdot)$ with respect to the basis $(e_1, \ldots, e_6)$ and observe that for the nonzero vector $v = e_4 - \frac{b_{14}}{b_{15}} e_5 + \frac{b_{13}}{b_{15}} e_6$ we have $h(v, v) = 0$. Therefore, h cannot be positive definite and, as a consequence, it is not possible to find a coupled SU(3)-structure on $\mathfrak{n}_4$.

For the Lie algebras $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$ we can give an explicit example of coupled SU(3)-structure. Consider on $\mathfrak{n}_9$ the forms

$$\begin{aligned}
\omega &= -\frac{3}{2}e^{12} - \frac{1}{4}e^{14} - e^{15} - e^{24} + \frac{1}{2}e^{26} - \frac{1}{2}e^{35} - e^{36} + e^{56},\\
\sigma &= \frac{\sqrt{15}\sqrt[4]{2}}{4}e^{123} + \frac{\sqrt{15}\sqrt[4]{2}}{8}e^{234} - \frac{\sqrt{15}\sqrt[4]{2}}{8}e^{125} + \frac{\sqrt{15}\sqrt[4]{2}}{8}e^{134}\\
&\quad + \frac{\sqrt{15}\sqrt[4]{2}}{4}e^{135} - \frac{\sqrt{15}\sqrt[4]{2}}{4}e^{146} + \frac{\sqrt{15}\sqrt[4]{2}}{4}e^{236} + \frac{\sqrt{15}\sqrt[4]{2}}{4}e^{345}.
\end{aligned}$$

We have

$$\omega \wedge \sigma = 0, \quad \omega^3 \neq 0, \quad \lambda(\sigma) = -\frac{225}{64}, \quad d\omega = -\frac{4}{\sqrt{15}\sqrt[4]{2}}\sigma,$$

in particular (ω, σ) is a compatible pair of stable forms. The associated almost complex structure $J = J_\sigma$ has the following matrix expression with respect to the basis $(e_1, \ldots, e_6)$:

$$J = \begin{bmatrix} 0 & 0 & -\sqrt{2} & 0 & 0 & 0\\ \sqrt{2} & 0 & 0 & -\sqrt{2} & 0 & 0\\ \frac{\sqrt{2}}{2} & 0 & 0 & 0 & 0 & 0\\ 0 & \frac{\sqrt{2}}{2} & -\sqrt{2} & 0 & 0 & 0\\ \sqrt{2} & 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} & 0 & \sqrt{2}\\ -\frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{3\sqrt{2}}{2} & 0 & -\frac{\sqrt{2}}{2} & 0 \end{bmatrix}$$

and it is easy to check that $J^*\sigma \wedge \sigma = \frac{2}{3}\omega^3$, i.e., the pair (ω, σ) is normalized.

The inner product $h(\cdot, \cdot) = \omega(J\cdot, \cdot)$ is given with respect to the basis $(e_1, \dots, e_6)$ by

$$h = \begin{bmatrix} \frac{5\sqrt{2}}{2} & \frac{\sqrt{2}}{8} & \frac{\sqrt{2}}{4} & -\sqrt{2} & 0 & \sqrt{2} \\ \frac{\sqrt{2}}{8} & \frac{5\sqrt{2}}{8} & -\frac{\sqrt{2}}{4} & 0 & \frac{\sqrt{2}}{4} & 0 \\ \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{4} & \frac{7\sqrt{2}}{4} & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \\ -\sqrt{2} & 0 & \frac{\sqrt{2}}{4} & \sqrt{2} & 0 & 0 \\ 0 & \frac{\sqrt{2}}{4} & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} & 0 \\ \sqrt{2} & 0 & \frac{\sqrt{2}}{2} & 0 & 0 & \sqrt{2} \end{bmatrix}.$$

and it is positive definite. Therefore, we can conclude that (ω, σ) is a coupled SU(3)-structure on $\mathfrak{n}_9$.

For $\mathfrak{n}_{28}$ consider the pair of compatible, normalized, stable forms

$$\left(\omega = e^{12} + e^{34} - e^{56}, \quad \sigma = e^{136} - e^{145} - e^{235} - e^{246}\right). \tag{1}$$

This pair defines a coupled SU(3)-structure with $d\omega = -\sigma$. Moreover, the associated inner product

$$h = (e^1)^2 + \cdots + (e^6)^2$$

is a nilsoliton with

$$\mathrm{Ric}(h) = -3I + 2\,\mathrm{diag}(1, 1, 1, 1, 2, 2).$$

Summarizing our results, we can conclude that $\mathfrak{n}_9$ and $\mathfrak{n}_{28}$ are, up to isomorphisms, the only six-dimensional nilpotent Lie algebras admitting a coupled SU(3)-structure.

