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# t-Structures and cotilting modules over commutative noetherian rings

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**Abstract** For a commutative noetherian ring $R$, we establish a bijection between the resolving subcategories consisting of finitely generated $R$-modules of finite projective dimension and the compactly generated t-structures in the unbounded derived category $\mathcal{D}(R)$ that contain $R[1]$ in their heart. Under this bijection, the t-structures $(\mathcal{U}, \mathcal{V})$ such that the aisle $\mathcal{U}$ consists of objects with homology concentrated in degrees $< n$ correspond to the $n$-cotilting classes in Mod-$R$. As a consequence of these results, we prove that the little finitistic dimension findim$R$ of $R$ equals an integer $n$ if and only if the direct sum $\bigoplus_{k=0}^{n} E_k(R)$ of the first $n+1$ terms in a minimal injective coresolution $0 \to R \to E_0(R) \to E_1(R) \to \cdots$ of $R$ is an injective cogenerator of Mod-$R$.

**Keywords** t-Structure · Tilting module · Cotilting module · Resolving subcategory · Finitistic dimension · Gorenstein-injective · Gorenstein-flat

**Mathematics Subject Classification (2010)** 13D09 · 13D05 · 16E30

## 1 Introduction

Aim of this note is to give a unified approach to several recent classification results over a commutative noetherian ring $R$: the classification of compactly generated t-structures in the unbounded derived category $\mathcal{D}(R)$ given in [2], the classification of tilting and cotilting classes in the module category Mod-$R$ from [7], and the classification of resolving subcategories of the category $\mathcal{P}^{<\infty}$ of finitely generated $R$-modules of finite projective dimension achieved in [14].

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Our main result (Theorem 3.3) establishes a bijection between the resolving subcategories of $\mathcal{P}^{<\infty}$ and the compactly generated t-structures in $\mathcal{D}(R)$ that contain $R[1]$ in their heart. Under this bijection, the resolving subcategories consisting of modules of projective dimension bounded by an integer $n$ correspond to t-structures $(\mathcal{U}, \mathcal{V})$ such that the heart $\mathcal{U} \cap \mathcal{V}$ contains $R[1]$ and the aisle $\mathcal{U}$ consists of objects with homology concentrated in degrees $< n$. We thus recover the classification of tilting and cotilting classes in Mod-$R$ from [7], see Corollary 3.5. The main tool for obtaining these correspondences is a parametrization by descending sequences of subsets of Spec$(R)$ closed under specialization, called filtrations by supports in [2]. Such parametrization is also used in [28] to classify the compactly generated co-t-structures in $\mathcal{D}(R)$.

We further show that every compactly generated t-structure in $\mathcal{D}(R)$ that contains $R[1]$ in the heart is cogenerated by a module (Proposition 3.6). Then the associated filtration by supports is also determined by this module (Proposition 3.7). This allows to prove that the little finitistic dimension findim $R$ of $R$ equals an integer $n$ if and only if the first $n+1$ terms in a minimal injective coresolution $0 \to R \to E_0(R) \to E_1(R) \to \cdots$ of $R$ yield an injective cogenerator $I = \bigoplus_{k=0}^{n} E_k(R)$ of Mod-$R$ (Theorem 4.1). Furthermore, it leads to a homological characterization of Gorenstein-injective and Gorenstein-flat modules over Gorenstein rings (Corollary 4.3).

The paper is organized as follows. In Sect. 2 we collect some preliminaries on subsets of Spec$(R)$, t-structures and (co)tilting modules. Section 3 is devoted to our classification results. Section 4 contains the applications to finitistic dimension and Gorenstein-injective and Gorenstein-flat modules.

## 2 Preliminaries

### 2.1 Notation

Throughout this note, $R$ will be a commutative noetherian ring, and Spec$(R)$ will denote the spectrum of $R$ with the Zariski topology, where the closed sets are those of the form $V(I) = \{\mathbf{p} \in \text{Spec}(R) \mid \mathbf{p} \supseteq I\}$ for some subset $I \subseteq R$. For $\mathbf{p} \in \text{Spec}(R)$, we denote by $R_{\mathbf{p}}$ the localization of $R$ at $\mathbf{p}$, and by $k(\mathbf{p}) = R_{\mathbf{p}}/\mathbf{p}_{\mathbf{p}}$ the residue field.

Given a class $\mathcal{S}$ of right modules, we denote:

$$\mathcal{S}^{\perp} = \{M \in \text{Mod}-R \mid \text{Ext}^i_R(S, M) = 0 \text{ for all} S \in \mathcal{S} \text{ and } i \geq 1\},$$
$${}^{\perp}\mathcal{S} = \{M \in \text{Mod}-R \mid \text{Ext}^i_R(M, S) = 0 \text{ for all } S \in \mathcal{S} \text{ and } i \geq 1\}.$$

If $\mathcal{S} = \{S\}$ is a singleton, we shorten the notation to $S^{\perp}$ and ${}^{\perp}S$. A similar notation is used for the classes of modules orthogonal with respect to the Tor functor:

$$\mathcal{S}^{\intercal} = \{M \in R-\text{Mod} \mid \text{Tor}^R_i(S, M) = 0 \text{ for all } S \in \mathcal{S} \text{ and } i \geq 1\}.$$

Given a class $\mathcal{U}$ of complexes in the derived category $\mathcal{D}(R)$ of $R$ (see 2.4), we denote:

$$\mathcal{U}^o = \{X \in \mathcal{D}(R) \mid \text{Hom}_{\mathcal{D}(R)}(U, X) = 0 \text{ for all } U \in \mathcal{U}\},$$
$${}^o\mathcal{U} = \{X \in \mathcal{D}(R) \mid \text{Hom}_{\mathcal{D}(R)}(X, U) = 0 \text{ for all } U \in \mathcal{U}\}.$$

Given a module $M$, we denote by Add$M$, respectively Prod$M$, the class of all modules that are isomorphic to direct summands of direct sums, respectively of direct products, of copies of $M$.

2.2 Subsets closed under specialization

A subset $Y \subseteq \operatorname{Spec}(R)$ is said to be *closed under specialization* (or specialization-closed) if it contains $V(\mathbf{p})$ for any $\mathbf{p} \in Y$. In other words, $Y$ is a upper set in the poset $(\operatorname{Spec}(R), \subseteq)$. Work of Gabriel [17] establishes a one-to-one correspondence between the subsets of $\operatorname{Spec}(R)$ closed under specialization and the hereditary torsion pairs in Mod$-R$. More precisely, every specialization-closed subset $Y \subseteq \operatorname{Spec}(R)$ determines a hereditary torsion pair $(\mathcal{T}(Y), \mathcal{F}(Y))$, where:

$$\begin{aligned}\mathcal{T}(Y) &= \{M \in \text{Mod}-R \mid \operatorname{Supp} M \subseteq Y\} \\ &= \{M \in \text{Mod}-R \mid \operatorname{Hom}_R(M, E(R/\mathbf{q})) = 0 \text{ for all } \mathbf{q} \notin Y\} \\ \mathcal{F}(Y) &= \{M \in \text{Mod}-R \mid \operatorname{Ass} M \cap Y = \emptyset\} \\ &= \{M \in \text{Mod}-R \mid \operatorname{Hom}_R(R/\mathbf{p}, M) = 0 \text{ for all } \mathbf{p} \in Y\}\end{aligned}$$

In particular, $\mathcal{T}(Y)$ contains all $E(R/\mathbf{p})$ with $\mathbf{p} \in Y$, and $\mathcal{F}(Y)$ contains all $E(R/\mathbf{q})$ with $\mathbf{q} \notin Y$.

**Definition 2.1** [2] A *filtration by supports* of $\operatorname{Spec}(R)$ is a map $\Phi : \mathbb{Z} \longrightarrow \mathcal{P}(\operatorname{Spec}(R))$ such that each $\Phi(i)$ is a subset of $\operatorname{Spec}(R)$ closed under specialization and $\Phi(i) \supseteq \Phi(i+1)$ for all $i \in \mathbb{Z}$.

2.3 Associated primes

Given $M \in \text{Mod}-R$,

$$0 \longrightarrow M \longrightarrow E_0(M) \longrightarrow E_1(M) \longrightarrow E_2(M) \longrightarrow \cdots$$

will stand for the minimal injective coresolution, and the image of $E_{i-1}(M) \to E_i(M)$ for $i \geq 1$ will be denoted by $\mho_i(M)$. We set $\mho_0(M) = M$. Moreover, we denote by $\operatorname{Ass} M$ the set of all associated primes of $M$, and by $\operatorname{Supp} M$ the support of $M$.

If $M \in \text{Mod}-R$, $\mathbf{p} \in \operatorname{Spec}(R)$ and $i \geq 0$, the *Bass invariant* $\mu_i(\mathbf{p}, M)$ is defined as the number of direct summands isomorphic to $E(R/\mathbf{p})$ in a decomposition of $E_i(M)$ into indecomposable direct summands, that is,

$$E_i(M) = \bigoplus_{\mathbf{p} \in \operatorname{Spec}(R)} E(R/\mathbf{p})^{(\mu_i(\mathbf{p}, M))}.$$

The relation of associated primes to these invariants is subsumed by the following lemma relying on work by Bass.

**Lemma 2.2** [7, 1.3 and 4.1] *Let $M$ be an $R$-module, $\mathbf{p} \in \operatorname{Spec}(R)$ and $i \geq 0$. Then*

$$\mu_i(\mathbf{p}, M) = \dim_{k(\mathbf{p})} \operatorname{Ext}^i_{R_\mathbf{p}}(k(\mathbf{p}), M_\mathbf{p}),$$

*and we have the following equivalences:*

$$\mathbf{p} \in \operatorname{Ass} \mho_i(M) \iff \mathbf{p} \in \operatorname{Ass} E_i(M) \iff \mu_i(\mathbf{p}, M) \neq 0.$$

*Moreover, if $Y \subseteq \operatorname{Spec}(R)$ is specialization closed, then $\mu_i(\mathbf{p}, M) = 0$ for each $\mathbf{p} \in Y$ if and only if $\operatorname{Ext}^i_R(R/\mathbf{p}, M) = 0$ for each $\mathbf{p} \in Y$.*

**Lemma 2.3** [7, 2.9] *Let $R$ be a (not necessarily commutative) left noetherian ring, and let $0 \neq U \in R$–mod and $n \geq 0$ such that $\operatorname{Ext}^i_R(U, R) = 0$ for all $i = 0, 1, \ldots, n$. Then we have:*

*(i)* $\operatorname{proj.dim}_R(\operatorname{Tr}\Omega^n(U)) = n+1$;
*(ii)* $\operatorname{Ext}^n_R(U,-)$ *and* $\operatorname{Tor}^R_1(\operatorname{Tr}\Omega^n(U),-)$ *are isomorphic functors.*
*(iii)* $\operatorname{Ext}^1_R(\operatorname{Tr}\Omega^n(U),-)$ *and* $\operatorname{Tor}^R_n(-,U)$ *are isomorphic functors.*

The following fact is standard.

**Lemma 2.4** [12, Proposition VI.2.5] *If $M$ is a finitely generated module with* $\operatorname{proj.dim}(M) = n < \infty$*, then* $\operatorname{Ext}^n_R(M,R) \neq 0$.

