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Bounds for Green's functions on noncompact hyperbolic Riemann orbisurfaces of finite volume

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Abstract In Jorgenson and Kramer (Compos Math 142:679–700, 2006) derived bounds for the canonical Green's function and the hyperbolic Green's function defined on a compact hyperbolic Riemann surface. In this article, we extend these bounds to noncompact hyperbolic Riemann orbisurfaces of finite volume and of genus greater than zero, which can be realized as a quotient space of the action of a Fuchsian subgroup of first kind on the hyperbolic upper half-plane.

Keywords Greens functions · Arakelov theory · Modular curves · Hyperbolic heat kernels

Mathematics Subject Classification 14G40 · 30F10 · 11F72 · 30C40

1 Introduction

Notation Let X be a noncompact hyperbolic Riemann orbisurface of finite volume $\mathrm{vol}_{\mathrm{hyp}}(X)$ with genus $g_X \geq 1$, and can be realized as the quotient space $\Gamma_X\backslash\mathbb{H}$, where $\Gamma_X \subset \mathrm{PSL}_2(\mathbb{R})$ is a Fuchsian subgroup of the first kind acting on the hyperbolic upper half-plane $\mathbb{H}$, via fractional linear transformations. Let $\mathcal{P}_X$ and $\mathcal{E}_X$ denote the set of cusps and the set of elliptic fixed points of Γ_X, respectively. Put $\overline{X} = X \cup \mathcal{P}_X$. Then, $\overline{X}$ admits the structure of a Riemann surface.

Let $\mu_{\mathrm{hyp}}(z)$ denote the (1,1)-form associated to hyperbolic metric, which is the natural metric on X, and of constant negative curvature minus one. Let $\mu_{\mathrm{shyp}}(z)$ denote the rescaled hyperbolic metric $\mu_{\mathrm{hyp}}(z)/\,\mathrm{vol}_{\mathrm{hyp}}(X)$, which measures the volume of X to be one.

The Riemann surface $\overline{X}$ is embedded in its Jacobian variety $\mathrm{Jac}(\overline{X})$ via the Abel-Jacobi map. Then, the pull back of the flat Euclidean metric by the Abel-Jacobi map is called the

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canonical metric, and the (1,1)-form associated to it is denoted by $\widehat{\mu}_{\mathrm{can}}(z)$. We denote its restriction to X by $\mu_{\mathrm{can}}(z)$.

For $\mu = \mu_{\mathrm{shyp}}(z)$ or $\mu_{\mathrm{can}}(z)$, let $g_{X,\mu}(z,w)$ defined on $X \times X$ denote the Green's function associated to the metric μ. The Green's function $g_{X,\mu}(z,w)$ is uniquely determined by the differential equation (which is to be interpreted in terms of currents)

$$d_z d_z^c g_{X,\mu}(z,w) + \delta_w(z) = \mu(z), \tag{1}$$

with the normalization condition

$$\int_X g_{X,\mu}(z,w)\mu(z) = 0.$$

The Green's function $g_{X,\mathrm{can}}(z,w)$ associated to the canonical metric $\mu_{\mathrm{can}}(z)$ is called the canonical Green's function. Similarly the Green's function $g_{X,\mathrm{hyp}}(z,w)$ associated to the (rescaled) hyperbolic metric $\mu_{\mathrm{shyp}}(z)$ is called the hyperbolic Green's function.

From differential Eq. (1), we can deduce that for a fixed $w \in X$, as a function in the variable z, both the Green's functions $g_{X,\mathrm{can}}(z,w)$ and $g_{X,\mathrm{hyp}}(z,w)$ are log-singular at $z = w$. Recall that $\mu_{\mathrm{hyp}}(z)$ is singular at the cusps and at the elliptic fixed points, and $\mu_{\mathrm{can}}(z)$ the pull back of the smooth and flat Euclidean metric is smooth on X. Hence, from the elliptic regularity of the $d_z d_z^c$ operator, it follows that $g_{X,\mathrm{hyp}}(z,w)$ is log log-singular at the cusps, and $g_{X,\mathrm{can}}(z,w)$ remains smooth at the cusps.

From a geometric perspective, it is very interesting to compare the two metrics $\mu_{\mathrm{hyp}}(z)$ and $\mu_{\mathrm{can}}(z)$, and study the difference of the two Green's functions

$$g_{X,\mathrm{hyp}}(z,w) - g_{X,\mathrm{can}}(x,w). \tag{2}$$

on compact subsets of X.

In [10], Jorgenson and Kramer have already established these tasks, when X is a compact Riemann surface devoid of elliptic fixed points. They proved a key-identity that relates the hyperbolic metric $\mu_{\mathrm{hyp}}(z)$ and the canonical metric $\mu_{\mathrm{can}}(z)$ via the hyperbolic heat kernel. Using the key-identity, they expressed the difference (2) in terms of integrals which involve only the hyperbolic heat kernel and the hyperbolic metric. This allowed them to derive bounds for the difference (2) in terms of invariants coming from the hyperbolic geometry of X, namely, the injectivity radius of X and the first non-zero eigenvalue $\lambda_{X,1}$ of the hyperbolic Laplacian Δ_{hyp} acting on smooth functions defined on X.

In [2], we extend the key-identity from [10] to cusps and elliptic fixed points at the level of currents. This relation serves as a starting point for extending the bounds for the canonical and the hyperbolic Green's function from [10] to noncompact hyperbolic Riemann orbisurfaces of finite volume.

In this article, using the key-identity from [2] and by extending the methods used in [10], we study the difference (2) on compact subsets of X, and as an application, we derive upper bounds for the canonical Green's function $g_{X,\mathrm{can}}(z,w)$ on X. Our bounds are similar to the ones derived in [10].

Statement of main results We now describe our results for the modular curve $Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}$. However, our results hold true for any noncompact hyperbolic Riemann orbisurface of finite volume and of genus greater than zero. Let $N \in \mathbb{N}_{>0}$ be such that the modular curve $Y_0(N)$ has genus $g_{Y_0(N)} \geq 1$. Let $0 < \varepsilon < 1$ be small enough such that it satisfies the conditions elucidated in Notation 4.1.

For any cusp $p \in \mathcal{P}_{Y_0(N)}$, let $U_{N,\varepsilon}(p)$ denote an open coordinate disk of radius ε around the cusp p. For any elliptic fixed point $\mathfrak{e} \in \mathcal{E}_{Y_0(N)}$, let $U_{N,\varepsilon}(\mathfrak{e})$ denote an open coordinate disk around the elliptic fixed point $\mathfrak{e}$, which is as described in condition (3) in Notation 4.1. Put

$$Y_0(N)_\varepsilon = Y_0(N) \Big\backslash \bigg(\bigcup_{p \in \mathcal{P}_{Y_0(N)}} U_\varepsilon(p) \cup \bigcup_{\mathfrak{e} \in \mathcal{E}_{Y_0(N)}} U_\varepsilon(\mathfrak{e}) \bigg).$$

For any $\delta > 0$ and a fixed $z, w \in X$, identifying $Y_0(N)$ with its fundamental domain, we define the set

$$S_{\Gamma_{Y_0(N)}}(\delta; z, w) = \Big\{ \gamma \in \mathcal{H}(\Gamma_0(N)) \cup \{\mathrm{id}\} \Big| \, d_{\mathbb{H}}(z, \gamma w) < \delta \Big\},$$

where $\mathcal{H}(\Gamma_0(N))$ denotes the hyperbolic elements of $\Gamma_0(N)$. Furthermore, let $g_{\mathbb{H}}(z, w)$ denote the free-space Green's function defined on $\mathbb{H} \times \mathbb{H}$, which is given by the formula

$$g_{\mathbb{H}}(z, w) = \log \left| \frac{z - \overline{w}}{z - w} \right|^2.$$

From [17], recall that the first non-zero eigenvalue of the hyperbolic Laplacian Δ_{hyp} satisfies the lower bound $\lambda_{Y_0(N),1} \geq 3/16$. With notation as above, for any $\delta > 0$, using the dependence of the genus $g_{Y_0(N)}$, the number of cusps $|\mathcal{P}_{Y_0(N)}|$, and the number of elliptic fixed points $|\mathcal{E}_{Y_0(N)}|$ in terms of N from pp. 22–25 in [18], we derive the following estimates

$$\begin{aligned} &\sup_{z,w \in Y_0(N)_\varepsilon} \left| g_{Y_0(N),\mathrm{can}}(z, w) - g_{Y_0(N),\mathrm{hyp}}(z, w) \right| \\ &\quad = O_{\varepsilon,\delta}\left(\frac{\left(|\mathcal{P}_{Y_0(N)}| + |\mathcal{E}_{Y_0(N)}| \right)}{g_{Y_0(N)}} \left(1 + \frac{1}{\lambda_{Y_0(N),1}} \right) \right) = O_{\varepsilon,\delta}(1); \end{aligned} \tag{3}$$

$$\begin{aligned} &\sup_{z,w \in Y_0(N)_\varepsilon} \left| g_{Y_0(N),\mathrm{can}}(z, w) - \sum_{\gamma \in S_{\Gamma_{Y_0(N)}}(\delta; z, w)} g_{\mathbb{H}}(z, \gamma w) \right| \\ &\quad = O_{\varepsilon,\delta}\left(\frac{\left(|\mathcal{P}_{Y_0(N)}| + |\mathcal{E}_{Y_0(N)}| \right)}{g_{Y_0(N)}} \left(1 + \frac{1}{\lambda_{Y_0(N),1}} \right) \right) = O_{\varepsilon,\delta}(1). \end{aligned} \tag{4}$$

We even derive bounds for the canonical Green's function $g_{Y_0(N),\mathrm{can}}(z, w)$ at cusps and at elliptic fixed points.

Arithmetic significance In 1974, in [1], Arakelov defined an intersection theory for divisors on an arithmetic surface by incorporating the associated compact Riemann surface with its complex analytic geometry. The contribution at infinity is calculated by using canonical Green's functions defined on the corresponding Riemann surfaces.

In [7], Edixhoven et al. devised an algorithm which for a given prime ℓ, computes the Galois representations modulo ℓ associated to a fixed modular form of arbitrary weight, in time polynomial in ℓ.

To show that the complexity of the algorithm is polynomial in ℓ, they needed an upper bound for the canonical Green's function associated to the compactified modular surface $X_1(\ell)$, and the upper bound provided by Merkl (also published in [7]) proved sufficient.

Bounds for the canonical Green's function from [10] when restricted to $X_1(\ell)$ yield better bounds than the ones derived by Merkl.

In 2011, in [5], while extending the algorithm of Edixhoven–Couveignes–de Jong, following the methods of Merkl, Bruin has derived bounds for the canonical Green's function, which for a given modular curve $Y_0(N)$ are of the form $O(N^2)$, which will appear as [6].

Furthermore, using the bounds of Bruin for the canonical Green's function, Javanpeykar has derived bounds for various Arakelovian invariants like the Faltings delta function and Faltings height function in [9].

Our bounds for the canonical Green's function are stronger than the ones derived by Bruin, and are optimally derived by following the methods from [10]. Furthermore, our bounds for the canonical Green's function $g_{X,\mathrm{can}}(z,w)$ at cusps are essential for calculating the Faltings height of any modular curve X. We are hopeful that our results together with [9] will lead to better bounds for the Arakelovian invariants considered in [9].

It is to be mentioned that using a different method, we have computed bounds for the canonical Green's function $g_{X,\mathrm{can}}(z,w)$ at cusps in [3]. Although the bounds computed in [3] are more explicit, their dependence on N for a modular curve $Y_0(N)$ is not known.

This article also completes the program of Jorgenson and Kramer of estimating Arakelovian invariants of modular curves via techniques coming from global analysis and theory of heat kernels. However it would be interesting to study Edixhoven–Couveignes–de Jong's algorithm from [7], using our bounds for the canonical Green's function, and we hope our bounds lead to a better complexity for the algorithm.

Moreover, for any noncompact hyperbolic Riemann orbisurface $X=\Gamma_X\backslash\mathbb{H}$, we have studied the convergence of the following series

$$\sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z),\quad \sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z),\quad \int_X\left(\sum_{\gamma\in\mathcal{H}(\Gamma_X)} K_{\mathbb{H}}(t;z,\gamma z)-\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)dt, \tag{5}$$

where $\mathcal{P}(\Gamma_X)$, $\mathcal{E}(\Gamma_X)$, and $\mathcal{H}(\Gamma_X)$ denote the parabolic, elliptic, and hyperbolic elements of Γ_X, respectively, and the quantity $K_{\mathbb{H}}(t;z,w)$ denotes the hyperbolic heat kernel on $\mathbb{H}\times\mathbb{H}$. We have also studied the behavior of the above stated series at the cusps and at the elliptic fixed points. We believe that this analysis helps in the generalization of the work of Jorgenson and Kramer from [10] and [11] to noncompact hyperbolic Riemann orbisurfaces and to higher dimensions.

Organization of the paper In the first section, we set up our notation, introduce basic notions, and results. In Sect. 2, we prove convergence of the automorphic functions mentioned in (5). In Sect. 3, using the existing bounds for the heat kernel from [10], we derive bounds for the hyperbolic Green's function $g_{X,\mathrm{hyp}}(z,w)$ on compact subsets of X, and then extend these bounds to the neighborhoods of cusps and elliptic fixed points. In Sect. 4, using the convergence results from Sect. 2, and bounds for the hyperbolic Green's function, we derive bounds for the canonical Green's function $g_{X,\mathrm{can}}(z,w)$ on compact subsets of X, and then extend these bounds to the neighborhoods of cusps and elliptic fixed points. Finally, in Sect. 5, we extend our bounds to certain sequences of admissible noncompact Riemann orbisurfaces to prove estimates (3) and (4).

2 Background material

In this section, we recall the basic notions and results required for next sections.

Let $\Gamma_X\subset\mathrm{PSL}_2(\mathbb{R})$ be a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$. Let X be the quotient space $\Gamma_X\backslash\mathbb{H}$, and let $g_X\geq 1$ denote the genus of X. The quotient space X admits the structure of a Riemann orbisurface.

Let $\mathcal{P}_X$ and $\mathcal{E}_X$ denote the finite set of cusps and finite set of elliptic fixed points of X, respectively. For $\mathfrak{e} \in \mathcal{E}_X$, let $m_{\mathfrak{e}}$ denote the order of $\mathfrak{e}$; for $p \in \mathcal{P}_X$, put $m_p = \infty$; for $z \in X \backslash \mathcal{E}_X$, put $m_z = 1$. Let $\overline{X}$ denote $\overline{X} = X \cup \mathcal{P}_X$.

Locally, away from cusps and elliptic fixed points, we identity $\overline{X}$ with its universal cover $\mathbb{H}$, and hence, denote the points on $\overline{X} \backslash (\mathcal{P}_X \cup \mathcal{E}_X)$ by the same letter as the points on $\mathbb{H}$.

Structure of $\overline{X}$ as a Riemann surface The quotient space $\overline{X}$ admits the structure of a compact Riemann surface. We refer the reader to section 1.8 in [16], for the details regarding the structure of $\overline{X}$ as a compact Riemann surface. For the convenience of the reader, we recall the coordinate functions for the neighborhoods of cusps and elliptic fixed points.

Let $p \in \mathcal{P}_X$ be a cusp, and let $U(p)$ denote a coordinate disk around the cusp p. Then, for any $w \in U(p)$, the coordinate function $\vartheta_p(w)$ for the open coordinate disk $U(p)$ is given by

$$\vartheta_p(w) = e^{2\pi i \sigma_p^{-1} w},$$

where σ_p is a scaling matrix of the cusp p satisfying the following relations

$$\sigma_p i\infty = p \quad \text{and} \quad \sigma_p^{-1} \Gamma_{X,p} \sigma_p = \langle \gamma_\infty \rangle, \quad \text{where } \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \Gamma_{X,p} = \langle \gamma_p \rangle \qquad (6)$$

denotes the stabilizer of the cusp p with generator γ_p.

Similarly, let $\mathfrak{e} \in \mathcal{E}_X$ be an elliptic fixed point, and let $U(\mathfrak{e})$ denote a coordinate disk around the elliptic fixed point $\mathfrak{e}$. Then, for any $w \in U(\mathfrak{e})$, the coordinate function $\vartheta_{\mathfrak{e}}(w)$ for the open coordinate disk $U(\mathfrak{e})$ is given by

$$\vartheta_{\mathfrak{e}}(w) = \left(\frac{w - \mathfrak{e}}{w - \overline{\mathfrak{e}}} \right)^{m_{\mathfrak{e}}}.$$

Hyperbolic metric We denote the (1,1)-form corresponding to the hyperbolic metric of X, which is compatible with the complex structure on X and has constant negative curvature equal to minus one, by $\mu_{\mathrm{hyp}}(z)$. Locally, away from elliptic fixed points, as we identity X with $\mathbb{H}$, for $z \in X \backslash \mathcal{E}_X$, the hyperbolic metric is given by

$$\mu_{\mathrm{hyp}}(z) = \frac{i}{2} \cdot \frac{dz \wedge d\overline{z}}{\mathrm{Im}(z)^2}.$$

Let $\mathrm{vol}_{\mathrm{hyp}}(X)$ be the volume of X with respect to the hyperbolic metric μ_{hyp}. It is given by the formula

$$\mathrm{vol}_{\mathrm{hyp}}(X) = 2\pi \left(2g - 2 + |\mathcal{P}_X| + \sum_{\mathfrak{e} \in \mathcal{E}_X} \left(1 - \frac{1}{m_{\mathfrak{e}}} \right) \right).$$

The hyperbolic metric $\mu_{\mathrm{hyp}}(z)$ is singular at the cusps and at the elliptic fixed points, and the rescaled hyperbolic metric

$$\mu_{\mathrm{shyp}}(z) = \frac{\mu_{\mathrm{hyp}}(z)}{\mathrm{vol}_{\mathrm{hyp}}(X)}$$

measures the volume of X to be one.

Locally, for $z \in X$, the hyperbolic Laplacian Δ_{hyp} on X is given by

$$\Delta_{\mathrm{hyp}} = -y^2 \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) = -4y^2 \left(\frac{\partial^2}{\partial z \partial \overline{z}} \right).$$

Recall that $d = \left(\partial + \overline{\partial}\right)$, $d^c = \frac{1}{4\pi i}\left(\partial - \overline{\partial}\right)$, and $dd^c = -\frac{\partial\overline{\partial}}{2\pi i}$. So for any smooth function f on X, we have

$$-4\pi\, d_z d_z^c f(z) = \Delta_{\mathrm{hyp}}(f)\, \mu_{\mathrm{hyp}}(z). \tag{7}$$

Canonical metric Let $S_2(\Gamma_X)$ denote the $\mathbb{C}$-vector space of cusp forms of weight 2 with respect to Γ_X equipped with the Petersson inner-product. Let $\{f_1, \ldots, f_{g_X}\}$ denote an orthonormal basis of $S_2(\Gamma_X)$ with respect to the Petersson inner product. Then, the (1,1)-form $\mu_{\mathrm{can}}(z)$ corresponding to the canonical metric of X is given by

$$\mu_{\mathrm{can}}(z) = \frac{i}{2g_X} \sum_{j=1}^{g_X} \left| f_j(z) \right|^2 dz \wedge d\overline{z}.$$

The canonical metric $\mu_{\mathrm{can}}(z)$ remains smooth at the cusps and at the elliptic fixed points, and measures the volume of X to be one.

Put

$$d_X = \sup_{z \in X} \frac{\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{shyp}}(z)}. \tag{8}$$

As the canonical metric $\mu_{\mathrm{can}}(z)$ remains smooth at the cusps and at the elliptic fixed points, and the hyperbolic metric is singular at these points, the quantity d_X is well-defined.

Canonical Green's function For $z, w \in X$, the canonical Green's function $g_{X,\mathrm{can}}(z, w)$ is defined as the solution of the differential equation (which is to be interpreted in terms of currents)

$$d_z d_z^c\, g_{X,\mathrm{can}}(z, w) + \delta_w(z) = \mu_{\mathrm{can}}(z), \tag{9}$$

with the normalization condition

$$\int_X g_{X,\mathrm{can}}(z, w)\, \mu_{\mathrm{can}}(z) = 0.$$

From Eq. (9), it follows that $g_{X,\mathrm{can}}(z, w)$ has a log-singularity at $z = w$, i.e., for $z, w \in X$, it satisfies

$$\lim_{w \to z} \left(g_{X,\mathrm{can}}(z, w) + \log |\vartheta_z(w)|^2 \right) = O_z(1). \tag{10}$$

Parabolic Eisenstein series For $z \in X$ and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the parabolic Eisenstein series $\mathcal{E}_{X,\mathrm{par},p}(z, s)$ corresponding to a cusp $p \in \mathcal{P}_X$ is defined by the series

$$\mathcal{E}_{X,\mathrm{par},p}(z, s) = \sum_{\eta \in \Gamma_{X,p} \backslash \Gamma_X} \mathrm{Im}\left(\sigma_p^{-1} \eta z\right)^s.$$

The series converges absolutely and locally uniformly for $\mathrm{Re}(s) > 1$ (as a function in the variable z, for a fixed s). It admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$, and the Laurent expansion at $s = 1$ is of the form

$$\mathcal{E}_{X,\mathrm{par},p}(z, s) = \frac{1}{(s-1)\,\mathrm{vol}_{\mathrm{hyp}}(X)} + \kappa_{X,p}(z) + O_z(s-1), \tag{11}$$

where $\kappa_{X,p}(z)$ the constant term of $\mathcal{E}_{X,\mathrm{par},p}(z, s)$ at $s = 1$ is called Kronecker's limit function (see Chapter 6 of [8]).

For $z \in X$, and $p, q \in \mathcal{P}_X$, the Kronecker's limit function $\kappa_{X,p}(\sigma_q z)$ satisfies the following equation (see Theorem 1.1 of [14] for the proof)

$$\kappa_{X,p}(\sigma_q z) = \sum_{n<0} k_{p,q}(n) e^{2\pi i n \bar{z}} + \delta_{p,q} \operatorname{Im}(z) + k_{p,q}(0) - \frac{\log\big(\operatorname{Im}(z)\big)}{\operatorname{vol}_{\mathrm{hyp}}(X)} + \sum_{n>0} k_{p,q}(n) e^{2\pi i n z}, \tag{12}$$

with Fourier coefficients $k_{p,q}(n) \in \mathbb{C}$.

For $p, q \in \mathcal{P}_X$, as $z \in X$ approaches q, the Eisenstein series $\mathcal{E}_{X,\mathrm{par},p}(z,s)$ corresponding to the cusp $p \in \mathcal{P}_X$ satisfies the following equation (see Corollary 3.5 in [8])

$$\begin{aligned}\mathcal{E}_{X,\mathrm{par},p}(z,s) &= \delta_{p,q} \operatorname{Im}(\sigma_q^{-1} z)^s + \alpha_{p,q}(s) \operatorname{Im}(\sigma_q^{-1} z)^{1-s} \\ &\quad + O\Big(\big(1 + \operatorname{Im}(\sigma_q^{-1} z)^{-\operatorname{Re}(s)}\big) e^{-2\pi \operatorname{Im}(\sigma_q^{-1} z)}\Big),\end{aligned} \tag{13}$$

where the Fourier coefficient $\alpha_{p,q}(s)$ is given by equation (3.21) in [8].

