Math. Z. (2014) 278:1179–1193
DOI 10.1007/s00209-014-1351-1

Mathematische Zeitschrift

Ampleness of canonical divisors of hyperbolic normal projective varieties

Fei Hu · Sheng Meng · De-Qi Zhang

Received: 14 April 2014 / Accepted: 29 June 2014 / Published online: 9 August 2014

Abstract Let X be a projective variety which is algebraic Lang hyperbolic. We show that Lang's conjecture holds (one direction only): X and all its subvarieties are of general type and the canonical divisor K_X is ample at smooth points and Kawamata log terminal points of X, provided that K_X is $\mathbb{Q}$-Cartier, no Calabi–Yau variety is algebraic Lang hyperbolic and a weak abundance conjecture holds.

Keywords Algebraic Lang hyperbolic variety · Ample canonical divisor

Mathematics Subject Classification 32Q45 · 14E30

1 Introduction

We work over the field $\mathbb{C}$ of complex numbers. A variety X is *Brody hyperbolic* (resp. *algebraic Lang hyperbolic*) if every holomorphic map $V \to X$, where V is the complex line $\mathbb{C}$ (resp. V is an abelian variety), is a constant map. Since an abelian variety is a complex torus, Brody hyperbolicity implies algebraic Lang hyperbolicity. When X is a compact complex variety, Brody hyperbolicity is equivalent to the usual Kobayashi hyperbolicity (cf. [13]).

In the first part (Theorem 1.4 and its consequences 3.7, 3.8) of this paper, we let X be a normal projective variety and aim to show the ampleness of the *canonical divisor* K_X *of* X, assuming that X is algebraic Lang hyperbolic. We allow X to have arbitrary singularities and

F. Hu · S. Meng · D.-Q. Zhang (✉)
Department of Mathematics, National University of Singapore,
10 Lower Kent Ridge Road, Singapore 119076, Singapore
e-mail: matzdq@nus.edu.sg

F. Hu
e-mail: hf@nus.edu.sg

S. Meng
e-mail: ms@nus.edu.sg

assume only that X is $\mathbb{Q}$-*Gorenstein* (so that the ampleness of K_X is well-defined), i.e., K_X is $\mathbb{Q}$-*Cartier:* mK_X is a Cartier divisor for some positive integer m.

For related work, it was proven in [18] that a 3-dimensional hyperbolic smooth projective variety X has ample K_X unless X is a Calabi–Yau manifold where every non-zero effective divisor is ample. The authors of [7] proved the ampleness of K_X when X is a smooth projective threefold having a Kähler metric of negative holomorphic sectional curvature; they also generalized the results to higher dimensions with some additional conditions.

In the second part of the paper (Theorem 1.5 and its more general form Theorem 3.1), we make some contributions toward Lang's conjecture in Corollary 1.6, where even the normality of X is not assumed. Our approach is to take a projective resolution of X and run the relative Minimal Model Program (MMP) over X. We use only the frame work of MMP, but not its detailed technical part. Certain mild singularities occur naturally along the way. See [12, Definitions 2.34 and 2.37] for definitions of *canonical*, *Kawamata log terminal* (klt), and *divisorial log terminal* (dlt) *singularities*.

In the last part (Proposition 1.7 and its more general form Theorem 3.2), we try to avoid assuming conjectures.

We now state two conjectures. Conjecture 1.1 below is long standing. When $\dim X \le 2$, it is true by the classification of complex surfaces and the following:

Fact ($*$). A (smooth) $K3$ surface has infinitely many (singular) elliptic curves; see [16, Theorem in Appendix] or Proposition 2.2.

In Conjecture 1.1, the conclusion means the existence of at least one non-constant holomorphic map $f : V \to X$ from an abelian variety V, but does *not* require the image $f(V)$ (or the union of such images) to be Zariski-dense in X. This does not seem sufficient for our purpose to show the non-existence of subvariety X' of Kodaira dimension zero in an algebraic Lang hyperbolic variety W as in Corollary 1.6 below (see 1.9, and think about a proof of the non-hyperbolicity of every normal $K3$ surface using the Fact ($*$) above). Fortunately, we are able to show in Corollary 1.6 (or Theorems 1.5 and 3.1) that the normalization X of $X' \subseteq W$ is a Calabi–Yau variety and hence f composed with the finite morphism $X \to X' \subseteq W$ produces a non-constant holomorphic map from the abelian variety V, thus deducing a contradiction to the hyperbolicity of W.

Conjecture 1.1 *Let X be an absolutely minimal Calabi–Yau variety (cf. 2.1). Suppose further that every birational morphism $X \to Y$ onto a normal projective variety is an isomorphism. Then X is not algebraic Lang hyperbolic.*

We need the result below about *nef reduction map* and *nef dimension*. A meromorphic map $f : X \dashrightarrow Y$ between complex varieties is *almost holomorphic* if it is well defined on a Zariski dense open subset U of X and the map $f_{|U} : U \to Y$ has compact connected general fibres.

Theorem 1.2 (cf. [1, Theorem 2.1]) *Let L be a nef $\mathbb{Q}$-Cartier divisor on a normal projective variety X. Then there exists an almost holomorphic, dominant rational map $f : X \dashrightarrow Y$ with connected fibres, called a "nef reduction map" such that*

(1) L is numerically trivial on all compact fibres F of f with $\dim F = \dim X - \dim Y$*;*
(2) for every general point $x \in X$ and every irreducible curve C passing through x with $\dim f(C) > 0$*, we have $L.C > 0$.*

The map f is unique up to birational equivalence of Y. We call $\dim Y$ *the "nef dimension" of L and denote it as $n(L)$.*

Proof See [1] for the proof. □

Next we state Conjecture 1.3. We stress that 1.3 without the extra "Hyp(A)" is the usual abundance conjecture and *stronger* than our one here. When K_X is nef and big or when $\dim X \leq 3$, both Conjectures 1.3 (1) and 1.3 (2) (and their log versions, even without the extra Hyp(A)) are true; see [12, Theorem 3.3, §3.13], or Proposition 2.2.

Conjecture 1.3 *Let X be an n-dimensional minimal normal projective variety, i.e., the canonical divisor K_X is a nef $\mathbb{Q}$-Cartier divisor. Assume Hyp(A): the nef dimension $n(K_X)$ satisfies $n(K_X) = n$.*

(1) If X has at worst klt singularities, then K_X is semi-ample, i.e., the linear system $|mK_X|$ is base-point free for some $m > 0$.
(2) If X has at worst canonical singularities and $K_X \not\equiv 0$ (not numerically zero), then the Kodaira dimension $\kappa(X) > 0$.

Theorems 1.4 and 1.5 below are our main results. When X has at worst klt singularities, Theorem 1.4 below follows from the MMP and has been generalized to the quasi-projective case in [14]. In Theorem 1.4, we do not impose any condition on the singularities of X, except the $\mathbb{Q}$-Cartierness of K_X. This assumption is necessary to formulate the conclusion that K_X be ample. Without assuming Conjecture 1.1 or 1.3 as in Theorem 1.4, we can at least say that K_X is movable or nef in codimension-one (cf. Remark 1.8). See also Corollaries 3.7 and 3.8 when $\dim X \leq 3$.

Theorem 1.4 *Let X be a $\mathbb{Q}$-Gorenstein normal projective variety which is algebraic Lang hyperbolic. Assume that Conjecture 1.1 holds for all varieties birational to X, or to any subvariety of X. Further, assume that Conjecture 1.3 (1) holds for all varieties birational to X.*

Then K_X is ample at smooth points and klt points of X. To be precise, there is a birational morphism $f_c : X_c \to X$ such that X_c has at worst klt singularities, K_{X_c} is ample, and $E_c := f_c^* K_X - K_{X_c}$ is an effective and f_c-exceptional divisor with $f_c(E_c) \subseteq \mathrm{Nklt}(X)$, the non-klt locus of X.