## 2.4 Derived functors

In this section we remind the reader of three classical derived functors which we will use frequently in the paper. Let $\mathcal{H}(R)$ denote the *homotopy category* of $R$. Its objects are the chain complexes of $R$-modules and the $R$-module of morphisms $\operatorname{Hom}_{\mathcal{H}(R)}(X,Y)$ is the factor $Hom_{C(R)}(X,Y)/\mathcal{N}(X,Y)$, where $\mathcal{N}(X,Y)$ is the submodule consisting of the null-homotopic maps. Then it is well-known that $\mathcal{H}(R)$ is a triangulated category, with the canonical shift $[1] : \mathcal{H}(R) \longrightarrow \mathcal{H}(R)$ as suspension functor. Moreover, the *derived category* of $R$, denoted $\mathcal{D}(R)$, is the localization of $\mathcal{H}(R)$ with respect the class of quasi-isomorphisms. Then we have a canonical triangulated functor $q : \mathcal{H}(R) \longrightarrow \mathcal{D}(R)$, which has both a left adjoint $p$ and a right adjoint $i$, called the *(homotopically) projective resolution* and *(homotopically) injective resolution*, respectively. More concretely, the unit of the adjunction $(p,q)$ and the counit of $(q,i)$ are isomorphisms, and if $\pi : p\circ q \longrightarrow 1_{\mathcal{H}(R)}$ and $\iota : 1_{\mathcal{H}(R)} \longrightarrow i\circ q$ are the counit and the unit of the respective adjunctions, then the morphisms $\pi_X : P_X := (p\circ q)(X) \longrightarrow X$ and $\iota_X : X \longrightarrow (i\circ q)(X) =: I_X$ are called the (homotopically) projective and injective resolutions of $X$, respectively. Both $\pi_X$ and $\iota_X$ are quasi-isomorphisms and $P_X$ and $I_X$ are uniquely determined by $X$, up to isomorphism in $\mathcal{H}(R)$.

*Example 2.5* If $M$ is an $R$-module, then we identify $M$ with the stalk complex $M[0]$ concentrated in degree zero. Let

$$\cdots \to P^{-n} \to \cdots \to P^{-1} \to P^0 \stackrel{\pi}{\to} M \to 0$$

be a projective resolution and denote by $P_M$ the complex

$$\cdots \to P^{-n} \to \cdots \to P^{-1} \to P^0 \to 0 \cdots$$

Then the map $\pi$ induces an obvious quasi-isomorphism $\pi_M : P_M \longrightarrow M$, which is the homotopically projective resolution of $M$. The dual is true for an injective resolution of $M$.

When $F : \mathcal{H}(R) \longrightarrow \mathcal{H}(R)$ is a triangulated functor, its left and right derived functor are, respectively, the compositions

$$LF : \mathcal{D}(R) \stackrel{p}{\longrightarrow} \mathcal{H}(R) \stackrel{f}{\longrightarrow} \mathcal{H}(R) \stackrel{q}{\longrightarrow} D(R)$$
$$RF : \mathcal{D}(R) \stackrel{i}{\longrightarrow} \mathcal{H}(R) \stackrel{f}{\longrightarrow} \mathcal{H}(R) \stackrel{q}{\longrightarrow} D(R).$$

Suppose that $X$ is a complex of $R$-modules. Then the total tensor product and the total Hom give functors $X \otimes_R - : C(R) \longrightarrow C(R)$ and $\operatorname{Hom}_R(X,-) : C(R) \longrightarrow C(R)$ which preserve null-homotopy and, hence, induce corresponding functors $X\otimes_R - : \mathcal{H}(R) \longrightarrow \mathcal{H}(R)$ and $\operatorname{Hom}_R(X,-) : \mathcal{H}(R) \longrightarrow \mathcal{H}(R)$. These two functors turn out to be triangulated and their left and right derived functors are denoted $X \otimes^{\mathbf{L}}_R - : \mathcal{D}(R) \longrightarrow \mathcal{D}(R)$ and $\operatorname{RHom}_R(X,-) : \mathcal{D}(R) \longrightarrow \mathcal{D}(R)$, respectively.

We will also need a contravariant version of the latter functor. Namely, the contravariant Hom on $C(R)$ also preserves null-homotopy and induces a triangulated contravariant functor $\mathrm{Hom}_R(-, X) : \mathcal{H}(R) \longrightarrow \mathcal{H}(R)$. Its right derived functor is denoted by $\mathrm{RHom}_R(-, X) : \mathcal{D}(R) \longrightarrow \mathcal{D}(R)$ and it is, by definition, the composition

$$\mathcal{D}(R) \xrightarrow{p} \mathcal{H}(R) \xrightarrow{\mathrm{Hom}_R(-,X)} \mathcal{H}(R) \xrightarrow{q} \mathcal{D}(R).$$

It turns out that the assignments $(X, Y) \mapsto \mathrm{RHom}_R(-, Y)(X)$ and $(X, Y) \mapsto \mathrm{RHom}_R(X, -)(Y)$ are naturally isomorphic triangulated bifunctors, contravariant in the first and covariant in the second variable. We will write $\mathrm{RHom}_R(X, Y)$ to denote the image of the pair $(X, Y)$ by either of these two functors.

We will be especially interested in the contravariant RHom when $X = R$, in which case we will simply write $M^* = \mathrm{RHom}_R(M, R)$. One easily sees that we have a canonical triangulated natural transformation $\sigma : 1_{\mathcal{D}(R)} \longrightarrow (-)^{**}$.

The following result summarizes some well-known properties of these functors that we shall use in the paper. We sketch a proof for the convenience of the reader.

**Proposition 2.6** *Let $X$ and $M$ be chain complexes of $R$-modules, where the second one is a compact object of $\mathcal{D}(R)$. The following assertions hold:*

*i) The pair $(X \otimes_R^{\mathbf{L}} -, \mathrm{RHom}_R(X, -))$ is an adjoint pair of triangulated functors;*
*ii) $M^*$ is a compact object of $\mathcal{D}(R)$ and the morphism $\sigma_M : M \longrightarrow M^{**}$ is an isomorphism.*
*iii) There are natural isomorphisms of triangulated functors $M \otimes_R^{\mathbf{L}} - \cong \mathrm{RHom}_R(M^*, -)$ and $\mathrm{RHom}_R(M, -) \cong M^* \otimes_R^{\mathbf{L}} -$.*

*Proof* Assertion i) is standard, even in much more general contexts (see [22, Section 6.2]). For assertions ii) and iii), note that saying that $M$ is compact in $\mathcal{D}(R)$ is equivalent to saying that $M$ is quasi-isomorphic to a bounded complex of finitely generated projective $R$-modules (see [29, Section 6]). So we assume in the sequel that $M = P : ...0 \to P^k \longrightarrow P^{k+1} \longrightarrow \cdots \longrightarrow P^m \longrightarrow 0...$ is a complex of finitely generated projective $R$-modules. But then if $p : \mathcal{D}(R) \longrightarrow \mathcal{H}(R)$ is the projetive resolution functor and $\pi : p \circ q \longrightarrow 1_{\mathcal{H}(R)}$ is the counit map, we get that $\pi_P : (p \circ q)(P) \longrightarrow P$ is an isomorphism in $\mathcal{H}(R)$. This implies that $P^* = \mathrm{RHom}_R(-, R)(P) = \mathrm{Hom}_R(P, R)$, which is just the complex obtained from $P$ by applying the usual contravariant functor $\mathrm{Hom}_R(-, R) : \text{Mod-}R \longrightarrow \text{Mod-}R$. Then $P^*$ is clearly a compact object. The morphism $\sigma_P : P \longrightarrow P^{**}$ is then the obvious one, which is an isomorphism even in $\mathcal{C}(R)$. Then assertion ii) holds. For iii), the second natural isomorphism is proven in [22, Lemma 6.2 (a)], in a more general context, and the first follows by using ii). □

## 2.5 t-Structures

**Definition 2.7** A subcategory $\mathcal{U}$ of $\mathcal{D}(R)$ is said to be *suspended* when it is closed under extensions and $\mathcal{U}[1] \subseteq \mathcal{U}$. If, in addition, the equality $\mathcal{U} = {}^o(\mathcal{U}^o)$ holds, and for each $X \in \mathcal{D}(R)$ there is triangle

$$U \longrightarrow X \longrightarrow V \longrightarrow U[1],$$

with $U \in \mathcal{U}$ and $V \in \mathcal{U}^o$, then we will say that the pair $(\mathcal{U}, \mathcal{U}^o[1])$ is a *t-structure* in $\mathcal{D}(R)$. In this case $\mathcal{U}$ is called the *aisle*, $\mathcal{U}^o$ the *co-aisle*, and $\mathcal{U} \cap \mathcal{U}^o[1]$ the *heart* of the t-structure.

We denote by $(\mathcal{D}^{\leq 0}, \mathcal{D}^{\geq 0})$ the standard t-structure of $R$. More generally, for each integer $n$, we will denote by $D^{\leq n}$ (resp. $D^{\geq n}$) the subcategory consisting of the complexes $X$ such that $H^i(X) = 0$ for all $i > n$ (resp. $i < n$), and we set $D^{<n} = D^{\leq n-1}$ and $D^{>n} = D^{\geq n+1}$.

Given a class $\mathcal{X}$ of objects in $\mathcal{D}(R)$, there is always a smallest suspended subcategory of $\mathcal{D}(R)$ containing $\mathcal{S}$ and closed under taking coproducts, called the *cocomplete suspended subcategory generated by* $\mathcal{X}$. In general, a suspended subcategory of $\mathcal{D}(R)$ need not be an aisle, even if it is closed under taking coproducts in $\mathcal{D}(R)$. However, we have the following fundamental fact:

**Theorem 2.8** [3] *Let $\mathcal{X}$ be a set of objects of $\mathcal{D}(R)$. Then the cocomplete suspended subcategory generated by $\mathcal{X}$ is an aisle of $\mathcal{D}(R)$. Its corresponding coaisle consists of those $Y \in \mathcal{D}(R)$ such that* $\mathrm{Hom}_{\mathcal{D}(R)}(X[i], Y) = 0$, *for all $i \geq 0$ and $X \in \mathcal{X}$.*

Due to the theorem above, when $\mathcal{X}$ is a set, the cocomplete suspended subcategory generated by $\mathcal{X}$ will also be called the *aisle generated by* $\mathcal{X}$, and it will be denoted by aisle($\mathcal{X}$). We will also say that the associated t-structure is generated by $\mathcal{X}$. We will say that a t-structure (or its aisle) is *compactly generated* when it is generated by a set of compact objects. The following is a fundamental fact that we shall frequently use in the paper.

**Theorem 2.9** [2, Theorem 3.11] *There is a bijective correspondence between:*

*(i) Compactly generated t-structures in $\mathcal{D}(R)$*
*(ii) Filtrations by supports of* $\mathrm{Spec}(R)$

*The correspondence from (i) to (ii) is given by $(\mathcal{U}, \mathcal{U}^o[1]) \mapsto \Phi_{\mathcal{U}}$, where*

$$\Phi_{\mathcal{U}}(i) = \{\mathbf{p} \in \mathrm{Spec}(R) \colon R/\mathbf{p}[-i] \in \mathcal{U}\},$$

*while the correspondence from (ii) to (i) maps $\Phi$ onto the t-structure $(\mathcal{U}_\Phi, \mathcal{U}_\Phi{}^o[1])$ whose aisle is*

$$\mathcal{U}_\Phi = aisle(R/\mathbf{p}[-i] \colon i \in \mathbb{Z}, \mathbf{p} \in \Phi(i)) = \{X \in \mathcal{D}(R) \colon \mathrm{Supp}\, H^i(X) \subseteq \Phi(i), \text{ for all } i \in \mathbb{Z}\}.$$

It was recently shown in [28] that filtrations by supports of $\mathrm{Spec}(R)$ also parametrize the compactly generated co-t-structures in $\mathcal{D}(R)$.