We have just provided a coupled SU(3)-structure on $\mathfrak{n}_{28}$ whose associated inner product is a nilsoliton, we claim that this is the unique case among all six-dimensional nilpotent Lie algebras. It is clear that to prove the previous assertion it suffices to show that $\mathfrak{n}_9$ does not admit any coupled SU(3)-structure inducing a nilsoliton inner product. In order to do this, we consider an orthonormal basis $(e_1, \dots, e_6)$ of $\mathfrak{n}_9$ whose dual basis satisfies the structure equations

$$\left(0, 0, 0, \frac{\sqrt{5}}{2}e^{12}, e^{14} - e^{23}, \frac{\sqrt{5}}{2}e^{15} + e^{34}\right)$$

(by the results of [30] and [39], these are, up to isomorphisms, the structure equations for which the considered inner product on $\mathfrak{n}_9$ is a nilsoliton). As we did before, consider a generic 2-form ω, the 3-form $\sigma = c(d\omega)$, evaluate $\lambda(\sigma)$ and impose that it is negative. Then compute J_σ and the matrix associated to $h(\cdot, \cdot) = \omega(J_\sigma\cdot, \cdot)$ with respect to the considered basis. Since h has to be the restriction to $\mathfrak{n}_9$ of an Einstein inner product defined on $\mathfrak{n}_9 \oplus \mathbb{R}e_7$ and since the latter is unique up to scaling, we have to impose that the symmetric matrix associated to h is a multiple of the identity. Solving the associated equations we find that $\lambda(\sigma)$ has to be zero, which is a contradiction. □

Starting from a six-dimensional nilpotent Lie algebra $\mathfrak{n}$ endowed with a coupled SU(3)-structure, it is possible to construct a locally conformal calibrated G_2-structure on the rank-one solvable extension $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_7$ under some extra hypothesis. Let $\hat{d}$ denote the exterior derivative on $\mathfrak{n}$ and d denote the exterior derivative on $\mathfrak{s}$. Observe that given a k-form $\theta \in \Lambda^k(\mathfrak{n}^*)$ we have

$$d\theta = \hat{d}\theta + \rho \wedge e^7$$

for some $\rho \in \Lambda^k(\mathfrak{n}^*)$.

Proposition 4.2 *Let* $\mathfrak{n}$ *be a six-dimensional, nilpotent Lie algebra endowed with a coupled* SU(3)*-structure* (ω, σ) *with* $\hat{d}\omega = c\sigma$, $c \in \mathbb{R} - \{0\}$. *Consider on its rank one solvable extension* $\mathfrak{s} = \mathfrak{n} \oplus \mathbb{R}e_7$ *the* G_2*-structure defined by* $\varphi = \omega \wedge e^7 + \sigma$, *where the closed 1-form* e^7 *is the dual of* e_7. *Then the* G_2*-structure is locally conformal calibrated with* $\tau_1 = \frac{1}{3}ce^7$ *if and only if* $d\sigma = -2c\sigma \wedge e^7$.

Proof Suppose that $d\sigma = -2c\sigma \wedge e^7$, we can write $d\omega = \hat{d}\omega + \gamma \wedge e^7$ for some 2-form $\gamma \in \Lambda^2(\mathfrak{n}^*)$. We obtain $d\varphi = ce^7 \wedge \varphi$. Then, φ is locally conformal calibrated with $\tau_1 = \frac{1}{3}ce^7$.

Conversely, suppose that φ is locally conformal calibrated with $\tau_1 = \frac{1}{3}ce^7$. Then we have $d\varphi = ce^7 \wedge \varphi$. Moreover, we know that $d\sigma = \hat{d}\sigma + \beta \wedge e^7 = \beta \wedge e^7$ for some 3-form $\beta \in \Lambda^3(\mathfrak{n}^*)$, since σ is $\hat{d}$-closed. We then have

$$d\varphi = d\omega \wedge e^7 + d\sigma = e^7 \wedge (-c\sigma - \beta)$$

and comparing this with the previous expression of $d\varphi$ we obtain

$$e^7 \wedge (-c\sigma - \beta) = ce^7 \wedge \varphi = e^7 \wedge (c\sigma)$$

from which follows $\beta = -2c\sigma$. □

Now we will construct an Einstein locally conformal calibrated G_2-structure on a rank-one extension of the Lie algebra $\mathfrak{n}_{28}$ (Lie algebra of the 3-dimensional complex Heisenberg group) endowed with the coupled SU(3)-structure (1).