In particular, $f_c = \mathrm{id}$ and K_X is ample, if X has at worst klt singularities.

The normality of X is not assumed in Theorem 1.5 below which is a special case of Theorem 3.1 by letting $g : X \to W$ there be the identity map $\mathrm{id}_X : X \to X$. When $\dim X \leq 3$, Case (3) below does not occur.

Theorem 1.5 *Let X be an algebraic Lang hyperbolic projective variety of dimension n. Assume either $n \leq 3$ or Conjecture 1.3 (2) (resp. either $n \leq 3$ or Conjecture 1.3 (2) without the extra Hyp(A)) holds for all varieties birational to X.*

Then there is a birational surjective morphism $g_m : X_m \to X$ such that X_m is a minimal variety with at worst canonical singularities and one of the following is true.

(1) *K_{X_m} is ample. Hence both X_m and X are of general type.*
(2) *$g_m : X_m \to X$ is the normalization map. X_m is an absolutely minimal Calabi–Yau variety with $\dim X_m \geq 3$.*
(3) *There is an almost holomorphic map $\tau : X_m \dashrightarrow Y$ (resp. a holomorphic map $\tau : X_m \to Y$) such that its general fibre F is an absolutely minimal Calabi–Yau variety with $3 \leq \dim F < \dim X_m$, and $(g_m)_{|F} : F \to g_m(F) \subset X$ is the normalization map.*

Given a projective variety W, let $\widetilde{W} \stackrel{\sigma}{\to} W$ be a projective resolution. We define the *albanese variety of* W as $\mathrm{Alb}(W) := \mathrm{Alb}(\widetilde{W})$, which is independent of the choice of the resolution $\widetilde{W}$, since every two resolutions of W are dominated by a third one and the albanese variety, being an abelian variety, contains no rational curves. We define the *albanese (rational) map* $\mathrm{alb}_W : W \dashrightarrow \mathrm{Alb}(W)$ as the composition

$$W \stackrel{\sigma^{-1}}{\dashrightarrow} \widetilde{W} \stackrel{\mathrm{alb}_{\widetilde{W}}}{\longrightarrow} \mathrm{Alb}(\widetilde{W}).$$

One direction of Lang's [13, Conjecture 5.6] follows from Conjectures 1.1 and 1.3 (2). See Remark 1.8 (6) for the other direction.

Corollary 1.6 *Let W be an algebraic Lang hyperbolic projective variety of dimension n. Assume either $n \le 3$ or Conjecture 1.3 (2) holds for all varieties of dimension $\le n$. Then we have:*

(1) *If Conjecture 1.1 holds for all varieties of dimension $\le n$, then W and all its subvarieties are of general type.*
(2) *If the albanese map $\mathrm{alb}_W : W \dashrightarrow \mathrm{Alb}(W)$ has general fibres of dimension ≤ 2, then W is of general type.*

Without assuming Conjecture 1.3 (or 1.1), we have the following (see Theorem 3.2 for a generalization). For a singular projective variety Z, we define the *Kodaira dimension* $\kappa(Z)$ as $\kappa(\widetilde{Z})$ (cf. [12, §7.73]) for some (or equivalently any) projective resolution $\widetilde{Z} \to Z$.

Proposition 1.7 *Let X be an algebraic Lang hyperbolic projective variety. Assume one of the following conditions.*

(i) *X has maximal albanese dimension, i.e., the albanese map $\mathrm{alb}_X : X \dashrightarrow \mathrm{Alb}(X)$ is generically finite (but not necessarily surjective).*
(ii) *The Kodaira dimension $\kappa(X) \ge \dim X - 2$.*
(iii) *$\kappa(X) \ge \dim X - 3$, and Conjecture 1.1 holds in dimension three.*

Then X is of general type.

Remark 1.8 (1) In Theorem 1.4, by the equality $f_c^* K_X = K_{X_c} + E_c$ and the ampleness of K_{X_c}, the *exceptional locus* $\mathrm{Exc}(f_c)$ (the subset of X_c along which f_c is not isomorphic) is contained in $\mathrm{Supp}\, E_c$. Indeed, if C is an f_c-contractible curve, then $0 = C.f_c^* K_X = C.K_{X_c} + C.E_c > C.E_c$, so $C \subseteq \mathrm{Supp}\, E_c$. This and the effectivity of E_c justify the phrasal: K_X is ample outside $f_c(E_c)$.

(2) Without assuming Conjecture 1.1 or 1.3, the proof of Theorem 1.4 (Claim 3.3 and the equality (2) above it) shows that $(f')^* K_X = K_{X'} + E'$ with $K_{X'}$ nef and $E' \ge 0$ f'-exceptional. Hence $K_X = f'_* K_{X'}$ is movable, or nef in codimension-one.
(3) Let $X_2 \to X_1$ be a finite morphism (but not necessarily surjective). If X_1 is Brody hyperbolic or algebraic Lang hyperbolic then so is X_2. The converse is not true.
(4) Every algebraic Lang hyperbolic projective variety X_1 is absolutely minimal in the sense of 2.1, i.e., every birational map $h : X_2 \dashrightarrow X_1$ from a normal projective variety X_2 with at worst klt singularities, is a well defined morphism. This result was proved by S. Kobayashi when X_2 is nonsingular. Indeed, let Z be a resolution of the graph of h such that we have birational surjective morphisms $p_i : Z \to X_i$ satisfying $h \circ p_2 = p_1$. Then every fibre $p_2^{-1}(x_2)$ is rationally chain connected by [6, Corollary 1.5] and hence $p_1(p_2^{-1}(x_2))$ is a point since hyperbolic X_1 contains no rational curve. Thus h can be extended to a well defined morphism by [8, Proof of Lemma 14], noting that X_2 is normal and p_2 is surjective, and using the Stein factorization.

(5) If Y is an algebraic Lang hyperbolic Calabi–Yau variety (like X_m and F in Theorem 1.5 (2) and (3), respectively), then every birational morphism $h : Y \to Z$ onto a normal projective variety is an isomorphism. Indeed, by [11, Corollary 1.5], Z has only canonical singularities. Thus the exceptional locus $\mathrm{Exc}(h)$ is covered by rational curves by [6, Corollary 1.5]. Since Y is hyperbolic and hence has no rational curve, we have $\mathrm{Exc}(h) = \emptyset$ and hence h is an isomorphism, Z being normal and by the Stein factorization.

(6) Consider the converse of Corollary 1.6, i.e., the other direction of Lang [13, Conjecture 5.6], but with the assumption that every non-uniruled projective variety has a minimal model with at worst canonical singularities and that abundance Conjecture 1.3 (2) holds. To be precise, supposing that a projective variety W and all its subvarieties are of general type, we see that W is algebraic Lang hyperbolic. Indeed, let $f : V \to W$ be a morphism from an abelian variety V and let $V \to X \to f(V)$ be its Stein factorization, where $V \to X$ has a connected general fibre F and $X \to f(V)$ is a finite morphism. Since V is non-uniruled, so is F. Hence $\kappa(F) \geq 0$ by the assumption. The assumption and Iitaka's $C_{n,m}$ also imply that $0 = \kappa(V) \geq \kappa(F) + \kappa(X) \geq \kappa(X) \geq \kappa(f(V)) = \dim f(V)$ (cf. [9, Corollary 1.2]). Hence f is a constant map.