Of course, a filtration by supports $\Phi$ with $\Phi(i) = \mathrm{Spec}(R)$ for all $i < 0$ corresponds to a decreasing sequence $(Y_0, Y_1, ..., Y_k, ...)$ of specialization-closed subsets $Y_i = \Phi(i)$ of $\mathrm{Spec}(R)$. Extending the notation from [7], we consider the subcategory

$$\mathcal{C}_{(Y_0, Y_1, \ldots, Y_k, \ldots)}$$

consisting of the modules $M$ such that $\mathrm{Ext}^i_R(R/\mathbf{p}, M) = 0$ for all $i \geq 0$ and $\mathbf{p} \in Y_i$. We then get:

**Lemma 2.10** *Let $\Phi$ be a filtration by supports such that $\Phi(i) = \mathrm{Spec}(R)$ for all $i < 0$. Then*

$$\mathcal{U}_\Phi{}^o \cap \text{Mod-}R = \mathcal{C}_{(Y_0, Y_1, \ldots, Y_k, \ldots)},$$

*and this class consists of all modules $M$ satisfying one of the following equivalent conditions:*

*(1) $E_i(M)$ belongs to the torsion-free class $\mathcal{F}(Y_i)$ for all $i \geq 0$;*
*(2) $\mathrm{Ass}\, E_i(M) \cap Y_i = \emptyset$ for all $i \geq 0$,*

*where $0 \to M \to E_0(M) \to E_1(M) \to \cdots E_k(M) \to \cdot$ is a minimal injective (co)resolution.*

*Proof* By definition, $\mathcal{U}_\Phi$ is the aisle generated by the stalk complexes $R/\mathbf{p}[-i]$ with $i \in \mathbb{Z}$ and $\mathbf{p} \in \Phi(i) = Y_i$. Then a module $M$ is in ${\mathcal{U}_\Phi}^o$ if and only if the following equality holds

$$0 = \mathrm{Hom}_{\mathcal{D}(R)}(R/\mathbf{p}[-i][j], M) = \mathrm{Ext}_R^{i-j}(R/\mathbf{p}, M)$$

for all $i \in \mathbb{Z}$, $\mathbf{p} \in \Phi(i)$ and $j \geq 0$. This is clearly equivalent to saying that $\mathrm{Ext}_R^k(R/\mathbf{p}, M) = 0$ for all $k = 0, 1, \cdots, i$, $\mathbf{p} \in Y_i$, and $i \geq 0$. But, due to the fact that $Y_{i-1} \supseteq Y_i$ for all $i > 0$, this condition is clearly equivalent to saying that $M \in \mathcal{C}_{(Y_0, Y_1, \cdots, Y_k, \cdots)}$.

Condition (1) and (2) in the statement are equivalent by definition of $\mathcal{F}(Y_i)$. Moreover, condition (2) means by Lemma 2.2 that $\mathrm{Ext}_R^i(R/\mathbf{p}, M) = 0$ for all $\mathbf{p} \in Y_i$ and $i \geq 0$, whence $M \in \mathcal{C}_{(Y_0, Y_1, \cdots, Y_k, \cdots)}$. □

The following auxiliary results will be useful later on.

**Lemma 2.11** *Let* $(\mathcal{U}, \mathcal{U}^o[1])$ *be a compactly generated t-structure in* $\mathcal{D}(R)$ *and let* $0 \to M \to Y^0 \to Y^1 \to \ldots \to Y^n \to \ldots$ *be an exact sequence in* Mod-$R$. *If all the stalk complexes* $Y^n[-n]$ *are in* $\mathcal{U}^o$, *then* $M = M[0]$ *is in* $\mathcal{U}^o$.

*In particular, if* $Y$ *is a module in* $\mathcal{U}^o$ *and* $M$ *is a module which admits a* Prod$Y$*-coresolution, then* $M = M[0]$ *is in* $\mathcal{U}^o$.

*Proof* The final statement is a consequence of the first one since $\mathcal{U}^o$ is closed under taking products. To prove the first assertion, consider the complex

$$Y^\bullet : \quad \cdots 0 \to Y^0 \to Y^1 \to \cdots \to Y^n \to \cdots,$$

so that we have a quasi-isomorphism $M = M[0] \longrightarrow Y^\bullet$, and fix any compact object $X \in \mathcal{U}$. Then there is an integer $n \geq 0$ such that $X \in D^{\leq n}$. On the other hand, we have the triangle

$$\sigma^{>n} Y^\bullet \longrightarrow Y^\bullet \longrightarrow \sigma^{\leq n} Y^\bullet \longrightarrow \sigma^{>n} Y^\bullet[1],$$

coming from the stupid truncation at $n$. By hypothesis, we have $\mathrm{Hom}_{\mathcal{D}(R)}(X, Y^i[-i]) = 0$, for all $i = 0, 1, \cdots, n$. This implies that $\mathrm{Hom}_{\mathcal{D}(R)}(X, \sigma^{\leq n} Y^\bullet) = 0$ since $\sigma^{\leq n} Y^\bullet$ is a finite iterated extension of the stalk complexes $Y^i[-i]$, with $0 \leq i \leq n$. Moreover, we have $\mathrm{Hom}_{\mathcal{D}(R)}(X, \sigma^{>n} Y^\bullet) = 0$ because $X \in D^{\leq n}$ and $\sigma^{>n} Y^\bullet \in D^{>n}$. By applying the cohomological functor $\mathrm{Hom}_{\mathcal{D}(R)}(X, -)$ to the triangle above, we conclude that $\mathrm{Hom}_{\mathcal{D}(R)}(X, Y^\bullet) = 0$. Since this is true for any compact object $X$ in $\mathcal{U}$ it follows that $M \cong Y^\bullet \in \mathcal{U}^0$. □

**Lemma 2.12** *Let* $\mathcal{Y} \subset \mathcal{D}(R)$ *be any class of objects. The t-structure* $(\mathcal{U}, \mathcal{U}^o[1])$ *in* $\mathcal{D}(R)$ *generated by all compact objects* $X$ *of* $\mathcal{D}(R)$ *such that* $\mathrm{Hom}_{\mathcal{D}(R)}(X[j], Y) = 0$ *for all* $j \geq 0$ *and* $Y \in \mathcal{Y}$ *coincides with the following t-structures in* $\mathcal{D}(R)$*:*

*(i) the one generated by all stalk complexes* $R/\mathbf{p}[-n]$, $n \in \mathbb{Z}$, *such that* $\mathbf{p} \in \mathrm{Spec}(R)$ *and* $\mathrm{Hom}_{\mathcal{D}(R)}(R/\mathbf{p}[-n+j], Y) = 0$, *for all* $j \geq 0$ *and* $Y \in \mathcal{Y}$*;*

*(ii) the one generated by all stalk complexes* $M[-n]$, $n \in \mathbb{Z}$, *such that* $M \in$ mod-$R$ *and* $\mathrm{Hom}_{\mathcal{D}(R)}(M[-n+j], Y) = 0$, *for all* $j \geq 0$ *and* $Y \in \mathcal{Y}$.

*Moreover,* $\mathcal{U}$ *is the largest compactly generated aisle such that* $\mathcal{Y} \subset \mathcal{U}^o$.

*Proof* Of course, $\mathcal{U}$ is a compactly generated aisle such that $\mathcal{Y} \subset \mathcal{U}^o$. Any other such aisle is generated by a set of compact objects $X$ satisfying $\mathrm{Hom}_{\mathcal{D}(R)}(X[j], Y) = 0$ for all $j \geq 0$ and $Y \in \mathcal{Y}$, and it is therefore contained in $\mathcal{U}$. In particular, this applies to the aisle $\mathcal{V}$ of any of the t-structures in (i) or in (ii), which are compactly generated by [2, Theorem 3.10]. On the other hand, in both cases $\mathcal{V}^o \subset \mathcal{U}^o$ by [2, Proposition 3.7], so $\mathcal{U} = \mathcal{V}$. □

**Definition 2.13** Let $\mathcal{Y} \subset \mathcal{D}(R)$ be any class of objects. We say that a compactly generated t-structure $(\mathcal{U}, \mathcal{U}^o[1])$ of $\mathcal{D}(R)$ is *cogenerated* by $\mathcal{Y}$ if it is the t-structure of Lemma 2.12.

Observe that the aisle of the compactly generated t-structure cogenerated by a class $\mathcal{Y}$ is, in general, properly contained in the suspended subcategory $\{X \in \mathcal{D}(R)\colon \operatorname{Hom}_{\mathcal{D}(R)}(X[i], Y) = 0$, for all $i \geq 0$ and all $Y \in \mathcal{Y}\}$, as shown by the example below.

*Example 2.14* If $R = \mathbb{Z}$ then the compactly generated t-structure cogenerated by $\mathbb{Z}$ has as associated filtration by supports the one given as $\Psi(i) = \operatorname{Spec}(\mathbb{Z})$, for $i < 0$, and in non-negative degrees by $\Psi(0) = \{p\mathbb{Z}\colon p \neq 0\}$ and $\Psi(i) = \emptyset$, for $i > 0$. The aisle of this t-structure consists of the $X \in D^{\leq 0}(\mathbb{Z})$ such that $H^0(X)$ is a torsion abelian group (see Theorem 2.9). However we have $\operatorname{Hom}_{D(\mathbb{Z})}(\mathbb{Q}[i], \mathbb{Z}) = 0$, for all $i \geq 0$.

### 2.6 Tilting and cotilting modules

**Definition 2.15** [4,13] A module $T$ is *tilting* provided that

(T1) $T$ has finite projective dimension.
(T2) $\operatorname{Ext}^i_R(T, T^{(\kappa)}) = 0$ for all $i \geq 1$ and all cardinals $\kappa$.
(T3) There is an exact sequence $0 \to R \to T_0 \to T_1 \to \cdots \to T_r \to 0$ where $T_0, T_1, \ldots, T_r \in \operatorname{Add}T$.

The class $T^{\perp}$ is called the *tilting class* induced by $T$. Given an integer $n \geq 0$, a tilting module as well as its associated class are called *n-tilting* provided the projective dimension of $T$ is at most $n$.

Dually, a module $C$ is *cotilting* provided that

(C1) $C$ has finite injective dimension.
(C2) $\operatorname{Ext}^i_R(C^{\kappa}, C) = 0$ for all $i \geq 1$ and all cardinals $\kappa$.
(C3) There is an exact sequence $0 \to C_r \to \cdots \to C_1 \to C_0 \to W \to 0$ where $W$ is an injective cogenerator of Mod$-R$ and $C_0, C_1, \ldots C_r \in \operatorname{Prod}C$.

The class ${}^{\perp}C$ is the *cotilting class* induced by $C$. Again, if the injective dimension of $C$ is at most $n$, we call $C$ and ${}^{\perp}C$ an *n-cotilting* module and class, respectively.

If $T$ is an $n$-tilting right $R$-module, then the character module

$$C = T^+ = \operatorname{Hom}_{\mathbb{Z}}(T, \mathbb{Q}/\mathbb{Z}) \tag{1}$$

is an $n$-cotilting left $R$-module by [6, Proposition 2.3]. In fact, the assignment $T \mapsto T^+$ induces a bijection between equivalence classes of tilting and equivalence classes of cotilting modules, a statement that may fail for non-noetherian rings, see [11]. More precisely, this bijection relates tilting and cotilting classes with resolving subcategories.