Elliptic Eisenstein series Let $\mathfrak{e} \in \mathcal{E}_X$ be an elliptic fixed point of order $m_{\mathfrak{e}}$ with stabilizer subgroup $\Gamma_{X,\mathfrak{e}}$. Let $\sigma_{\mathfrak{e}}$ be a scaling matrix of $\mathfrak{e}$ satisfying the conditions

$$\sigma_{\mathfrak{e}} i = \mathfrak{e} \quad \text{and} \quad \sigma_{\mathfrak{e}}^{-1} \Gamma_{X,\mathfrak{e}} \sigma_{\mathfrak{e}} = \langle \gamma_i \rangle, \quad \text{where } \gamma_i = \begin{pmatrix} \cos(\pi/m_{\mathfrak{e}}) & \sin(\pi/m_{\mathfrak{e}}) \\ -\sin(\pi/m_{\mathfrak{e}}) & \cos(\pi/m_{\mathfrak{e}}) \end{pmatrix}. \tag{14}$$

Let $\rho(z)$ denote the hyperbolic distance $d_{\mathbb{H}}(z,i)$. Then, for $z \in X$ and $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, the elliptic Eisenstein series $\mathcal{E}_{X,\mathrm{ell},\mathfrak{e}}(z,s)$ corresponding to an elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$ is defined by the series

$$\mathcal{E}_{X,\mathrm{ell},\mathfrak{e}}(z,s) = \sum_{\eta \in \Gamma_{X,\mathfrak{e}} \backslash \Gamma_X} \sinh^{-s}\big(\rho(\sigma_{\mathfrak{e}}^{-1} \eta z)\big).$$

The series converges absolutely and locally uniformly for $\operatorname{Re}(s) > 1$ and $z \neq \mathfrak{e}$ (as a function in the variable z, for a fixed s, see [15]). From its definition, as $z \in X \backslash \mathcal{E}_X$ approaches an elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$, for any $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, we find

$$\mathcal{E}_{X,\mathrm{ell},\mathfrak{e}}(z,s) - \sinh^{-s}\big(\rho(\sigma_{\mathfrak{e}}^{-1} z)\big) = O_z(1). \tag{15}$$

Moreover, for any $z \in X$, $s \in \mathbb{C}$ with $\operatorname{Re}(s) > 1$, and any cusp $p \in \mathcal{P}_X$, it follows that

$$\lim_{z \to p} \mathcal{E}_{X,\mathrm{ell},\mathfrak{e}}(z,s) = 0. \tag{16}$$

Space of square-integrable functions Let $L^2(X)$ denote the space of square integrable functions on X with respect to the hyperbolic (1, 1)-form $\mu_{\mathrm{hyp}}(z)$. There exists a natural inner-product $\langle \cdot, \cdot \rangle$ on $L^2(X)$ given by

$$\langle f, g \rangle = \int_X f(z) \overline{g(z)}\, \mu_{\mathrm{hyp}}(z),$$

where $f, g \in L^2(X)$, making $L^2(X)$ into a Hilbert space.

Furthermore, every $f \in L^2(X)$ admits the spectral expansion

$$f(z) = \sum_{n=0}^{\infty} \langle f, \varphi_{X,n}(z) \rangle \varphi_{X,n}(z) + \frac{1}{4\pi} \sum_{p \in \mathcal{P}_X} \int_{-\infty}^{\infty} \big\langle f, \mathcal{E}_{X,\mathrm{par},p}(z, 1/2 + ir) \big\rangle \mathcal{E}_{X,\mathrm{par},p}(z, 1/2 + ir) dr, \tag{17}$$

where $\{\varphi_{X,n}(z)\}$ denotes the set of orthonormal eigenfunctions for the discrete spectrum of Δ_{hyp}, and $\{\mathcal{E}_{X,\mathrm{par},p}(z, 1/2 + ir)\}$ denotes the set of eigenfunctions for the continuous spectrum of Δ_{hyp}, with $\mathcal{E}_{X,\mathrm{par},p}(z, s)$ denoting the parabolic Eisenstein series for the cusp $p \in \mathcal{P}_X$.

The eigenfunctions $\{\varphi_{X,n}(z)\}$ corresponding to the discrete spectrum can all be chosen to be real-valued, and for the rest of this article we continue to assume so.

Heat Kernels For $t \in \mathbb{R}_{>0}$ and $z, w \in \mathbb{H}$, the hyperbolic heat kernel $K_{\mathbb{H}}(t; z, w)$ on $\mathbb{R}_{>0} \times \mathbb{H} \times \mathbb{H}$ is given by the formula

$$K_{\mathbb{H}}(t; z, w) = \frac{\sqrt{2}e^{-t/4}}{(4\pi t)^{3/2}} \int_{d_{\mathbb{H}}(z,w)}^{\infty} \frac{re^{-r^2/4t}}{\sqrt{\cosh(r) - \cosh(d_{\mathbb{H}}(z, w))}} dr, \tag{18}$$

where $d_{\mathbb{H}}(z, w)$ is the hyperbolic distance between z and w.

For $t \in \mathbb{R}_{>0}$ and $z, w \in X$, the hyperbolic heat kernel $K_{X,\mathrm{hyp}}(t; z, w)$ on $\mathbb{R}_{>0} \times X \times X$ is defined as

$$K_{X,\mathrm{hyp}}(t; z, w) = \sum_{\gamma \in \Gamma_X} K_{\mathbb{H}}(t; z, \gamma w).$$

For notational brevity, we denote $K_{X,\mathrm{hyp}}(t; z, w)$ by $K_{X,\mathrm{hyp}}(t; z)$, when $z = w$.

The hyperbolic heat kernel $K_{X,\mathrm{hyp}}(t; z, w)$ admits the spectral expansion

$$K_{X,\mathrm{hyp}}(t; z, w) = \sum_{n=0}^{\infty} \varphi_{X,n}(z)\varphi_{X,n}(w)e^{-\lambda_{X,n}t} + \frac{1}{4\pi} \sum_{p \in \mathcal{P}_X} \int_{-\infty}^{\infty} \mathcal{E}_{X,\mathrm{par},p}(z, 1/2 + ir)\mathcal{E}_{X,\mathrm{par},p}(w, 1/2 - ir)e^{-(r^2+1/4)t} dr, \tag{19}$$

where $\lambda_{X,n}$ denotes the eigenvalue of the normalized eigenfunction $\varphi_{X,n}(z)$ and $(r^2 + 1/4)$ is the eigenvalue of the eigenfunction $\mathcal{E}_{X,\mathrm{par},p}(z, 1/2 + ir)$, as above.

Let $\mathcal{P}(\Gamma_X)$, $\mathcal{E}(\Gamma_X)$, and $\mathcal{H}(\Gamma_X)$ (here id is not treated as a parabolic element) denote the sets of parabolic, elliptic, and hyperbolic elements of the Fuchsian subgroup Γ_X, respectively. For $t \in \mathbb{R}_{\geq 0}$ and $z \in X$, put

$$PK_{X,\mathrm{hyp}}(t; z) = \sum_{\gamma \in \mathcal{H}(\Gamma_X)} K_{\mathbb{H}}(t; z, \gamma z), \quad EK_{X,\mathrm{hyp}}(t; z) = \sum_{\gamma \in \mathcal{E}(\Gamma_X)} K_{\mathbb{H}}(t; z, \gamma z)$$

$$HK_{X,\mathrm{hyp}}(t; z) = \sum_{\gamma \in \mathcal{P}(\Gamma_X)} K_{\mathbb{H}}(t; z, \gamma z).$$

As the hyperbolic heat kernel $K_{X,\mathrm{hyp}}(t; z)$ is a sum of the above three series, the convergence of each of the above series follows from the convergence of the hyperbolic heat kernel $K_{X,\mathrm{hyp}}(t; z)$ and the fact that $K_{\mathbb{H}}(t; z, \gamma z)$ is positive for all $t \in \mathbb{R}_{\geq 0}$, $z \in \mathbb{H}$, and $\gamma \in \Gamma_X$.

Selberg constant The hyperbolic length of the closed geodesic determined by a primitive non-conjugate hyperbolic element $\gamma \in \mathcal{H}(\Gamma_X)$ on X is given by

$$\ell_\gamma = \inf\{d_{\mathbb{H}}(z, \gamma z) | \, z \in \mathbb{H}\}.$$

The length of the shortest geodesic ℓ_X on X is given by

$$\ell_X = \inf\big\{d_{\mathbb{H}}(z, \gamma z) \big| \, \gamma \in \mathcal{H}(\Gamma_X), \; \gamma \text{ hyperbolic}, \; z \in \mathbb{H}\big\}.$$

From the definition, it is clear that $\ell_X > 0$.

For $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the Selberg zeta function associated to X is defined as

$$Z_X(s) = \prod_{\gamma \in \mathcal{H}(\Gamma_X)} Z_\gamma(s), \quad \text{where} \quad Z_\gamma(s) = \prod_{n=0}^{\infty} \big(1 - e^{(s+n)\ell_\gamma}\big).$$

The Selberg zeta function $Z_X(s)$ admits a meromorphic continuation to all $s \in \mathbb{C}$, with zeros and poles characterized by the spectral theory of the hyperbolic Laplacian. Furthermore, $Z_X(s)$ has a simple zero at $s = 1$, and the following constant is well-defined

$$c_X = \lim_{s \to 1} \left(\frac{Z_X'(s)}{Z_X(s)} - \frac{1}{s-1} \right). \tag{20}$$

For $t \in \mathbb{R}_{\geq 0}$, the hyperbolic heat trace is given by the integral

$$H\mathrm{Tr}\, K_{X,\mathrm{hyp}}(t) = \int_X HK_{X,\mathrm{hyp}}(t; z)\, \mu_{\mathrm{hyp}}(z).$$

The convergence of the integral follows from the celebrated Selberg trace formula. Furthermore, from Lemma 4.2 in [12], we have the following relation

$$\int_0^\infty \big(H\mathrm{Tr}\, K_{X,\mathrm{hyp}}(t) - 1\big) dt = c_X - 1. \tag{21}$$

Bounds on heat kernels For the rest of this article, we fix a $0 < t_0 < 1$. Then, there exist constants c_0 and c_∞ such that for $0 < t < t_0$ and $\eta \geq 0$, we have

$$K_{\mathbb{H}}(t; \eta) \leq \frac{c_0}{4\pi t} e^{-\eta^2/(4t)};$$

furthermore, for $t \geq t_0$ and $\eta \geq 0$, we get

$$K_{\mathbb{H}}(t; \eta) \leq c_\infty e^{-t/4}. \tag{22}$$

The above two formulae follow directly from the expression for the heat kernel $K_{\mathbb{H}}(t; \eta)$ stated in Eq. (18).

Definition 2.1 We fix a constant $0 < \beta < 1/4$, such that for $t \geq t_0$ and a fixed $\eta \geq 0$, the function

$$e^{\beta t} K_{\mathbb{H}}(t; \eta) \tag{23}$$

is a monotone decreasing function in the variable t.

Furthermore, there exists a $\delta_0 > 0$, such that for $\eta > \delta_0$ and a fixed $0 < t \leq t_0$, the function $K_{\mathbb{H}}(t; \eta)$ is a monotone decreasing function in the variable η. We now fix a δ_X satisfying $\delta_X > \max\{\delta_0, \, 4\ell_X + 5\}$.

As a function in the variable z, the sum $EK_{X,\mathrm{hyp}}(t_0, z) + HK_{X,\mathrm{hyp}}(t_0; z)$ remains bounded on X and also at the cusps. So we put

$$C_X^{HK} = \max_{z \in X} \big(K_{\mathbb{H}}(t_0; z) + EK_{X,\mathrm{hyp}}(t_0; z) + HK_{X,\mathrm{hyp}}(t_0; z) \big).$$

Automorphic Green's function For $z, w \in \mathbb{H}$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 0$, the free-space Green's function $g_{\mathbb{H},s}(z, w)$ is defined as

$$g_{\mathbb{H},s}(z, w) = g_{\mathbb{H},s}(u(z, w)) = \frac{\Gamma(s)^2}{\Gamma(2s)} u^{-s} F(s, s; 2s, -1/u),$$

where $u = u(z, w) = |z - w|^2 / (4 \,\mathrm{Im}(z) \,\mathrm{Im}(w))$ and $F(s, s; 2s, -1/u)$ is the hypergeometric function.

For $z, w \in \mathbb{H}$ with $z \neq w$ and $s = 1$, we put $g_{\mathbb{H}}(z, w) = g_{\mathbb{H},1}(z, w)$, and by substituting $s = 1$ in the definition of $g_{\mathbb{H},s}(z, w)$, we get

$$g_{\mathbb{H}}(z, w) = \log \left(1 + \frac{1}{u(z, w)} \right) = \log \left| \frac{z - \overline{w}}{z - w} \right|^2 \geq 0. \tag{24}$$

Using the formula from equation (1.3) in [8], we get

$$\cosh(d_{\mathbb{H}}(z, w)) = 1 + 2u(z, w) \Longrightarrow g_{\mathbb{H}}(z, w) = \log \left(1 + \frac{1}{\sinh^2 \big(d_{\mathbb{H}}(z, w)/2 \big)} \right). \tag{25}$$

Furthermore, for $z, w \in \mathbb{H}$ with $z \neq w$, we have the following relation

$$g_{\mathbb{H}}(z, w) = \int_0^\infty K_{\mathbb{H}}(t; z, w) dt. \tag{26}$$

For $z, w \in X$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the automorphic Green's function $g_{X,\mathrm{hyp},s}(z, w)$ is defined as

$$g_{X,\mathrm{hyp},s}(z, w) = \sum_{\gamma \in \Gamma_X} g_{\mathbb{H},s}(z, \gamma w).$$

The series converges absolutely and locally uniformly for $z \neq w$ and $\mathrm{Re}(s) > 1$ (as a function in the variables z and w, for a fixed s, see Chapter 5 in [8]).

For $z, w \in X$ with $z \neq w$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the automorphic Green's function satisfies the following properties (see Chapters 5 and 6 in [8]):

(1) The automorphic Green's function $g_{X,\mathrm{hyp},s}(z, w)$ admits a meromorphic continuation to all $s \in \mathbb{C}$ with a simple pole at $s = 1$ with residue $4\pi / \mathrm{vol}_{\mathrm{hyp}}(X)$, and the Laurent expansion at $s = 1$ is of the form

$$g_{X,\mathrm{hyp},s}(z, w) = \frac{4\pi}{s(s-1)\,\mathrm{vol}_{\mathrm{hyp}}(X)} + g_{X,\mathrm{hyp}}^{(1)}(z, w) + O_{z,w}(s - 1),$$

where $g_{X,\mathrm{hyp}}^{(1)}(z, w)$ is the constant term of $g_{X,\mathrm{hyp},s}(z, w)$ at $s = 1$.

(2) Let $p, q \in \mathcal{P}_X$ be two cusps. Put

$$C_{p,q} = \min \left\{ c > 0 \,\middle|\, \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \sigma_p^{-1} \Gamma_X \sigma_q \right\}, \quad C_{p,p} = C_p.$$

Then, for $z, w \in X$ with $\mathrm{Im}(z) > \mathrm{Im}(w)$ and $\mathrm{Im}(z)\,\mathrm{Im}(w) > C_{p,q}^{-2}$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, the automorphic Green's function admits the Fourier expansion

$$g_{\mathrm{hyp},s}(\sigma_p z, \sigma_q w) = \frac{4\pi\,\mathrm{Im}(z)^{1-s}}{2s-1}\mathcal{E}_{\mathrm{par},p}(\sigma_q w, s) + \delta_{p,q}\sum_{n\neq 0}\frac{1}{|n|}W_s(nz)\overline{V_{\overline{s}}(nw)} + O\big(e^{-2\pi(\mathrm{Im}(z)-\mathrm{Im}(w))}\big), \tag{27}$$

where $W_s(z)$ and $V_s(z)$ denote the Whittaker functions, which are given by equations (1.26) and (1.36) in [8], respectively. This equation has been proved as Lemma 5.4 in [8], and one of the terms was wrongly estimated in the proof of the lemma. We have corrected this error, and stated the corrected equation.

The space $C_{\ell,\ell\ell}(X)$ Let $C_{\ell,\ell\ell}(X)$ denote the set of complex-valued functions $f : X \to \mathbb{P}^1(\mathbb{C})$, which admit the following type of singularities at finitely many points $\mathrm{Sing}(f) \subset X$, and are smooth away from $\mathrm{Sing}(f)$:

(1) If $s \in \mathrm{Sing}(f)$, then as z approaches s, the function f satisfies

$$f(z) = c_{f,s}\log|\vartheta_s(z)| + O_z(1), \tag{28}$$

for some $c_{f,s} \in \mathbb{C}$.

(2) As z approaches a cusp $p \in \mathcal{P}_X$, the function f satisfies

$$f(z) = c_{f,p}\log\big(-\log|\vartheta_p(z)|\big) + O_z(1), \tag{29}$$

for some $c_{f,p} \in \mathbb{C}$.

Hyperbolic Green's function For $z, w \in X$ and $z \neq w$, the hyperbolic Green's function is defined as

$$g_{X,\mathrm{hyp}}(z, w) = 4\pi\int_0^\infty\bigg(K_{X,\mathrm{hyp}}(t; z, w) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg)dt.$$

For $z, w \in X$ with $z \neq w$, the hyperbolic Green's function satisfies the following properties:

(1) For $z, w \in X$, the hyperbolic Green's function is uniquely determined by the differential equation (which is to be interpreted in terms of currents)

$$d_z d_z^c\, g_{X,\mathrm{hyp}}(z, w) + \delta_w(z) = \mu_{\mathrm{shyp}}(z), \tag{30}$$

with the normalization condition

$$\int_X g_{X,\mathrm{hyp}}(z, w)\,\mu_{\mathrm{hyp}}(z) = 0. \tag{31}$$

(2) From Eq. (30), it follows that $g_{X,\mathrm{hyp}}(z, w)$ admits a log-singularity at $z = w$, i.e., for $z, w \in X$, it satisfies

$$\lim_{w\to z}\big(g_{X,\mathrm{hyp}}(z, w) + \log|\vartheta_z(w)|^2\big) = O_z(1). \tag{32}$$

(3) For $z, w \in X$ and $z \neq w$, we have

$$g_{X,\mathrm{hyp}}(z, w) = g_{X,\mathrm{hyp}}^{(1)}(z, w) = \lim_{s\to 1}\bigg(g_{X,\mathrm{hyp},s}(z, w) - \frac{4\pi}{s(s-1)\,\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg). \tag{33}$$

The above properties follow from the properties of the heat kernel $K_{X,\mathrm{hyp}}(t; z, w)$ or from the properties of the automorphic Green's function $g_{X,\mathrm{hyp},s}(z, w)$.

(4) From Proposition 2.1 in [2], (or from Proposition 2.4.1 in [4]) for a fixed $w \in X$, and for $z \in X$ with $\mathrm{Im}(\sigma_p^{-1}z) > \mathrm{Im}(\sigma_p^{-1}w)$, and $\mathrm{Im}(\sigma_p^{-1}z)\,\mathrm{Im}(\sigma_p^{-1}w) > C_p^{-2}$, we have

$$g_{X,\mathrm{hyp}}(z,w) = 4\pi\kappa_{X,p}(w) - \frac{4\pi}{\mathrm{vol_{hyp}}(X)} - \frac{4\pi\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)}{\mathrm{vol_{hyp}}(X)} \\ - \log\big|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\big|^2 + O\big(e^{-2\pi(\mathrm{Im}(\sigma_p^{-1}z) - \mathrm{Im}(\sigma_p^{-1}w))}\big), \tag{34}$$

i.e., for a fixed $w \in X$, as $z \in X$ approaches a cusp $p \in \mathcal{P}_X$, we have

$$g_{X,\mathrm{hyp}}(z,w) = -\frac{4\pi\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)}{\mathrm{vol_{hyp}}(X)} + O_{z,w}(1) = -\frac{4\pi\log\big(-\log|\vartheta_p(z)|\big)}{\mathrm{vol_{hyp}}(X)} + O_{z,w}(1).$$

(5) For any $f \in C_{\ell,\ell\ell}(X)$ and for any fixed $w \in X\backslash\mathrm{Sing}(f)$, from Corollary 2.5 in [2] (or from Corollary 3.1.8 in [4]), we have the equality of integrals

$$\int_X g_{X,\mathrm{hyp}}(z,w)d_zd_z^c f(z) + f(w) + \sum_{s\in\mathrm{Sing}(f)} \frac{c_{f,s}}{2}\, g_{X,\mathrm{hyp}}(s,w) = \int_X f(z)\,\mu_{\mathrm{shyp}}(z). \tag{35}$$

An auxiliary identity From Definition 8.1 in [13], for $z \in X\backslash\mathcal{E}_X$, we have the following relation

$$4\pi\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{X,\mathrm{hyp}}(t;z)dt = \sum_{\gamma\in\Gamma_X\backslash\{\mathrm{id}\}} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z).$$

Furthermore, from Lemmas 5.2 and 6.3, Proposition 7.3, the right-hand side of above equation remains bounded at the cusps and at the elliptic fixed points. Hence, as in [2], we extend Definition 8.1 in [13] and the above relation to cusps and elliptic fixed points to conclude that the following quantity is well-defined on X and remains bounded at the cusps and at the elliptic fixed points

$$\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{X,\mathrm{hyp}}(t;z)dt.$$

Definition 2.2 For notational brevity, put

$$C_{X,\mathrm{hyp}} = \int_X\int_X g_{X,\mathrm{hyp}}(\zeta,\xi)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{X,\mathrm{hyp}}(t;\zeta)dt\bigg) \\ \times\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{X,\mathrm{hyp}}(t;\xi)dt\bigg)\,\mu_{\mathrm{hyp}}(\xi)\,\mu_{\mathrm{hyp}}(\zeta).$$

From Proposition 2.8 in [2] (or from Proposition 2.6.4 in [4]), for $z, w \in X$, we have

$$g_{X,\mathrm{hyp}}(z,w) - g_{X,\mathrm{can}}(z,w) = \phi_X(z) + \phi_X(w), \tag{36}$$

where from Remark 2.16 in [2] (or from Corollary 3.2.7 in [4]), the function $\phi_X(z)$ is given by the formula

$$\phi_X(z) = \frac{1}{2g_X}\int_X g_{X,\mathrm{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\mathrm{hyp}}\, K_{X,\mathrm{hyp}}(t;\zeta)dt\bigg)\,\mu_{\mathrm{hyp}}(\zeta) - \frac{C_{X,\mathrm{hyp}}}{8g_X^2}. \tag{37}$$

Key-identity From Corollary 2.15 in [2] (or from Corollary 3.2.5 in [4]), for any $f \in C_{\ell,\ell\ell}(X)$, we have following identity, which is a generalization of Theorem 3.4 from [10] to cusps and elliptic fixed points at the level of currents

$$g\int_X f(z)\,\mu_{\mathrm{can}}(z) = \left(\frac{1}{4\pi}+\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)\int_X f(z)\,\mu_{\mathrm{hyp}}(z)+\frac{1}{2}\int_X f(z)\left(\int_0^\infty \Delta_{\mathrm{hyp}} K_{X,\mathrm{hyp}}(t;z)dt\right)\mu_{\mathrm{hyp}}(z). \tag{38}$$

3 Certain convergence results

In this section, we prove the absolute and uniform convergence of certain series, and compute their asymptotics at cusps and at elliptic fixed points. The analysis of this section allows us to decompose the integrals involved in (37) into expressions, which we will bound in Sect. 4.

3.1 Parabolic case

Definition 3.1 For $z \in \mathbb{H}$, put

$$P_X(z) = \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z).$$

For any $z \in \mathbb{H}$ and $\gamma \in \mathrm{SL}_2(\mathbb{R})$, from the definition of $u(z,w)$, it follows that $u(\gamma z, w) = u(z,\gamma^{-1}w)$. Using which and Eq. (24), we arrive at $g_{\mathbb{H}}(\gamma z, w) = g_{\mathbb{H}}(z,\gamma^{-1}w)$. Furthermore, for any $\gamma_0 \in \Gamma_X$, we have $\gamma_0^{-1}\mathcal{P}(\Gamma_X)\gamma_0 = \mathcal{P}(\Gamma_X)$. So, for any $\gamma_0 \in \Gamma_X$ and $z \in \mathbb{H}$, observe that

$$P_X(\gamma_0 z) = \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(\gamma_0 z,\gamma\gamma_0 z) = \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma_0^{-1}\gamma\gamma_0 z)$$
$$= \sum_{\gamma\in\left(\gamma_0^{-1}\mathcal{P}(\Gamma_X)\gamma_0\right)} g_{\mathbb{H}}(z,\gamma z) = \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z),$$

which implies that the function $P_X(z)$ is invariant under the action of Γ_X, and hence, defines a function on X (recall that $\mathrm{id} \notin \mathcal{P}(\Gamma_X)$).