1.9 Comments about the proofs

In our proofs, neither the existence of minimal model nor the termination of MMP is assumed. Let W be an algebraic Lang hyperbolic projective variety. To show that every subvariety X of W is of general type, one key observation is the existence of a birational model X' of X with $K_{X'}$ relative nef over W, by using the main Theorem 1.2 in [2]. *$K_{X'}$ is indeed nef* since W is hyperbolic (cf. Lemma 2.5 or 2.6). One natural approach is to take a general fibre F (which may not even be normal) of an Iitaka (rational) fibration of X (assuming $\kappa(X) \geq 0$) and prove that F has a minimal model F_m. Next, one tries to show that $q(F_m) = 0$ and F_m is a Calabi–Yau variety and then tries to use Conjecture 1.1 to produce a non-hyperbolic subvariety S of F_m, but this does not guarantee the same on $F \subset X$ (to contradict the hyperbolicity of X) because such $S \subseteq F_m$ might be contracted to a point on F. In our approach, we are able to show that the *normalization of F is a Calabi–Yau variety, which is the key of the proofs*. It would not help even if one assumes the smoothness of the ambient space W since its subvariety X may not be smooth, or at least normal or Cohen–Macaulay to define the canonical divisor K_X meaningfully to pull back or push forward.

2 Preliminary results

2.1 Convention, notation and terminology

In this paper, by hyperbolic we mean algebraic Lang hyperbolic.

(i) We use the notation and terminology in the book of Hartshorne and the book [12].
(ii) Given two morphisms $g_i : Y_i \to Z$ $(i = 1, 2)$ between varieties, a rational map $Y_1 \dashrightarrow Y_2$ is said to be *a map over* Z, if the composition $Y_1 \dashrightarrow Y_2 \stackrel{g_2}{\to} Z$ coincides with $g_1 : Y_1 \to Z$.
(iii) For a rational map $h : X \dashrightarrow Y$, we take a birational resolution $\pi : W \to X$ of the indeterminacy of h such that the composition $h \circ \pi$ is a well defined morphism: $h_1 : W \to Y$. For a point $y \in Y$, we defined the *fibre* $h^{-1}(y)$ as $\pi(h_1^{-1}(y))$. This

definition does not depend on the choice of the resolution π of h, since every two such resolutions are dominated by a third one.

(iv) For a singular projective variety Z, we define the *Kodaira dimension* $\kappa(Z)$ as $\kappa(\widetilde{Z})$ (cf. [12, §7.73]) for some (or equivalently any) projective resolution $\widetilde{Z} \to Z$. When $\kappa(\widetilde{Z}) \geq 0$, there is a *(rational) Iitaka fibration*, unique up to birational equivalence, $I_Z : Z \dashrightarrow Y$ such that its very general fibre F has $\kappa(F) = 0$ and that $\dim Y = \kappa(Z)$.

(v) For two Weil $\mathbb{Q}$-divisors D_i on a normal variety X, if $m(D_1 - D_2) \sim 0$ (linear equivalence) for some integer $m > 0$, we say that D_1 and D_2 are $\mathbb{Q}$-linearly equivalent and denote this relation as: $D_1 \sim_{\mathbb{Q}} D_2$.

(vi) Let X be a normal projective variety. X is a *Calabi–Yau variety* if X has at worst canonical singularities, its canonical divisor is $\mathbb{Q}$-linearly equivalent to zero: $K_X \sim_{\mathbb{Q}} 0$, and the *irregularity* $q(X) := h^1(X, \mathcal{O}_X) = 0$. If this is the case, X has Kodaira dimension $\kappa(X) = 0$.

(vii) A projective variety X is *of general type* if some (equivalently every) projective resolution X' of X has maximal Kodaira dimension: $\kappa(X') = \dim X'$.

(viii) A $\mathbb{Q}$-Gorenstein variety X is *minimal* if the canonical divisor K_X is *numerically effective* (=*nef*). A projective variety X_1 is *absolutely minimal* if every birational map $h : X_2 \dashrightarrow X_1$ from a normal projective variety X_2 with at worst klt singularities, is a well defined morphism.

Proposition 2.2 (1) *Let X be a projective surface. Then either it has infinitely many rational curves or elliptic curves, or it is of general type, or it is birational to a simple abelian surface.*

(2) *Let Y be a normal projective surface such that $K_Y \sim_{\mathbb{Q}} 0$ and Y is birational to an abelian surface A. Then Y is isomorphic to A.*

(3) *Let Z be a normal projective surface with $K_Z \sim_{\mathbb{Q}} 0$. Then Z is not algebraic Lang hyperbolic. In particular, Conjecture 1.1 holds when dimension ≤ 2.*

(4) *In dimension ≤ 3, both Conjectures* 1.3 (1) *and (2) even without the extra Hyp(A) (and even for log canonical pairs) hold.*

(5) *Both Conjectures* 1.3 (1) *and* (2) *even without the extra Hyp(A) hold for varieties of general type.*

(6) *Let X be a variety with maximal albanese dimension, i.e.,* $\dim \mathrm{alb}_X(X) = \dim X$. *If X has only canonical singularities and K_X is nef, then K_X is semi-ample. In particular, Conjecture* 1.3 (2) *even without the extra Hyp(A) holds for varieties with maximal albanese dimension.*

Proof (1) It is well known that every Enrique surface has an elliptic fibration. By [16, Theorem in Appendix], every $K3$ surface has infinitely many singular elliptic curves. Thus (1) follows from the classification of algebraic surfaces.

(2) Take a common resolution Z of Y and A, i.e., let $p : Z \to A$ and $q : Z \to Y$ be two biraitonal morphisms. Write $K_Z = p^*K_A + E_p = E_p$ where $E_p \geq 0$ is p-exceptional and Supp E_p is equal to $\mathrm{Exc}(p)$, the exceptional locus of p. Write $K_Z = q^*K_Y + E_1 - E_2 \sim_{\mathbb{Q}} E_1 - E_2$ where both $E_i \geq 0$ are q-exceptional and there is no common irreducible component of E_1 and E_2.

Equating the two expressions of K_Z, we get $E_1 \sim_{\mathbb{Q}} E_2 + E_p$. Since E_1 is q-exceptional, its Iitaka D-dimension is zero, so $E_1 = E_2 + E_p$. Thus $\mathrm{Exc}(p) = \mathrm{Supp}\, E_p \subseteq \mathrm{Supp}\, E_1 \subseteq \mathrm{Exc}(q)$. Hence there is a birational surjective morphism $h : A \to Y$ such that $q = h \circ p$:

$$Z \xrightarrow{\;p\;} A \xrightarrow{\;h\;} Y, \qquad \overset{q}{\curvearrowright}\ (q : Z \to Y).$$

If $h : A \to Y$ is not an isomorphism, then it contracts a curve C on A to a point on Y. Clearly, $C^2 < 0$. By the genus formula, $2g(C) - 2 = C^2 + C.K_A = C^2 < 0$. So $C \cong \mathbb{P}^1$. This contradicts the fact that there is no rational curve on the abelian variety A. Thus h is an isomorphism.

(3) Since $K_Z \sim_{\mathbb{Q}} 0$, Z is not of general type. By (1), either Z is birational to an abelian surface, or Z has infinitely many rational or elliptic curves. In the first case, Z is an abelian surface by (2). Thus Z is not algebraic Lang hyperbolic in all cases.
(4) We refer to [12, §3.13] for its proof or references.
(5) This follows from the base point freeness result for nef and big canonical divisors of klt varieties (cf. [12, Theorem 3.3]).
(6) It is proven in [5, Theorem 3.6].