**Definition 2.16** A subclass $\mathcal{S}$ of mod$-R$ is said to be *resolving* in case $\mathcal{S}$ is closed under extensions, direct summands, kernels of epimorphisms, and $R \in \mathcal{S}$.

**Theorem 2.17** [18, 5.2.23],[7, 4.2] *There are bijective correspondences between*

*(i) n-tilting classes in* Mod$-R$*,*
*(ii) n-cotilting classes in* Mod$-R$*,*
*(iii) resolving subclasses $\mathcal{S}$ of* mod$-R$ *consisting of modules of projective dimension $\leq n$.*

*The correspondence between (i) and (iii) is given by the assignments*

$$\mathcal{T} \mapsto {}^{\perp}\mathcal{T} \cap \text{mod}-R \text{ and } \mathcal{S} \mapsto \mathcal{S}^{\perp}$$

*while the bijection between (ii) and (iii) is established by the map*

$$\mathcal{C} \mapsto {}^{\intercal}\mathcal{C} \cap \text{mod}-R \text{ and } \mathcal{S} \mapsto \mathcal{C} = \mathcal{S}^{\intercal}$$

## 3 Classification results

We now want to focus on the resolving subcategories of mod$-R$ consisting of modules of finite projective dimension. To this end, we first show that any t-structure gives rise to a resolving subcategory.

**Lemma 3.1** *Let* $(\mathcal{U}, \mathcal{U}^o[1])$ *be a t-structure in* $\mathcal{D}(R)$*. Then the classes* $\mathcal{C} = \mathcal{U}^o \cap \text{Mod}-R$ *and* $\mathcal{S} = {}^{\intercal}\mathcal{C} \cap \text{mod}-R$ *are closed under direct summands, extensions, and kernels of epimorphisms. In particular,* $\mathcal{S}$ *is a resolving subcategory of* mod$-R$.

*Proof* The only nontrivial property to check in both cases is the closure under kernels of epimorphisms. By the long exact sequence of Tor, one immediately sees that any class of modules of the form ${}^{\intercal}\mathcal{Y}$, and hence also ${}^{\intercal}\mathcal{Y} \cap$ mod-$R$, is closed under taking kernels of epimorphisms. Furthermore, if $0 \to L \longrightarrow M \longrightarrow N \to 0$ is an exact sequence, with $M$ and $N$ in $\mathcal{C}$, then we have a triangle

$$N[-1] \longrightarrow L \longrightarrow M \longrightarrow N$$

in $\mathcal{D}(R)$ where $N[-1]$ and $M$ are in $\mathcal{U}^o$. It follows that $L \in \mathcal{U}^o$ and so $L \in \mathcal{C}$. □

*Remark 1* Lemma 3.1 allows to define a map $F$ that assigns a resolving subcategory of mod$-R$ to any compactly generated t-structure in $\mathcal{D}(R)$. Observe however that $F$ is not injective. In fact, given a compactly generated t-structure of the form $(\mathcal{U}_\Phi, \mathcal{U}_\Phi{}^o[1])$ for some filtration by supports $\Phi$, the class $\mathcal{C}$ above has the form

$$\mathcal{C} = \{M \in \text{Mod}-R \mid R\Gamma_{\Phi(i)}(M) \in \mathcal{D}^{>i} \quad \text{for all } i \in \mathbb{Z}\}$$

where $R\Gamma_{\Phi(i)}(M)$ is computed on the injective coresolution $\ldots 0 \to E_0(M) \to E_1(M) \to \ldots$ of $M$, see [2]. So, for any choice of $\Phi$ we obtain $R\Gamma_{\Phi(i)}(M) \in \mathcal{D}^{\geq 0} \subset \mathcal{D}^{>i}$ for all $i < 0$. In other words, $\mathcal{C}$ does not determine the values of $\Phi$ for negative $i$, and different choices of $\Phi$ yield the same resolving subcategory $\mathcal{S} = {}^{\intercal}\mathcal{C} \cap \text{mod}-R$.

We will need the following result.

**Lemma 3.2** [6, 1.3] *With the notation of Lemma 3.1 we have*
*(1) If* $R \in \mathcal{U}^o$*, then* ${}^{\intercal}\mathcal{C} = \{M \in \text{Mod}-R \mid \text{Tor}_1^R(M, C) = 0$ *for all* $C \in \mathcal{C}\}$.
*(2) If* $\mathcal{S}$ *is a resolving subcategory of* mod$-R$*, then* ${}^{\intercal}(\mathcal{S}^{\intercal})$ *consists of the modules that are direct limits of modules in* $\mathcal{S}$ *and* $\mathcal{S} = {}^{\intercal}(\mathcal{S}^{\intercal}) \cap \text{mod}-R$.

We can now state our classification of the resolving subcategories of $\mathcal{P}^{<\infty}$. For a related result see [14].

We denote by $\mathcal{P}^{<\infty}$ the category of all finitely generated modules of finite projective dimension, and by $\mathcal{P}^{\leq n}$ the category of all finitely generated modules of projective dimension at most $n$.

**Theorem 3.3** *There are bijective correspondences between*

*(i)* *resolving subcategories of* $\mathcal{P}^{<\infty}$;
*(ii)* *filtrations by supports* $\Phi$ *with* $\Phi(i) = \mathrm{Spec}(R)$ *for* $i < 0$ *and* $\mathrm{Ass}\, E_i(R) \cap \Phi(i) = \emptyset$ *for* $i \geq 0$;
*(iii)* *compactly generated t-structures* $(\mathcal{U}, \mathcal{V})$ *in* $\mathcal{D}(R)$ *such that* $R[1]$ *is contained in the heart.*

*The correspondence between (i) and (ii) is given by the assignments* $\mathcal{S} \mapsto \Phi_{\mathcal{S}}$ *and* $\Phi \mapsto \mathcal{S}_{\Phi}$, *where*

$$\Phi_{\mathcal{S}}(i) = \bigcup_{S \in \mathcal{S}} \mathrm{Supp}\,(\mathrm{Ext}_R^{i+1}(S, R)) \; for\ each\ i \geq 0, \quad and \quad \Phi_{\mathcal{S}}(i) = \mathrm{Spec}(R) \quad for\ i < 0;$$

$$\mathcal{S}_{\Phi} = \{S \in \mathcal{P}^{<\infty} : \mathrm{Supp}\,(\mathrm{Ext}_R^{i+1}(S, R)) \subseteq \Phi(i), for\ all\ i \geq 0\}.$$

*The correspondence between (i) and (iii) is given by the assignment*

$$F : (\mathcal{U}, \mathcal{U}^o[1]) \mapsto \mathcal{S} = {}^{\mathsf{T}}(\mathcal{U}^o \cap \text{Mod-}R) \cap \text{mod-}R.$$

*Its inverse* $G$ *assigns to* $\mathcal{S}$ *the t-structure generated by* $\{\mathrm{RHom}_R(S, R)[1] \mid S \in \mathcal{S}\}$.

*Proof* We divide the proof in several steps.

Step 1: We start out by proving that the bijection in Theorem 2.9 restricts to a bijection between (ii) and (iii). Indeed, it follows from the description of $\mathcal{U}_{\Phi}$ in Theorem 2.9 that $\Phi(i) = \mathrm{Spec}(R)$ for each $i < 0$ if and only if $D^{<0} \subseteq \mathcal{U}_{\Phi}$. Since $D^{<0}$ is the aisle of $\mathcal{D}(R)$ generated by $R[1]$, the latter means that $R[1]$ is contained in the aisle $\mathcal{U}_{\Phi}$. Furthermore, by Lemma 2.10, the condition $\mathrm{Ass}\, E_i(R) \cap \Phi(i) = \emptyset$ for $i \geq 0$ means that $R$ is contained in the coaisle $\mathcal{U}_{\phi}{}^{o}$. So the filtrations by supports $\Phi$ as in (ii) correspond to compactly generated t-structures with $R[1]$ in the heart.

Step 2: Next, we show that $F$ is a well-defined map. Let $(\mathcal{U}, \mathcal{U}^o[1])$ be a t-structure as in (iii) and let $\Phi$ be the corresponding filtration by supports. Set

$$\mathcal{C} = \mathcal{U}^o \cap \text{Mod-}R, \; \mathcal{S} = {}^{\mathsf{T}}\mathcal{C} \cap \text{mod-}R.$$

We already know from Lemma 3.1 that $\mathcal{S}$ is resolving. Moreover, by Lemma 2.10, an $R$-module $C$ is in $\mathcal{C}$ if and only if $\mathrm{Ext}^i_R(R/\mathbf{p}, C) = 0$ for all $i \geq 0$ and $\mathbf{p} \in \Phi(i)$. The fact that $R[1]$ is in the heart of the t-structure implies that $R \in \mathcal{C}$. We infer from Lemma 2.3 that $\mathrm{proj.dim}(Tr\Omega^i(R/\mathbf{p})) \leq i+1$ and the functors $\mathrm{Tor}_1^R(Tr\Omega^i(R/\mathbf{p}), -)$ and $\mathrm{Ext}^i_R(R/\mathbf{p}, -)$ are naturally isomorphic for all $i \geq 0$ and $\mathbf{p} \in \Phi(i)$. This gives a new characterization of the modules in $\mathcal{C}$, namely:

$$C \in \mathcal{C} \text{ if and only if } \mathrm{Tor}_1^R(Tr\Omega^i(R/\mathbf{p}), C) = 0 \quad \text{for all } i \geq 0 \text{ and } \mathbf{p} \in \Phi(i).$$

We set

$$\mathcal{X} = \{Tr\Omega^i(R/\mathbf{p}) \mid i \geq 0, \mathbf{p} \in \Phi(i)\}$$

and we claim that $\mathcal{S}$ coincides with the smallest resolving subcategory $\tilde{\mathcal{X}}$ of mod-$R$ containing $\mathcal{X}$. This will give that $\mathcal{S} \subset \mathcal{P}^{<\infty}$, because $\mathcal{P}^{<\infty}$ is resolving and contains $\mathcal{X}$.

To prove the claim, we first combine the description of $\mathcal{C}$ above with Lemma 3.2(1) to obtain $\mathcal{X} \subset {}^{\mathsf{T}}\mathcal{C} \cap \text{mod-}R = \mathcal{S}$, hence $\tilde{\mathcal{X}} \subset \mathcal{S}$ and $\mathcal{S}^{\mathsf{T}} \subset \tilde{\mathcal{X}}^{\mathsf{T}}$. On the other hand, $\tilde{\mathcal{X}}^{\mathsf{T}} \subset \mathcal{C}$, because every $C \in \tilde{\mathcal{X}}^{\mathsf{T}} \subset \mathcal{X}^{\mathsf{T}}$ satisfies $\mathrm{Tor}_1^R(X, C) = 0$ for all $X \in \mathcal{X}$. Since $\mathcal{C} \subset \mathcal{S}^{\mathsf{T}}$ by definition of $\mathcal{S}$, we conclude that

$$\mathcal{S}^{\mathsf{T}} = \tilde{\mathcal{X}}^{\mathsf{T}} = \mathcal{C},$$

and therefore $\tilde{\mathcal{X}} = \mathcal{S}$ by Lemma 3.2(2).