Lemma 3.2 *For $z \in X$, the series $P_X(z)$ converges absolutely and locally uniformly.*

Proof We have the following decomposition of parabolic elements of Γ_X

$$\mathcal{P}(\Gamma_X) = \bigcup_{p\in\mathcal{P}_X}\bigcup_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\left(\eta^{-1}\Gamma_{X,p}\eta\backslash\{\mathrm{id}\}\right) = \bigcup_{p\in\mathcal{P}_X}\bigcup_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\bigcup_{n\neq0}\{\eta^{-1}\gamma_p^n\eta\},$$

where γ_p is a generator of the stabilizer subgroup $\Gamma_{X,p}$ of the cusp $p \in \mathcal{P}_X$. This implies that formally, we have

$$P_X(z) = \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z) = \sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\sum_{n\neq0} g_{\mathbb{H}}(z,\eta^{-1}\gamma_p^n\eta z)$$
$$= \sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\sum_{n\neq0} g_{\mathbb{H}}(\eta z,\gamma_p^n\eta z) = \sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X} P_{\mathrm{gen},p}(\eta z), \tag{39}$$

where $P_{\mathrm{gen},p}(z)=\sum_{n\neq 0}g_{\mathbb{H}}(z,\gamma_p^n z)$. We first prove the absolute convergence of the function $P_{\mathrm{gen},p}(z)$. From the definition of $g_{\mathbb{H}}(z,w)$ as given in (24), for any cusp $p\in\mathcal{P}_X$, observe that

$$\begin{aligned}P_{\mathrm{gen},p}(z)&=\sum_{n\neq 0}g_{\mathbb{H}}(\sigma_p^{-1}z,\gamma_\infty^n\sigma_p^{-1}z)=\sum_{n\neq 0}\log\left(\frac{4\,\mathrm{Im}(\sigma_p^{-1}z)^2+n^2}{n^2}\right)\\&\leq 2\log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2+1\right)+2\int_1^\infty\log\left(\frac{4\,\mathrm{Im}(\sigma_p^{-1}z)^2+t^2}{t^2}\right)dt\\&=4\pi\,\mathrm{Im}(\sigma_p^{-1}z)-8\,\mathrm{Im}(\sigma_p^{-1}z)\tan^{-1}\left(\frac{1}{2\,\mathrm{Im}(\sigma_p^{-1}z)}\right)\leq 32\,\mathrm{Im}(\sigma_p^{-1}z)^2,\end{aligned}\tag{40}$$

where σ_p is a scaling matrix associated to the cusp $p\in\mathcal{P}_X$ as in (6) (for the details regarding the computation of the last inequality, we refer the reader to Proposition 4.2.3 in [4]). This proves the absolute convergence of the function $P_{\mathrm{gen},p}(z)$.

Hence, combining Eq. (39) with inequality (40), we arrive at the estimate

$$P_X(z)\leq 32\sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\mathrm{Im}(\sigma_p^{-1}\eta z)^2=32\sum_{p\in\mathcal{P}_X}\mathcal{E}_{X,\mathrm{par},p}(z,2),$$

which proves the locally uniform convergence of the series $P_X(z)$. Furthermore, each term of the series $P_X(z)$ is positive, hence, it converges absolutely. □

Lemma 3.3 *As $z\in X$ approaches a cusp $p\in\mathcal{P}_X$, the function $P_X(z)$ satisfies the estimate*

$$P_X(z)=4\pi\,\mathrm{Im}(\sigma_p^{-1}z)-\log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2\right)+O_z(1).$$

Proof Let $z\in X$ approach a cusp $p\in\mathcal{P}_X$. From Eq. (39), we obtain the decomposition

$$P_X(z)=\sum_{\substack{q\in\mathcal{P}_X\\ q\neq p}}\sum_{\eta\in\Gamma_{X,q}\backslash\Gamma_X}P_{\mathrm{gen},q}(\eta z)+\sum_{\substack{\eta\in\Gamma_{X,p}\backslash\Gamma_X\\ \eta\neq\mathrm{id}}}P_{\mathrm{gen},p}(\eta z)+P_{\mathrm{gen},p}(z).\tag{41}$$

We now estimate the right-hand side of the above equation term by term. Using inequality (40), we derive the following upper bounds for the first and second terms

$$\sum_{\substack{q\in\mathcal{P}_X\\ q\neq p}}\sum_{\eta\in\Gamma_{X,q}\backslash\Gamma_X}P_{\mathrm{gen},q}(\eta z)\leq 32\sum_{\substack{q\in\mathcal{P}_X\\ q\neq p}}\sum_{\eta\in\Gamma_{X,q}\backslash\Gamma_X}\mathrm{Im}(\sigma_q^{-1}\eta z)^2=32\sum_{\substack{q\in\mathcal{P}_X\\ q\neq p}}\mathcal{E}_{X,\mathrm{par},q}(z,2);\tag{42}$$

$$\sum_{\substack{\eta\in\Gamma_{X,p}\backslash\Gamma_X\\ \eta\neq\mathrm{id}}}P_{\mathrm{gen},p}(\eta z)\leq 32\sum_{\substack{\eta\in\Gamma_{X,p}\backslash\Gamma_X\\ \eta\neq\mathrm{id}}}\mathrm{Im}(\sigma_p^{-1}\eta z)^2=32\big(\mathcal{E}_{\mathrm{par},p}(z,2)-\mathrm{Im}(\sigma_p^{-1}z)^2\big).\tag{43}$$

So using the above upper bounds, for $z\in X$ approaching $p\in\mathcal{P}_X$, from Eq. (13), we have the following estimate for the first and second terms

$$\sum_{\substack{q\in\mathcal{P}_X\\ q\neq p}}\sum_{\eta\in\Gamma_{X,q}\backslash\Gamma_X}P_{\mathrm{gen},q}(\eta z)+\sum_{\substack{\eta\in\Gamma_{X,p}\backslash\Gamma_X\\ \eta\neq\mathrm{id}}}P_{\mathrm{gen},p}(\eta z)=O\left(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\right).\tag{44}$$

As $z \in X$ approaches $p \in \mathcal{P}_X$, we are now left to investigate the behavior of the third term

$$P_{\mathrm{gen},p}(z) = \sum_{n\neq 0} g_{\mathbb{H}}(\sigma_p^{-1}z, \gamma_\infty^n\sigma_p^{-1}z)$$
$$= \lim_{w\to z}\lim_{s\to 1}\Bigg(\sum_{n=-\infty}^{\infty} g_{\mathbb{H},s}(\sigma_p^{-1}w, \gamma_\infty^n\sigma_p^{-1}z) - g_{\mathbb{H},s}(\sigma_p^{-1}z, \sigma_p^{-1}w)\Bigg). \quad (45)$$

From Lemma 5.1 in Chapter 5 of [8], for $\mathrm{Im}(\sigma_p^{-1}z) > \mathrm{Im}(\sigma_p^{-1}w)$, and $s \in \mathbb{C}$ with $\mathrm{Re}(s) > 1$, we have

$$\sum_{n=-\infty}^{\infty} g_{\mathbb{H},s}(\sigma_p^{-1}w, \gamma_\infty^n\sigma_p^{-1}z) = \frac{4\pi}{2s-1}\mathrm{Im}(\sigma_p^{-1}w)^s\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}$$
$$+ \sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)}. \quad (46)$$

Substituting the above expression in Eq. (45), we get

$$P_{\mathrm{gen},p}(z) = 4\pi\,\mathrm{Im}(\sigma_p^{-1}z) + \lim_{w\to z}\lim_{s\to 1}\Bigg(\sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)}$$
$$- g_{\mathbb{H},s}(\sigma_p^{-1}z, \sigma_p^{-1}w)\Bigg). \quad (47)$$

From the Proof of Lemma 5.4 in [8] (there is a slight error in the calculation of this lemma, which has been corrected in Corollary 1.9.5 in [4]), we have the estimate

$$\sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)}$$
$$= -\log\left|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\right|^2 + O\left(e^{-2\pi(\mathrm{Im}(\sigma_p^{-1}z) - \mathrm{Im}(\sigma_p^{-1}w))}\right).$$

Using the estimate stated in above equation, we compute

$$\lim_{w\to z}\lim_{s\to 1}\Bigg(\sum_{n\neq 0}\frac{1}{|n|}W_s(n\sigma_p^{-1}z)\overline{V_{\overline{s}}(n\sigma_p^{-1}w)} - g_{\mathbb{H},s}(\sigma_p^{-1}z, \sigma_p^{-1}w)\Bigg)$$
$$= -\log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2\right) + O_z(1). \quad (48)$$

Combining Eqs. (47) and (48), we arrive at the estimate

$$P_{\mathrm{gen},p}(z) = \lim_{w\to z}\left(-\log\left|1 - e^{2\pi i(\sigma_p^{-1}z - \sigma_p^{-1}w)}\right|^2 - \log\left|\frac{\sigma_p^{-1}z - \overline{\sigma_p^{-1}w}}{\sigma_p^{-1}z - \sigma_p^{-1}w}\right|^2\right) + O_z(1)$$
$$= 4\pi\,\mathrm{Im}(\sigma_p^{-1}z) - \log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2\right) + O_z(1), \quad (49)$$

which along with the estimate obtained in Eq. (44) completes the proof of the proposition.

□

Remark 3.4 From Lemma 5.2 in [13], the following series

$$\sum_{\gamma\in\mathcal{P}(\Gamma_X)} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z)$$

converges absolutely and uniformly for all $z \in X$, and the above series remains bounded at the cusps of X. Furthermore, from the absolute and locally uniform convergence of the series $P_X(z)$, and the uniform convergence of the above series, we have the following relations

$$\sum_{\gamma \in \mathcal{P}(\Gamma_X)} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z, \gamma z) = \Delta_{\mathrm{hyp}}\, P_X(z) = \sum_{p \in \mathcal{P}_X} \sum_{\eta \in \Gamma_{X,p} \backslash \Gamma_X} \Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},p}(\eta z),$$

$$\Delta_{\mathrm{hyp}}\, P_{\mathrm{gen},p}(z) = \sum_{n \neq 0} \Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(\sigma_p^{-1} z, \gamma_\infty^n \sigma_p^{-1} z) = 2\left(\frac{2\pi\, \mathrm{Im}(\sigma_p^{-1} z)}{\sinh(2\pi\, \mathrm{Im}(\sigma_p^{-1} z))}\right)^2 - 2. \quad (50)$$

Put

$$C_{X,\mathrm{par}}^{\mathrm{aux}} = \sup_{z \in X} \big| \Delta_{\mathrm{hyp}}\, P_X(z) \big|. \quad (51)$$

3.2 Elliptic case

Definition 3.5 For $z \in \mathbb{H}$, put

$$E_X(z) = \sum_{\gamma \in \mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z, \gamma z).$$

Using similar arguments as in Definition 3.1, we can conclude that the function $E_X(z)$ is Γ_X-invariant and hence, defines a function on X.

Lemma 3.6 *For $z \in X \backslash \mathcal{E}_X$, the series $E_X(z)$ converges absolutely and locally uniformly, and as $z \in X$ approaches an elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$, we have*

$$E_X(z) = -\frac{m_{\mathfrak{e}} - 1}{m_{\mathfrak{e}}} \log |\vartheta_{\mathfrak{e}}(z)|^2 + O_z(1). \quad (52)$$

Furthermore, the function $E_X(z)$ is zero at the cusps.

Proof We have the following decomposition of elliptic elements of Γ_X

$$\mathcal{E}(\Gamma_X) = \bigcup_{\mathfrak{e} \in \mathcal{E}_X} \bigcup_{\eta \in \Gamma_{X,\mathfrak{e}} \backslash \Gamma_X} \{\eta^{-1} \Gamma_{X,\mathfrak{e}} \eta \backslash \{\mathrm{id}\}\} = \bigcup_{\mathfrak{e} \in \mathcal{E}_X} \bigcup_{\eta \in \Gamma_{X,\mathfrak{e}} \backslash \Gamma_X} \bigcup_{n=1}^{m_{\mathfrak{e}}-1} \{\eta^{-1} \gamma_{\mathfrak{e}}^n \eta\},$$

where $\Gamma_{X,\mathfrak{e}}$ denotes the stabilizer subgroup of the elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$, and $\gamma_{\mathfrak{e}}$ denotes a generator of $\Gamma_{X,\mathfrak{e}}$. Using the above decomposition, formally we have

$$\begin{aligned} E_X(z) &= \sum_{\gamma \in \mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z, \gamma z) = \sum_{\mathfrak{e} \in \mathcal{E}_X} \sum_{\eta \in \Gamma_{X,\mathfrak{e}} \backslash \Gamma_X} \sum_{n=1}^{m_{\mathfrak{e}}-1} g_{\mathbb{H}}(z, \eta^{-1} \gamma_{\mathfrak{e}}^n \eta z) \\ &= \sum_{\mathfrak{e} \in \mathcal{E}_X} \sum_{\eta \in \Gamma_{X,\mathfrak{e}} \backslash \Gamma_X} \sum_{n=1}^{m_{\mathfrak{e}}-1} g_{\mathbb{H}}(\sigma_{\mathfrak{e}}^{-1} \eta z, \gamma_i^n \sigma_{\mathfrak{e}}^{-1} \eta z), \end{aligned} \quad (53)$$

where $\sigma_{\mathfrak{e}}$ denotes a scaling matrix of the elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$ as given in (14). Now for any $\mathfrak{e} \in \mathcal{E}_X$, $0 < n \leq m_{\mathfrak{e}} - 1$, and $\eta \in \Gamma_{X,\mathfrak{e}} \backslash \Gamma_X$, let $w = u + iv$ denote $\sigma_{\mathfrak{e}}^{-1} \eta z$. Using formula (24) and the relation

$$u^2 + v^2 + 1 = 2v \cosh(\rho(w)),$$

where $\rho(u)$ denotes $d_{\mathbb{H}}(z,i)$ the hyperbolic distance between the points z and i, we compute

$$\begin{aligned} g_{\mathbb{H}}(w,\gamma_i^n w) &= \log\left|\frac{-\sin(n\pi/m_{\mathfrak{e}})(|w|^2+1)+\cos(n\pi/m_{\mathfrak{e}})(w-\overline{w})}{-\sin(n\pi/m_{\mathfrak{e}})(w^2+1)}\right|^2 \\ &= \log\left(\frac{\sin^2(n\pi/m_{\mathfrak{e}})\cosh^2(\rho(w))+\cos^2(n\pi/m_{\mathfrak{e}})}{\sin^2(n\pi/m_{\mathfrak{e}})\cosh^2(\rho(w))-\sin^2(n\pi/m_{\mathfrak{e}})}\right) \\ &= \log\left(1+\frac{1}{\sin^2(n\pi/m_{\mathfrak{e}})\sinh^2(\rho(w))}\right) \leq \frac{1}{\sin^2(n\pi/m_{\mathfrak{e}})\sinh^2(\rho(w))}. \end{aligned} \tag{54}$$

Put

$$c_{X,\mathrm{ell}} = \max\left\{1/\sin^2(n\pi/m_{\mathfrak{e}})\,\middle|\,\mathfrak{e}\in\mathcal{E}_X,\,0\ <n\leq m_{\mathfrak{e}}-1\right\}. \tag{55}$$

Then, from decomposition (53) and inequality (54), we derive

$$E_X(z) \leq \sum_{\mathfrak{e}\in\mathcal{E}_X}\sum_{\eta\in\Gamma_{X,\mathfrak{e}}\backslash\Gamma_X}\sum_{n=1}^{m_{\mathfrak{e}}-1}\frac{c_{X,\mathrm{ell}}}{\sinh^2(\rho(\sigma_{\mathfrak{e}}^{-1}\eta z))} = c_{X,\mathrm{ell}}\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1)\,\mathcal{E}_{X,\mathrm{ell},\mathfrak{e}}(z,2), \tag{56}$$

which proves the locally uniform convergence of the series $E_X(z)$. Furthermore, each term of the series $E_X(z)$ is positive, hence, it converges absolutely. The asymptotic relation stated in (52) follows trivially from decomposition (53).

Moreover, for any $z, w \in \mathbb{H}$ with $z \neq w$, any $\gamma \in \Gamma_X\backslash\mathcal{P}(\Gamma_X)$, and any cusp $p \in \mathcal{P}_X$, observe that

$$\lim_{z\to p} g_{\mathbb{H}}(z,\gamma w) = 0.$$

From the above relation, it trivially follows that the function $E_X(z)$ is zero at the cusps. □

Remark 3.7 From Lemma 3.6, it follows that the function $E_X(z)$ admits log-singularities at elliptic fixed points, and is zero at the cusps. So we can conclude that $E_X(z) \in C_{\ell,\ell\ell}(X)$ with $\mathrm{Sing}(E_X(z)) = \mathcal{E}_X$ and $c_{E_X,\mathfrak{e}} = -2(m_{\mathfrak{e}}-1)/m_{\mathfrak{e}}$, for any $\mathfrak{e}\in\mathcal{E}_X$.

From Lemma 6.3 in [13], the following series

$$\sum_{\gamma\in\mathcal{E}(\Gamma_X)}\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \leq 0$$

converges absolutely and uniformly for all $z \in X$, and the above series remains bounded at the cusps. Furthermore, from the absolute and locally uniform convergence of the series $E_X(z)$, and the uniform convergence of the above series, we have the following relation

$$\Delta_{\mathrm{hyp}}\, E_X(z) = \sum_{\gamma\in\mathcal{E}(\Gamma_X)}\Delta_{\mathrm{hyp}}\, g_{\mathbb{H}}(z,\gamma z) \leq 0. \tag{57}$$

3.3 Hyperbolic case

Definition 3.8 For $z \in X$, put

$$H_X(z) = 4\pi\int_0^{\infty}\left(HK_{X,\mathrm{hyp}}(t;z) - \frac{1}{\mathrm{vol_{hyp}}(X)}\right)dt. \tag{58}$$

The function $H_X(z)$ is invariant under the action of Γ_X, and hence, defines a function on X.

Proposition 3.9 *The function $H_X(z)$ is well-defined on X. Moreover it satisfies*

$$H_X(z) = \lim_{w\to z}\big(g_{X,\mathrm{hyp}}(z,w) - g_{\mathbb{H}}(z,w)\big) - E_X(z) - P_X(z). \tag{59}$$

Proof From Lemmas 3.2, 3.6, we know that the series

$$P_X(z) = \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z) = \sum_{\gamma\in\mathcal{P}(\Gamma_X)} 4\pi\int_0^\infty K_{\mathbb{H}}(t;z,\gamma z)dt,$$
$$E_X(z) = \sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z) = \sum_{\gamma\in\mathcal{E}(\Gamma_X)} 4\pi\int_0^\infty K_{\mathbb{H}}(t;z,\gamma z)dt.$$

converge absolutely for all $z\in X$, respectively. So, we can interchange summation and integration in the above integrals. Moreover, the integral

$$\int_0^\infty \left(K_{X,\mathrm{hyp}}(t;z) - K_{\mathbb{H}}(t;0) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)dt \tag{60}$$

converges for all $z\in X$. So we can write

$$\begin{aligned} H_X(z) &= 4\pi\int_0^\infty \left(HK_{X,\mathrm{hyp}}(t;z) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)dt \\ &= 4\pi\int_0^\infty \left(K_{X,\mathrm{hyp}}(t;z) - K_{\mathbb{H}}(t;0) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \sum_{\gamma\in\mathcal{E}(\Gamma_X)} K_{\mathbb{H}}(t;z,\gamma z)\right. \\ &\qquad \left. - \sum_{\gamma\in\mathcal{P}(\Gamma_X)} K_{\mathbb{H}}(t;z,\gamma z)\right)dt \\ &= 4\pi\int_0^\infty \left(K_{X,\mathrm{hyp}}(t;z) - K_{\mathbb{H}}(t;0) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)dt - E_X(z) - P_X(z), \end{aligned} \tag{61}$$

which proves the convergence of the function $H_X(z)$.

From the convergence of the integral in (60), and an application of Fatou's lemma from real analysis, we can interchange limit and integration in the following expression to derive

$$\lim_{w\to z}\big(g_{X,\mathrm{hyp}}(z,w) - g_{\mathbb{H}}(z,w)\big) = 4\pi\int_0^\infty \left(K_{X,\mathrm{hyp}}(t;z) - K_{\mathbb{H}}(t;0) - \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right)dt. \tag{62}$$

Combining Eqs. (61) and (62) proves Eq. (59). □

In the following proposition, we describe the behavior of the automorphic function $H_X(z)$ at the cusps.

Proposition 3.10 *As $z\in X$ approaches a cusp $p\in\mathcal{P}_X$, we have*

$$E_X(z) + H_X(z) = \frac{8\pi\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} + 4\pi k_{p,p}(0) + O\big(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\big),$$

where $k_{p,p}(0)$ is the zeroth Fourier coefficient in the Fourier expansion of Kronecker's limit function $\kappa_{X,p}(z)$ associated to the cusp $p\in\mathcal{P}_X$ (see Eq. (12)).

Proof Combining Eqs. (59) and (41), we have

$$E_X(z)+H_X(z)=\lim_{w\to z}\Bigg(g_{X,\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\Bigg)\\ -\sum_{\substack{q\in\mathcal{P}_X\\ q\neq p}}\sum_{\eta\in\Gamma_{X,q}\backslash\Gamma_X}P_{\mathrm{gen},q}(\eta z)-\sum_{\substack{\eta\in\Gamma_{X,p}\backslash\Gamma_X\\ \eta\neq\mathrm{id}}}P_{\mathrm{gen},p}(\eta z).$$

We now estimate the right-hand side of the above equation term by term. As $z\in X$ approaches the cusp $p\in\mathcal{P}_X$, from Eq. (44), we arrive at the estimate

$$E_X(z)+H_X(z)=\lim_{w\to z}\Bigg(g_{X,\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\Bigg)+O\big(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\big). \tag{63}$$

We are now left to compute the asymptotics of the limit

$$\lim_{w\to z}\Bigg(g_{\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\Bigg)\\ =\lim_{w\to z}\lim_{s\to 1}\Bigg(g_{\mathrm{hyp},s}(z,w)-\frac{4\pi}{s(s-1)\,\mathrm{vol}_{\mathrm{hyp}}(X)}-\sum_{n=-\infty}^{\infty}g_{\mathbb{H},s}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\Bigg). \tag{64}$$

As $z\in X$ approaches $p\in\mathcal{P}_X$, combining estimates (27) and (46), we have

$$g_{X,\mathrm{hyp},s}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H},s}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)=\frac{4\pi\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{X,\mathrm{par},p}(w,s)\\ -\frac{4\pi}{2s-1}\,\mathrm{Im}(\sigma_p^{-1}w)^s\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big).$$

Using the above expression, we find that the right-hand side of limit (64) can be written as

$$\lim_{w\to z}\lim_{s\to 1}\Bigg(\frac{4\pi\,\mathrm{Im}(\sigma_p^{-1}z)^{1-s}}{2s-1}\mathcal{E}_{X,\mathrm{par},p}(w,s)-\frac{4\pi}{(s-1)\,\mathrm{vol}_{\mathrm{hyp}}(X)}\Bigg)\\ +\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}-4\pi\,\mathrm{Im}(\sigma_p^{-1}z)+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big).$$

To evaluate the above limit, we compute the Laurent expansions of $\mathcal{E}_{\mathrm{par},p}(w,s)$, $\mathrm{Im}(\sigma_p^{-1}z)^{1-s}$, and $(2s-1)^{-1}$ at $s=1$. The Laurent expansions of $\mathrm{Im}\,(\sigma_p^{-1}z)^{1-s}$ and $(2s-1)^{-1}$ at $s=1$ are easy to compute, and are of the form

$$\mathrm{Im}\,(\sigma_p^{-1}z)^{1-s}=1-(s-1)\log\big(\mathrm{Im}\,(\sigma_p^{-1}z)\big)+O\big((s-1)^2\big),\\ \frac{1}{2s-1}=1-2(s-1)+O\big((s-1)^2\big).$$

Using the Laurent expansion of the Eisenstein series $\mathcal{E}_{\mathrm{par},p}(w,s)$ from Eq. (11), and combining it with above expressions, we compute

$$\lim_{w\to z}\bigg(g_{\mathrm{hyp}}(z,w)-\sum_{n=-\infty}^{\infty}g_{\mathbb{H}}(\sigma_p^{-1}w,\gamma_\infty^n\sigma_p^{-1}z)\bigg)=4\pi\kappa_{X,p}(z)-4\pi\,\mathrm{Im}(\sigma_p^{-1}z)$$
$$-\frac{4\pi\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)}{\mathrm{vol_{hyp}}(X)}-\frac{4\pi}{\mathrm{vol_{hyp}}(X)}+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big). \quad (65)$$

From the Fourier expansion of Kronecker's limit function $\kappa_{X,p}(z)$ described in (12), we have

$$\kappa_{X,p}(z)=\mathrm{Im}(\sigma_p^{-1}z)+k_{p,p}(0)-\frac{\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)}{\mathrm{vol_{hyp}}(X)}+O\big(e^{-2\pi\,\mathrm{Im}(\sigma_p^{-1}z)}\big).$$

As $z\in X$ approaches $p\in\mathcal{P}_X$, substituting the above estimate in the right-hand side of Eq. (65), and combining it with Eq. (60), we arrive at

$$E_X(z)+H_X(z)=-\frac{8\pi\log\big(\mathrm{Im}(\sigma_p^{-1}z)\big)}{\mathrm{vol_{hyp}}(X)}-\frac{4\pi}{\mathrm{vol_{hyp}}(X)}+4\pi k_{p,p}(0)+O\big(\mathrm{Im}(\sigma_p^{-1}z)^{-1}\big),$$

which completes the proof of the proposition. □

Remark 3.11 As the function $E_X(z)$ is zero at the cusps, from Proposition 3.10, we can conclude that $H_X(z)$ has log log-growth at the cusps. Moreover, the function $H(z)$ remains smooth for all $z\in X$. Hence, $H_X(z)\in C_{\ell,\ell\ell}(X)$ with $\mathrm{Sing}(H_X(z))=\emptyset$.