□

The result below is just [9, Theorem 8.3]; see also [9, Lemma 8.1] and [8, Theorem 1] for the assertion (1).

Lemma 2.3 *Let X be a normal projective variety with only canonical singularities and $K_X \sim_{\mathbb{Q}} 0$. Suppose that the irregularity $q(X) > 0$. Then we have:*

(1) The albanese map $\mathrm{alb}_X : X \to A := \mathrm{Alb}(X)$ is a surjective morphism, where $\dim A = q(X)$.
(2) There is an étale morphism $B \to A$ from another abelian variety B such that the fibre product $X \times_A B \cong Z \times B$ for some variety Z.
(3) X is covered by images of abelian varieties $\{z\} \times B$ ($z \in Z$).

Lemma 2.4 *Let X be a normal projective variety of dimension n such that K_X is $\mathbb{Q}$-Cartier. Suppose that X is not uniruled and $K_X \equiv 0$ (numerically). Then X has at worst canonical singularities and $K_X \sim_{\mathbb{Q}} 0$.*

Proof Let $\gamma : \widetilde{X} \to X$ be a projective resolution and write $K_{\widetilde{X}} = \gamma^* K_X + (E_1 - E_2) \equiv E_1 - E_2$ such that $E_i \geq 0$ ($i = 1, 2$) are γ-exceptional and have no common components. Since X and hence $\widetilde{X}$ are non-uniruled, $K_{\widetilde{X}}$ is pseudo-effective by [3, Theorem 2.6]. Let $K_{\widetilde{X}} = P_1 + N_1$ be the σ-decomposition in [17, ChIII, §1.b], which is also called the Zariski decomposition in codimension-one. Here P_1 is the movable part and N_1 the negative part which is an effective divisor. Then $E_1 \equiv P_1 + (N_1 + E_2)$. Since RHS $-(N_1 + E_2)$ is movable, the negative part of LHS which is E_1, satisfies $E_1 \leq N_1 + E_2$ (cf. [17, ChIII, Proposition 1.14]). Thus $(N_1 + E_2 - E_1)$ and also P_1 are pseudo-effective divisors, but their sum is numerically equivalent to zero. Take general members H_i ($1 \leq i \leq n - 1$) in a linear system $|H|$ with H a very ample divisor on $\widetilde{X}$. Then

$$0 = H^{n-1}.(P_1 + N_1 + E_2 - E_1) = H^{n-1}.P_1 + H^{n-1}.(N_1 + E_2 - E_1).$$

Hence $H^{n-1}.P_1 = 0 = H^{n-1}.(N_1 + E_2 - E_1)$. Thus $0 = (N_1 + E_2 - E_1) \cap (H_1 \cap \cdots \cap H_{n-1})$. Since $N_1 + E_2 - E_1$ is an effective divisor and the restriction to a subvariety of an ample divisor is still an ample divisor, we get $N_1 + E_2 - E_1 = 0$. Thus $N_1 + E_2 = E_1$. Since E_i

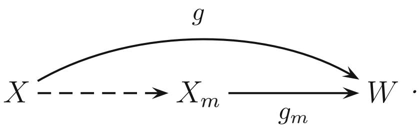
have no common components, either $E_2 = 0$, or $E_1 = 0$ (and hence $N_1 = E_2 = 0$). So $E_2 = 0$ and hence $K_{\tilde{X}} = \gamma^* K_X + E_1$ with $E_1 \geq 0$. Therefore, X has at worst canonical singularities by definition. This together with $K_X \equiv 0$ imply that $K_X \sim_{\mathbb{Q}} 0$ by [9, Theorem 8.2]. □

Lemma 2.5 *Let W be an algebraic Lang hyperbolic projective variety, V a projective variety with at worst klt singularities, and $h : V \to W$ a morphism such that $V \to h(V)$ is generically finite. Assume that K_V is relatively nef over W. Then K_V is nef.*

Proof Suppose the contrary that K_V is not nef and hence there is a K_V-negative extremal rational curve C by the cone theorem [12, Theorem 3.7]. Since W is hyperbolic and hence contains no rational curve, C must be contracted by $V \to W$. So $K_V.C < 0$ for a curve $C \subset V$ contracted by $V \to W$. This contradicts the relative nefness of K_V over W. Hence K_V is nef. This proves the lemma. □

Lemma 2.6 *Let W be an algebraic Lang hyperbolic projective variety, X a projective variety and $g : X \to W$ a morphism such that $X \to g(X)$ is generically finite.*

Then there is a birational map $X \dashrightarrow X_m$ over W, i.e., there is a (generically finite) morphism $g_m : X_m \to W$ such that the natural composition $X \dashrightarrow X_m \stackrel{g_m}{\to} W$ coincides with $g : X \to W$ (and hence $g_m(X_m) = g(X)$):



Further, X_m has at worst canonical singularities; the canonical divisor K_{X_m} is nef; and K_{X_m} is also relatively ample over W.

Proof Since $g(X)$ is also hyperbolic, replacing W by $g(X)$, we may assume that g is surjective (and generically finite). Take a projective resolution $X' \to X$. Since the relative dimension of X' over W is zero, the canonical divisor $K_{X'}$ (and indeed every divisor on X') is relative big over W. The main Theorem 1.2 in [2] says that X' has a log canonical model X_m over W, so X_m has at worst canonical singularities and K_{X_m} is relative ample over W. This X_m is obtained from a log terminal model of X' over W followed by a birational morphism over W using the relative-base point freeness result for relative nef and big divisors; see [2, Theorem 1.2, Definition 3.6.7, Theorem 3.9.1]. We note that [2] considers log pairs, while ours is the pure case; so the smoothness of X' implies that the log terminal (resp. log canonical) model of X' over W has at worst terminal (resp. canonical) singularities. By Lemma 2.5, K_{X_m} is nef. This proves the lemma. □

Remark 2.7 (1) By the proof, every subvariety S of X (with $g_{|S}$ generically finite) or of hyperbolic W has a minimal model S_m with only canonical singularities.

(2) Assume $n(K_{X_m}) = \dim X_m \geq 1$ and Conjecture 1.3 (2) holds. Then the Kodaira dimension $\kappa(X_m) > 0$. By [10, Theorem 7.3], K_{X_m} is "good" (or abundant). So it is semi-ample by [10, Theorem 1.1], which has a new proof by Fujino.

(3) Suppose that Y is a normal projective variety birational to the X in Lemma 2.6 and K_Y is $\mathbb{Q}$-Cartier. Then K_Y is pseudo-effective. Indeed, let $\sigma : Y' \to Y$ be a resolution. Since $g : X \to W$ is generically finite and W is hyperbolic, X and hence Y and Y' are non-uniruled. By [3, Theorem 2.6], $K_{Y'}$ is pseudo-effective. Hence $K_Y = \sigma_* K_{Y'}$ is pseudo-effective.

3 Proof of Theorems

In this section, we prove results in Introduction, and Theorems 3.1 and 3.2 which imply Theorem 1.5 and Proposition 1.7, respectively. We also prove Corollaries 3.7 and 3.8, all in dimension ≤ 3, where we do not assume Conjecture 1.1 or 1.3.

When $\dim X \le 3$, Case (3) below does not occur.