Step 3: Now we show that $G$ is well defined. Suppose that $\mathcal{S} \subset \mathcal{P}^{<\infty}$ is a resolving subcategory. For each $S \in \mathcal{S}$, we have a quasi-isomorphism $P = P_S \longrightarrow S$, where $P$ is a perfect complex, i.e. a bounded complex of finitely generated projective $R$-modules. Then $S^* := \mathrm{RHom}_R(S, R) = \mathrm{Hom}_R(P, R)$ is a compact object of $\mathcal{D}(R)$, and the t-structure $G(\mathcal{S}) := (\mathcal{U}, \mathcal{U}^o[1])$ of $\mathcal{D}(R)$ generated by the $S^*[1]$ with $S \in \mathcal{S}$ is compactly generated. Moreover, due to Proposition 2.6, for each $j \geq 0$ and $S \in \mathcal{S}$ we have isomorphisms

$$\begin{aligned}\mathrm{Hom}_{\mathcal{D}(R)}(S^*[1][j], R) &= \mathrm{Hom}_{\mathcal{D}(R)}(S^*, R[-1-j]) \\ &= H^{-1-j}(\mathrm{RHom}_R(S^*, R)) \cong H^{-1-j}(S) = 0,\end{aligned}$$

showing that $R \in \mathcal{U}^o$. On the other hand, we have that $R \in \mathcal{S}$ and $R^* \cong R$, which gives that $R[1] \in \mathcal{U}$. It follows that $R[1]$ is in the heart of $G(\mathcal{S})$, so that the assigment $\mathcal{S} \mapsto G(\mathcal{S})$ is a well-defined map from (i) to (iii).

Step 4: We claim that $(F \circ G)(\mathcal{S}) = \mathcal{S}$ for any resolving subcategory $\mathcal{S} \subset \mathcal{P}^{<\infty}$. Indeed, if $G(\mathcal{S}) = (\mathcal{U}_{\mathcal{S}}, {\mathcal{U}_{\mathcal{S}}}^o[1])$, then $\mathcal{C}_{\mathcal{S}} := {\mathcal{U}_{\mathcal{S}}}^o \cap \text{Mod-}R$ consists of the modules $C$ such that $0 = \mathrm{Hom}_{\mathcal{D}(R)}(S^*[1][j], C) = H^{-1-j}(\mathrm{RHom}_R(S^*, C)) \cong H^{-1-j}(S \otimes^{\mathbf{L}}_R C)$ for all $S \in \mathcal{S}$ and $j \geq 0$ (see Proposition 2.6). That is, a module $C$ is in $\mathcal{C}_{\mathcal{S}}$ if and only if $\mathrm{Tor}_i^R(S, C) = 0$, for all $S \in \mathcal{S}$ and $i > 0$. So $\mathcal{C}_{\mathcal{S}} = \mathcal{S}^{\intercal}$ and the map $F$ takes $(\mathcal{U}_{\mathcal{S}}, {\mathcal{U}_{\mathcal{S}}}^o[1])$ to ${}^{\intercal}(\mathcal{S}^{\intercal}) \cap \text{mod-}R = \mathcal{S}$ by Lemma 3.2(2).

Step 5: Let us prove that each t-structure $(\mathcal{U}, \mathcal{U}^o[1])$ as in (iii) is of the form $G(\mathcal{S})$ for some resolving subcategory $\mathcal{S} \subseteq \mathcal{P}^{<\infty}$. Combined with Step 4, this will yield that the maps $F$ and $G$ are mutually inverse.

To this end, we consider the full subcategory $\mathcal{S} \subseteq \mathcal{P}^{<\infty}$ consisting of the modules $S \in \mathcal{P}^{<\infty}$ such that $S^*[1] \in \mathcal{U}$. This is a resolving subcategory of mod-$R$ due to the fact that $\mathcal{U}$ is an aisle of $\mathcal{D}(R)$ containing $R[1] = R^*[1]$ and that the contravariant triangulated functor $\mathrm{RHom}_R(-, R) : \mathcal{D}(R) \longrightarrow \mathcal{D}(R)$ takes short exact sequences in Mod-$R$ to triangles in $\mathcal{D}(R)$. In order to check that $G(\mathcal{S}) = (\mathcal{U}, \mathcal{U}^o[1])$, we just need to find a set $\mathcal{X} \subseteq \mathcal{S}$ such that $(\mathcal{U}, \mathcal{U}^o[1])$ is the t-structure of $\mathcal{D}(R)$ generated by the complexes $X^*[1]$ with $X \in \mathcal{X}$.

Let $\Phi$ be the filtration by supports associated to $(\mathcal{U}, \mathcal{U}^o[1])$. We have already seen in Step 2 that $\mathrm{proj.dim}(Tr\Omega^i(R/\mathbf{p})) \leq i + 1$ when $\mathbf{p} \in \Phi(i)$ and $i \geq 0$. Let us fix $i \geq 0$ and $\mathbf{p} \in \Phi(i)$ and also a projective resolution

$$\cdots \to P^{-i-1} \to P^{-i} \to \cdots \to P^{-1} \to P^0 \to R/\mathbf{p} \to 0,$$

with all the $P^j$ finitely generated. Then we get a projective resolution

$$0 \to (P^0)^* \to (P^{-1})^* \to \cdots \to (P^{-i})^* \to (P^{-i-1})^* \to Tr\Omega^i(R/\mathbf{p}) \to 0,$$

where $M^* := Hom_R(M, R)$, for each $R$-module $M$. The complex

$$Q : \ldots 0 \to (P^0)^* \to (P^{-1})^* \to \cdots \to (P^{-i})^* \to (P^{-i-1})^* \to 0 \ldots$$

with $(P^{-1-i-})^* = Q^0$ in degree zero is a perfect complex, and there is an obvious quasi-isomorphism $Q \longrightarrow Tr\Omega^i(R/\mathbf{p})$. It follows that $\mathrm{RHom}_R(Tr\Omega^i(R/\mathbf{p}), R) = \mathrm{Hom}_R(Q, R)$. But the canonical map $M \longrightarrow M^{**}$ is an isomorphism whenever $M$ is finitely generated projective. So, in the category $C(R)$ of complexes, the complex $\mathrm{Hom}_R(Q, R)$ is canonically isomorphic to the complex

$$\cdots 0 \to P^{-i-1} \to P^{-i} \to \cdots \to P^{-1} \to P^0 \to 0 \cdots$$

with $P^0$ in degree $i + 1$. Thus we have an isomorphism $\sigma^{\geq -i-1}P[-i-1] \cong \mathrm{RHom}_R(Tr\Omega^i(R/\mathbf{p}), R)$ in $\mathcal{D}(R)$, where $\sigma^{\geq -i-1}P$ denotes the stupid truncation at $-i-1$ of the projective resolution $P$ of $R/\mathbf{p}$. Then $(Tr\Omega^i(R/\mathbf{p}))^*[1] = \mathrm{RHom}_R(Tr\Omega^i(R/\mathbf{p}), R)[1]$ is a complex having homology concentrated in degrees $-1$ and $i$, and these homologies are $\Omega^{i+2}(R/\mathbf{p})$ and $R/\mathbf{p}$, respectively. Now the triangle

$$\Omega^{i+2}(R/\mathbf{p})[1] \longrightarrow (Tr\Omega^i(R/\mathbf{p}))^*[1] \longrightarrow R/\mathbf{p}[-i] \xrightarrow{+}$$

in $\mathcal{D}(R)$ shows that $(Tr\Omega^i(R/\mathbf{p}))^*[1] \in \mathcal{U}$, because the left vertex is in $D^{<0} \subset \mathcal{U}$ and also $R/\mathbf{p}[-i] \in \mathcal{U}$. We conclude that $Tr\Omega^i(R/\mathbf{p}) \in \mathcal{S}$ for all $i \geq 0$ and $\mathbf{p} \in \Phi(i)$.

Taking the subset $\mathcal{X} = \{Tr\Omega^i(R/\mathbf{p}) : i \geq 0, \mathbf{p} \in \Phi(i)\} \cup \{R\}$ of $\mathcal{S}$, we see that $\mathrm{aisle}(X^*[1] : X \in \mathcal{X})$ is contained in $\mathcal{U}$ and contains $R[1]$ and therefore also $\mathcal{D}^{<0}$. In particular, $\mathrm{aisle}(X^*[1] : X \in \mathcal{X})$ contains all complexes of the form $\Omega^{i+2}(R/\mathbf{p})[2]$, and thus also all $R/\mathbf{p}[-i]$, which can be seen by shifting the triangle above. Hence we obtain the desired equality of aisles in $\mathcal{D}(R)$:

$$\mathrm{aisle}(X^*[1] : X \in \mathcal{X}) = \mathrm{aisle}(S^*[1] : S \in \mathcal{S}) = \mathrm{aisle}(R/\mathbf{p}[-i]) : i \in \mathbb{Z}, \mathbf{p} \in \Phi(i)) = \mathcal{U}.$$

Step 6: It remains to verify that the bijection between (i) and (ii) is as indicated in the statement of the theorem. Let again $\mathcal{S} \subseteq \mathcal{P}^{<\infty}$ be a resolving subcategory, $G(\mathcal{S}) = (\mathcal{U}, \mathcal{U}^o[1])$ its associated t-structure, and $\Phi$ the associated filtration by supports. If $S \in \mathcal{S}$ then $S^*[1] \in \mathcal{U}$, hence $\mathrm{Supp}\,(\mathrm{Ext}_R^{i+1}(S, R)) = \mathrm{Supp}\,(H^i(S^*[1])) \subseteq \Phi(i)$ for all $i \geq 0$. So the filtration by supports $\Phi_{\mathcal{S}}$ given in the statement satisfies $\Phi_{\mathcal{S}}(i) \subseteq \Phi(i)$. Conversely, if $\mathbf{p} \in \Phi(i)$, then we have seen in Step 5 that $Tr\Omega^i(R/\mathbf{p}) \in \mathcal{S}$ and that the homology module of $(Tr\Omega^i(R/\mathbf{p}))^*[1]$ in degree $i$, which is $\mathrm{Ext}_R^{i+1}(Tr\Omega^i(R/\mathbf{p}), R)$, is isomorphic to $R/\mathbf{p}$, thus $\mathbf{p} \in \Phi_{\mathcal{S}}(i)$. So we conclude that $\Phi = \Phi_{\mathcal{S}}$. Take now a filtration by supports $\Phi$ as in (ii) and let $(\mathcal{U}_\Phi, \mathcal{U}_\Phi^o[1])$ be its associated t-structure. Step 5 shows that the associated resolving subcategory $\mathcal{S}_\Phi := F[(\mathcal{U}_\Phi, \mathcal{U}_\Phi^o[1])]$ is given by

$$\mathcal{S}_\Phi = \{S \in \mathcal{P}^{<\infty} : S^*[1] \in \mathcal{U}_\Phi\}.$$

But $S^*[1] = \mathrm{RHom}_R(S, R)[1] \in \mathcal{U}_\Phi$ if and only if $\mathrm{Supp}\,(H^i(S^*[1])) = \mathrm{Supp}\,(\mathrm{Ext}_R^{i+1}(S, R)) \subseteq \Phi(i)$ for all $i \geq 0$. Therefore $\mathcal{S}_\Phi$ has the stated form. □

**Definition 3.4** Let $\Phi$ be a filtration by supports and $n$ a natural number. We say that $\Phi$ is *concentrated in* $0, \ldots, n-1$ if $\Phi(i) = \mathrm{Spec}(R)$ for all $i < 0$ and $\Phi(i) = \emptyset$ for all $i \geq n$.

A filtration by supports which is concentrated in $0, \ldots, n-1$ is determined by a finite decreasing sequence $(Y_0, \ldots, Y_{n-1})$ of subsets of $\mathrm{Spec}(R)$ closed under specialization. The corresponding class of modules

$$\mathcal{C}_{(Y_0,\ldots,Y_{n-1})} = \{M \in \mathrm{Mod}{-}R \mid \mathrm{Ass}\, E_i(M)) \cap Y_i = \emptyset \text{ for all } 0 \leq i < n\}$$

turns out to be an $n$-cotilting class provided that it contains $R$. We thus recover the classification of tilting and cotilting modules given in [7, Theorems 3.7 and 4.2] and determine the corresponding t-structures.