Furthermore, from Eq. (21), it follows that

$$\int_X H_X(z)\,\mu_{\mathrm{hyp}}(z)=4\pi(c_X-1). \quad (66)$$

Using Eq. (59), we get

$$\Delta_{\mathrm{hyp}}\,P_X(z)+\Delta_{\mathrm{hyp}}\,E_X(z)+\Delta_{\mathrm{hyp}}\,H_X(z)=\Delta_{\mathrm{hyp}}\lim_{w\to z}\big(g_{X,\mathrm{hyp}}(z,w)-g_{\mathbb{H}}(z,w)\big).$$

Since the integral

$$4\pi\int_0^\infty\bigg(K_{X,\mathrm{hyp}}(t;z,z)-K_{\mathbb{H}}(t;0)-\frac{1}{\mathrm{vol_{hyp}}(X)}\bigg)dt,$$

as well as the integral of the derivatives of the integrand are absolutely convergent, we can take the Laplace operator Δ_{hyp} inside the integral. So we find

$$\Delta_{\mathrm{hyp}}\,P_X(z)+\Delta_{\mathrm{hyp}}\,E_X(z)+\Delta_{\mathrm{hyp}}\,H_X(z)=4\pi\int_0^\infty\Delta_{\mathrm{hyp}}\,K_{X,\mathrm{hyp}}(t;z)dt. \quad (67)$$

Corollary 3.12 *For any $z\in X\backslash\mathcal{E}_X$, we have*

$$\phi_X(z)=\frac{\big(H_X(z)+E_X(z)\big)}{2g_X}+\frac{1}{8\pi g_X}\int_X g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)$$
$$-\sum_{\mathfrak{e}\in\mathcal{E}_X}\frac{m_{\mathfrak{e}}-1}{2g_X m_{\mathfrak{e}}}\,g_{X,\mathrm{hyp}}(z,\mathfrak{e})-\frac{C_{X,\mathrm{hyp}}}{8g_X^2}-\frac{2\pi(c_X-1)}{g_X\,\mathrm{vol_{hyp}}(X)}-\frac{1}{2g_X}\int_X E_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta).$$

Proof Using formula (7), and combining Eqs. (37) and (67), we have

$$\phi_X(z) = \frac{1}{2g_X}\int_X g_{X,\mathrm{hyp}}(z,\zeta)\big(-d_\zeta d_\zeta^c\big(E_X(\zeta)+H_X(\zeta)\big)\big) + \frac{1}{8\pi g_X}\int_X g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(z) - \frac{C_{X,\mathrm{hyp}}}{8g_X^2}. \tag{68}$$

From Remarks 3.7 and 3.11, we know that the functions $E_X(z)$ and $H_X(z)$ both belong to $C_{\ell,\ell\ell}(X)$ with $\mathrm{Sing}(E_X(z)) = \mathcal{E}_X$ and $\mathrm{Sing}(H_X(z)) = \emptyset$, respectively. Hence, from Eq. (35), for any $z \in X\backslash\mathcal{E}_X$, we have the following relations

$$-\int_X g_{X,\mathrm{hyp}}(z,\zeta)d_\zeta d_\zeta^c E_X(\zeta) = \frac{E_X(z)}{2g_X} - \sum_{\mathfrak{e}\in\mathcal{E}_X}\frac{m_{\mathfrak{e}}-1}{2g_X m_{\mathfrak{e}}}\,g_{X,\mathrm{hyp}}(z,\mathfrak{e}) - \frac{1}{2g_X}\int_X E_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta),$$

$$-\int_X g_{X,\mathrm{hyp}}(z,\zeta)d_\zeta d_\zeta^c H_X(\zeta) = \frac{H_X(z)}{2g_X} - \frac{1}{2g_X}\int_X H_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta).$$

Substituting the above two equations in Eq. (68) and using relation (66) completes the proof of the corollary. □

4 Bounds for hyperbolic Green's function

In this section, we derive bounds for the hyperbolic Green's functions on compact subsets of X, and in the neighborhoods of cusps and elliptic fixed points.

We begin by defining a compact subset Y_ε, for some $0<\varepsilon<1$, and we adapt the existing bounds for the hyperbolic heat kernel from [10]. We then use these bounds to bound the hyperbolic Green's function both on the compact subset Y_ε, and in the neighborhood of cusps and elliptic fixed points.

4.1 Bounds for hyperbolic Green's function

Notation 4.1 For any $\delta > 0$ and a fixed $z, w \in X$, identifying X with its fundamental domain, we define the set

$$S_{\Gamma_X}(\delta; z, w) = \big\{\gamma \in \mathcal{H}(\Gamma_X)\cup\{\mathrm{id}\}\big|\, d_{\mathbb{H}}(z,\gamma w) < \delta\big\}.$$

Let $0 < \varepsilon < \min\{1, \ell_X\}$ be any number such that the following conditions holds true:

(1) For any cusp $p \in \mathcal{P}_X$, let $U_\varepsilon(p)$ denote an open coordinate disk of radius ε around p. Then, we have $\mathrm{Im}(\sigma_p^{-1}z) \geq \mathrm{Im}(\sigma_p^{-1}\gamma z)$, where σ_p is a scaling matrix of the cusp p. Furthermore, for $p, q \in \mathcal{P}_X$ and $p \neq q$, we have

$$U_\varepsilon(p)\cap U_\varepsilon(q) = \emptyset.$$

(2) For any elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$, let $U_\varepsilon(\mathfrak{e})$ denote an open coordinate disk around $\mathfrak{e}$ such that $d_{\mathbb{H}}(z,\mathfrak{e}) = \varepsilon$ for all $z \in \partial U_\varepsilon(\mathfrak{e})$. Furthermore for $\mathfrak{e}, \mathfrak{f} \in \mathcal{E}_X$ and $\mathfrak{e} \neq \mathfrak{f}$, we have

$$U_\varepsilon(\mathfrak{e})\cap U_\varepsilon(\mathfrak{f}) = \emptyset.$$

(3) For any elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$, $z \in \partial U_\varepsilon(\mathfrak{e})$ and $\gamma \in \Gamma_X$, we have

$$d_{\mathbb{H}}(z,\gamma\mathfrak{e}) \geq \varepsilon.$$

Furthermore, for any $p \in \mathcal{P}_X$ and any $\mathfrak{e} \in \mathcal{E}_X$, we have

$$U_\varepsilon(p) \cap U_\varepsilon(\mathfrak{e}) = \emptyset.$$

We fix an ε satisfying the above three conditions and put

$$Y_\varepsilon = X \Big\backslash \bigg(\bigcup_{p\in\mathcal{P}_X} U_\varepsilon(p) \cup \bigcup_{\mathfrak{e}\in\mathcal{E}_X} U_\varepsilon(\mathfrak{e}) \bigg), \quad Y_\varepsilon^{\mathrm{par}} = X \Big\backslash \bigg(\bigcup_{p\in\mathcal{P}_X} U_\varepsilon(p) \bigg),$$
$$Y_\varepsilon^{\mathrm{ell}} = X \Big\backslash \bigg(\bigcup_{\mathfrak{e}\in\mathcal{E}_X} U_\varepsilon(\mathfrak{e}) \bigg).$$

Furthermore, for any cusp $p \in \mathcal{P}_X$, any elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$, put

$$Y_{\varepsilon,p}^{\mathrm{par}} = X \backslash U_\varepsilon(p), \quad Y_{\varepsilon,\mathfrak{e}}^{\mathrm{ell}} = X \backslash U_\varepsilon(\mathfrak{e}),$$

respectively. For brevity of notation, we identify the fundamental domains associated to the compact subsets Y_ε, $Y_\varepsilon^{\mathrm{par}}$, and $Y_\varepsilon^{\mathrm{ell}}$ again by the same symbols.

The computations carried out in the following two remarks will come handy in the calculations that follow.

Lemma 4.2 *Let $\mathfrak{e} \in \mathcal{E}_X$ be an elliptic fixed point. Then, for any $\gamma \in \Gamma_X$, and $z \in \partial U_\varepsilon(\mathfrak{e})$, we have the following upper bound*

$$\sinh^2\big(d_{\mathbb{H}}(z,\gamma z)/2\big) \leq 7\coth(\varepsilon/2)\sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big). \tag{69}$$

Proof For $z \in \partial U_\varepsilon(\mathfrak{e})$ and any $\gamma \in \Gamma_X$, from condition (3), which the fixed ε satisfies, we have

$$d_{\mathbb{H}}(z,\gamma\mathfrak{e}) \geq \varepsilon \Longrightarrow \frac{\sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)}{\sinh^2(\varepsilon/2)} \geq 1; \tag{70}$$

$$\begin{aligned} d_{\mathbb{H}}(z,\gamma z) \leq d_{\mathbb{H}}(z,\gamma\mathfrak{e}) + d_{\mathbb{H}}(\gamma z,\gamma\mathfrak{e}) = d_{\mathbb{H}}(z,\gamma\mathfrak{e}) + \varepsilon &\Longrightarrow \sinh^2\big(d_{\mathbb{H}}(z,\gamma z)/2\big) \\ &\leq \sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big). \end{aligned} \tag{71}$$

For any $z \in \partial U_\varepsilon(\mathfrak{e})$ and $\gamma \in \Gamma_X$, observe that

$$\begin{aligned} \sinh^2\big((d_{\mathbb{H}}(z,\gamma\mathfrak{e})+\varepsilon)/2\big) &= \sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\cosh^2(\varepsilon/2) \\ &+ \cosh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\sinh^2(\varepsilon/2) + \sinh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\cosh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\sinh(\varepsilon) \\ &= 2\sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\cosh^2(\varepsilon/2) + \sinh^2(\varepsilon/2) \\ &+ \sinh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\cosh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\sinh(\varepsilon). \end{aligned} \tag{72}$$

Using inequality (70) and the fact that $\sinh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big) \leq \cosh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)$, we estimate the second and third terms on the right-hand side of above equation

$$\begin{aligned} &\sinh^2(\varepsilon/2) + \sinh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\cosh\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\sinh(\varepsilon) \\ &\leq \sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big) + \frac{\sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)}{\sinh^2(\varepsilon/2)}\sinh(\varepsilon) + \sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\sinh(\varepsilon). \end{aligned}$$

Combining Eq. (72) with the above inequality, and using the fact that $0 < \varepsilon < 1$ (which implies that $0 < \sinh(\varepsilon/2) + \cosh(\varepsilon/2) < 2$, and $1 < \cosh(\varepsilon/2) < \cot(\varepsilon/2)$), we find

$$\begin{aligned}\sinh^2\big((d_{\mathbb{H}}(z,\gamma\mathfrak{e})+\varepsilon)/2\big) &\leq \sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\big(1+2\cosh^2(\varepsilon/2)+2\coth(\varepsilon/2)+\sinh(\varepsilon)\big)\\ &\leq \sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big)\big(3\coth(\varepsilon/2)+2\cosh(\varepsilon/2)\big(\sinh(\varepsilon/2)+\cosh(\varepsilon/2)\big)\big)\\ &\leq 7\coth(\varepsilon/2)\sinh^2\big(d_{\mathbb{H}}(z,\gamma\mathfrak{e})/2\big). \end{aligned} \tag{73}$$

Finally combining the above upper bound with inequality (70) completes the proof of the lemma. □

Lemma 4.3 *Let $\mathfrak{e} \in \mathcal{E}_X$ be an elliptic fixed point. Then, for any $\gamma \in \Gamma_X$, $z \in \partial U_{\varepsilon/2}(\mathfrak{e})$, and $w \in \partial U_{\varepsilon}(\mathfrak{e})$, we have the following upper bound*

$$\sinh^2\big(d_{\mathbb{H}}(z,\gamma z)/2\big) \leq 14\coth(\varepsilon/4)\sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big). \tag{74}$$

Proof For any $\gamma \in \Gamma_X$, $z \in \partial U_{\varepsilon/2}(\mathfrak{e})$, and $w \in \partial U_{\varepsilon}(\mathfrak{e})$, from the choice of ε (i.e., condition (3) which the fixed ε satisfies), we have

$$\begin{aligned} d_{\mathbb{H}}(z,\gamma w) + d_{\mathbb{H}}(z,\mathfrak{e}) \geq d_{\mathbb{H}}(\gamma w,\mathfrak{e}) &\Longrightarrow d_{\mathbb{H}}(z,\gamma w)\\ \geq \varepsilon/2 \Longrightarrow \frac{\sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big)}{\sinh^2(\varepsilon/4)} &\geq 1; \end{aligned} \tag{75}$$

$$\begin{aligned} d_{\mathbb{H}}(z,\gamma z) \leq d_{\mathbb{H}}(z,\gamma w) + d_{\mathbb{H}}(\gamma w,\gamma z) &\leq d_{\mathbb{H}}(z,\gamma w) + \varepsilon\\ \Longrightarrow \sinh^2\big(d_{\mathbb{H}}(z,\gamma z)/2\big) &\leq \sinh^2\big((d_{\mathbb{H}}(z,\gamma w)+\varepsilon)/2\big). \end{aligned} \tag{76}$$

Using computation (72) from Lemma 4.2, we have

$$\begin{aligned}\sinh^2\big((d_{\mathbb{H}}(z,\gamma w)+\varepsilon)/2\big) &= 2\sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big)\cosh^2(\varepsilon/2)\\ &+ \sinh^2(\varepsilon/2)+\sinh\big(d_{\mathbb{H}}(z,\gamma w)/2\big)\cosh\big(d_{\mathbb{H}}(z,\gamma w)/2\big)\sinh(\varepsilon).\end{aligned}$$

Using inequality (75), and the fact that $\sinh\big(d_{\mathbb{H}}(z,\gamma w)/2\big) \leq \cosh\big(d_{\mathbb{H}}(z,\gamma w)/2\big)$, we arrive at

$$\begin{aligned}&\sinh^2\big((d_{\mathbb{H}}(z,\gamma w)+\varepsilon)/2\big)\\ &\leq \sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big)\bigg(2\cosh^2(\varepsilon/2) + \frac{\sinh^2(\varepsilon/2)}{\sinh^2(\varepsilon/4)} + \sinh(\varepsilon) + \frac{\sinh(\varepsilon)}{\sinh^2(\varepsilon/4)}\bigg)\\ &= \sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big)\Big(2\cosh^2(\varepsilon/2)+4\cosh^2(\varepsilon/4)+\sinh(\varepsilon)+4\coth(\varepsilon/4)\cosh(\varepsilon/2)\Big)\end{aligned}$$

Using the fact that $0 < \varepsilon < 1$ (which implies that $\cosh^2(\varepsilon/4) \leq \cosh^2(\varepsilon/2)$, $\cosh(\varepsilon/2) \leq 1.13$, $\sinh(\varepsilon) \leq 1.18$, and $1 < \coth(\varepsilon/4)$), we arrive at the following estimate

$$\sinh^2\big((d_{\mathbb{H}}(z,\gamma w)+\varepsilon)/2\big) \leq 14\coth(\varepsilon/4)\sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big),$$

which together with inequality (76) completes the proof of the lemma. □

Definition 4.4 From Eqs. (13) and (15), it follows that the following quantities are well-defined

$$C_{X,\mathrm{par}} = \sup_{z\in X}\sum_{p\in\mathcal{P}_X}\big(\mathcal{E}_{X,\mathrm{par},p}(z,2) - \mathrm{Im}(\sigma_p^{-1}z)^2\big), \tag{77}$$

$$C_{X,\mathrm{ell}} = \sup_{z\in X} c_{X,\mathrm{ell}}\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1)\big(\mathcal{E}_{X,\mathrm{ell},\mathfrak{e}}(z,2) - \sinh^{-2}\big(\rho(\sigma_{\mathfrak{e}}^{-1}z)\big)\big). \tag{78}$$

Lemma 4.5 *We have the following upper bounds*

$$\sup_{z\in Y_\varepsilon^{\mathrm{par}}} P_X(z) \leq -6|\mathcal{P}_X| \log\varepsilon + 32C_{X,\mathrm{par}} \tag{79}$$

$$\sup_{z\in Y_\varepsilon^{\mathrm{ell}}} E_X(z) \leq -\sum_{\mathfrak{e}\in\mathcal{E}_X} (m_{\mathfrak{e}}-1)\log\big(\tanh^2(\varepsilon)/c_{X,\mathrm{ell}}\big) + C_{X,\mathrm{ell}}. \tag{80}$$

Proof Combining estimate (77) with the estimates from the Proof of Lemma 3.3 (estimate (43)), we arrive at the following upper bound

$$\sup_{z\in Y_\varepsilon^{\mathrm{par}}} P_X(z) \leq 32 \sum_{p\in\mathcal{P}_X}\Big(\mathrm{Im}(\sigma_p^{-1}z)^2 + 32\big(\mathcal{E}_{X,\mathrm{par},p}(z,2) - \mathrm{Im}(\sigma_p^{-1}z)^2\big)\Big)$$
$$\leq -\frac{16|\mathcal{P}_X|\log\varepsilon}{\pi} + 32C_{X,\mathrm{par}} \leq -6|\mathcal{P}_X|\log\varepsilon + 32C_{X,\mathrm{par}},$$

which proves (79).

Combining estimate (78) with the estimates from the proof of Lemma 3.6 (estimates (54) and (56)), and using the fact that $c_{X,\mathrm{ell}} \geq 1$, we arrive at the following estimate

$$\sup_{z\in Y_\varepsilon^{\mathrm{ell}}} E_X(z) \leq \sup_{z\in Y_\varepsilon^{\mathrm{ell}}} \sum_{\mathfrak{e}\in\mathcal{E}_X}\sum_{n=1}^{m_{\mathfrak{e}}-1} \log\bigg(1+\frac{1}{\sin^2(n\pi/m_{\mathfrak{e}})\sinh^2(\rho(\sigma_{\mathfrak{e}}^{-1}z))}\bigg)$$
$$+\sup_{z\in Y_\varepsilon^{\mathrm{ell}}} c_{X,\mathrm{ell}}\sum_{\mathfrak{e}\in\mathcal{E}_X}\Big((m_{\mathfrak{e}}-1)\big(\mathcal{E}_{X,\mathrm{ell},\mathfrak{e}}(z,2)-\sinh^{-2}\big(\rho(\sigma_{\mathfrak{e}}^{-1}z)\big)\big)\Big)$$
$$\leq \sup_{z\in Y_\varepsilon^{\mathrm{ell}}}\Big(-\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1)\log\big(\tanh^2(\rho(\sigma_{\mathfrak{e}}^{-1}z))/c_{X,\mathrm{ell}}\big)\Big) + C_{X,\mathrm{ell}}. \tag{81}$$

For any $\mathfrak{e}\in\mathcal{E}_X$, from condition (2) which the fixed ε satisfies, we find

$$\sup_{z\in Y_\varepsilon^{\mathrm{ell}}}\Big(-\log\big(\tanh^2(\rho(\sigma_{\mathfrak{e}}^{-1}z))/c_{X,\mathrm{ell}}\big)\Big) = \sup_{z\in Y_\varepsilon^{\mathrm{ell}}}\Big(-\log\big(\tanh^2(d_{\mathbb{H}}(z,\mathfrak{e}))/c_{X,\mathrm{ell}}\big)\Big)$$
$$\leq \sup_{z\in\partial U_\varepsilon(\mathfrak{e})}\Big(-\log\big(\tanh^2(d_{\mathbb{H}}(z,\mathfrak{e}))/c_{X,\mathrm{ell}}\big)\Big) = -\log\big(\tanh^2(\varepsilon)/c_{X,\mathrm{ell}}\big). \tag{82}$$

Combining inequalities (81) and (82), establishes upper bound (80). □

Definition 4.6 With notation as in Sect. 1, for any $\delta \geq \delta_X$, $\alpha > 0$, and $z, w \in Y_\varepsilon$, put

$$K_{X,\mathrm{hyp}}^{\alpha,\delta}(t;z,w)$$
$$= K_{X,\mathrm{hyp}}(t;z,w) - \sum_{n:\,0\leq\lambda_{X,n}<\alpha}\varphi_{X,n}(z)\varphi_{X,n}(w)e^{-\lambda_{X,n}t} - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} K_{\mathbb{H}}(t;d_{\mathbb{H}}(z,\gamma w)).$$

The following theorem is an adaption of Lemma 4.2 in [10] to the case where X admits cusps and elliptic fixed points.

Theorem 4.7 *For any $\alpha\in(0,\lambda_{X,1})$, $\delta\geq\delta_X$, and $z,w\in Y_\varepsilon$, we have the following upper bounds:*

(a) *For* $0 < t < t_0$, *then*

$$\left|K^{\alpha,\delta}_{X,\mathrm{hyp}}(t;z,w)\right| \leq \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{c_0\sinh(\ell_X)\sinh(\delta)}{8\delta^2\sinh^2(\ell_X/2)} + \frac{c_0 e^{2\ell_X}}{2\pi\sinh^2(\ell_X/2)} + \sum_{\gamma\in\mathcal{P}(\Gamma_X)} K_{\mathbb{H}}(t;z,\gamma w) + \sum_{\gamma\in\mathcal{E}(\Gamma_X)} K_{\mathbb{H}}(t;z,\gamma w); \tag{83}$$

(b) *If* $t \geq t_0$, *then*

$$\left|K^{\alpha,\delta}_{X,\mathrm{hyp}}(t;z,w)\right| \leq \frac{1}{2}\big(PK_{X,\mathrm{hyp}}(t;z) + PK_{X,\mathrm{hyp}}(t;w)\big) + e^{-\beta(t-t_0)}\,C^{HK}_X + \frac{c_\infty\sinh(\delta+\ell_X)\,e^{-t/4}}{\sinh(\ell_X)}. \tag{84}$$

Proof For any $\alpha \in (0,\lambda_{X,1})$, $\delta \geq \delta_X$, $z, w \in Y_\varepsilon$, and $0 < t < t_0$, adapting the arguments from the Proof of Lemma 4.2 in [10], we have

$$\left|K^{\alpha,\delta}_{X,\mathrm{hyp}}(t;z,w)\right| \leq \frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \sum_{\gamma\notin S_{\Gamma_X}(\delta;z,w)} K_{\mathbb{H}}(t;z,\gamma w) + \sum_{\gamma\in\mathcal{P}(\Gamma_X)} K_{\mathbb{H}}(t;z,\gamma w) + \sum_{\gamma\in\mathcal{E}(\Gamma_X)} K_{\mathbb{H}}(t;z,\gamma w).$$

Estimate (83) now follows from restricting the arguments from the same proof to hyperbolic elements of Γ_X, and from the observation that the length of the shortest geodesic ℓ_X corresponds to the injectivity radius r_X in the Proof of Lemma 4.2 in [10].