Theorem 3.1 *Let W be an algebraic Lang hyperbolic projective variety, X a projective variety of dimension n and $g : X \to W$ a morphism such that $X \to g(X)$ is generically finite. Assume either $n \le 3$ or Conjecture 1.3* (2) *(resp. either $n \le 3$ or Conjecture 1.3 (2) without the extra Hyp(A)) holds for all varieties birational to X.*

Then there is a birational map $X \dashrightarrow X_m$ over W, i.e., there is a morphism $g_m : X_m \to W$ such that the composition $X \dashrightarrow X_m \stackrel{g_m}{\to} W$ coincides with $g : X \to W$ (and hence $g_m(X_m) = g(X)$):

$$X \dashrightarrow X_m \xrightarrow{g_m} W, \qquad g : X \to W \; .$$

Further, X_m is a minimal variety with at worst canonical singularities; K_{X_m} is relatively ample over W; and one of the following is true.

(1) *K_{X_m} is ample. Hence both X_m and X are of general type.*
(2) *X_m is an absolutely minimal Calabi–Yau variety of dimension $n \ge 3$, and $g_m : X_m \to g_m(X_m) = g(X) \subseteq W$ is a finite morphism.*
(3) *There is an almost holomorphic map $\tau : X_m \dashrightarrow Y$ (resp. a holomorphic map $\tau : X_m \to Y$) such that its general fibre F is an absolutely minimal Calabi–Yau variety with $3 \le \dim F < \dim X_m$, and $(g_m)_{|F} : F \to g_m(F) \subset g_m(X_m) = g(X) \subseteq W$ is a finite morphism. The Kodaira dimension $\kappa(X) \le \dim Y \le n - 3$.*

In Theorem 3.2 below, Conjecture 1.1 or 1.3 is not assumed. $\mathrm{a}(W)$ denotes (the Zariski-closure of) the image $\mathrm{Im}(\mathrm{alb}_W : W \dashrightarrow \mathrm{Alb}(W))$ of the albanese map. Since $\mathrm{Alb}(W)$ is generated by $\mathrm{a}(W)$, and $\dim \mathrm{Alb}(W) = q(\widetilde{W}) = \frac{1}{2} b_1(\widetilde{W})$ for any projective resolution $\widetilde{W}$ of W, the condition (iii) in Theorem 3.2 is satisfied if $n = 4$ and $q(\widetilde{W}) > 0$.

For related work, the authors of [7] also considered albanese map for smooth W and used classical results of Ueno, [8, Theorem 1], etc., while we use [5,9,10].

Theorem 3.2 *Let X be an algebraic Lang hyperbolic projective variety of dimension n. Assume one of the following conditions holds.*

(i) *X has maximal albanese dimension, i.e.,* $\dim \mathrm{a}(X) = \dim X$.
(ii) *The Kodaira dimension $\kappa(X) \ge n - 3$.*
(iii) $\dim \mathrm{a}(X) \ge n - 3$ *and* $\kappa(\mathrm{a}(X)) \ge n - 4$.

Then one of the following is true.

(1) *There is a birational surjective morphism $g_m : X_m \to X$ such that X_m has at worst canonical singularities, K_{X_m} is ample and hence both X_m and X are of general type.*
(2) *$\kappa(X) \in \{n - 3, n - 4\}$, and X is covered by subvarieties whose normalizations are absolutely minimal Calabi–Yau varieties of dimension three.*

We prove Theorem 1.4. Let $f'' : X'' \to X$ be a dlt blowup with $E_{f''}$ the reduced f''-exceptional divisor (cf. [4, Theorem 10.4]). Namely, X'' is $\mathbb{Q}$-factorial, $(X'', E_{f''})$ is dlt (and hence X'' is klt) and

$$(f'')^* K_X = K_{X''} + E'' \tag{1}$$

where E'' is f''-exceptional and satisfies $E'' \geq E_{f''}$.

Since f'' is birational, $K_{X''}$ is relative big over X. By [2, Theorem 1.2, Definition 3.6.7], there is a birational map $\sigma : X'' \dashrightarrow X'$ over X, such that σ^{-1} does not contract any divisor, X', like X'', has only $\mathbb{Q}$-factorial klt singularities and $K_{X'}$ is relatively nef over X via a birational morphism $f' : X' \to X$. Pushing forward the equality (1) above by σ_*, we get

$$(f')^* K_X = K_{X'} + E' \tag{2}$$

where $E' := \sigma_* E'' \geq \sigma_* E_{f''} = E_{f'}$ and $E_{f'}$ is the reduced f'-exceptional divisor. Since $K_{X'}$ is relatively f'-nef over X, our $K_{X'}$ is nef by Lemma 2.5:

Claim 3.3 *$K_{X'}$ is nef.*

We continue the proof of Theorem 1.4. Let $\tau : X' \dashrightarrow Y$ be a nef reduction of the nef divisor $K_{X'}$, and $n(K_{X'}) := \dim Y$ the nef dimension of $K_{X'}$; let F be a general (compact) fibre of τ; then $K_F = (K_{X'})_{|F}$ is numerically trivial (cf. Theorem 1.2).

Lemma 3.4 *Assume the hypotheses of Theorem* 1.4. *For the X' and $\tau : X' \dashrightarrow Y$ defined above, it is impossible that* $\dim Y = 0$.

Proof Consider the case $\dim Y = 0$. Then $K_{X'} \equiv 0$ (numerically zero). Since X is hyperbolic, X and hence X' are non-uniruled. By Lemma 2.4, X' has at worst canonical singularities, and $K_{X'} \sim_{\mathbb{Q}} 0$; the same hold for X, noting that $K_X = f'_* K_{X'} \sim_{\mathbb{Q}} 0$ (cf. [11, Corollary 1.5]).

We claim that X is a Calabi–Yau variety. We only need to show that the irregularity $q(X) = 0$. Suppose the contrary that $q(X) > 0$. Then, by Lemma 2.3, X is covered by images of abelian varieties of dimension equal to $q(X)$. This contradicts the hyperbolicity of X. Therefore, $q(X) = 0$. Hence X is a Calabi–Yau variety. This contradicts the hyperbolicity of X, Remark 1.8 and the assumed Conjecture 1.1. This proves Lemma 3.4. □

Lemma 3.5 *Assume the hypotheses of Theorem* 1.4. *For the X' and $\tau : X' \dashrightarrow Y$ defined preceding Lemma* 3.4, *it is impossible that* $1 \leq \dim Y < \dim X'$.

Proof Consider the case $1 \leq \dim Y < \dim X'$. A general fibre F of $\tau : X' \dashrightarrow Y$ satisfies $1 \leq \dim F = \dim X - \dim Y < \dim X$. Also $K_F \equiv 0$. Since X and hence the general fibre F of $\tau : X' \dashrightarrow Y$ are not covered by rational curves by the hyperbolicity of X, F is not uniruled. By Lemma 2.4, F has at worst canonical singularities and $K_F \sim_{\mathbb{Q}} 0$.

Factor the birational map $X' \supset F \to f'(F) \subset X$ as $F \to F^n \to f'(F)$, where $F \to F^n$ is a birational morphism and $F^n \to f'(F)$ is the normalization. By [11, Corollary 1.5], F^n has only canonical singularities and $K_{F^n} \sim_{\mathbb{Q}} 0$.