**Corollary 3.5** *Let n be a natural number. There are bijective correspondences between*

*(i) resolving subcategories of* $\mathcal{P}^{\leq n}$*;*
*(ii) n-cotilting classes in* Mod$-R$*;*
*(iii) n-tilting classes in* Mod$-R$*;*
*(iv) filtrations by supports* $\Phi$ *concentrated in* $0, \ldots, n-1$ *with* $\mathrm{Ass}\, E_i(R) \cap \Phi(i) = \emptyset$ *for* $i \geq 0$*;*
*(v) compactly generated t-structures* $(\mathcal{U}, \mathcal{U}^o[1])$ *in* $\mathcal{D}(R)$ *such that* $R[1]$ *is contained in the heart and* $\mathcal{U} \subseteq D^{<n}$.

*Hereby, a filtration* $\Phi$ *given by the decreasing sequence* $(Y_0, \ldots, Y_{n-1})$ *with associated t-structure* $(\mathcal{U}_\Phi, \mathcal{U}_\Phi{}^o[1])$ *corresponds to the cotilting class* $\mathcal{C}_{(Y_0,\ldots,Y_{n-1})} = \mathcal{U}_\Phi{}^o \cap \text{Mod-}R$.

*Proof* By Theorem 3.3, we know that if $\mathcal{S} \subseteq \mathcal{P}^{\leq n}$ is a resolving subcategory, then the filtration $\Phi_\mathcal{S}$ associated to it is concentrated in degrees $0, 1, \ldots, n-1$. Conversely, if $\Phi$ is a filtration by supports as in (iv), then the resolving subcategory $\mathcal{S}_\Phi$ of $\mathcal{P}^{<\infty}$ associated to it by Theorem 3.3 has the property $\text{Supp}\,(\text{Ext}_R^{i+1}(S, R)) = \emptyset$, and hence $\text{Ext}_R^{i+1}(S, R) = 0$ for all $i \geq n$ and all $S \in \mathcal{S}$. But since $S$ has finite projective dimension, this means that $\text{proj.dim}(S) \leq n$ for all $S \in \mathcal{S}$, cf. Lemma 2.4.

Furthermore, under the bijection in Theorem 2.9, a filtration by supports with $\Phi(i) = \emptyset$ for $i \geq n$ corresponds to a t-structure with aisle $\mathcal{U} \subset \mathcal{D}^{\leq n-1}$, or equivalently, with $\mathcal{D}^{\geq n} \subset \mathcal{U}^o$.

The remaining bijections are established in Theorem 2.17, which also states that the bijection between (i) and (ii) maps a resolving subcategory $\mathcal{S}$ onto the cotilting class $\mathcal{S}^\intercal$. Now, if $\Phi$ is a filtration given by $(Y_0, \ldots, Y_{n-1})$, then the corresponding resolving subcategory is $\mathcal{S} = {}^\intercal\mathcal{C} \cap \text{mod-}R$ where $\mathcal{C} = \mathcal{U}_\Phi{}^o \cap \text{Mod-}R = \mathcal{C}_{(Y_0,\ldots,Y_{n-1})}$ by Lemma 2.10, and $\mathcal{S}^\intercal = \mathcal{C}$, as shown in Step 2 of the proof of Theorem 3.3. This proves the last statement. □

*Remark 2* Let $C$ be a 1-cotilting module and let $(\mathcal{U}_\Phi, \mathcal{U}_\Phi{}^o[1])$ be the t-structure corresponding to $\mathcal{C} = {}^\perp C$ under the bijection of Corollary 3.5. Then $\mathcal{C}$ is the torsion-free class in the hereditary torsion pair $(\mathcal{T}(Y), \mathcal{F}(Y))$ defined by the subset $Y = \Phi(0) \subset \text{Spec}(R)$, and $(\mathcal{U}_\Phi, \mathcal{U}_\Phi{}^o[1])$ coincides with the t-structure from [19,23]:

$$\mathcal{U}_\phi = \{X^\cdot \in \mathcal{D}(R) \mid H^0(X^\cdot) \in \mathcal{T}(Y), H^i(X^\cdot) = 0 \text{ for } i > 0\},$$
$$\mathcal{U}_\phi{}^o[1] = \{X^\cdot \in (R) \mid H^{-1}(X^\cdot) \in \mathcal{F}(Y), H^i(X^\cdot) = 0 \text{ for } i < -1\}.$$

We now show that the compactly generated t-structures of Theorem 3.3 have a somewhat surprising property, namely, they are cogenerated by a module (see Definition 2.13).

**Proposition 3.6** *Let* $(\mathcal{U}, \mathcal{U}^o[1])$ *be a compactly generated t-structure in* $\mathcal{D}(R)$ *such that* $R[1]$ *is in its heart. Then* $(\mathcal{U}, \mathcal{U}^o[1])$ *is cogenerated by a pure-injective module C (viewed a stalk complex* $C[0]$*). Moreover, if n is a natural number such that* $\mathcal{U} \subseteq D^{<n}$*, then C can be chosen to be any n-cotilting module such that* $\mathcal{U}^o \cap \text{Mod-}R = {}^\perp C$.

*Proof* (1) Set $\mathcal{C} = \mathcal{U}^o \cap \text{Mod-}R$. We first prove that $(\mathcal{U}, \mathcal{U}^o[1])$ coincides with the compactly generated t-structure $(\mathcal{U}', \mathcal{U}'^o[1])$ cogenerated by $\mathcal{C}$. Indeed, $R[1]$ is in the heart of $(\mathcal{U}', \mathcal{U}'^o[1])$ because $R \in \mathcal{C}$. Then Theorem 3.3 applies to this t-structure as well. With the notation of that theorem and its proof, we have that $F[(\mathcal{U}', \mathcal{U}'^o[1])] =: \mathcal{S}'$ consists of the modules $S \in \mathcal{P}^{<\infty}$ such that $S^*[1] = \text{RHom}_R(S, R)[1] \in \mathcal{U}'$. Since each $S^*[1]$ is compact the latter condition is equivalent to the following equality

$$0 = \text{Hom}_{\mathcal{D}(R)}(S^*[1][i], C) = H^{-1-i}(\text{RHom}_R(S^*, C)) = H^{-1-i}(S \otimes_R^{\mathbf{L}} C)$$
$$= Tor_{1+i}^R(S, C),$$

for all $i \geq 0$ and all $C \in \mathcal{C}$. We conclude that a module in $\mathcal{P}^{<\infty}$ belongs to $\mathcal{S}'$ if and only if it is contained in ${}^\intercal\mathcal{C} \cap \text{mod-}R = F[(\mathcal{U}, \mathcal{U}^o[1])]$. We then get $(\mathcal{U}', \mathcal{U}'^o[1]) = (\mathcal{U}, \mathcal{U}^o[1])$.

Set now $\mathcal{S} = {}^\intercal\mathcal{C} \cap \text{mod-}R = F[(\mathcal{U}, \mathcal{U}^o)]$ and recall that $\mathcal{S}^\intercal = \mathcal{C}$, as shown in Step 2 of the proof of Theorem 3.3.

(2) In the particular case when $\mathcal{U} \subseteq D^{<n}$, the class $\mathcal{S}$ is a resolving subcategory consisting of modules of projective dimension less or equal than $n$, and $\mathcal{C}$ is the corresponding $n$-cotilting class, cf. Corollary 3.5. Let $C$ be an $n$-cotilting module such that $\mathcal{C} = {}^\perp C$, and $(\mathcal{U}_C, \mathcal{U}_C{}^o[1])$ be the compactly generated t-structure cogenerated by $C$. It is well-known that each module $M \in \mathcal{C}$ admits $\text{Prod}(C)$-coresolution:

$$0 \to M \longrightarrow C^{H_0} \to C^{H_1} \to \cdots \to C^{H_k} \to \cdots$$

for some sets $H_k$, cf. [18, 8.1.5]. By Lemma 2.11, it follows that $\mathcal{C} \subset \mathcal{U}_C{}^o$, and from (1) and Lemma 2.12 we infer $\mathcal{U}_C \subseteq \mathcal{U}$. On the other hand, the inclusion $\{C\} \subset \mathcal{C}$ implies that the compactly generated aisle cogenerated by $\mathcal{C}$ is contained in the one cogenerated by $C$. Hence $\mathcal{U} = \mathcal{U}_C$, and $(\mathcal{U}, \mathcal{U}^o[1])$ is cogenerated by $C$. Note that the module $C$ is pure-injective by [10,30].

(3) In the general case, we consider the class $\mathcal{T} = \mathcal{S}^\perp$. This class is definable (cf. [24, Section 2.1]), and so there is a closed subset $\mathbb{U}$ of the Ziegler spectrum of $R$ such that each module in $\mathcal{T}$ is isomorphic to a pure submodule of a direct product of modules in $\mathbb{U}$, see [24, Theorem 2.11]. We then put $T = \prod_{U \in \mathbb{U}} U$, so that each module in $\mathcal{T}$ is isomorphic to a pure submodule of a product of copies of $T$. By [5, 9.12], the class $\mathcal{C} = \mathcal{S}^\intercal$ is the dual definable class of $\mathcal{T}$, and therefore the duality $(-)^+ = \mathrm{Hom}_{\mathbb{Z}}(-, \mathbb{Q}/\mathbb{Z})$ yields $\mathcal{T}^+ = \{T'^+ : T' \in \mathcal{T}\} \subseteq \mathcal{C}$ and $\mathcal{C}^+ \subseteq \mathcal{T}$. For the reader's convenience, we recall that this is shown by the usual Ext-Tor relations

$$ET1)\ \mathrm{Ext}^i_R(S, T')^+ = \mathrm{Tor}^R_i(S, T'^+)$$
$$ET2)\ \mathrm{Tor}^R_i(S, C)^+ = \mathrm{Ext}^i_R(S, C^+),$$

holding for all $i \geq 0$ and all $R$-modules $S, T', C$ with $S$ finitely generated (e.g. see [18, 1.2.11]). Let now $C \in \mathcal{C}$ be arbitrary. We then have a pure monomorphism $u : C^+ \rightarrowtail T^H$, for some set $H$. It induces a split epimorphism $(T^H)^+ \twoheadrightarrow C^{++}$, which shows that $C$ is isomorphic to a pure submodule of $(T^H)^+$ since the canonical map $C \longrightarrow C^{++}$ and any split monomorphism are pure monomorphisms. We fix a pure monomorphism $v : C \rightarrowtail (T^H)^+$. Since definable subcategories are closed under pure epimorphic images, $\mathrm{Coker}(v)$ is in $\mathcal{C}$. We include an argument for the reader's convenience: the map $1_S \otimes v : S \otimes_R C \longrightarrow S \otimes_R (T^H)^+$ is a monomorphism, which implies that $\mathrm{Tor}^R_1(S, \mathrm{Coker}(v)) = 0$ for all $S \in \mathcal{S}$, since both $C$ and $(T^H)^+$ are in $\mathcal{C} = \mathcal{S}^\intercal$. We conclude that there are sets $H_i$ $(i \geq 0)$ together with an exact sequence

$$0 \to C \to (T^{H_0})^+ \to (T^{H_1})^+ \to \cdots \to (T^{H_i})^+ \to \cdots$$

Since the compactly generated t-structure $(\mathcal{U}, \mathcal{U}^o[1])$ is cogenerated by $\mathcal{C}$, an argument already used in (2) shows that it is also cogenerated by the class $\mathcal{C}_T = \{(T^H)^+ : H \text{ is a set}\}$.