For notational brevity, put

$$K(t;z) = \sum_{n=1}^{\infty}\varphi_{X,n}(z)\varphi_{X,n}(w)e^{-\lambda_{X,n}t} + \frac{1}{4\pi}\sum_{p\in\mathcal{P}_X}\int_0^\infty \left|\mathcal{E}_{\mathrm{X,par,p}}\left(z,1/2+ir\right)\right|^2 e^{-(r^2+1/4)t}dr.$$

For $t \geq t_0$, again from the Proof of Lemma 4.2 in [10], we have

$$\begin{aligned}\left|K^{\alpha,\delta}_{X,\mathrm{hyp}}(t;z,w)\right| &\leq \frac{1}{2}\big(K(t;z) + K(t;w)\big) + \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} K_{\mathbb{H}}(t;d_{\mathbb{H}}(z,\gamma w))\\ &\leq \frac{1}{2}\big(K_{X,\mathrm{hyp}}(t;z) + K_{X,\mathrm{hyp}}(t;w)\big) + \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} K_{\mathbb{H}}(t;d_{\mathbb{H}}(z,\gamma w)).\end{aligned}$$

Adapting the arguments from the Proof of Lemma 4.2 in [10] to $\mathcal{H}(\Gamma_X)$, we find

$$\sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} K_{\mathbb{H}}(t;d_{\mathbb{H}}(z,\gamma w)) \leq \frac{c_\infty\sinh(\delta+\ell_X)\,e^{-t/4}}{\sinh(\ell_X)}.$$

Now it suffices to show that

$$\begin{aligned}K_{X,\mathrm{hyp}}(t;z) &= PK_{X,\mathrm{hyp}}(t;z) + \big(K_{\mathbb{H}}(t;0) + EK_{X,\mathrm{hyp}}(t;z) + HK_{X,\mathrm{hyp}}(t;z)\big)\\ &\leq PK_{X,\mathrm{hyp}}(t;z) + e^{-\beta(t-t_0)}\,C^{HK}_X.\end{aligned}$$

As in the Proof of Lemma 4.2 in [10], put

$$h(t;z) = e^{\beta t}\big(K_{\mathbb{H}}(t;0) + EK_{X,\mathrm{hyp}}(t;z) + HK_{X,\mathrm{hyp}}(t;z)\big). \tag{85}$$

From Eq. (23), for a fixed $z \in Y_\varepsilon$, it follows that for all $t \geq t_0$, the function $h(t; z)$ is a monotone decreasing function in t. Hence, following arguments as in the Proof of Lemma 4.2 in [10], we arrive at

$$\begin{aligned}&\big(K_{\mathbb{H}}(t;0) + EK_{X,\mathrm{hyp}}(t;z) + HK_{X,\mathrm{hyp}}(t;z)\big)\\ &\quad \leq e^{-\beta(t-t_0)}\big(K_{\mathbb{H}}(t_0;0) + EK_{X,\mathrm{hyp}}(t_0;z) + HK_{X,\mathrm{hyp}}(t_0;z)\big) \leq e^{-\beta(t-t_0)}\, C_X^{HK},\end{aligned}$$

which completes the proof of the lemma. □

Proposition 4.8 *For any $\alpha \in (0, \lambda_{X,1})$, $\delta > 0$, and $z, w \in Y_\varepsilon$, we have the following upper bound*

$$\Big| g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big| \leq B_{X,\varepsilon,\alpha,\delta},$$

where for $\delta \geq \delta_X$, we have

$$\begin{aligned}B_{X,\varepsilon,\alpha,\delta} = 4\pi\bigg(&\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{c_0\sinh(\ell_X)\sinh(\delta)}{8\delta^2\sinh^2(\ell_X/2)} + \frac{c_0 e^{2\ell_X}}{2\pi\sinh^2(\ell_X/2)}\\ &+ \frac{4c_\infty\sinh(\delta+\ell_X)}{\sinh(\ell_X)} + \frac{C_X^{HK}}{\beta}\bigg)\\ &+ 7\,|\mathcal{P}_X|\,(\log\varepsilon)^2 + 41\,C_{X,\mathrm{par}}\\ &+ 14\coth\big(\varepsilon/4\big)\bigg(-\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1)\,\log\big(\tanh^2(\varepsilon/2)/c_{X,\mathrm{ell}}\big) + C_{X,\mathrm{ell}}\bigg);\end{aligned}$$

and for $\delta \leq \delta_X$, we have

$$B_{X,\varepsilon,\alpha,\delta} = B_{X,\varepsilon,\alpha,\delta_X} + \frac{\sinh(\delta_X+\ell_X)}{\sinh(\ell_X)}\big|\log\big(\tanh^2(\delta/2)\big)\big|.$$

Proof For any $\alpha \in (0, \lambda_{X,1})$, $\delta > 0$, and $z, w \in Y_\varepsilon$, we have

$$\Big| g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big| = \int_0^{t_0}\big|K_{\mathrm{hyp}}^{\alpha,\delta}(t;z,w)\big|dt + \int_{t_0}^{\infty}\big|K_{\mathrm{hyp}}^{\alpha,\delta}(t;z,w)\big|dt.$$

From Theorem 4.7, and using the fact that the heat kernel $K_{\mathbb{H}}(t;\eta)$ is positive for all $t \geq 0$ and $\eta \geq 0$, and that $0 < t_0 < 1$, we have the following inequality

$$\begin{aligned}&\Big| g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big|\\ &\leq \sup_{z,w\in Y_\varepsilon}\bigg(P_X(z) + \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) + \sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w)\bigg)\\ &\quad + 4\pi\bigg(\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{c_0\sinh(\ell_X)\sinh(\delta)}{8\delta^2\sinh^2(\ell_X/2)} + \frac{c_0 e^{2\ell_X}}{2\pi\sinh^2(\ell_X/2)} + \frac{4c_\infty\sinh(\delta+\ell_X)}{\sinh(\ell_X)} + \frac{C_X^{HK}}{\beta}\bigg).\end{aligned}$$

For $z, w \in Y_\varepsilon$, we are left to bound the term

$$P_X(z) + \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) + \sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w). \tag{86}$$

From upper bound (79), we have the following upper bound for the first term

$$\sup_{z\in Y_{\varepsilon}} P_X(z) \leq \sup_{z\in Y_{\varepsilon}^{\mathrm{par}}} P_X(z) \leq -6\,|\mathcal{P}_X|\,\log\varepsilon + 32\,C_{X,\mathrm{par}}. \tag{87}$$

Now, for $z \in Y_{\varepsilon/2}^{\mathrm{par}}$, a fixed $w \in Y_{\varepsilon}^{\mathrm{par}}$, and $z \neq w$, observe that

$$\Delta_{\mathrm{hyp}} \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) = 0;$$

from Eq. (50), for $z = w$, we find that

$$\Delta_{\mathrm{hyp}} \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z) = \Delta_{\mathrm{hyp}}\, P_X(z) \leq 0.$$

Hence, for $z \in Y_{\varepsilon/2}^{\mathrm{par}}$, and a fixed $w \in Y_{\varepsilon}^{\mathrm{par}}$, the second term in expression (86) is a superharmonic function in the variable z. So from the maximum principle for superharmonic functions, we deduce that

$$\sup_{z,w\in Y_{\varepsilon}} \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) \leq \sup_{\substack{z\in Y_{\varepsilon/2}^{\mathrm{par}}\\ w\in Y_{\varepsilon}^{\mathrm{par}}}} \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) \leq \sup_{\substack{z\in\partial U_{\varepsilon/2}(p)\\ w\in Y_{\varepsilon}^{\mathrm{par}}}} \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w),$$

for some cusp $p \in \mathcal{P}_X$. From the definition of $g_{\mathbb{H}}(z,w)$ from (24) and from condition (1) which the fixed ε satisfies, for any $\gamma \in \Gamma_X$, $z \in \partial U_{\varepsilon/2}(p)$ and $w \in Y_{\varepsilon}^{\mathrm{par}}$, we derive

$$\begin{aligned} g_{\mathbb{H}}(z,\gamma w) &= g_{\mathbb{H}}(\sigma_p^{-1}z,\sigma_p^{-1}\gamma w) = \log\left(1 + \frac{4\,\mathrm{Im}(\sigma_p^{-1}z)\,\mathrm{Im}(\sigma_p^{-1}\gamma w)}{|\sigma_p^{-1}z - \sigma_p^{-1}\gamma w|^2}\right)\\ &\leq \log\left(1 + \frac{4\,\mathrm{Im}(\sigma_p^{-1}z)^2}{\big(\mathrm{Im}(\sigma_p^{-1}z) - \mathrm{Im}(\sigma_p^{-1}\gamma w)\big)^2}\right) \leq \frac{4\,\mathrm{Im}(\sigma_p^{-1}z)^2}{(\log 2)^2} \leq 9\,\mathrm{Im}(\sigma_p^{-1}z)^2, \end{aligned}$$

where σ_p is a scaling matrix for the cusp $p \in \mathcal{P}_X$. Using the above inequality, we arrive at

$$\begin{aligned} \sup_{\substack{z\in\partial U_{\varepsilon/2}(p)\\ w\in Y_{\varepsilon}^{\mathrm{par}}}} \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) &\leq \sup_{z\in\partial U_{\varepsilon/2}(p)} 9 \sum_{\gamma\in\mathcal{P}(\Gamma_X)} \mathrm{Im}(\sigma_p^{-1}\gamma z)^2 = \sup_{z\in\partial U_{\varepsilon/2}(p)} 9 \sum_{p\in\mathcal{P}_X} \mathrm{Im}(\sigma_p^{-1}z)^2\\ &+ \sup_{z\in\partial U_{\varepsilon/2}(p)} 9 \sum_{p\in\mathcal{P}_X} \big(\mathcal{E}_{\mathrm{X,par,p}}(z,2) - \mathrm{Im}(\sigma_p^{-1}z)^2\big) \leq |\mathcal{P}_X|\,\big(\log(\varepsilon/2)\big)^2 + 9\,C_{X,\mathrm{par}}. \end{aligned} \tag{88}$$

Hence, combining upper bounds (87) and (88), and using the fact that $0 < \varepsilon < 1$ (which implies that $-\log\varepsilon \leq (\log(\varepsilon/2)^2)$), we arrive at the following upper bound for the first two terms in expression (86)

$$P_X(z) + \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) \leq 7\,|\mathcal{P}_X|\,\big(\log(\varepsilon/2)\big)^2 + 41\,C_{X,\mathrm{par}}. \tag{89}$$

For $z \in Y_{\varepsilon/2}^{\mathrm{ell}}$, a fixed $w \in Y_{\varepsilon}^{\mathrm{ell}}$, and $z \neq w$, observe that

$$\Delta_{\mathrm{hyp}} \sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) = 0;$$

from Eq. (57), for $z = w$, we find that

$$\Delta_{\mathrm{hyp}} \sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma z) \leq 0.$$

Hence, for $z \in Y^{\mathrm{ell}}_{\varepsilon/2}$, and a fixed $w \in Y^{\mathrm{ell}}_{\varepsilon}$, the third term in the expression (86) is a superharmonic function in the variable z. So from the maximum principle for superharmonic functions, we deduce that

$$\sup_{z,w\in Y_\varepsilon}\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) \leq \sup_{\substack{z\in\partial Y^{\mathrm{ell}}_{\varepsilon/2}\\ w\in Y^{\mathrm{ell}}_{\varepsilon,\mathfrak{e}}}}\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) = \sup_{\substack{z\in\partial U_{\varepsilon/2}(\mathfrak{e})\\ w\in Y^{\mathrm{ell}}_{\varepsilon,\mathfrak{e}}}}\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w),$$

for some elliptic fixed point $\mathfrak{e}\in\mathcal{E}_X$. Similarly for $w \in Y^{\mathrm{ell}}_{\varepsilon,\mathfrak{e}}$ and a fixed $z \in U_{\varepsilon/2}(\mathfrak{e})$, the third term in expression (86) is a superharmonic function in the variable w. Hence, we arrive at

$$\sup_{\substack{z\in\partial U_{\varepsilon/2}(\mathfrak{e})\\ w\in Y^{\mathrm{ell}}_{\varepsilon,\mathfrak{e}}}}\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) = \sup_{\substack{z\in\partial U_{\varepsilon/2}(\mathfrak{e})\\ w\in\partial U_{\varepsilon}(\mathfrak{e})}}\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w).$$

From Eq. (25), recall that

$$\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) = \sum_{\gamma\in\mathcal{E}(\Gamma_X)}\log\bigg(1+\frac{1}{\sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big)}\bigg).$$

Combining upper bound (74) from Lemma 4.3 with upper bound (80), for any $\gamma \in \Gamma_X$, $z \in \partial U_{\varepsilon/2}(\mathfrak{e})$, and $w \in \partial U_{\varepsilon}(\mathfrak{e})$, we derive

$$\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma w) \leq \sum_{\gamma\in\mathcal{E}(\Gamma_X)}\log\bigg(1+\frac{14\coth(\varepsilon/4)}{\sinh^2\big(d_{\mathbb{H}}(z,\gamma z)/2\big)}\bigg) \leq \sup_{z\in\partial U_{\varepsilon/2}(\mathfrak{e})} 14\coth(\varepsilon/4)\, E(z)$$
$$\leq 14\coth\big(\varepsilon/4\big)\bigg(-\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1)\log\big(\tanh^2(\varepsilon/2)/c_{X,\mathrm{ell}}\big)+C_{X,\mathrm{ell}}\bigg).$$

Combining the above inequality with upper bound (89) completes the proof of the proposition. □

Notation 4.9 For the rest of this article, put

$$\widetilde{\varepsilon} = 2\log\bigg(\frac{1+\sqrt{1+\big(3\log(\varepsilon/2)\big)^2}}{3\log(\varepsilon/2)}\bigg). \tag{90}$$

Corollary 4.10 *For any $\alpha \in (0,\lambda_{X,1})$, $\delta \in (0,\widetilde{\varepsilon})$, $z \in \partial Y^{\mathrm{par}}_{\varepsilon/2}$, and $w \in Y_\varepsilon$, we have the following upper bound*

$$\big|\, g_{X,\mathrm{hyp}}(z,w)\big| \leq B_{X,\varepsilon/2,\alpha,\delta}\,.$$

Proof Without loss of generality, we may assume that $z \in \partial U_{\varepsilon/2}(p)$, for some cusp $p \in \mathcal{P}_X$. For any $\gamma \in \Gamma_X$, $z \in \partial U_{\varepsilon/2}(p)$, and $w \in Y_\varepsilon$, recall that

$$u(z,\gamma w) = \sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big) = \frac{|z-\gamma w|^2}{4\,\mathrm{Im}(z)\,\mathrm{Im}(\gamma w)} \geq \frac{|\,\mathrm{Im}(z)-\mathrm{Im}(\gamma w)|^2}{4\,\mathrm{Im}(z)\,\mathrm{Im}(\gamma w)}. \tag{91}$$

From condition (1), which the fixed ε satisfies, we derive

$$\sinh^2\big(d_{\mathbb{H}}(z,\gamma w)/2\big) \geq \frac{\big(\log(\varepsilon)-\log(\varepsilon/2)\big)^2}{4\big(\log(\varepsilon/2)\big)^2} \Longrightarrow \sinh\big(d_{\mathbb{H}}(z,\gamma w)/2\big) \geq \frac{1}{3\log(\varepsilon/2)}.$$

From the above inequality, it follows that for any $\gamma \in \Gamma_X$, $z \in \partial U_{\varepsilon/2}(p)$, and $w \in Y_\varepsilon$, we get $d_{\mathbb{H}}(z, \gamma w) \geq \widetilde{\varepsilon}$. Now for any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in (0, \widetilde{\varepsilon})$, from Proposition 4.8, we arrive at

$$\sup_{\substack{z\in\partial U_{\varepsilon/2}(p)\\ w\in Y_\varepsilon}} \Big| g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big| \leq \sup_{z,w\in Y_{\varepsilon/2}} \big| g_{X,\mathrm{hyp}}(z,w)\big| \leq B_{X,\varepsilon/2,\alpha,\delta},$$

which completes the proof of the corollary. □

Corollary 4.11 *Let $\mathfrak{e} \in \mathcal{E}_X$ be an elliptic fixed point. Then, for any $\alpha \in (0, \lambda_{X,1})$, $\delta \in (0, \varepsilon)$, and $z \in Y_\varepsilon$, we have the following upper bound*

$$\big| g_{X,\mathrm{hyp}}(z,\mathfrak{e})\big| \leq B_{X,\varepsilon,\alpha,\delta}\,.$$

Proof For any $\alpha \in (0, \lambda_{X,1})$, $\delta \in (0, \varepsilon)$, and $z \in Y_\varepsilon$, from condition (3) which the fixed ε satisfies, we find

$$\Big| g_{X,\mathrm{hyp}}(z,\mathfrak{e}) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,\mathfrak{e})} g_{\mathbb{H}}(z,\gamma\mathfrak{e})\Big| = \big| g_{X,\mathrm{hyp}}(z,\mathfrak{e})\big|.$$

Following similar arguments as in the Proof of Proposition 4.8, we get

$$\big| g_{X,\mathrm{hyp}}(z,\mathfrak{e})\big| \leq \sup_{z\in Y_\varepsilon}\Bigg(P_X(z) + \sum_{\gamma\in\mathcal{P}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma\mathfrak{e}) + \sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma\mathfrak{e})\Bigg)$$
$$+\,4\pi\Bigg(\frac{1}{\mathrm{vol}_{\mathrm{hyp}}(X)} + \frac{c_0\sinh(\ell_X)\sinh(\delta)}{8\delta^2\sinh^2(\ell_X/2)} + \frac{c_0 e^{2\ell_X}}{2\pi\sinh^2(\ell_X/2)} + \frac{4c_\infty\sinh(\delta+\ell_X)}{\sinh(\ell_X)} + \frac{C_X^{HK}}{\beta}\Bigg).$$

We estimate the first two terms on the right-hand side of above inequality by the same quantities as in the Proof of Proposition 4.8. For the third term, from similar arguments as in the Proof of Proposition 4.8, and using the upper bound from Lemma 4.2 (i.e., estimate (69)), we derive

$$\begin{aligned}\sup_{z\in Y_\varepsilon}\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma\mathfrak{e}) &= \sup_{z\in\partial U_\varepsilon(\mathfrak{e})}\sum_{\gamma\in\mathcal{E}(\Gamma_X)} g_{\mathbb{H}}(z,\gamma\mathfrak{e})\\ &\leq \sup_{z\in\partial U_\varepsilon(\mathfrak{e})}\sum_{\gamma\in\mathcal{E}(\Gamma_X)}\log\Bigg(1+\frac{7\coth(\varepsilon/2)}{\sinh^2\big(d_{\mathbb{H}}(z,\gamma z)/2\big)}\Bigg)\\ &\leq \sup_{z\in\partial U_\varepsilon(\mathfrak{e})} 7\coth(\varepsilon/2)\,E(z) \leq \sup_{z\in\partial U_{\varepsilon/2}(\mathfrak{e})} 14\coth(\varepsilon/4)\,E(z),\end{aligned}$$

which can be bounded again by the same estimate as in the Proof of Proposition 4.8. Hence, we deduce that for hypothesis as in the statement of the corollary, we have the same bound for $\big| g_{X,\mathrm{hyp}}(z,\mathfrak{e})\big|$ as in Proposition 4.8, i.e., $B_{X,\varepsilon,\alpha,\delta}$, which completes the proof of the corollary. □

Corollary 4.12 *Let $p \in \mathcal{P}_X$ be any cusp. Then, for any $\alpha \in (0, \lambda_{X,1})$, $\delta > 0$, $z \in Y_\varepsilon^{\mathrm{par}}$, and $w \in U_\varepsilon(p)$, we have*

$$g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\Bigg(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\Bigg) + h_{\delta,p}(z,w),$$

where $h_{\delta,p}(z,w)$ is a harmonic function in the variable $w \in U_\varepsilon(p)$, which satisfies the following upper bound

$$\sup_{z\in U_\varepsilon(p)} \big|h_{\delta,p}(z,w)\big| \leq B_{X,\varepsilon,\alpha,\delta}\,.$$

Proof For any $\delta > 0$, a fixed $z \in Y_\varepsilon^{\mathrm{par}}$, and $w \in U_\varepsilon(p)$, both the functions

$$g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\,, \quad -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right)$$

are solutions of differential Eq. (30). So we find that

$$g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) + h_{\delta,p}(z,w),$$

where $h_{\delta,p}(z,w)$ is a harmonic function in the variable $z \in U_\varepsilon(p)$.

As $h_{\delta,p}(z,w)$ is a harmonic function, $|h_{\delta,p}(z,w)|$ is a subharmonic function. So for a fixed $z \in Y_\varepsilon^{\mathrm{par}}$, from the maximum principle for subharmonic functions and Proposition 4.8, we arrive at the upper bound

$$\sup_{w\in U_\varepsilon(p)} \big|h_{\delta,p}(z,w)\big| = \sup_{w\in\partial U_\varepsilon(p)} \big|h_{\delta,p}(z,w)\big| = \bigg|g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_\Gamma(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\bigg| \leq B_{\varepsilon,\alpha,\delta},$$

for any $\alpha \in (0,\lambda_{X,1})$ and $\delta > 0$. The proof of the corollary follows from the fact that the upper bound derived above does not depend on the fixed $z \in Y_\varepsilon^{\mathrm{par}}$. □

Corollary 4.13 *Let $p, q \in \mathcal{P}_X$ and $p \neq q$ be two cusps. Then, for any $\alpha \in (0,\lambda_{X,1})$, $\delta > 0$, $z \in U_\varepsilon(p)$, and $w \in U_\varepsilon(q)$, we have*

$$\begin{aligned} &g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) \\ &\quad = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_q(w)|}{\log\varepsilon}\right) + h_{\delta,p,q}(z,w), \end{aligned}$$

where $h_{\delta,p,q}(z,w)$ is a harmonic function in both the variables $z \in U_\varepsilon(p)$ and $w \in U_\varepsilon(q)$, which satisfies the following upper bound

$$\sup_{\substack{z\in U_\varepsilon(p)\\ z\in U_\varepsilon(q)}} \big|h_{\delta,p,q}(z,w)\big| \leq B_{X,\varepsilon,\alpha,\delta}\,.$$

Proof The proof of the corollary follows from similar arguments as in Corollary 4.12. □

Corollary 4.14 *Let $p \in \mathcal{P}_X$ be any cusp. Then, for any $\alpha \in (0,\lambda_{X,1})$, $\delta > 0$, and $z, w \in U_\varepsilon(p)$, we have*

$$\begin{aligned} &g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z,\gamma w) - \sum_{\gamma\in\Gamma_{X,p}} g_{\mathbb{H}}(z,\gamma w) \\ &\quad = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right) + h_{\delta,p,p}(z,w), \end{aligned}$$

where $h_{\delta,p,p}(z,w)$ is a harmonic function in both the variables $z \in U_{\varepsilon}(p)$ and $w \in U_{\varepsilon}(q)$, which satisfies the following upper bound

$$\sup_{z,w\in U_{\varepsilon}(p)} \left| h_{\delta,p,p}(z,w) \right| \leq B_{X,\varepsilon,\alpha,\delta} \,. \tag{92}$$

Proof For $z, w \in U_{\varepsilon}(p)$, the hyperbolic Green's function satisfies the differential Eq. (30). For $z, w \in U_{\varepsilon}(p)$, put

$$h(z,w) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) - \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right) + \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z,\gamma w) + \sum_{\gamma\in\Gamma_{X,p}} g_{\mathbb{H}}(z,\gamma w).$$

Observe that for $z \neq w$, $d_z d_z^c h(z,w) = \mu_{\mathrm{shyp}}(z)$. So, if we show that both the functions $h(z,w)$ and $g_{X,\mathrm{hyp}}(z,w)$ admit the same type of singularity when $z = w$ on $U_{\varepsilon}(p)$, we can conclude that

$$g_{X,\mathrm{hyp}}(z,w) = h(z,w) + h_{\delta,p,p}(z,w),$$

where $h_{\delta,p,p}(z,w)$ is a harmonic function in both the variables $z, w \in U_{\varepsilon}(p)$. Moreover, from similar arguments as in Corollary 4.12, we can conclude that the function $h_{\delta,p,p}(z,w)$ satisfies the asserted upper bound (92).

For any $z \in U_{\varepsilon}(p)$, from Eqs. (36) and (10), we find that

$$\begin{aligned}\lim_{w\to z}\left(g_{X,\mathrm{hyp}}(z,w) + \log|\vartheta_z(w)|^2\right) &= \lim_{w\to z}\left(g_{X,\mathrm{can}}(z,w) + \log|\vartheta_z(w)|^2\right) + 2\phi_X(z) \\ &= -\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) + O_z(1),\end{aligned}$$

where the contribution from the term $O_z(1)$ is a smooth function which remains bounded for all $z \in U_{\varepsilon}(p)$ and for $z = p$.