If $q(F^n) > 0$, by Lemma 2.3, F^n and hence $f'(F)$ and X are covered by images of abelian varieties of dimension equal to $q(F^n)$, contradicting the hyperbolicity of X. Thus $q(F^n) = 0$, so F^n is a Calabi–Yau variety. By the assumed Conjecture 1.1 and Remark 1.8, there is a non-constant holomorphic map $V \to F^n$ from an abelian variety V, which, combined with the (birational and) finite map $F^n \to f'(F)$, produces a non-constant holomorphic map $V \to X$, contradicting the hyperbolicity of X. This proves Lemma 3.5. □

By the two lemmas above, we are left with the case $\dim Y = \dim X'$. Namely, the nef dimension $n(K_{X'}) = \dim X'$. By the assumed abundance Conjecture 1.3 (1), $K_{X'}$ is semi-ample. Hence $\Phi_{|sK_{X'}|}$, for some $s > 0$, is a morphism onto a normal variety X_c, with connected fibres, and there is an ample $\mathbb{Q}$-divisor H_c on X_c such that $K_{X'} \sim_{\mathbb{Q}} \Phi^*_{|sK_{X'}|} H_c$. Clearly, this map which is now holomorphic, is (up to birational equivalence) a nef reduction of $K_{X'}$ and also denoted as $\tau : X' \to X_c$. In other words, $Y = X_c$, $K_{X'}$ is big (and nef), and τ is birational. Pushing forward the equality $K_{X'} \sim_{\mathbb{Q}} \tau^* H_c$ by τ_*, we get $K_{X_c} \sim_{\mathbb{Q}} H_c$ and hence $K_{X'} \sim_{\mathbb{Q}} \tau^* K_{X_c}$ (so that τ is a crepant birational morphism) with K_{X_c} an ample $\mathbb{Q}$-divisor. Since X' is klt and τ is crepant, X is also klt. By [6, Corollary 1.5], every fibre of $\tau : X' \to X_c$ is rationally chain connected and hence is contracted to a point by the birational morphism $f' : X' \to X$ due to the hyperbolicity of X and the absence of rational curves on X. Thus f' factors through τ, i.e., there is a birational morphism $f_c : X_c \to X$ such that $f' = f_c \circ \tau$ (cf. [8, Proof of Lemma 14]). In summary, we have the following commutative diagram:

$$
\begin{array}{ccccc}
X'' & \overset{\sigma}{\dashrightarrow} & X' & \xrightarrow{\phi_{|sK_{X'}|}} & X_c \\
{\scriptstyle f''}\downarrow & & {\scriptstyle f'}\downarrow & & {\scriptstyle f_c}\downarrow \\
X & = & X & = & X
\end{array}
$$

Pushing forward the equality (2) above by τ_*, we get

$$f_c^* K_X = K_{X_c} + E_c.$$

Here $E_c := \tau_* E' \geq \tau_* E_{f'} = E_{f_c}$ and E_{f_c} is the reduced f_c-exceptional divisor. So the image $f_c(E_c)$ is contained in $\mathrm{Nklt}(X)$, the non-klt locus of X, which is a Zariski-closed subset of X consisting of exactly the non-klt points of X.

This proves Theorem 1.4; *see Remark* 1.8 *(1) for the final part.*

If we do not assume Conjecture 1.1 in Theorem 1.4, the argument above actually shows:

Remark 3.6 Let X be a $\mathbb{Q}$-Gorenstein normal projective variety of dimension n which is algebraic Lang hyperbolic. Assume either $n \leq 3$ or Conjecture 1.3 (1) holds for all varieties of dimension $\leq n$. Then either K_X is ample at smooth points and klt points of X as detailed in Theorem 1.4; or X is a Calabi–Yau variety of dimension $n \geq 3$; or X is covered by subvarieties $\{F'_t\}$ whose normalizations are Calabi–Yau varieties of the same dimension k with $3 \leq k < n$.

Since Conjecture 1.1 is true for surfaces, and the abundance Conjecture 1.3 is known in dimension ≤ 3, we can and will soon prove the following consequences of Theorem 1.4.

Corollary 3.7 *Let X be a $\mathbb{Q}$-Gorenstein normal projective surface which is algebraic Lang hyperbolic. Then X is of general type and the canonical divisor K_X is ample.*

We can not remove the second alternative below even when X is smooth, because, for instance, we do not know, at the moment, the non-hyperbolicity of a general smooth Calabi–Yau n-fold and a Hyperkähler n-fold when $n > 2$.

Corollary 3.8 *Let X be a $\mathbb{Q}$-Gorenstein normal projective threefold which is algebraic Lang hyperbolic. Then either the canonical divisor K_X is ample at the smooth points and klt points of X; or X is a Calabi–Yau variety.*

3.9. **Proof of Corollaries 3.7 and 3.8**

We use the fact that Conjecture 1.1 holds in dimension ≤ 2, and both Conjectures 1.3 (1) and (2) (even without the extra Hyp(A)) hold in dimension ≤ 3 (cf. Proposition 2.2).

Corollary 3.7 is a consequence of Theorem 1.4 and the observation: if $f_c : X_c \to X$ is a birational morphism between $\mathbb{Q}$-Gorenstein normal projective surfaces and K_{X_c} is ample, then $K_X = (f_c)_* K_{X_c}$ is also ample, by using Nakai–Moishezon ampleness criterion and the projection formula.

For Corollary 3.8, we follow the argument for the proof of Theorem 1.4. Thus we have to consider Cases (I) $\dim Y = 0$, (II) $0 < \dim Y < \dim X$ and (III) $\dim Y = \dim X$. In Case (I), X has been proven to be a Calabi–Yau variety of dimension three; for this purpose, Conjecture 1.1 was not used. In Case (II), a contradiction has been deduced utilizing Conjecture 1.1 in dimension ≤ 2. In Case (III), using the proven abundance conjecture in dimension 3, we get the first possibility in the conclusion part of Corollary 3.8. This proves Corollary 3.8.

3.10. **Proof of Theorems 1.5 and 3.1**

If we let $g : X \to W$ in Theorem 3.1 be the identity map $\mathrm{id}_X : X \to X$, we get Theorem 1.5; we also use the observation that a birational finite morphism from a normal variety like X_m or F in Theorem 3.1 is just the normalization map. Thus we have only to (and are going to) prove Theorem 3.1.

We may assume Conjecture 1.3 (2) (the case without the extra Hyp(A) is similar and indeed easier); for varieties of dimension ≤ 3, this assumption is automatically satisfied by Proposition 2.2.

Theorem 3.1 is clearly true when $\dim X = 1$. So we may assume that $n = \dim X \geq 2$. We apply Lemma 2.6 and let the birational map $X \dashrightarrow X_m$ over W and $g_m : X_m \to W$ be as there, where X_m has only canonical singularities and K_{X_m} is relatively ample over W and is also nef. Since $g : X \to W$ is generically finite, so is $g_m : X_m \to W$.

Let $\tau : X_m \dashrightarrow Y$ be a nef reduction of the nef divisor K_{X_m}. For our $\tau : X_m \dashrightarrow Y$ and $g_m : X_m \to W$ here (with W algebraic Lang hyperbolic) we closely follow the argument of Theorem 1.4 for $\tau : X' \dashrightarrow Y$ and $f' : X' \to X$ there (with X algebraic Lang hyperbolic), but we do not assume Conjecture 1.1 here.

Suppose that $\dim Y = 0$, i.e., $n(K_{X_m}) = 0$, or $K_{X_m} \equiv 0$. Now Lemma 3.4 is applicable under the current weaker assumption. Precisely, due to the lack of the assumption of Conjecture 1.1 here, instead of the contradiction there, we have that X_m is a Calabi–Yau variety. The relative ampleness of K_{X_m} over W implies that $g_m : X_m \to g_m(X_m) = g(X) \subseteq W$ is a finite morphism. This and the hyperbolicity of W imply that X_m is hyperbolic. Further, X_m is absolutely minimal (cf. Remark 1.8). By Proposition 2.2, $\dim X_m \geq 3$. So Case 3.1 (2) occurs.