Since $T^+$ is pure-injective, our task is reduced to prove that the compactly generated t-structure cogenerated by $C := T^+$ and the one cogenerated by the class $\mathcal{C}_T$ coincide. Bearing in mind that both aisles contain $D^{<0}$ and using Lemma 2.12, it is enough to prove that if $M$ is a finitely generated (=presented) module and $i \geq 0$ is an integer, then the following implication holds for all sets $H$ and all integers $j \geq 0$:

$$\mathrm{Ext}^{i-j}_R(M, T^+) = \mathrm{Hom}_{\mathcal{D}(R)}(M[-i][j], T^+) = 0 \Longrightarrow$$
$$\mathrm{Ext}^{i-j}_R(M, (T^H)^+) = \mathrm{Hom}_{\mathcal{D}(R)}(M[-i][j], (T^H)^+) = 0.$$

By the equality ET2) above, we see that the implication above holds true if and only if the following one holds for all sets $H$ and all integers $k = 0, 1, \ldots, i$:

$$\mathrm{Tor}^R_k(M, T) = 0 \Longrightarrow \mathrm{Tor}^R_k(M, T^H) = 0.$$

But the latter is clearly true, due to the fact that $M$ admits a projective resolution with finitely generated terms and that products are exact in Mod-$R$. □

When a compactly generated t-structure is cogenerated by a module, the associated filtration by supports is also determined by that module, as the following result shows.

**Proposition 3.7** *Let $C$ be an $R$-module, and $\Phi_C$ be the filtration by supports associated by Theorem 2.9 to the compactly generated t-structure of $\mathcal{D}(R)$ cogenerated by $C$. The following assertions are equivalent for a prime ideal $\mathbf{p}$ and an integer $i \geq 0$.*

*(1)* $\mathbf{p} \in \operatorname{Spec}(R) \setminus \Phi_C(i)$.
*(2) There are* $0 \leq k \leq i$ *and* $\mathbf{q} \in \operatorname{Ass} E_k(C)$ *such that* $\mathbf{p} \subseteq \mathbf{q}$.
*(3)* $\operatorname{Ext}^k_R(R/\mathbf{p}, C) \neq 0$, *for some* $k = 0, 1, \ldots, i$.

*Proof* 1) $\Longleftrightarrow$ 3) By Lemma 2.12, we know that

$$\begin{aligned}\Phi_C(i) &= \{\mathbf{p} \in \operatorname{Spec}(R) \mid 0 = \operatorname{Hom}_{\mathcal{D}(R)}(R/\mathbf{p}[-i][j], C) \\ &= \operatorname{Ext}^{i-j}_R(R/\mathbf{p}, C) \quad \text{for all integers } j \geq 0\}.\end{aligned}$$

From this the equivalence of (1) and (3) is obvious.

(3) $\Longrightarrow$ (2) If $\operatorname{Ext}^k_R(R/\mathbf{p}, C) \neq 0$ then $\operatorname{Hom}_R(R/\mathbf{p}, E_k(C)) \neq 0$. In particular, there are a $\mathbf{q} \in \operatorname{Ass} E_k(C)$ and a nonzero homomorphism $f : R/\mathbf{p} \longrightarrow E(R/\mathbf{q})$. By the essentiality of $R/\mathbf{q}$ in $E(R/\mathbf{q})$, we have an element $x \in R/\mathbf{p}$ such that $0 \neq f(x) \in R/\mathbf{q}$. This implies that $\mathbf{p} \subseteq \operatorname{ann}_R(f(x)) = \mathbf{q}$.

(2) $\Longrightarrow$ (1) Let $\mathbf{p}$, $i$ and $\mathbf{q}$ be as in assertion (2). Suppose that $\mathbf{p} \in \Phi_C(i)$. Then $\mathbf{q} \in \Phi_C(i)$ since $\Phi_C(i)$ is closed under specialization. But then we also have $\operatorname{Ext}^k_R(R/\mathbf{q}, C) = 0$, for $0 \leq k \leq i$. This is a contradiction (see Lemma 2.2) for we have $\mu_k(\mathbf{q}, C) \neq 0$. □

**Corollary 3.8** *If $C$ is a $n$-cotilting module with $\mathcal{C} = {}^{\perp}C$, and $(Y_0, Y_1, \ldots, Y_{n-1})$ is the decreasing sequence of subsets of $\operatorname{Spec}(R)$ closed under specialization which is associated to $\mathcal{C}$ under the bijection in Corollary 3.5, then for any $0 \leq i < n$*

$$Y_i = \left\{\mathbf{p} \in \operatorname{Spec}(R) \;\middle|\; V(\mathbf{p}) \cap \operatorname{Ass}\left(\bigoplus_{k=0}^{i} E_k(C)\right) = \emptyset\right\}.$$

*Moreover, the injective dimension of $C$ equals the least integer $m$ such that $Y_m = \emptyset$.*

*Proof* By Propositions 3.6 and 3.7 we have that $Y_i = \Phi_C(i)$ has the stated form.

Moreover, if $C$ has injective dimension $m$, then the corresponding resolving subcategory $\mathcal{S}$ is contained in $\mathcal{P}^{\leq m}$ by Theorem 2.17. So, using Theorem 3.3 and Lemma 2.4, we see that $Y_i = \Phi_{\mathcal{S}}(i) = \bigcup_{S \in \mathcal{S}} \operatorname{Supp}(\operatorname{Ext}^{i+1}_R(S, R)) = \emptyset$ if and only if $i \geq m$. □

*Example 3.9* Let $\Psi = \Phi_R$ be the filtration by supports associated to the compactly generated t-structure cogenerated by $R$. By Proposition 3.7, $\Psi$ is given by $\Psi(i) = \operatorname{Spec}(R)$ for $i < 0$, and

$$\Psi(i) = \{\mathbf{p} \in \operatorname{Spec}(R)\colon \operatorname{Ext}^k_R(R/\mathbf{p}, R) = 0 \text{ for all } 0 \leq k \leq i\} \quad \text{for } i \geq 0.$$

Then $\Psi$ is the largest filtration by supports satisfying condition (ii) of Theorem 3.3, and the corresponding resolving subcategory is $\mathcal{S}_\Psi = \mathcal{P}^{<\infty}$.

In fact, let $\Phi : \mathbb{Z} \longrightarrow \mathcal{P}(\operatorname{Spec}(R))$ be a filtration by supports such that $\Phi(i) = \operatorname{Spec}(R)$ for $i < 0$ and $\Phi(i) \cap \operatorname{Ass}(E_i(R)) = \emptyset$ for $i \geq 0$, and let $G_i = \operatorname{Spec}(R) \setminus \Psi(i)$ for each $i \geq 0$. We have to show that $\Phi(i) \cap G_i = \emptyset$ for all $i \geq 0$. Suppose there is $\mathbf{p} \in \Phi(i) \cap G_i$. By Theorem 3.3, the t-structure associated to $\Phi$ contains $R[1]$ in its heart, so that we have $R \in \mathcal{U}_\Phi{}^o$. Then $\operatorname{Ext}^k_R(R/\mathbf{p}, R) = \operatorname{Hom}_{\mathcal{D}(R)}(R/\mathbf{p}[-k], R) = 0$ since $\mathbf{p} \in \Phi(k)$ for all $0 \leq k \leq i$. But this contradicts the fact that $\mathbf{p} \in G_i$.

*Example 3.10* Consider the resolving subcategory $\mathcal{P}^{\leq n}$ of mod-$R$ given by all finitely generated modules of projective dimension bounded by $n$. By Theorem 2.17, it corresponds to the smallest $n$-tilting class $\mathcal{T}_n = (\mathcal{P}^{\leq n})^{\perp}$ and to the smallest $n$-cotilting class $\mathcal{C}_n = (\mathcal{P}^{\leq n})^{\intercal}$. Under the bijection in Corollary 3.5, these classes are associated to the maximal choice of the sequence $(Y_0, \ldots, Y_{n-1})$, which is given by

$$Y_i = \left\{ \mathbf{p} \in \mathrm{Spec}(R) \mid V(\mathbf{p}) \cap \mathrm{Ass}\left( \bigoplus_{k=0}^{i} E_k(R) \right) = \emptyset \right\}$$

for $0 \leq i < n$. This follows as above from Proposition 3.7.

We now turn to a property of filtrations by supports which is studied in [2, Theorem 6.9]. It is a necessary condition for the associated t-structure to restrict to the full subcategory $D^b_{fg}(R)$ of bounded complexes with finitely generated homologies, and it is also sufficient in case $R$ admits a dualizing complex.

**Definition 3.11** [2] A filtration by supports $\Phi$ satisfies the *weak Cousin condition* if the following property holds true: if $\mathbf{p}$ and $\mathbf{q}$ are prime ideals with $\mathbf{p}$ maximal under $\mathbf{q}$ and $\mathbf{q} \in \Phi(i)$ for some $i \in \mathbb{Z}$, then $\mathbf{p} \in \Phi(i-1)$.

**Corollary 3.12** *Let $\Phi : \mathbb{Z} \longrightarrow \mathrm{Spec}(R)$ be a filtration by supports whose associated compactly generated t-structure is cogenerated by the module $C$. The following assertions are equivalent:*

*(1) $\Phi$ satisfies the weak Cousin condition*
*(2) If $\mathbf{p}$, $\mathbf{q}$ are two prime ideals of $R$ such that $\mathbf{p}$ is maximal under $\mathbf{q}$, then the condition*

$$V(\mathbf{p}) \cap \mathrm{Ass}\,(\bigoplus_{k=0}^{i-1} E_k(C)) \neq \emptyset \textit{ implies } V(\mathbf{q}) \cap \mathrm{Ass}\,(\bigoplus_{k=0}^{i} E_k(C)) \neq \emptyset.$$

*Proof* Suppose that $\mathbf{p}, \mathbf{q}$ are prime ideals such that $\mathbf{p}$ is maximal under $\mathbf{q}$. We need to prove that if $\mathbf{q} \in \Phi_C(i)$ then $\mathbf{p} \in \Phi_C(i-1)$. This is equivalent to prove that if $\mathbf{p} \in \mathrm{Spec}(R) \setminus \Phi_C(i-1)$ then $\mathbf{q} \in \mathrm{Spec}(R) \setminus \Phi_C(i)$. By Proposition 3.7, the latter is exactly condition (2). □

*Example 3.13* (1) If $C$ is a 1-cotilting module, then the associated filtration by supports $\Phi_C$ is concentrated in 0 and trivially satisfies the weak Cousin condition.

(2) We will say that $R$ is *n-Gorenstein* when $\mathrm{inj.dim}(R) = n$ or, equivalently, when the Krull dimension of $R$ is $n$ and $\mathrm{inj.dim}(R) < \infty$, cf. [9, Corollary 3.4]. If $m \leq n$, then by [25, Theorem 18.8] the smallest $m$-cotilting class $\mathcal{C}_m$ from Example 3.10 corresponds to the sequence $(Y_0, \ldots, Y_{m-1})$ where $Y_i = \{\mathbf{p} \in \mathrm{Spec}(R) \mid \mathbf{p} \text{ has height } > i\}$ for any $0 \leq i < m$. This sequence trivially satisfies the weak Cousin condition.