Now observe that

$$\begin{aligned}\lim_{w\to z}\left(h(z,w) + \log|\vartheta_z(w)|^2\right) &= -\frac{8\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(z)|}{\log\varepsilon}\right) \\ &+ \lim_{w\to z}\left(\sum_{\gamma\in\Gamma_{X,p}\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z,\gamma w) + g_{\mathbb{H}}(z,w) + \log|\vartheta_z(w)|^2\right) + O_z(1),\end{aligned} \tag{93}$$

where the contribution from the term $O_z(1)$ is a smooth function which remains bounded for all $z \in U_{\varepsilon}(p)$ and for $z = p$. For $z \in U_{\varepsilon}(p)$, from Eq. (49) from Proof of Lemma 3.3, and from the definition of $g_{\mathbb{H}}(z,w)$, i.e., Eq. (24), the second term on the right-side of Eq. (93) simplifies to give

$$\begin{aligned}&\lim_{w\to z}\left(\sum_{\gamma\in\Gamma_{X,p}\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z,\gamma w) + g_{\mathbb{H}}(z,w) + \log|\vartheta_p(w) - \vartheta_p(z)|^2\right) \\ &= P_{\mathrm{gen},p}(z) - 4\pi\,\mathrm{Im}(\sigma_p^{-1}z) + \lim_{w\to z}\left(g_{\mathbb{H}}(\sigma_p^{1}z, \sigma_p^{-1}w) + \log\left|1 - e^{2\pi i(w-z)}\right|^2\right) \\ &= P_{\mathrm{gen},p}(z) - 4\pi\,\mathrm{Im}(\sigma_p^{-1}z) + \log\left(4\,\mathrm{Im}(\sigma_p^{-1}z)^2\right) + \log(4\pi^2) = O_z(1),\end{aligned}$$

which together with Eq. (93) completes the proof of the corollary. □

Corollary 4.15 *Let $\mathfrak{e}, \mathfrak{f} \in \mathcal{E}_X$ and $\mathfrak{e} \neq \mathfrak{f}$ be two elliptic fixed points. Then, for any $\alpha \in (0, \lambda_{X,1})$, $\delta > 0$, $z \in U_\varepsilon(\mathfrak{e})$, and $w \in U_\varepsilon(\mathfrak{f})$, we have*

$$g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w) = -\frac{4\pi \log\big(1 - |\vartheta_{\mathfrak{e}}(z)|^{2/m_{\mathfrak{e}}}\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi \log\big(1 - |\vartheta_{\mathfrak{f}}(w)|^{2/m_{\mathfrak{f}}}\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} + h_{\delta,\mathfrak{e},\mathfrak{f}}(z,w),$$

where $h_{\delta,\mathfrak{e},\mathfrak{f}}(z,w)$ is a harmonic function in both the variables $z \in U_\varepsilon(\mathfrak{e})$ and $w \in U_\varepsilon(\mathfrak{e})$, which satisfies the following upper bound

$$\sup_{\substack{z \in U_\varepsilon(\mathfrak{e}) \\ w \in U_\varepsilon(\mathfrak{f})}} \Big| h_{\delta,\mathfrak{e},\mathfrak{f}}(z,w) \Big| \leq B_{X,\varepsilon,\alpha,\delta};$$

furthermore, for $z, w \in U_\varepsilon(\mathfrak{e})$, we have

$$g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma \in S_{\Gamma_X}(\delta;z,w)\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z,\gamma w) - \sum_{\gamma \in \Gamma_{X,\mathfrak{e}}} g_{\mathbb{H}}(z,\gamma w) = -\frac{4\pi \log\big(1 - |\vartheta_{\mathfrak{e}}(z)|^{2/m_{\mathfrak{e}}}\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{4\pi \log\big(1 - |\vartheta_{\mathfrak{e}}(w)|^{2/m_{\mathfrak{e}}}\big)}{\mathrm{vol}_{\mathrm{hyp}}(X)} + h_{\delta,\mathfrak{e},\mathfrak{e}}(z,w),$$

where $h_{\delta,\mathfrak{e},\mathfrak{e}}(z,w)$ is a harmonic function in both the variables $z, w \in U_\varepsilon(\mathfrak{e})$, which satisfies the following upper bound

$$\sup_{z \in U_\varepsilon(\mathfrak{e})} \Big| h_{\delta,\mathfrak{e},\mathfrak{e}}(z,w) \Big| \leq B_{X,\varepsilon,\alpha,\delta};$$

Proof The proof of the corollary follows from arguments similar to the ones employed in the proofs of Corollaries 4.13 and 4.14. □

Remark 4.16 In order to understand the dependence of our bounds for the hyperbolic Green's function on ε, it suffices to analyze the dependence of $B_{X,\varepsilon,\alpha,\delta}$ on ε. From the formula for $B_{X,\varepsilon,\alpha,\delta}$ from Proposition 4.8, and the asymptotics of the functions $\coth(x)$ and $\log(\tanh(x))$ at $x = 0$, we arrive at the following estimate for $B_{X,\varepsilon,\alpha,\delta}$

$$B_{X,\varepsilon,\alpha,\delta} = O_X\big(\varepsilon^{-2}\big).$$

5 Bounds for canonical Green's function

In this section, we obtain bounds for the canonical Green's function on the compact subset Y_ε of X. From Eq. (36), to derive bounds for the canonical Green's function $g_{X,\mathrm{can}}(z,w)$, it suffices to derive bounds for the function $\phi_X(z)$, and for the hyperbolic Green's function $g_{X,\mathrm{hyp}}(z,w)$. From last section, we have bounds for $g_{X,\mathrm{hyp}}(z,w)$, and it remains to bound the function $\phi_X(z)$. Recall that from Corollary 3.12, we have

$$\phi_X(z) = \frac{\big(H_X(z) + E_X(z)\big)}{2g_X} + \frac{1}{8\pi g_X}\int_X g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(z) - \sum_{\mathfrak{e} \in \mathcal{E}_X} \frac{m_{\mathfrak{e}} - 1}{2g_X m_{\mathfrak{e}}}\, g_{X,\mathrm{hyp}}(z,\mathfrak{e}) - \frac{C_{X,\mathrm{hyp}}}{8g_X^2} - \frac{2\pi(c_X - 1)}{g_X\,\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{1}{2g_X}\int_X E_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta). \tag{94}$$

Using analysis from the Sects. 2 and 3, it is easy to bound almost all the quantities involved in the above expression for $\phi_X(z)$ excepting the integral

$$\frac{1}{8\pi g_X}\int_X g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(z),$$

which we now accomplish.

Lemma 5.1 *For $z\in Y_\varepsilon$, we have the equality of integrals*

$$\begin{aligned}&\int_X g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\\ &\quad=4\pi\,P_X(z)-4\pi\int_{Y^{\mathrm{par}}_{\varepsilon/2}}P_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta)\\ &\qquad+4\pi\sum_{p\in\mathcal{P}_X}\bigg(\int_{\partial U_{\varepsilon/2}(p)}g_{X,\mathrm{hyp}}(z,\zeta)d^c_\zeta P_X(\zeta)-\int_{\partial U_{\varepsilon/2}(p)}P_X(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg)\\ &\qquad+\sum_{p\in\mathcal{P}_X}\int_{U_{\varepsilon/2}(p)}g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta).\end{aligned}$$

Proof Observe that we have the following decomposition

$$\begin{aligned}&\int_X g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\\ &\quad=-4\pi\int_X g_{X,\mathrm{hyp}}(z,\zeta)d_\zeta d^c_\zeta P_X(\zeta)\\ &\quad=-4\pi\int_{Y^{\mathrm{par}}_{\varepsilon/2}}g_{X,\mathrm{hyp}}(z,\zeta)d_\zeta d^c_\zeta P_X(\zeta)+\sum_{p\in\mathcal{P}_X}\int_{U_{\varepsilon/2}(p)}g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta).\end{aligned}\tag{95}$$

Let $U_r(z)$ denote an open coordinate disk of radius r around $z\in Y_\varepsilon$ with r small enough such that $U_r(z)\subsetneq Y^{\mathrm{par}}_{\varepsilon/2}$. From Eq. (30) and from Stokes's theorem, we have

$$\begin{aligned}&-\int_{Y^{\mathrm{par}}_{\varepsilon/2}}g_{X,\mathrm{hyp}}(z,\zeta)d_\zeta d^c_\zeta P_X(\zeta)+\int_{Y^{\mathrm{par}}_{\varepsilon/2}}P_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta)\\ &\quad=\lim_{r\to0}\bigg(-\int_{Y^{\mathrm{par}}_{\varepsilon/2}\backslash U_r(z)}g_{X,\mathrm{hyp}}(z,\zeta)d_\zeta d^c_\zeta P_X(\zeta)+\int_{Y^{\mathrm{par}}_{\varepsilon/2\backslash U_r(z)}}P_X(\zeta)d_\zeta d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg)\\ &\quad=\lim_{r\to0}\bigg(\int_{\partial U_r(z)}g_{X,\mathrm{hyp}}(z,\zeta)d^c_\zeta P_X(\zeta)-\int_{\partial U_r(z)}P_X(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg)\\ &\qquad+\sum_{p\in\mathcal{P}_X}\bigg(\int_{\partial U_{\varepsilon/2}(p)}g_{X,\mathrm{hyp}}(z,\zeta)d^c_\zeta P_X(\zeta)-\int_{\partial U_{\varepsilon/2}(p)}P_X(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg).\end{aligned}\tag{96}$$

Using the fact that the function $P_X(\zeta)$ is smooth at z, and as ζ approaches z, the hyperbolic Green's function $g_{X,\mathrm{hyp}}(z,\zeta)$ satisfies

$$g_{X,\mathrm{hyp}}(z,\zeta)=-\log|\vartheta_z(\zeta)|^2+O_z(1),$$

we derive that

$$\lim_{r\to0}\bigg(\int_{\partial U_r(z)}g_{X,\mathrm{hyp}}(z,\zeta)d^c_\zeta P_X(\zeta)-\int_{\partial U_r(z)}P_X(\zeta)d^c_\zeta g_{\mathrm{hyp}}(z,\zeta)\bigg)=P_X(z).$$

Combining the above equation with Eqs. (95) and (96) completes the proof of the lemma. □

Corollary 5.2 *For any $z \in Y_{\varepsilon}^{\mathrm{par}}$, we have*

$$\phi_X(z) = \frac{\big(P_X(z)+E_X(z)+H_X(z)\big)}{2g_X} + \frac{1}{8\pi g_X}\sum_{p\in\mathcal{P}_X}\int_{U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)$$
$$+\frac{1}{2g_X}\sum_{p\in\mathcal{P}_X}\bigg(\int_{\partial U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)d_\zeta^c P_X(\zeta) - \int_{\partial U_{\varepsilon/2}(p)} P_X(\zeta)d_\zeta^c g_{\mathrm{hyp}}(z,\zeta)\bigg)$$
$$-\frac{2\pi(c_X-1)}{g_X\,\mathrm{vol}_{\mathrm{hyp}}(X)} - \frac{1}{2g_X}\int_{Y_{\varepsilon/2}^{\mathrm{par}}} P_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta) - \frac{C_{X,\mathrm{hyp}}}{8g_X^2} + \sum_{\mathfrak{e}\in\mathcal{E}_X}\frac{m_{\mathfrak{e}}-1}{2g_X m_{\mathfrak{e}}}\,g_{X,\mathrm{hyp}}(z,\mathfrak{e})$$
$$-\frac{1}{2g_X}\int_X E_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta). \tag{97}$$

Proof The proof of the corollary follows directly from combining Eq. (94) and Lemma 5.1. □

Lemma 5.3 *For any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in (0, \ell_X)$, we have the following upper bound*

$$\sup_{z\in Y_\varepsilon}\frac{\big|P_X(z)+E_X(z)+H_X(z)\big|}{2g_X} \le \frac{B_{X,\varepsilon/2,\alpha,\delta}}{2g_X}.$$

Proof For any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in (0, \ell_X)$, from Eq. (59), we have

$$\sup_{z\in Y_\varepsilon}\big|P_X(z)+E_X(z)+H_X(z)\big|$$
$$= \sup_{z\in Y_\varepsilon}\lim_{w\to z}\Big|g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big|$$
$$\le \sup_{z\in Y_{\varepsilon/2}}\lim_{w\to z}\Big|g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Big|,$$

and the proof of the lemma follows from Proposition 4.8. □

Proposition 5.4 *For any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in (0, \widetilde{\varepsilon})$, we have the following upper bound*

$$\frac{1}{8\pi g_X}\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}_X}\bigg|\int_{U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|$$
$$\le -\frac{|\mathcal{P}_X|\,C_{X,\mathrm{par}}^{\mathrm{aux}}}{4g_X\,\log(\varepsilon/2)}\bigg(B_{X,\varepsilon/2,\alpha,\delta} + \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\bigg).$$

Proof Observe the inequality

$$\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}_X}\bigg|\int_{U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg| \le \sup_{\zeta\in X}\big|\,\Delta_{\mathrm{hyp}}\,P_X(\zeta)\big|$$
$$\times\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}_X}\bigg|\int_{U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|$$
$$= C_{X,\mathrm{par}}^{\mathrm{aux}}\bigg(\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}_X}\bigg|\int_{U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\bigg|\bigg). \tag{98}$$

For any $p \in \mathcal{P}_X$, $z \in Y_\varepsilon$, and $\zeta \in U_{\varepsilon/2}(p)$, from arguments as in Corollary 4.12, we have

$$g_{X,\mathrm{hyp}}(z,\zeta) = -\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)} \log\left(\frac{\log|\vartheta_p(\zeta)|}{\log(\varepsilon/2)}\right) + g_p(z,\zeta), \tag{99}$$

where $g_p(z,\zeta)$ is a harmonic function in the variable ζ. From maximum principle for harmonic functions and from Corollary 4.10, we have the following upper bound

$$\begin{aligned} \sup_{\substack{z\in Y_\varepsilon \\ \zeta\in U_{\varepsilon/2}(p)}} \left|g_p(z,\zeta)\right| &= \sup_{\substack{z\in Y_\varepsilon \\ \zeta\in \partial U_{\varepsilon/2}(p)}} \left|g_p(z,\zeta)\right| = \sup_{\substack{z\in Y_\varepsilon \\ \zeta\in \partial U_{\varepsilon/2}(p)}} \left|g_{X,\mathrm{hyp}}(z,\zeta)\right| \\ &\leq \sup_{\substack{z\in Y_\varepsilon \\ \zeta\in \partial Y^{\mathrm{par}}_{\varepsilon/2}}} \left|g_{X,\mathrm{hyp}}(z,\zeta)\right| \leq B_{X,\varepsilon/2,\alpha,\delta}, \end{aligned} \tag{100}$$

for any $\alpha \in (0,\lambda_{X,1})$ and $\delta \in (0,\widetilde{\varepsilon})$.

For any $p \in \mathcal{P}_X$, we make the following computations

$$\int_{U_{\varepsilon/2}(p)} \mu_{\mathrm{hyp}}(\zeta) = \int_0^{\varepsilon/2}\int_0^{2\pi} \frac{r\,dr\,d\theta}{(r\log r)^2} = 2\pi\int_0^{\varepsilon/2}\frac{d(\log r)}{(\log r)^2} = -\frac{2\pi}{\log(\varepsilon/2)},$$

$$\begin{aligned} \int_{U_{\varepsilon/2}(p)} \log\big(-\log|\vartheta_p(\zeta)|\big)\,\mu_{\mathrm{hyp}}(\zeta) &= \int_0^{\varepsilon/2}\int_0^{2\pi} \frac{r\log\big(-\log r\big)dr\,d\theta}{(r\log r)^2} \\ = 2\pi\int_0^{\varepsilon/2}\frac{\log\big(-\log r\big)d(\log r)}{(\log r)^2} &= -\frac{2\pi\big(\log\big(-\log(\varepsilon/2)\big)+1\big)}{\log(\varepsilon/2)}. \end{aligned}$$

For any $p \in \mathcal{P}_X$, using inequality (100), and the above computations, we derive

$$\left|\int_{U_{\varepsilon/2}(p)} g_p(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\right| \leq -\frac{2\pi\,B_{X,\varepsilon/2,\alpha,\delta}}{\log(\varepsilon/2)}, \tag{101}$$

$$\begin{aligned} &\left|\int_{U_{\varepsilon/2}(p)} \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(\zeta)|}{\log(\varepsilon/2)}\right)\right|\mu_{\mathrm{hyp}}(\zeta) \\ &= \int_{U_{\varepsilon/2}(p)} \frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{-\log|\vartheta_p(\zeta)|}{-\log(\varepsilon/2)}\right)\mu_{\mathrm{hyp}}(\zeta) = -\frac{8\pi^2}{\mathrm{vol}_{\mathrm{hyp}}(X)\log(\varepsilon/2)}. \end{aligned} \tag{102}$$

For any $p \in \mathcal{P}_X$, using Eq. (99), and the above computations (101) and (102), we arrive at

$$\left|\int_{U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)\,\mu_{\mathrm{hyp}}(\zeta)\right| \leq -\frac{2\pi}{\log(\varepsilon/2)}\left(B_{X,\varepsilon/2,\alpha,\delta}+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right) \tag{103}$$

Combining the above upper bound with inequality (98) completes the proof of the corollary. □

Remark 5.5 For any $z \in Y_\varepsilon$, combining Lemma 5.3 and Proposition 5.4, we obtain the following upper bound for the first line on the right-hand side of Eq. (97)

$$\frac{B_{X,\varepsilon/2,\alpha,\delta}}{2g_X} - \frac{|\mathcal{P}_X|\,C^{\mathrm{aux}}_{X,\mathrm{par}}}{4g_X\,\log(\varepsilon/2)}\left(B_{X,\varepsilon/2,\alpha,\delta}+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\right),$$

for any $\alpha \in (0,\lambda_{X,1})$ and $\delta \in \big(0,\min\{\ell_X,\widetilde{\varepsilon}\}\big)$.

Proposition 5.6 *For any $\alpha \in (0,\lambda_{X,1})$ and $\delta \in (0,\widetilde{\varepsilon})$, we have the following upper bound*

$$\frac{1}{2g_X}\sup_{z\in Y_\varepsilon}\sum_{p\in\mathcal{P}_X}\left|\int_{\partial U_{\varepsilon/2}(p)} g_{X,\mathrm{hyp}}(z,\zeta)d^c_\zeta P_X(\zeta)\right| \leq \frac{|\mathcal{P}_X|\;B_{X,\varepsilon/2,\alpha,\delta}}{2g_X}.$$

Proof From Corollary 4.10 and Stokes's theorem, we have the elementary estimate

$$\sup_{z\in Y_{\varepsilon}}\sum_{p\in\mathcal{P}_X}\bigg|\int_{\partial U_{\varepsilon/2}(p)}g_{X,\mathrm{hyp}}(z,\zeta)d_{\zeta}^{c}P_X(\zeta)\bigg|\leq\sup_{\substack{z\in Y_{\varepsilon}\\ \zeta\in\partial Y^{\mathrm{par}}_{\varepsilon/2}}}\big|g_{X,\mathrm{hyp}}(z,\zeta)\big|\cdot\Bigg(\sum_{p\in\mathcal{P}_X}\bigg|\int_{\partial U_{\varepsilon/2}(p)}d_{\zeta}^{c}P_X(\zeta)\bigg|\Bigg)$$
$$\leq B_{X,\varepsilon/2,\alpha,\delta}\cdot\Bigg(\sum_{p\in\mathcal{P}_X}\int_{\partial U_{\varepsilon/2}(p)}\big|d_{\zeta}d_{\zeta}^{c}P_X(\zeta)\big|\Bigg)\leq\frac{B_{X,\varepsilon/2,\alpha,\delta}}{4\pi}\cdot\bigg(\int_X\big|\,\Delta_{\mathrm{hyp}}P_X(\zeta)\big|\,\mu_{\mathrm{hyp}}(\zeta)\bigg)\tag{104}$$

for any $\alpha\in(0,\lambda_{X,1})$ and $\delta\in(0,\widetilde{\varepsilon})$.

Let $U_r(p)$ denote an open coordinate disk of radius r around a parabolic fixed point $p\in\mathcal{P}_X$. Put

$$Y_r^{\mathrm{par}}=X\backslash\bigcup_{p\in\mathcal{P}_X}U_r(p).$$

For every $z\in X$, from formula (50), we know that $\big|\,\Delta_{\mathrm{hyp}}P_X(\zeta)\big|=-\Delta_{\mathrm{hyp}}P_X(\zeta)$. Then, using Stokes's theorem, we find

$$\int_X\big|\,\Delta_{\mathrm{hyp}}P_X(\zeta)\big|\,\mu_{\mathrm{hyp}}(\zeta)=4\pi\lim_{r\to0}\int_{Y_r^{\mathrm{par}}}d_{\zeta}d_{\zeta}^{c}P_X(\zeta)$$
$$=4\pi\sum_{p\in\mathcal{P}_X}\lim_{r\to0}\int_{\partial U_r(p)}d_{\zeta}^{c}P_X(\zeta)=-4\pi\,|\mathcal{P}_X|\lim_{r\to0}\int_0^{2\pi}\frac{r}{2}\frac{\partial P_X(\zeta)}{\partial r}\frac{d\theta}{2\pi},\tag{105}$$

for any $p\in\mathcal{P}_X$. Now from Lemma 3.3, for any $z\in\partial U_r(p)$, we have

$$P_X(\zeta)=4\pi\,\mathrm{Im}(\sigma_p^{-1}\zeta)-\log\big(4\,\mathrm{Im}(\sigma_p^{-1}\zeta)^2\big)+O_{\zeta}(1)=-2\log r-2\log\big(-\log r\big)+O(1)$$
$$\Longrightarrow\frac{r}{2}\frac{\partial P_X(\zeta)}{\partial r}=-1-\frac{2}{r\log r}+O(r)\Longrightarrow-4\pi\,|\mathcal{P}_X|\lim_{r\to0}\int_0^{2\pi}\frac{r}{2}\frac{\partial P_X(\zeta)}{\partial r}\frac{d\theta}{2\pi}=4\pi\,|\mathcal{P}_X|.\tag{106}$$

Combining computations (105) and (106) with upper bound (104), completes the proof of the proposition. □

Proposition 5.7 *We have the following upper bound*

$$\frac{1}{2g_X}\sup_{z\in Y_{\varepsilon}}\sum_{p\in\mathcal{P}_X}\bigg|\int_{\partial U_{\varepsilon/2}(p)}P_X(\zeta)d_{\zeta}^{c}g_{X,\mathrm{hyp}}(z,\zeta)\bigg|\leq-\frac{3\,|\mathcal{P}_X|\log(\varepsilon/2)}{g_X}+\frac{16\,C_{X,\mathrm{par}}}{g_X}.$$

Proof Since $P(\zeta)$ is a non-negative function on X, using Stokes's theorem, we derive

$$\sup_{z\in Y_{\varepsilon}}\sum_{p\in\mathcal{P}_X}\bigg|\int_{\partial U_{\varepsilon/2}(p)}P_X(\zeta)d_{\zeta}^{c}g_{X,\mathrm{hyp}}(z,\zeta)\bigg|$$
$$\leq\sup_{\zeta\in Y^{\mathrm{par}}_{\varepsilon/2}}P_X(\zeta)\cdot\Bigg(\sup_{z\in Y_{\varepsilon}}\sum_{p\in\mathcal{P}_X}\bigg|\int_{\partial U_{\varepsilon/2}(p)}d_{\zeta}d_{\zeta}^{c}g_{X,\mathrm{hyp}}(z,\zeta)\bigg|\Bigg)$$
$$=\sup_{\zeta\in Y^{\mathrm{par}}_{\varepsilon/2}}P_X(\zeta)\cdot\Bigg(\sup_{z\in Y_{\varepsilon}}\sum_{p\in\mathcal{P}_X}\bigg|\int_{\partial U_{\varepsilon/2}(p)}\mu_{\mathrm{shyp}}(\zeta)\bigg|\Bigg)\leq\sup_{z\in Y^{\mathrm{par}}_{\varepsilon/2}}P_X(\zeta),$$

and the proof of the proposition follows directly from estimate (79). □

Remark 5.8 For any $z \in Y_{\varepsilon}$, combining Propositions 5.6 and 5.7, we obtain the following upper bound for the second line on the right-hand side of Eq. (97)

$$\frac{|\mathcal{P}_X|\, B_{X,\varepsilon/2,\alpha,\delta}}{2g_X} - \frac{3\,|\mathcal{P}_X|\log(\varepsilon/2)}{g_X} + \frac{16\,C_{X,\mathrm{par}}}{g_X} + \frac{2\pi\,|c_X-1|}{g_X\,\mathrm{vol_{hyp}}(X)},$$

for any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in (0, \widetilde{\varepsilon})$.