Suppose that $1 \leq n(K_{X_m}) < n$. Denote by F a general fibre of $\tau : X_m \dashrightarrow Y$. For this case, Lemma 3.5 is applicable even under the current weaker assumption. So F is a Calabi–Yau variety. The map $(g_m)_{|F} : F \to g_m(F) \subset g_m(X_m) = g(X) \subseteq W$ is a finite morphism, otherwise, a curve C in F is g_m-exceptional and hence $K_{X_m}.C > 0$, by the relative ampleness of K_{X_m} over W, contradicting the numerical triviality of $(K_{X_m})_{|F}$ entailing $K_{X_m}.C = 0$. This and the hyperbolicity of W imply that F is hyperbolic. Further, F is absolutely minimal (cf. Remark 1.8). By Proposition 2.2, $\dim F \geq 3$. So Case 3.1 (3) occurs. Indeed, for the final part, the well known Iitaka addition for Kodaira dimension implies that $\kappa(X) \leq \kappa(F) + \dim Y = \dim Y = n - \dim F \leq n - 3$.

Suppose that $n(K_{X_m}) = n$. Then K_{X_m} is semi-ample by Conjecture 1.3 (2) (cf. Remark 2.7). Hence K_{X_m} is nef and big, and there is a birational morphism $\gamma : X_m \to Z$ onto a normal variety Z such that $K_{X_m} \sim_{\mathbb{Q}} \gamma^* H$ for an ample $\mathbb{Q}$-divisor H on Z. Hence both X_m and X are of general type. Since γ is birational, the projection formula implies that $K_Z = \gamma_* K_{X_m} \sim_{\mathbb{Q}} H$. Hence K_Z is ample and $K_{X_m} \sim_{\mathbb{Q}} \gamma^* K_Z$. Thus Z, like X_m, has only canonical singularities. If γ is not an isomorphism, then it has a positive-dimensional fibre. By [6, Corollary 1.5], every fibre of γ is rationally chain connected. So we may assume that γ contracts a rational curve C on X_m to a point on Z. Since W is hyperbolic, $g_m : X_m \to X \subseteq W$ contracts the rational curve C on X_m to a point on W. This and $C.K_{X_m} = C.\gamma^* K_Z = \gamma_* C.K_Z = 0$ contradict the relative ampleness of K_{X_m} over W. So $\gamma : X_m \to Z$ is an isomorphism. Hence K_{X_m} is ample. Thus Case 3.1 (1) occurs. This proves Theorem 3.1.

3.11. **Proof of Corollary 1.6**

For the assertion (1), we apply Theorem 3.1 to the inclusion map $g : X \hookrightarrow W$ for a projective subvariety X of W. By Theorem 3.1, either Case 3.1 (1) occurs and hence X is of general type, or Case 3.1 (2) or (3) occurs. We use the the notation there: birational morphism $g_m : X_m \to X$, etc. If Case 3.1 (2) (resp. (3)) occurs, by the assumed Conjecture 1.1 and Remark 1.8, there is a non-constant holomorphic map from an abelian variety V to X_m (resp. to F), thus, combined with the (birational and) finite morphism

$$g_m : X_m \to g_m(X_m) = g(X) = X \subseteq W$$

(resp. $(g_m)_{|F} : F \to g_m(F) \subset g_m(X_m) = g(X) = X \subseteq W$), this map gives a non-constant holomorphic map from V to W, contradicting the hyperbolicity of W. This proves the assertion (1).

For the assertion (2), the case $\dim W = 1$ is clear. We may assume that $\dim W \geq 2$. We apply Theorem 3.1 and let $g : X \to W$ be the identity map $\mathrm{id}_X : X \to X$, with $W = X$. Hence there is a birational morphism $g_m : X_m \to X$ such that 3.1 (1), (2) or (3) occurs. We use the following known fact (cf. [9, Lemma 8.1], or Lemma 2.3):

Fact 3.12 *If a projective variety V has only canonical (or more generally rational) singularities, then the albanese map* $\mathrm{alb}_V : V \to \mathrm{Alb}(V)$ *is a well defined morphism and* $\dim \mathrm{Alb}(V) = q(V) = h^1(V, \mathcal{O}_V)$.

In Case 3.1 (1), X is of general type.

In Case 3.1 (2), the irregularity $q(X_m) = 0$, and hence $\mathrm{Alb}(X) = \mathrm{Alb}(X_m)$ is a point. Thus X itself is the fibre of alb_X. Hence $\dim X \leq 2$, by the assumption. Thus $\dim X = 2$, by the extra assumption that $X = W$ has dimension at least two. By Proposition 2.2, either X has infinitely many rational or elliptic curves, or it is birational to an abelian surface (and hence $q(X_m) = 2$), or it is of general type. Since $X = W$ is hyperbolic and $q(X_m) = 0$, $X = W$ is of general type.

In Case 3.1 (3), by the above fact, $\mathrm{alb}_{X_m} : X_m \to \mathrm{Alb}(X_m) = \mathrm{Alb}(X)$ is a well defined morphism. Since the general fibre F of the nef reduction map $\tau : X_m \dashrightarrow Y$ which is almost holomorphic, is a Calabi–Yau variety, we have $q(F) = 0$, so $\mathrm{Alb}(F)$ is a point. By the universality of the albanese map, the composition $F \hookrightarrow X_m \to \mathrm{Alb}(X_m)$ factors through $F \to \mathrm{Alb}(F)$. So $\mathrm{alb}_{X_m} : X_m \to \mathrm{Alb}(X_m)$ maps F to a point. Hence $g_m(F) \subset X$ is contained in a general fibre G of $\mathrm{alb}_X : X \overset{g_m^{-1}}{\dashrightarrow} X_m \to \mathrm{Alb}(X_m) = \mathrm{Alb}(X)$. Indeed, alb_X factors as

$$X \overset{g_m^{-1}}{\dashrightarrow} X_m \overset{\tau}{\dashrightarrow} Y \overset{\eta}{\dashrightarrow} \mathrm{Alb}(X_m) = \mathrm{Alb}(X)$$

for some rational map η, by applying [8, Lemma 14] to the domain of the almost holomorphic map τ. Now $\dim G \geq \dim g_m(F) = \dim F \geq 3$, with the last inequality shown in Theorem 3.1. This contradicts the assumption. This proves Corollary 1.6.

3.13. **Proof of Proposition 1.7 and Theorem 3.2**

We prove Theorem 3.2. We first assume that X has maximal albanese dimension. Since abundance conjecture holds for varieties with maximal albanese dimension (cf. Proposition 2.2), we can apply Theorem 1.5 to our X. So Case 1.5 (1), (2) or (3) occurs. We use the notation $g_m : X_m \to X$ there, where K_{X_m} is relatively ample over X, and is also nef. By [5, Theorem 3.6], K_{X_m} is semi-ample. So there exist a holomorphic map $\tau := \Phi_{|sK_{X_m}|} : X_m \to Y$ onto a normal variety Y, for some $s > 0$, and an ample divisor H on Y such that $K_{X_m} \sim_{\mathbb{Q}} \tau^* H$. We can take this τ as a nef reduction of K_{X_m} in Theorem 1.5. Let F be a general fibre of τ.

If Case 1.5 (1) occurs, we are in Case 3.2 (1).