Example 4.3 Consider $\mathfrak{n}_{28}$ and the metric rank-one solvable extension $\mathfrak{s} = \mathfrak{n}_{28} \oplus \mathbb{R}e_7$ with structure equations

$$\left(\frac{1}{2}e^{17}, \frac{1}{2}e^{27}, \frac{1}{2}e^{37}, \frac{1}{2}e^{47}, e^{13} - e^{24} + e^{57}, e^{14} + e^{23} + e^{67}, 0\right).$$

The associated solvable Lie group S is not unimodular and so it does not admit any compact quotient [35]. Consider on $\mathfrak{n}_{28}$ the coupled SU(3)-structure (ω, σ) given by (1) with the nilsoliton associated inner product

$$h = (e^1)^2 + \cdots + (e^6)^2.$$

Then the inner product on $\mathfrak{s}$

$$g = (e^1)^2 + \cdots + (e^7)^2$$

is Einstein with Ricci tensor $\mathrm{Ric}(g) = -3g$.

Since $d\sigma = 2\sigma \wedge e^7$, by the previous proposition we have a locally conformal calibrated G_2-structure on $\mathfrak{s}$ given by

$$\varphi = \omega \wedge e^7 + \sigma = e^{127} + e^{347} - e^{567} + e^{136} - e^{145} - e^{235} - e^{246}$$

and it is easy to show that $g_\varphi = g$. Then the corresponding solvmanifold (S, φ) is an example of non-compact homogeneous manifold endowed with an Einstein (non-flat) locally conformal calibrated G_2-structure.

Observe that the G_2-structure φ satisfies the conditions

$$d\varphi = -e^7 \wedge \varphi,$$
$$d * \varphi = -e^7 \wedge (3e^{1256} + 2e^{1234} + 3e^{3456}).$$

Then

$$\tau_1 = -\frac{1}{3}e^7,$$

as we expected from Proposition 4.2, and

$$\tau_2 = -\left(\frac{5}{3}e^{12} + \frac{5}{3}e^{34} + \frac{10}{3}e^{56}\right).$$

Moreover, the G_2-structure is not $*$-Einstein, since by direct computation with respect to the orthonormal basis $(e_1, \ldots, e_7)$ one has

$$\rho^* = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 22 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 22 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -6 \end{pmatrix}.$$

It is worth emphasizing here that, by [16], on seven-dimensional solvmanifolds there are no left-invariant calibrated G_2-structures inducing an Einstein non-flat metric. The previous example shows that the situation is different in the case of locally conformal calibrated G_2-structures.

We provide now a non-compact example of homogeneous manifold admitting an Einstein (non-flat) locally conformal parallel G_2-structure.

Example 4.4 The Einstein rank-one solvable extension of the six-dimensional abelian Lie algebra is the solvable Lie algebra with structure equations

$$(ae^{17}, ae^{27}, ae^{37}, ae^{47}, ae^{57}, ae^{67}, 0),$$

where a is a nonzero real number. The Riemannian metric

$$g = (e^1)^2 + \cdots + (e^7)^2$$

is Einstein with Ricci tensor given by $\mathrm{Ric}(g) = -6a^2 g$.

The 3-form

$$\varphi = -e^{125} - e^{136} - e^{147} + e^{237} - e^{246} + e^{345} - e^{567}$$

has stabilizer G_2, is such that $g_\varphi = g$ and satisfies the conditions

$$d\varphi = -3ae^{2467} + 3ae^{3457} - 3ae^{1257} - 3ae^{1367},$$
$$d * \varphi = 4ae^{23567} + 4ae^{12347} - 4ae^{14567}.$$

It is immediate to show that $\tau_1 = -ae^7$ and $\tau_0 \equiv 0$, $\tau_2 \equiv 0$, $\tau_3 \equiv 0$, that is, the G_2-structure φ is locally conformal parallel.

Acknowledgments The authors would like to thank Simon Salamon, Uwe Semmelmann and Luigi Vezzoni for useful comments on the paper. Aspects of the work were described at the Workshop "G_2 days"(King's and University College, London, 2012).

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