Proposition 5.9 *We have the following upper bound*

$$\frac{1}{2g_X}\left|\int_{Y^{\mathrm{par}}_{\varepsilon/2}} P_X(z)\,\mu_{\mathrm{shyp}}(z)\right| \leq -\frac{|\mathcal{P}_X|\log(\varepsilon/2)}{g_X}.$$

Proof Since $P_X(z)$ is a non-negative function on X, we have

$$\left|\int_{Y^{\mathrm{par}}_{\varepsilon/2}} P_X(z)\,\mu_{\mathrm{shyp}}(z)\right| \leq \int_{Y^{\mathrm{par}}_{\varepsilon/2,p}} P_X(z)\,\mu_{\mathrm{shyp}}(z) = \sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\int_{Y^{\mathrm{par}}_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\eta z)\,\mu_{\mathrm{shyp}}(z). \tag{107}$$

The interchange of summation and integration in the above equation is valid, provided that the latter series converges absolutely. As the function $P_X(z)$ is a non-negative function, to prove the absolute convergence of the latter series, it suffices to prove that

$$\sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\int_{Y^{\mathrm{par}}_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\eta z)\,\mu_{\mathrm{shyp}}(z) \leq -2\,|\mathcal{P}_X|\log(\varepsilon/2). \tag{108}$$

For every $p \in \mathcal{P}_X$, after making the substitution $z \mapsto \eta^{-1}\sigma_p z$, from the $\mathrm{PSL}_2(\mathbb{R})$-invariance of the metric $\mu_{\mathrm{shyp}}(z)$, from estimate (40) from Proof of Lemma 3.2, and using the fact that $2\pi \leq \mathrm{vol_{hyp}}(X)$, we get

$$\begin{aligned}
&\sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\int_{Y^{\mathrm{par}}_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\eta z)\,\mu_{\mathrm{shyp}}(z) = \sum_{p\in\mathcal{P}_X}\sum_{\eta\in\Gamma_{X,p}\backslash\Gamma_X}\int_{\sigma_p^{-1}\eta Y^{\mathrm{par}}_{\varepsilon/2,p}} P_{\mathrm{gen},p}(\sigma_p z)\,\mu_{\mathrm{shyp}}(z)\\
&= \frac{1}{\mathrm{vol_{hyp}}(X)}\sum_{p\in\mathcal{P}_X}\int_0^{-\log(\varepsilon/2)/2\pi}\int_0^1 P_{\mathrm{gen},p}(\sigma_p z)\frac{dxdy}{y^2}\\
&\leq \frac{1}{\mathrm{vol_{hyp}}(X)}\sum_{p\in\mathcal{P}_X}\int_0^{-\log(\varepsilon/2)/2\pi}\int_0^1 32y^2\frac{dxdy}{y^2} = -\frac{16\,|\mathcal{P}_X|\log(\varepsilon/2)}{\pi\,\mathrm{vol_{hyp}}(X)} \leq -2\,|\mathcal{P}_X|\log(\varepsilon/2),
\end{aligned}$$

which proves upper bound (108), and completes the proof of the proposition. □

Proposition 5.10 *We have the following upper bound*

$$\frac{\left|C_{X,\mathrm{hyp}}\right|}{8g_X^2} \leq \frac{2\pi\,(d_X+1)^2}{\lambda_{X,1}\,\mathrm{vol_{hyp}}(X)}.$$

Proof Recall that $C_{X,\mathrm{hyp}}$ is defined as

$$\begin{aligned}
C_{X,\mathrm{hyp}} = \int_X\int_X g_{X,\mathrm{hyp}}(\zeta,\xi)\left(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{X,\mathrm{hyp}}(t;\zeta)dt\right)\\
\times\left(\int_0^\infty \Delta_{\mathrm{hyp}}\,K_{X,\mathrm{hyp}}(t;\xi)dt\right)\mu_{\mathrm{hyp}}(\xi)\,\mu_{\mathrm{hyp}}(\zeta).
\end{aligned}$$

From formulae (36), (37), we have

$$\Delta_{\text{hyp}}\,\phi_X(z) = \frac{4\pi\;\mu_{\text{can}}(z)}{\mu_{\text{hyp}}(z)} - \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} \Longrightarrow \int_X \Delta_{\text{hyp}}\,\phi_X(z)\,\mu_{\text{hyp}}(z) = 0, \tag{109}$$

$$\phi_X(z) = \frac{1}{2g_X}\int_X g_{X,\text{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\text{hyp}}\,K_{X,\text{hyp}}(t;\zeta)dt\bigg)\,\mu_{\text{hyp}}(\zeta) - \frac{C_{X,\text{hyp}}}{8g_X^2},$$

respectively. So combining the above two equations, we get

$$\begin{aligned} &-\frac{1}{4\pi}\int_X \phi_X(z)\,\Delta_{\text{hyp}}\,\phi_X(z)\,\mu_{\text{hyp}}(z)\\ &\quad= -\frac{1}{2g_X}\int_X\int_X g_{X,\text{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\text{hyp}}\,K_{X,\text{hyp}}(t;\zeta)dt\bigg)\,\mu_{\text{hyp}}(\zeta)\,\mu_{\text{can}}(z). \end{aligned} \tag{110}$$

Observe that

$$\int_X g_{X,\text{hyp}}(z,\zeta)\bigg(\int_0^\infty \Delta_{\text{hyp}}\,K_{X,\text{hyp}}(t;\zeta)dt\bigg)\,\mu_{\text{hyp}}(\zeta) = 2g_X\phi_X(z) + \frac{C_{X,\text{hyp}}}{4g_X} \in C_{\ell,\ell\ell}(X).$$

So combining Eqs. (38) and (110), we derive

$$\begin{aligned} \int_X \phi_X(z)\,\Delta_{\text{hyp}}\,\phi_X(z)\,\mu_{\text{hyp}}(z) &= \frac{\pi}{g_X^2}\int_X\int_X g_{X,\text{hyp}}(z,\zeta)\,\bigg(\int_0^\infty \Delta_{\text{hyp}}\,K_{X,\text{hyp}}(t;\zeta)dt\bigg)\\ \times\bigg(\int_0^\infty \Delta_{\text{hyp}}\,K_{X,\text{hyp}}(t;z)dt\bigg)\,\mu_{\text{hyp}}(\zeta)\,\mu_{\text{hyp}}(z) &= \frac{\pi\,C_{X,\text{hyp}}}{g_X^2}. \end{aligned} \tag{111}$$

Using Eq. (109), we have

$$\sup_{z\in X}|\,\Delta_{\text{hyp}}\,\phi_X(z)| \leq \sup_{z\in X}\bigg|\frac{4\pi\;\mu_{\text{can}}(z)}{\text{vol}_{\text{hyp}}(X)\,\mu_{\text{shyp}}(z)}\bigg| + \frac{4\pi}{\text{vol}_{\text{hyp}}(X)} = \frac{4\pi\;(d_X+1)}{\text{vol}_{\text{hyp}}(X)}, \tag{112}$$

where d_X is as defined in (8). As the function $\phi_X(z)\in L^2(X)$, it admits a spectral expansion of the form (17). So from the arguments used to prove Proposition 4.1 in [11], we have

$$\bigg|\int_X \phi_X(z)\,\Delta_{\text{hyp}}\,\phi_X(z)\,\mu_{\text{hyp}}(z)\bigg| \leq \sup_{z\in X}\frac{|\,\Delta_{\text{hyp}}\,\phi_X(z)|^2}{\lambda_{X,1}}\int_X \mu_{\text{hyp}}(z). \tag{113}$$

Hence, from Eq. (111), and combining estimates (112) and (113), we arrive at the estimate

$$\begin{aligned} \big|C_{X,\text{hyp}}\big| &= \frac{g_X^2}{\pi}\bigg|\int_X \phi_X(z)\,\Delta_{\text{hyp}}\,\phi_X(z)\,\mu_{\text{hyp}}(z)\bigg|\\ &\leq \frac{g_X^2}{\pi\lambda_{X,1}}\int_X |\,\Delta_{\text{hyp}}\,\phi_X(z)|^2\,\mu_{\text{hyp}}(z) \leq \frac{16\pi g_X^2\,(d_X+1)^2}{\lambda_{X,1}\,\text{vol}_{\text{hyp}}(X)}, \end{aligned}$$

which completes the proof of the proposition. □

Lemma 5.11 *We have the following upper bound*

$$\frac{1}{2g_X}\int_X E_X(\zeta)\,\mu_{\text{shyp}}(\zeta) \leq \frac{5\,c_{X,\text{ell}}}{g_X\,\text{vol}_{\text{hyp}}(X)}\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1).$$

Proof For any $z \in X$ and Eq. (53), we have

$$\int_X E_X(\zeta)\,\mu_{\mathrm{shyp}}(\zeta) = \int_X \sum_{\mathfrak{e}\in\mathcal{E}_X}\sum_{\eta\in\Gamma_{X,\mathfrak{e}}\backslash\Gamma_X}\sum_{n=1}^{m_{\mathfrak{e}}-1} g_{\mathbb{H}}(\sigma_{\mathfrak{e}}^{-1}\eta z, \gamma_i^n\sigma_{\mathfrak{e}}^{-1}\eta z)\,\mu_{\mathrm{shyp}}(\zeta)$$
$$= \sum_{\mathfrak{e}\in\mathcal{E}_X}\sum_{\eta\in\Gamma_{X,\mathfrak{e}}\backslash\Gamma_X}\sum_{n=1}^{m_{\mathfrak{e}}-1}\int_X g_{\mathbb{H}}(\sigma_{\mathfrak{e}}^{-1}\eta z, \gamma_i^n\sigma_{\mathfrak{e}}^{-1}\eta z)\,\mu_{\mathrm{shyp}}(\zeta).$$

The interchange of summation and integration in the above equation is valid, provided that the latter series converges absolutely. As the function $E_X(z)$ is a non-negative function, to prove the absolute convergence of latter series, it suffices to prove

$$\sum_{\mathfrak{e}\in\mathcal{E}_X}\sum_{\eta\in\Gamma_{X,\mathfrak{e}}\backslash\Gamma_X}\sum_{n=1}^{m_{\mathfrak{e}}-1}\int_X g_{\mathbb{H}}(\sigma_{\mathfrak{e}}^{-1}\eta z, \gamma_i^n\sigma_{\mathfrak{e}}^{-1}\eta z)\,\mu_{\mathrm{shyp}}(\zeta) \le \frac{9\,c_{X,\mathrm{ell}}\,|\mathcal{E}_X|}{\mathrm{vol}_{\mathrm{hyp}}(X)}\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1). \tag{114}$$

For any $\mathfrak{e} \in \mathcal{E}_X$, $\gamma_i \in \Gamma_{X,\mathfrak{e}}$, and $\eta \in \Gamma_{X,\mathfrak{e}}\backslash\Gamma_X$, from computation (54), and from definition of constant $c_{X,\mathrm{ell}}$ in (55), we have

$$g_{\mathbb{H}}(\sigma_{\mathfrak{e}}^{-1}\eta z, \gamma_i^n\sigma_{\mathfrak{e}}^{-1}\eta z) = \log\left(1+\frac{1}{\sin^2(n\pi/m_{\mathfrak{e}})\sinh^2(\rho(\sigma_{\mathfrak{e}}^{-1}\eta z))}\right) \tag{115}$$
$$\le c_{X,\mathrm{ell}}\log\left(1+\frac{1}{\sinh^2(\rho(\sigma_{\mathfrak{e}}^{-1}\eta z))}\right). \tag{116}$$

Furthermore, recall that the hyperbolic metric $\mu_{\mathrm{hyp}}(z)$ in elliptic coordinates is given by

$$\mu_{\mathrm{hyp}}(z) = \sinh(\rho(z))d\rho\wedge d\theta.$$

From estimate (115), we find

$$\sum_{\mathfrak{e}\in\mathcal{E}_X}\sum_{\eta\in\Gamma_{X,\mathfrak{e}}\backslash\Gamma_X}\sum_{n=1}^{m_{\mathfrak{e}}-1}\int_X g_{\mathbb{H}}(\sigma_{\mathfrak{e}}^{-1}\eta z, \gamma_i^n\sigma_{\mathfrak{e}}^{-1}\eta z)\,\mu_{\mathrm{shyp}}(\zeta)$$
$$\le c_{X,\mathrm{ell}}\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_{\mathfrak{e}}-1)\sum_{\eta\in\Gamma_{X,\mathfrak{e}}\backslash\Gamma_X}\int_X \log\left(1+\frac{1}{\sinh^2(\rho(\sigma_{\mathfrak{e}}^{-1}\eta z))}\right)\mu_{\mathrm{shyp}}(z). \tag{117}$$

For every $\mathfrak{e} \in \mathcal{E}_X$, after making the substitution $z \mapsto \eta^{-1}\sigma_{\mathfrak{e}}z$, from the $\mathrm{PSL}_2(\mathbb{R})$-invariance of the metric $\mu_{\mathrm{shyp}}(z)$, we compute

$$\sum_{\eta\in\Gamma_{X,\mathfrak{e}}\backslash\Gamma_X}\int_X \log\left(1+\frac{1}{\sinh^2(\rho(\sigma_{\mathfrak{e}}^{-1}\eta z))}\right)\mu_{\mathrm{shyp}}(z)$$
$$= \int_0^\infty\int_0^{2\pi}\log\left(\coth^2(\rho(z))\right)\frac{\sinh(\rho(z))d\rho\wedge d\theta}{\mathrm{vol}_{\mathrm{hyp}}(X)} = \frac{4\pi\log 2}{\mathrm{vol}_{\mathrm{hyp}}(X)} \le \frac{9}{\mathrm{vol}_{\mathrm{hyp}}(X)},$$

which together with upper bound (117) proves upper bound (114), and completes the proof of the lemma. □

Remark 5.12 For any elliptic fixed point $\mathfrak{e} \in \mathcal{E}_X$, from Corollary 4.11, we have

$$\sup_{z\in Y_\varepsilon}\left(\sum_{\mathfrak{e}\in\mathcal{E}_X}\frac{m_\mathfrak{e}-1}{2g_X m_\mathfrak{e}}\left|g_{X,\mathrm{hyp}}(z,\mathfrak{e})\right|\right)\leq \sup_{z\in Y_{\varepsilon/2}}\left(\sum_{\mathfrak{e}\in\mathcal{E}_X}\frac{m_\mathfrak{e}-1}{2g_X m_\mathfrak{e}}\left|g_{X,\mathrm{hyp}}(z,\mathfrak{e})\right|\right)\leq \frac{|\mathcal{E}_X|\,B_{X,\varepsilon/2,\alpha,\delta}}{2g_X},$$

for any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in (0, \varepsilon)$. For any $z \in Y_\varepsilon^{\mathrm{par}}$, combining Propositions 5.9 and 5.10, and Lemma 5.11 with the above upper bound, we obtain the following upper bound for the third line on the right-hand side of Eq. (97)

$$\frac{|\mathcal{E}_X|\,B_{X,\varepsilon/2,\alpha,\delta}}{2g_X}-\frac{|\mathcal{P}_X|\log(\varepsilon/2)}{g_X}+\frac{5\,c_{X,\mathrm{ell}}}{g_X\,\mathrm{vol}_{\mathrm{hyp}}(X)}\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_\mathfrak{e}-1)+\frac{2\pi\,(d_X+1)^2}{\lambda_{X,1}\,\mathrm{vol}_{\mathrm{hyp}}(X)},$$

for any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in (0, \varepsilon)$.

Theorem 5.13 *For any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in \big(0, \min\{\varepsilon, \widetilde{\varepsilon}\}\big)$, we have the following upper bound*

$$\sup_{z\in Y_\varepsilon^{\mathrm{par}}}\left|\phi_X(z)\right|\leq C_{X,\varepsilon,\alpha,\delta},$$

where
$$C_{X,\varepsilon,\alpha,\delta}=\frac{B_{X,\varepsilon/2,\alpha,\delta}}{2g_X}\left(|\mathcal{P}_X|\left(1-\frac{C_{X,\mathrm{par}}^{\mathrm{aux}}}{2\,\log(\varepsilon/2)}\right)+|\mathcal{E}_X|+1\right)-\frac{4\,|\mathcal{P}_X|\,\log(\varepsilon/2)}{g_X}+\frac{16\,C_{X,\mathrm{par}}}{g_X}+\frac{5\,c_{X,\mathrm{ell}}}{g_X\,\mathrm{vol}_{\mathrm{hyp}}(X)}\sum_{\mathfrak{e}\in\mathcal{E}_X}(m_\mathfrak{e}-1)+\frac{2\pi\,(d_X+1)^2}{\lambda_{X,1}\,\mathrm{vol}_{\mathrm{hyp}}(X)}+\frac{2\pi\,|c_X-1|}{g_X\,\mathrm{vol}_{\mathrm{hyp}}(X)}-\frac{\pi\,|\mathcal{P}_X|\,C_{X,\mathrm{par}}^{\mathrm{aux}}}{g_X\,\mathrm{vol}_{\mathrm{hyp}}(X)\,\log(\varepsilon/2)}. \tag{118}$$

Proof The proof of the theorem follows from Corollary 5.2, and combining the upper bounds stated in Remarks 5.5, 5.8, and 5.12. □

Corollary 5.14 *Let $p \in \mathcal{P}_X$ be any cusp. Then, for any $\alpha \in (0, \lambda_{X,1})$, $\delta \in \big(0, \min\{\varepsilon, \widetilde{\varepsilon}\}\big)$, and $z \in U_\varepsilon(p)$, we have*

$$\phi_X(z)=-\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right)+\phi_p(z),$$

where $\phi_p(z)$ is a subharmonic function for $z \in U_\varepsilon(p)$, which satisfies the following upper bound

$$\sup_{z\in U_{\varepsilon(p)}}|\phi_p(z)|\leq C_{X,\varepsilon,\alpha,\delta}\,.$$

Proof For any $p \in \mathcal{P}_X$ and $z \in U_\varepsilon(p)$, using Eq. (36), we find

$$\Delta_{\mathrm{hyp}}\left(\phi_X(z)+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right)\right)=\frac{4\pi\,\mu_{\mathrm{can}}(z)}{\mu_{\mathrm{hyp}}(z)}\geq 0,$$

which implies that

$$\phi_p(z)=\left(\phi_X(z)+\frac{4\pi}{\mathrm{vol}_{\mathrm{hyp}}(X)}\log\left(\frac{\log|\vartheta_p(w)|}{\log\varepsilon}\right)\right)$$

is a subharmonic function. From Theorem 5.13 and maximum principle for subharmonic functions, we derive

$$\sup_{z\in U_{\varepsilon}(p)} |\phi_p(z)| = \sup_{z\in \partial U_{\varepsilon}(p)} |\phi_p(z)| = \sup_{z\in \partial U_{\varepsilon}(p)} |\phi(z)| \leq C_{X,\varepsilon,\alpha,\delta},$$

which completes the proof of the lemma. □

Corollary 5.15 *Let $\mathfrak{e} \in \mathcal{E}_X$ be any elliptic fixed point. Then, for any $\alpha \in (0, \lambda_{X,1})$, $\delta \in \big(0, \min\{\varepsilon, \widetilde{\varepsilon}\}\big)$, and $z \in U_{\varepsilon}(\mathfrak{e})$, we have*

$$\phi_X(z) = -\frac{4\pi \log\big(1 - |\vartheta_{\mathfrak{e}}(z)|^{2/m_{\mathfrak{e}}}\big)}{\mathrm{vol_{hyp}}(X)} + \phi_{\mathfrak{e}}(z),$$

where $\phi_{\mathfrak{e}}(z)$ is a subharmonic function on $z \in U_{\varepsilon}(\mathfrak{e})$, which satisfies the following upper bound

$$\sup_{z\in U_{\varepsilon}(\mathfrak{e})} |\phi_{\mathfrak{e}}(z)| \leq C_{X,\varepsilon,\alpha,\delta}\,.$$

Proof The proof of the corollary follows from similar arguments as in Corollary 5.14. □

Theorem 5.16 *For any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in \big(0, \min\{\varepsilon, \widetilde{\varepsilon}\}\big)$, we have the following upper bounds*

$$\sup_{z,w\in Y_{\varepsilon}} \big| g_{X,\mathrm{hyp}}(z,w) - g_{X,\mathrm{can}}(z,w)\big| \leq 2\, C_{X,\varepsilon,\alpha,\delta}; \tag{119}$$

$$\sup_{z,w\in Y_{\varepsilon}} \Bigg| g_{X,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Bigg| \leq 2\, C_{X,\varepsilon,\alpha,\delta} + B_{X,\varepsilon,\alpha,\delta}\,. \tag{120}$$

Proof Upper bound (119) follows directly from formula (36) and Theorem 5.13. From triangle inequality, for any $z, w \in Y_{\varepsilon}$, we have

$$\Bigg| g_{X,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Bigg| \leq \big| g_{X,\mathrm{can}}(z,w) - g_{X,\mathrm{hyp}}(z,w)\big|$$
$$+ \Bigg| g_{X,\mathrm{hyp}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Bigg|. \tag{121}$$

Hence, upper bound (120) follows directly from combining Theorem 5.13 and Proposition 4.8. □

Corollary 5.17 *Let $p, q \in \mathcal{P}_X$ and $p \neq q$ be two cusps. Then, for any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in \big(0, \min\{\varepsilon, \widetilde{\varepsilon}\}\big)$, we have the following upper bounds*

$$\sup_{\substack{z\in U_{\varepsilon}(p)\\ w\in U_{\varepsilon}(q)}} \Bigg| g_{X,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)} g_{\mathbb{H}}(z,\gamma w)\Bigg| \leq 2\, C_{X,\varepsilon,\alpha,\delta} + B_{X,\varepsilon,\alpha,\delta}; \tag{122}$$

$$\sup_{z,w\in U_{\varepsilon}(p)} \Bigg| g_{X,\mathrm{can}}(z,w) - \sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z,\gamma w) - \sum_{\gamma\in \Gamma_{X,p}} g_{\mathbb{H}}(z,\gamma w)\Bigg| \leq 2\, C_{X,\varepsilon,\alpha,\delta} + B_{X,\varepsilon,\alpha,\delta}\,. \tag{123}$$

Proof Upper bound (122) follows directly from triangle inequality (121), and combining Corollaries 4.13 and 5.14.

Similarly upper bound (123) follows directly from triangle inequality (121), and combining Corollaries 4.14 and 5.14. □

Remark 5.18 Let $p, q \in \mathcal{P}_X$ and $p \neq q$ be two cusps. Then, for any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in \big(0, \min \varepsilon, \widetilde{\varepsilon}\}\big)$, from upper bound (122), we have the following upper bound

$$\Bigg| g_{X,\mathrm{can}}(p,q) - \sum_{\gamma \in S_{\Gamma_X}(\delta; z,w)} g_{\mathbb{H}}(p, \gamma q) \Bigg| = \big| g_{X,\mathrm{can}}(p,q) \big| \leq 2\, C_{X,\varepsilon,\alpha,\delta} + B_{X,\varepsilon,\alpha,\delta}\,. \quad (124)$$

In an upcoming article, we will derive an upper bound for $g_{X,\mathrm{can}}(p,q)$ using a different method, and the upper bound does not depend on the choice of ε.

Corollary 5.19 *Let $\mathfrak{e}, \mathfrak{f} \in \mathcal{E}_X$ and $\mathfrak{e} \neq \mathfrak{f}$ be two elliptic fixed points. Then, for any $\alpha \in (0, \lambda_{X,1})$ and $\delta \in \big(0, \varepsilon, \widetilde{\varepsilon}\}\big)$, we have the following upper bounds*

$$\sup_{\substack{z \in U_\varepsilon(\mathfrak{e}) \\ w \in U_\varepsilon(\mathfrak{f})}} \Bigg| g_{X,\mathrm{can}}(z,w) - \sum_{\gamma \in S_{\Gamma_X}(\delta; z,w)} g_{\mathbb{H}}(z, \gamma w) \Bigg| \leq 2\, C_{X,\varepsilon,\alpha,\delta} + B_{X,\varepsilon,\alpha,\delta}$$

$$\sup_{z,w \in U_\varepsilon(\mathfrak{e})} \Bigg| g_{X,\mathrm{can}}(z,w) - \sum_{\gamma \in S_{\Gamma_X}(\delta; z,w)\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z, \gamma w) - \sum_{\gamma \in \Gamma_{X,\mathfrak{e}}} g_{\mathbb{H}}(z, \gamma w) \Bigg|$$
$$\leq 2\, C_{X,\varepsilon,\alpha,\delta} + B_{X,\varepsilon,\alpha,\delta}\,.$$

Proof The proof of the corollary follows from triangle inequality 121, and combining Corollaries 5.15 and 4.15. □

Remark 5.20 In order to understand the dependence of our bounds for the canonical Green's function on ε, it suffices to analyze the dependence of $B_{X,\varepsilon,\alpha,\delta}$ and $C_{X,\varepsilon,\alpha,\delta}$ on ε. From the formula for $C_{X,\varepsilon,\alpha,\delta}$ from Theorem 5.13, and the dependence of $B_{X,\varepsilon,\alpha,\delta}$ on ε from Remark 4.16, we arrive at the following estimate for $C_{X,\varepsilon,\alpha,\delta}$

$$C_{X,\varepsilon,\alpha,\delta} = O_X\big(\varepsilon^{-3}\big).$$

6 Bounds for families of modular curves

In this section, we investigate the bounds obtained in previous subsections for certain sequences of Riemann orbisurfaces similar to the study conducted in Section 5 of [10].