Consider Case 1.5 (3). Then $(g_m)_{|F} : F \to g_m(F)$ is the normalization map, and $F \subseteq X_m$ is a Calabi–Yau variety of dimension ≥ 3. Since X and hence X_m have maximal albanese dimension, i.e., $\dim \mathrm{a}(X_m) = \dim X_m$, we may assume that the restriction $(\mathrm{alb}_{X_m})_{|F}$ to a general fibre F of τ, is a generically finite morphism onto the image $F_a \subseteq \mathrm{Alb}(X)$. It is known that F_a, being a subvariety of an abelian variety, satisfies $\kappa(F_a) \geq 0$ with equality holding only when F_a is the translation of a subtorus. Now $0 = \kappa(F) \geq \kappa(F_a) \geq 0$. Thus $\kappa(F_a) = 0$ and hence F_a is an abelian variety. Therefore, $0 = q(F) \geq q(F_a) = \dim F_a = \dim F \geq 3$. This is a contradiction.

If Case 1.5 (2) occurs, we get a similar contradiction by the arguments above, with $F = X_m$.

Next we assume that $\kappa(X) \geq n - 3$. Let $I_X : X \dashrightarrow Y$ be the Iitaka fibration with F a very general fibre. So $\kappa(F) = 0$ and $\dim F = \dim X - \kappa(X) \leq 3$ by the assumption. Since F is a subvariety of the hyperbolic variety X, it is also hyperbolic.

If $\dim F = 0$, then X is of general type. Since abundance conjecture holds for varieties of general type (cf. Proposition 2.2), we can apply Theorem 1.5 to our X, and only Case 1.5 (1) there occurs. This fits Case 3.2 (1).

Consider the case $\dim F \in \{1, 2, 3\}$. Applying Theorem 1.5 to the hyperbolic F and noting that $\kappa(F) = 0$ and $1 \leq \dim F \leq 3$, only Case 1.5 (2) there occurs for F: the normalization of F is an absolutely minimal Calabi–Yau variety of dimension three, so $\kappa(X) = n - \dim F = n - 3$; also these F cover X. This fits Case 3.2 (2).

Finally, we assume that $\dim \mathrm{a}(X) \geq n - 3$ and $\kappa(\mathrm{a}(X)) \geq n - 4$. By the results obtained so far, we may add the extra assumptions: $n > \dim \mathrm{a}(X)$ and $\kappa(X) \leq n - 4$. Let G be an irreducible component of a general fibre of the albanese map $\mathrm{alb}_X : X \dashrightarrow \mathrm{a}(X) \subseteq \mathrm{Alb}(X)$. Then $1 \leq g := \dim G = \dim X - \dim \mathrm{a}(X) \leq 3$. The hyperbolicity of X implies that X and hence G are not uniruled. So G has a good minimal model in the sense of Kawamata [10], by the abundance theorem in dimension ≤ 3 (cf. [12, §3.13] or Proposition 2.2). In particular, $\kappa(G) \geq 0$. By Iitaka's $C_{n,n-g}$ proved in [9, Corollary 1.2], $\kappa(X) \geq \kappa(G) + \kappa(\mathrm{a}(X)) \geq 0 + 0$. By the assumptions,

$$n - 4 \geq \kappa(X) \geq \kappa(G) + \kappa(\mathrm{a}(X)) \geq \kappa(G) + n - 4 \geq n - 4.$$

Thus the inequalities above all become equalities. In particular, $\kappa(G) = 0$ and $\kappa(X) = n - 4$. Since $\kappa(X) \geq 0$, we have $n \geq 4$. As in the previous paragraph, applying Theorem 1.5 to G, the normalization of G is an absolutely minimal Calabi–Yau variety of dimension three; also these G cover X. This fits Case 3.2 (2).

This proves Theorem 3.2.

Now we prove Proposition 1.7. Set $n := \dim X$. Each of the three conditions in Proposition 1.7 implies one of the first two conditions in Theorem 3.2. Hence we can apply Theorem

3.2. We may assume that Case 3.2 (2) occurs. Thus X is covered by subvarieties F_t whose normalizations F_t^n are absolutely minimal Calabi–Yau varieties of dimension three. The algebraic Lang hyperbolicity of X implies the same for F_t and also F_t^n. This contradicts the assumed Conjecture 1.1 in dimension three (cf. Remark 1.8).

This proves Proposition 1.7.

Acknowledgments We would like to thank the referee for very careful reading, many suggestions to improve the paper and the insistence on clarity of exposition, and Kenji Matsuki for bringing [15] to our attention where Theorem 1.4 was proved in dimension 3. The last named-author is partially supported by an ARF of NUS.

References

1. Bauer, T., Campana, F., Eckl, T., Kebekus, S., Peternell, T., Rams, S., Szemberg, T., Wotzlaw, L.: A Reduction Map for Nef Line Bundles, Complex Geometry (Göttingen, 2000). Springer, Berlin, pp. 27–36 (2002)
2. Birkar, C., Cascini, P., Hacon, C.D., McKernan, J.: Existence of minimal models for varieties of log general type. J. Am. Math. Soc. **23**, 405–468 (2010)
3. Boucksom, S., Demailly, J.-P., Paun, M., Peternell, T.: The pseudo-effective cone of a compact Kähler manifold and varieties of negative Kodaira dimension. J. Algebr. Geom. **22**(2), 201–248 (2013)
4. Fujino, O.: Fundamental theorems for the log minimal model program. Publ. Res. Inst. Math. Sci. **47**(3), 727–789 (2011)
5. Fujino, O.: On maximal Albanese dimensional varieties. Proc. Jpn Acad. Ser. A Math. Sci. **89**(8), 92–95 (2013)
6. Hacon, C.D., McKernan, J.: On Shokurov's rational connectedness conjecture. Duke Math. J. **138**(1), 119–136 (2007)
7. Heier, G., Steven, SYLu: On the canonical line bundle and negative holomorphic sectional curvature. Math. Res. Lett. **17**(6), 1101–1110 (2010)
8. Kawamata, Y.: Characterization of abelian varieties. Compos. Math. **43**(2), 253–276 (1981)
9. Kawamata, Y.: Minimal models and the Kodaira dimension of algebraic fiber spaces. J. Reine Angew. Math. **363**, 1–46 (1985)
10. Kawamata, Y.: Pluricanonical systems on minimal algebraic varieties. Invent. Math. **79**(3), 567–588 (1985)
11. Kawamata, Y.: Crepant blowing-ups of 3-dimensional canonical singularities and its application to degeneration of surfaces. Ann. Math. **127**, 93–163 (1988)
12. Kollár, J., Mori, S.: Birational Geometry of Algebraic Varieties, Cambridge Tracts in Math., vol. 134. Cambridge University Press, Cambridge (1998)
13. Lang, S.: Hyperbolic and Diophantine analysis. Bull. Am. Math. Soc. (N.S.) **14**, 159–205 (1986)
14. Lu, S.S.Y., Zhang, D.-Q.: Positivity criteria for log canonical divisors and hyperbolicity. arXiv:1207.7346
15. Matsuki, K.: An approach to the abundance conjecture for 3-folds. Duke Math. J. **61**(1), 207–220 (1990)
16. Mori, S., Mukai, S.: The uniruledness of the moduli space of curves of genus 11, Algebraic Geometry (Tokyo/Kyoto, 1982). Lecture Notes in Math., vol. 1016, Springer, Berlin, pp. 334–353 (1983)
17. Nakayama, N.: Zariski-decomposition and abundance. MSJ Memoirs, vol. 14. Mathematical Society of Japan, Tokyo (2004)
18. Peternell, T.: Calabi–Yau manifolds and a conjecture of Kobayashi. Math. Z. **207**(2), 305–318 (1991)