We start by recalling the definition of an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite volume.

Definition 6.1 Let $\{X_N\}_{N \in \mathcal{N}}$ indexed by $N \in \mathcal{N} \subseteq \mathbb{N}$ be a set of non-compact hyperbolic Riemann orbisurfaces of finite volume of genus $g_N \geq 1$, which can be realized as a quotient space $\Gamma_{X_N}\backslash\mathbb{H}$, where Γ_{X_N} is a Fuchsian subgroup of the first kind acting by fractional linear transformations on the upper half-plane $\mathbb{H}$. We say that the sequence is *admissible* if it is one of the following two types:

(1) If $\mathcal{N} = \mathbb{N}$ and $N \in \mathcal{N}$, then X_{N+1} is a finite degree cover of X_N.

(2) For $N \in \mathbb{N}_{>0}$, let

$$Y_0(N) = \Gamma_0(N)\backslash\mathbb{H}, \quad Y_1(N) = \Gamma_1(N)\backslash\mathbb{H}, \quad Y(N) = \Gamma(N)\backslash\mathbb{H},$$

with the congruence subgroups $\Gamma_0(N)$, $\Gamma_1(N)$, $\Gamma(N)$, respectively. In each of the three cases above, let $\mathcal{N} \subseteq \mathbb{N}$ be such that $Y_0(N)$, $Y_1(N)$, $Y(N)$ has genus bigger than zero for $N \in \mathcal{N}$, respectively. We then consider here the families $\{X_N\}_{N\in\mathcal{N}}$ given by

$$\{Y_0(N)\}_{N\in\mathcal{N}}, \quad \{Y_1(N)\}_{N\in\mathcal{N}}, \quad \{Y(N)\}_{N\in\mathcal{N}}.$$

Denote by $q_{\mathcal{N}} \in \mathcal{N}$ the minimal element in Case (1), i.e., $q_{\mathcal{N}} = 0$; and the smallest prime in $\mathcal{N}$ in Case (2).

Remark 6.2 It is to be noted that the family of hyperbolic modular curves do not form a single tower of hyperbolic Riemann orbisurfaces, hence, the distinction in the above definition. However, they form a different structure which we call a net. We refer the reader to Section 5 of [11] for further details.

Notation 6.3 Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite volume. We fix an $0 < \varepsilon < 1$ satisfying the conditions elucidated in Notation 4.1 for the Riemann orbisurface $X_{q_{\mathcal{N}}}$.

Then, for any $N \in \mathcal{N}$, to emphasize the dependence on N, we denote the open coordinate disks around a cusp $p \in \mathcal{P}_{X_N}$ and an elliptic fixed point $\mathfrak{e} \in \mathcal{E}_{X_N}$ described in Notation 4.1 by $U_{N,\varepsilon}(p)$ and $U_{N,\varepsilon}(\mathfrak{e})$, respectively. Furthermore, we denote the compact subset Y_ε associated to the Riemann orbisurface X_N by $Y_{N,\varepsilon}$.

Lemma 6.4 *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite volume. Then, we have the following upper bounds:*

(1) For any $N \in \mathcal{N}$, we have

$$d_{X_N} = O_{X_{q_{\mathcal{N}}}}(1).$$

(2) For any $N \in \mathcal{N}$, we have

$$c_{X_N} = O_{X_{q_{\mathcal{N}}}}\left(\frac{g_{X_N}}{\lambda_{X_N,1}}\right).$$

(3) For any $N \in \mathcal{N}$, we have

$$\ell_{X_N} = O_{X_{q_{\mathcal{N}}}}(1).$$

(4) For any $N \in \mathcal{N}$, we have

$$C^{HK}_{X_N} = O_{X_{q_{\mathcal{N}}}}(1).$$

Proof The first three assertions follow directly from Lemma 5.3 of [10]. Assertion (4) follows from employing arguments similar to the ones used to prove assertion (d) in Lemma 5.3 of [10]. □

Notation 6.5 For $\Gamma \subset \mathrm{PSL}_2(\mathbb{R})$ a Fuchsian subgroup of the first kind, let $\mathcal{M}_{\mathrm{par}}(\Gamma)$ denote the set of maximal parabolic subgroups of Γ. Note that for $P \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, we have $P = \langle\gamma_P\rangle \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, where γ_P denotes a generator of the maximal parabolic subgroup P. Furthermore, there exists a scaling matrix σ_P satisfying the condition

$$\sigma_P^{-1}\gamma_P\sigma_P = \gamma_\infty, \quad \text{where } \gamma_\infty = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}. \tag{125}$$

Remark 6.6 Let Γ be a subgroup of finite index in $\Gamma_0 \subset \mathrm{PSL}_2(\mathbb{R})$, a Fuchsian subgroup of the first kind. Then, there is a bijection

$$\varphi : \mathcal{M}_{\mathrm{par}}(\Gamma) \longrightarrow \mathcal{M}_{\mathrm{par}}(\Gamma_0),$$

which is given as follows. For each $P \in \mathcal{M}_{\mathrm{par}}(\Gamma)$, there exists a maximal parabolic subgroup $P_0 \subset \Gamma_0$ containing P, and we set $\varphi(P) = P_0$; the inverse map is given by $\varphi^{-1}(P_0) = P_0 \cap \Gamma$.

Furthermore, the scaling matrices σ_{P_0} and σ_P of the parabolic subgroups P_0 and P, respectively, can be chosen such that they satisfy the relation

$$\sigma_{P_0} = \sigma_P \begin{pmatrix} 1/\sqrt{n_{P_0 P}} & 0 \\ 0 & \sqrt{n_{P_0 P}} \end{pmatrix}, \tag{126}$$

where $n_{P_0 P} = [P_0 : P]$.

Proposition 6.7 *Let $\{X_N\}_{N \in \mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite volume. Then, we have the following upper bounds:*

(1) For any $N \in \mathcal{N}$, we have

$$C_{X_N,\mathrm{par}} = O_{X_{q_\mathcal{N}}}(1).$$

(2) For any $N \in \mathcal{N}$, we have

$$C^{\mathrm{aux}}_{X_N,\mathrm{par}} = O_{X_{q_\mathcal{N}}}(1).$$

(3) For any $N \in \mathcal{N}$, we have

$$c_{X_N,\mathrm{ell}} = O_{X_{q_\mathcal{N}}}(1); \quad \frac{5\, c_{X_N,\mathrm{ell}}}{g_{X_N}\, \mathrm{vol_{hyp}}(X_N)} \sum_{\mathfrak{e} \in \mathcal{E}_{X_N}} (m_\mathfrak{e} - 1) = O_{X_{q_\mathcal{N}}}\left(\frac{|\mathcal{E}_{X_N}|}{g_{X_N}}\right).$$

(4) For any $N \in \mathcal{N}$, we have

$$C_{X,\mathrm{ell}} = O_{X_{q_\mathcal{N}}}(1).$$

Proof We first prove assertion (1) for $\{X_N\}_{N \in \mathcal{N}}$, an admissible sequence of Riemann orbisurfaces of type (1). In order to do so, we need to consider the pair of Riemann orbisurfaces X_N and $X_{q_\mathcal{N}}$, where X_N is a finite degree cover of $X_{q_\mathcal{N}}$.

For any $N \in \mathcal{N}$ and $X_N = \Gamma_{X_N} \backslash \mathbb{H}$, from Eq. (77), recall that

$$C_{X_N,\mathrm{par}} = \sup_{z \in X_N} \sum_{p \in \mathcal{P}_{X_N}} \big(\mathcal{E}_{X_N,\mathrm{par}}(z,2) - \mathrm{Im}(\sigma_p^{-1} z)^2\big).$$

Consider the set

$$\mathbb{P}(\Gamma_{X_N}) = \big\{\Gamma_{X_N,p} \mid p \in \mathcal{P}_{X_N}\big\},$$

where $\Gamma_{X_N,p}$ denotes the stabilizer subgroup of the cusp $p \in \mathcal{P}_{X_N}$. Keeping in mind that the set $\mathcal{P}_{X_N}$ is in bijection with the set of conjugacy classes of maximal parabolic subgroups of Γ_{X_N}, for any $z \in \mathbb{H}$, we have the equality

$$\bigcup_{p \in \mathcal{P}_{X_N}} \bigcup_{\substack{\eta \in \Gamma_{X_N,p} \backslash \Gamma_{X_N} \\ \eta \neq \mathrm{id}}} \eta^{-1} \Gamma_{X_N,p} \eta = \bigcup_{\substack{P \in \mathcal{M}_{\mathrm{par}}(\Gamma_{X_N}) \\ P \notin \mathbb{P}(\Gamma_{X_N})}} P$$

$$\Longrightarrow \sum_{p \in \mathcal{P}_{X_N}} \big(\mathcal{E}_{X_N,\mathrm{par}}(z,2) - \mathrm{Im}(\sigma_p^{-1} z)^2\big) = \sum_{\substack{P \in \mathcal{M}_{\mathrm{par}}(\Gamma_{X_N}) \\ P \notin \mathbb{P}(\Gamma_{X_N})}} \mathrm{Im}\,\big(\sigma_P^{-1} z\big)^2. \tag{127}$$

From Remark 6.6, we have a bijective map

$$\varphi_{N,q_{\mathcal{N}}}:\mathcal{M}_{\mathrm{par}}\big(\Gamma_{X_N}\big)\longrightarrow\mathcal{M}_{\mathrm{par}}\big(\Gamma_{X_{q_{\mathcal{N}}}}\big),$$

sending $P\in\mathcal{M}_{\mathrm{par}}(\Gamma_{X_N})$ to $P_0=\varphi_{N,q_{\mathcal{N}}}(P)\in\mathcal{M}_{\mathrm{par}}(\Gamma_{X_{q_{\mathcal{N}}}})$. Then, for $z\in\mathbb{H}$, using the relation stated in Eq. (126), we have

$$y_P=\mathrm{Im}(\sigma_P^{-1}z)=\begin{pmatrix}1/\sqrt{n_{P_0P}} & 0\\ 0 & \sqrt{n_{P_0P}}\end{pmatrix}\mathrm{Im}(\sigma_{P_0}^{-1}z)=\frac{y_{P_0}}{n_{P_0P}}, \tag{128}$$

where $n_{P_0P}=[P_0:P]$. For $z\in\mathbb{H}$, using relations (127) and (128), and the bijection between the sets $\mathcal{M}_{\mathrm{par}}(\Gamma_{X_N})$ and $\mathcal{M}_{\mathrm{par}}(\Gamma_{X_{q_{\mathcal{N}}}})$, we derive

$$\sum_{\substack{P\in\mathcal{M}_{\mathrm{par}}(\Gamma_{X_N})\\ P\notin\mathbb{P}(\Gamma_{X_N})}}\mathrm{Im}\left(\sigma_P^{-1}z\right)^2\leq\sum_{\substack{P_0\in\mathcal{M}_{\mathrm{par}}(\Gamma_{X_{q_{\mathcal{N}}}})\\ P_0\notin\mathbb{P}(\Gamma_{X_{q_{\mathcal{N}}}})}}\frac{\mathrm{Im}\left(\sigma_{P_0}^{-1}z\right)^2}{n_{P_0P}^2}\leq\sum_{\substack{P_0\in\mathcal{M}_{\mathrm{par}}(\Gamma_{X_{q_{\mathcal{N}}}})\\ P_0\notin\mathbb{P}(\Gamma_{X_{q_{\mathcal{N}}}})}}\mathrm{Im}\left(\sigma_{P_0}^{-1}z\right)^2,$$

using which, we deduce that

$$C_{X_N,\mathrm{par}}\leq C_{X_{q_{\mathcal{N}}},\mathrm{par}}=O_{X_{q_{\mathcal{N}}}}(1),$$

which proves assertion (1) for the case of an admissible sequence of type (1).

We now prove assertion (1) for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann orbisurfaces of type (2). We prove assertion (1) only for the sequence of modular curves $\{Y_0(N)\}_{N\in\mathcal{N}}$, as the proof extends with notational changes to the other sequences of modular curves $\{Y_1(N)\}_{N\in\mathcal{N}}$ and $\{Y(N)\}_{N\in\mathcal{N}}$.

For any $N\in\mathcal{N}$ the modular curve $Y_0(N)$ is a finite degree cover of $Y_0(1)=\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Extending our notation to the modular curve $Y_0(1)$, and adapting the arguments from the proof for admissible sequences of Riemann orbisurfaces of type (1), for $N\in\mathcal{N}$, we have

$$C_{Y_0(N),\mathrm{par}}=O(1),\Longrightarrow C_{Y_0(N),\mathrm{par}}=O_{Y_0(q_{\mathcal{N}})}(1).$$

This completes the proof for assertion (1).

For the case of admissible sequences of Riemann orbisurfaces of type (1), assertion (2) has been established as Proposition 5.4 in [13]. Using Proposition 5.4 from [13] and adapting the arguments from proof of assertion (1), trivially proves assertion (2) for the case of admissible sequences of Riemann orbisurfaces of type (2).

We first prove assertion (3) for $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann orbisurfaces of type (1). We again the consider a pair of Riemann orbisurfaces X_N and $X_{q_{\mathcal{N}}}$, where X_N is a finite degree cover of $X_{q_{\mathcal{N}}}$.

For any $N\in\mathcal{N}$, from Eq. (55), recall that

$$c_{X_N,\mathrm{ell}}=\max\left\{1/\sin^2(n\pi/m_{\mathfrak{e}})\big|\,\mathfrak{e}\in\mathcal{E}_{X_N},0\;<n\leq m_{\mathfrak{e}}-1\right\}.$$

Observe that

$$\left\{m_{\mathfrak{e}}\big|\,\mathfrak{e}\in\mathcal{E}_{X_N}\right\}\subseteq\left\{m_{\mathfrak{e}}\big|\,\mathfrak{e}\in\mathcal{E}_{X_{q_{\mathcal{N}}}}\right\},\quad\sum_{\mathfrak{e}\in\mathcal{E}_{X_N}}(m_{\mathfrak{e}}-1)\leq|\mathcal{E}_{X_N}|\sum_{\mathfrak{e}\in\mathcal{E}_{X_{q_{\mathcal{N}}}}}(m_{\mathfrak{e}}-1),$$

which along with the inequality $g_{X_N}\leq\mathrm{vol}_{\mathrm{hyp}}(X_N)$, trivially proves assertion (3) or admissible sequences of Riemann orbisurfaces of type (1).

Adapting similar arguments as the ones used to prove assertion (1) for admissible sequences of Riemann orbisurfaces of type (2), trivially proves assertion (3) for admissible sequences of Riemann orbisurfaces of type (2).

Assertion (4) follows easily from similar arguments as the ones used to prove assertions (1), (2), and (3). □

Proposition 6.8 *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite volume. Then, for any $N\in\mathcal{N}$, $\alpha\in(0,\lambda_{X_N,1})$, and $\delta>0$, we have the following estimate*

$$\sup_{z,w\in Y_{N,\varepsilon}}\Big|g_{X_N,\mathrm{hyp}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)\Big|=O_{X_{q_{\mathcal{N}}},\varepsilon,\alpha,\delta}(1).$$

Proof The proof of the proposition follows from similar arguments as the ones used to prove Theorem 5.5 in [10], and using Lemma 6.4 and Propositions 4.8 and 6.7. □

Theorem 6.9 *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite hyperbolic volume. Then, for any $N\in\mathcal{N}$, we have the following estimates*

$$\sup_{z,w\in Y_{N,\varepsilon}}\big|g_{X_N,\mathrm{can}}(z,w)-g_{X_N,\mathrm{hyp}}(z,w)\big|=O_{X_{q_{\mathcal{N}}},\varepsilon}\bigg(\frac{\big(|\mathcal{P}_{X_N}|+|\mathcal{E}_{X_N}|\big)}{g_{X_N}}\bigg(1+\frac{1}{\lambda_{X_N,1}}\bigg)\bigg);\tag{129}$$

$$\sup_{z,w\in Y_{N,\varepsilon}}\Big|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)\Big|$$
$$=O_{X_{q_{\mathcal{N}}},\varepsilon,\delta}\bigg(\frac{\big(|\mathcal{P}_{X_N}|+|\mathcal{E}_{X_N}|\big)}{g_{X_N}}\bigg(1+\frac{1}{\lambda_{X_N,1}}\bigg)\bigg).\tag{130}$$

Proof Estimate (129) follows from similar arguments as the ones used to prove Theorem 5.6 in [10], and using Lemma 6.4, and Propositions 5.16 and 6.7.

Estimate (130) follows from similar arguments as the ones used to prove Corollary 5.7 in [10], and using Proposition 6.8 and estimate (129). □

Corollary 6.10 *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite hyperbolic volume. For any $N\in\mathcal{N}$, let $p,q\in\mathcal{P}_{X_N}$ and $p\neq q$ be two cusps. Then, for any $\delta>0$, we have the following estimates*

$$\sup_{\substack{z\in U_{N,\varepsilon}(p)\\ w\in U_{N,\varepsilon}(q)}}\Big|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)\Big|$$
$$=O_{X_{q_{\mathcal{N}}},\varepsilon,\delta}\bigg(\frac{\big(|\mathcal{P}_{X_N}|+|\mathcal{E}_{X_N}|\big)}{g_{X_N}}\bigg(1+\frac{1}{\lambda_{X_N,1}}\bigg)\bigg);$$
$$\sup_{z,w\in U_{N,\varepsilon}(p)}\Big|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)\backslash\{\mathrm{id}\}}g_{\mathbb{H}}(z,\gamma w)-\sum_{\gamma\in\Gamma_{X_N,p}}g_{\mathbb{H}}(z,\gamma w)\Big|$$
$$=O_{X_{q_{\mathcal{N}}},\varepsilon,\delta}\bigg(\frac{\big(|\mathcal{P}_{X_N}|+|\mathcal{E}_{X_N}|\big)}{g_{X_N}}\bigg(1+\frac{1}{\lambda_{X_N,1}}\bigg)\bigg).$$

Proof The proof of the corollary follows directly from Corollary 5.17 and Theorem 6.9. □

Corollary 6.11 *Let $\{X_N\}_{N\in\mathcal{N}}$ be an admissible sequence of non-compact hyperbolic Riemann orbisurfaces of finite hyperbolic volume. For any $N\in\mathcal{N}$, let $\mathfrak{e},\mathfrak{f}\in\mathcal{E}_{X_N}$ and $\mathfrak{e}\neq\mathfrak{f}$ be two elliptic fixed points. Then, for any $\delta>0$, we have the following estimates*

$$\sup_{\substack{z\in U_{N,\varepsilon}(\mathfrak{e})\\ w\in U_{N,\varepsilon}(\mathfrak{f})}}\bigg|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)\bigg|$$
$$=O_{X_{q_\mathcal{N}},\varepsilon,\delta}\bigg(\frac{\big(|\mathcal{P}_{X_N}|+|\mathcal{E}_{X_N}|\big)}{g_{X_N}}\bigg(1+\frac{1}{\lambda_{X_N,1}}\bigg)\bigg);$$
$$\sup_{z,w\in U_{N,\varepsilon}(\mathfrak{e})}\bigg|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)\backslash\{\mathrm{id}\}}g_{\mathbb{H}}(z,\gamma w)-\sum_{\gamma\in\Gamma_{X_N,\mathfrak{e}}}g_{\mathbb{H}}(z,\gamma w)\bigg|$$
$$=O_{X_{q_\mathcal{N}},\varepsilon,\delta}\bigg(\frac{\big(|\mathcal{P}_{X_N}|+|\mathcal{E}_{X_N}|\big)}{g_{X_N}}\bigg(1+\frac{1}{\lambda_{X_N,1}}\bigg)\bigg).$$

Proof The proof of the corollary follows directly from Corollary 5.19 and Theorem 6.9. □

Remark 6.12 Consider the admissible sequence of modular curves $\{Y_0(N)\}_{N\in\mathcal{N}}$. For any $N\in\mathcal{N}$, the modular curve $Y_0(N)$ is a finite degree cover of $Y_0(1)=\mathrm{PSL}_2(\mathbb{Z})\backslash\mathbb{H}$. Furthermore, we have the following estimate for the genus $g_{Y_0(N)}$ of $Y_0(N)$

$$g_{Y_0(N)}=O\big(N\log N\big).$$

From Riemann–Hurwitz formula, we have the following estimates

$$\big[\mathrm{PSL}_2(\mathbb{Z}):\Gamma_0(N)\big]=O\big(g_{Y_0(N)}\big),\quad |\mathcal{P}_{Y_0(N)}|=O\big(N\log N\big),\quad |\mathcal{E}_{Y_0(N)}|=O_\epsilon\big(N^\epsilon\big),$$

for any $\epsilon>0$. We refer the reader to [18], pp. 22–25 for details of the above estimates.

Furthermore, from work of Selberg [17], we know that $\lambda_{Y_0(N),1}\geq 3/16$. All the above estimates also hold true for the other sequences of modular curves $\{Y_1(N)\}_{N\in\mathcal{N}}$ and $\{Y(N)\}_{N\in\mathcal{N}}$.

Corollary 6.13 *Let $\{X_N\}_{N\in\mathcal{N}}$, an admissible sequence of Riemann orbisurfaces of type (2). Then, for any $N\in\mathcal{N}$ and $\delta>0$, we have the following estimate*

$$\sup_{z,w\in Y_{N,\varepsilon}}\bigg|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_X}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)\bigg|=O_{X_{q_\mathcal{N}},\varepsilon,\delta}(1). \tag{131}$$

For any $N\in\mathcal{N}$, let $p,q\in\mathcal{P}_{X_N}$ and $p\neq q$ be two cusps. Then, for any $\delta>0$, we have the following estimates

$$\sup_{\substack{z\in U_{N,\varepsilon}(p)\\ w\in U_{N,\varepsilon}(q)}}\bigg|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)}g_{\mathbb{H}}(z,\gamma w)\bigg|=O_{X_{q_\mathcal{N}},\varepsilon,\delta}(1); \tag{132}$$

$$\sup_{z,w\in U_{N,\varepsilon}(p)}\bigg|g_{X_N,\mathrm{can}}(z,w)-\sum_{\gamma\in S_{\Gamma_{X_N}}(\delta;z,w)\backslash\{\mathrm{id}\}}g_{\mathbb{H}}(z,\gamma w)-\sum_{\gamma\in\Gamma_{X_N,p}}g_{\mathbb{H}}(z,\gamma w)\bigg|=O_{X_{q_\mathcal{N}},\varepsilon,\delta}(1). \tag{133}$$

For any $N \in \mathcal{N}$, let $\mathfrak{e}, \mathfrak{f} \in \mathcal{E}_{X_N}$ and $\mathfrak{e} \neq \mathfrak{f}$ be two elliptic fixed points. Then, for any $\delta > 0$, we have the following estimates

$$\sup_{\substack{z \in U_{N,\varepsilon}(\mathfrak{e}) \\ w \in U_{N,\varepsilon}(\mathfrak{f})}} \Bigg| g_{X_N,\mathrm{can}}(z,w) - \sum_{\gamma \in \mathcal{S}_{\Gamma_{X_N}}(\delta; z,w)} g_{\mathbb{H}}(z,\gamma w) \Bigg| = O_{X_{q_{\mathcal{N}}},\varepsilon,\delta}(1); \tag{134}$$

$$\sup_{z,w \in U_{N,\varepsilon}(\mathfrak{e})} \Bigg| g_{X_N,\mathrm{can}}(z,w) - \sum_{\gamma \in \mathcal{S}_{\Gamma_{X_N}}(\delta; z,w)\setminus\{\mathrm{id}\}} g_{\mathbb{H}}(z,\gamma w) - \sum_{\gamma \in \Gamma_{X_N,\mathfrak{e}}} g_{\mathbb{H}}(z,\gamma w) \Bigg| = O_{X_{q_{\mathcal{N}}},\varepsilon,\delta}(1). \tag{135}$$

Proof Estimate (131) follows directly from combining Remark (6.12) with Theorem 6.9. Estimates (132) and (133) follow directly from combining Remark (6.12) with Corollary 6.10. Estimates (134) and (135) follow directly from combining Remark (6.12) with Corollary 6.11. □

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