(3) Let $(R, \mathfrak{m}, k)$ be a regular local ring of Krull dimension 2. Then the only cotilting classes whose associated filtration satisfies the weak Cousin condition are the 1-cotilting classes and the class $\mathcal{C}_2$ of Example (2) above. Indeed, if $(Y_0, Y_1)$ is a sequence of specialization-closed subsets associated to such a 2-cotilting class, then $Y_1$ cannot contain a prime ideal of height 1, because the weak Cousin conditions would give some prime ideal of height 0 in $Y_0$, and this ideal would be in $\mathrm{Ass}\, E_0(R) \cap Y_0$, which is a contradiction. Therefore we have $Y_1 = \{\mathfrak{m}\}$ and $Y_0 = \{\mathbf{p} \in \mathrm{Spec}(R)\text{: height}(\mathbf{p}) > 0\}$, which proves the claim.

## 4 Finitistic dimension

Recall that the *little finitistic dimension* findim $R$ of a ring $R$ is defined as the supremum of the projective dimensions attained on $\mathcal{P}^{<\infty}$.

When $R$ is an $n$-Gorenstein ring, then findim $R = n$. Moreover, every indecomposable injective module is isomorphic to a direct summand of some $E_k(R)$ with $0 \le k \le n$, as proven in [21], cf. also [1,6,26,27].

The general result for a commutative noetherian ring reads as follows.

**Theorem 4.1** *The following statements are equivalent.*

*(1)* findim $R \le n$
*(2)* $I = \bigoplus_{k=0}^{n} E_k(R)$ *is an injective cogenerator of* Mod$-R$.
*(3)* findim $R_\mathfrak{m} \le n$ *for all maximal ideals* $\mathfrak{m}$.

*Moreover, the conditions above are satisfied if the following holds true:*

*(4) for every* $\mathbf{p} \in \operatorname{Spec}(R)$ *there is* $0 \le k \le n$ *such that* $\mu_k(\mathbf{p}, R) \ne 0$.

*If R is Cohen-Macaulay, then all conditions are equivalent.*

*Proof* We use the filtration $\Psi = \Phi_R$ from Example 3.9 corresponding to the resolving subcategory $\mathcal{S}_\Psi = \mathcal{P}^{<\infty}$. We know from Theorem 3.3 that

$$\Psi(i) = \bigcup_{S \in \mathcal{P}^{<\infty}} \operatorname{Supp}(\operatorname{Ext}_R^{i+1}(S, R)) \text{ for each } i \ge 0, \text{ and } \Psi(i) = \operatorname{Spec}(R) \quad \text{for } i < 0.$$

(1) $\Longleftrightarrow$ (2). The condition findim $R \le n$ states that proj. dim$(S) \le n$ for all $S \in \mathcal{P}^{<\infty}$. This means by Lemma 2.4 that $\operatorname{Ext}_R^{k+1}(S, R) = 0$ for all $k \ge n$ and $S \in \mathcal{P}^{<\infty}$, or equivalently, $\Psi(n) = \emptyset$. By Proposition 3.7, the latter amounts to saying that for each maximal ideal $\mathfrak{m}$ there is $0 \le k \le n$ such that $\mathfrak{m} \in \operatorname{Ass} E_k(R)$, or in other words, $E(R/\mathfrak{m})$ is a direct summand of $E_k(R)$. This clearly means that $I = \bigoplus_{k=0}^{n} E_k(R)$ is an injective cogenerator of Mod$-R$.

(2) $\Longrightarrow$ (3). When localizing at $\mathfrak{m}$ the minimal injective resolution of $R$, one gets the minimal injective resolution of $R_\mathfrak{m}$ in Mod-$R_\mathfrak{m}$. To see that, use the exactness of the localization functor and [25, Theorem 18.4]. On the other hand, the localization of an injective cogenerator in Mod-$R$ is an injective cogenerator in Mod-$R_\mathfrak{m}$. So this implication follows from the implication (2) $\Longrightarrow$ (1) by localizing at $\mathfrak{m}$.

(3) $\Longrightarrow$ (1). Let $S \in \mathcal{P}^{<\infty}$. If $\mathfrak{m}$ is a maximal ideal, then proj. $\dim_{R_\mathfrak{m}}(S_\mathfrak{m}) < \infty$ and so proj. $\dim_{R_\mathfrak{m}}(S_\mathfrak{m}) \le n$. It follows that

$$0 = \operatorname{Ext}_{R_\mathfrak{m}}^{k+1}(S_\mathfrak{m}, R_\mathfrak{m}) = \operatorname{Ext}_R^{k+1}(S, R)_\mathfrak{m}$$

for all $k \ge n$ and all maximal ideals $\mathfrak{m}$. Thus $\operatorname{Ext}_R^{k+1}(S, R) = 0$ for all $k \ge n$ and all $S \in \mathcal{P}^{<\infty}$, which again by Lemma 2.4 means findim $R \le n$.

(4) $\Longrightarrow$ (2). As shown above, condition (2) can be restated as saying that for each maximal ideal $\mathfrak{m}$ there is $0 \le k \le n$ such that $\mathfrak{m} \in \operatorname{Ass} E_k(R)$, which by Lemma 2.2 means $\mu_k(\mathfrak{m}, R) \ne 0$. Now the implication is straightforward.

We finally prove (1) $\Longrightarrow$ (4) assuming that $R$ is Cohen-Macaulay. By definition, each localization $R_\mathfrak{m}$ is Cohen-Macaulay and the proof gets reduced to the case when $R$ is local, which we assume from now on. Note that when $R$ is local, the equivalence of (1) and (2) asserts that findim $R = \operatorname{depth}(R)$, see the Remark below. In particular, for each $\mathbf{p} \in \operatorname{Spec}(R)$ we have

$$\operatorname{findim} R_\mathbf{p} = \operatorname{depth}(R_\mathbf{p}) = \min\{k \mid \operatorname{Ext}_{R_\mathbf{p}}^k(k(\mathbf{p}), R_\mathbf{p}) \ne 0\}.$$

Using the Cohen-Macaulay condition on $R$, and denoting the Krull-dimension by dim, we get:

$$\begin{aligned}\text{findim}\, R_{\mathbf{p}} &= \text{depth}(R_{\mathbf{p}}) \leq \dim(R_{\mathbf{p}}) = \text{height}(\mathbf{p}) \leq \text{height}(\mathfrak{m}) \\ &= \dim(R) = \text{depth}(R) = \text{findim}\, R\end{aligned}$$

But the equality $\mu_k(\mathbf{p}, R) = \dim_{k(\mathbf{p})} \text{Ext}^k_{R_{\mathbf{p}}}(k(\mathbf{p}), R_{\mathbf{p}})$ in Lemma 2.2 tells that $\text{depth}(R_{\mathbf{p}}) = k$ entails $\mu_k(\mathbf{p}, R) \neq 0$. So, findimR $\leq n$ implies assertion (4) by the inequality above. □

*Remark 3* The equivalence of (1) and (2) in the last theorem reproves a classical result implicit in [8] (see [20]) stating that $\text{findim}\, R = \text{depth}(R)$ whenever $R$ is a local commutative noetherian ring. The proposition also proves that $\text{findim}\, R = \text{Sup}\{\text{findim}\, R_{\mathfrak{m}} \colon \mathfrak{m}$ is a maximal ideal of $R\}$ and, in case $R$ is Cohen Macaulay, the corresponding equality holds with "maximal" replaced by "prime" ideal.

*Remark 4* The proof of Theorem 4.1 shows that if $\text{findim}\, R = n < \infty$, then every filtration by supports $\Phi$ as in (ii) of Theorem 3.3 is concentrated in $0, \dots, n-1$. With Corollary 3.5, one readily sees that $\text{findim}\, R < \infty$ if and only if there is a smallest (co)tilting class in Mod-$R$.

Condition (4) of last theorem is not equivalent to the other ones, as the following example shows. This example was communicated to us by Dolors Herbera, and we thank her for the help.

*Example 4.2* Let $R$ be the localization of $K[X, Y, Z]/(X^2, XY^2, XZ^2)$ at the maximal ideal $\mathfrak{m} = (X, Y, Z)/(X^2, XY^2, XZ^2)$. Denoting by $x, y, z$ the images of $X, Y, Z$ in $R$, we see that all elements of $\mathfrak{m}$ are zero divisors since $xyz$ annihilates $\mathfrak{m}$. It follows that $\text{findim}\, R = \text{depth}\, R = 0$. On the other hand, if $\mathbf{p}$ is the ideal of $R$ generated by $x$ and $y$, then $\mathbf{p}$ is prime and $R_{\mathbf{p}}$ is isomorphic to $K[X, Y, Z]_{\mathbf{q}}/(X^2, XY^2, XZ^2)K[X, Y, Z]_{\mathbf{q}}$, where $\mathbf{q}$ is the ideal of $K[X, Y, Z]$ generated by $X$ and $Y$. But due to the fact that $Z$ is invertible in $K[X, Y, Z]_{\mathbf{q}}$, we see that $(X^2, XY^2, XZ^2)K[X, Y, Z]_{\mathbf{q}} = XK[X, Y, Z]_{\mathbf{q}}$. So $R_{\mathbf{p}}$ is isomorphic to $K[X, Y, Z]_{\mathbf{q}}/XK[X, Y, Z]_{\mathbf{q}}$, which is a regular ring of Krull dimension 1. Therefore $\text{findim}\, R_{\mathbf{p}} = 1$.

Recall that a left module $M$ over an arbitrary ring $R$ is said to be *Gorenstein injective* (resp. *Gorenstein flat*) when there is an exact (=acyclic) complex $Y^\bullet$ of left $R$-modules such that each $Y^i$ is injective (resp. flat), $M$ is the image of the differential $Y^{-1} \longrightarrow Y^0$, and the complex of abelian groups $\text{Hom}_R(I, Y^\bullet) : \ldots\text{Hom}_R(I, Y^{n-1}) \longrightarrow \text{Hom}_R(I, Y^n) \longrightarrow \text{Hom}_R(I, Y^{n-1}) \longrightarrow \ldots$ (resp. $I \otimes_R Y^\bullet : \ldots I \otimes_R Y^{n-1} \longrightarrow I \otimes_R Y^n \longrightarrow I \otimes_R Y^{n+1} \longrightarrow \ldots$) is exact for each injective module $I$ (see [15] and [16]). When $R$ is a Gorenstein commutative ring, our previous results give new characterizations of these modules.

**Corollary 4.3** *If $R$ is an $n$-Gorenstein ring, $I = \bigoplus_{k=0}^n E_k(R)$ and $M \in \text{Mod}-R$, then*

*(1) $M$ is Gorenstein-flat if and only if, for each $0 \leq i < n$, the module $E_i(M)$ is a direct sum of indecomposable modules $E(R/\mathbf{p})$, where $\mathbf{p}$ is a prime ideal of height $\leq i$.*
*(2) $M$ is Gorenstein-injective if and only if $\text{Ext}^i_R(I, M) = 0$ for all $i > 0$, or equivalently, $\text{Tor}^R_i(R/\mathbf{p}, M) = 0$ for all $0 \leq i < n$ and all prime ideals $\mathbf{p}$ of height $> i$.*

*Proof* We know from [6, 3.2 and 3.4] that the class of all Gorenstein-injective modules coincides with $I^\perp$, and also with the smallest tilting class $(\mathcal{P}^{<\infty})^\perp = \mathcal{T}_n$ from Example 3.10, and dually, the class of all Gorenstein-flat modules coincides with the smallest cotilting class $(\mathcal{P}^{<\infty})^{\intercal} = \mathcal{C}_n$. So, both classes correspond to the filtration by supports from Example 3.13 (2), which is given by the decreasing sequence $(Y_0, \dots, Y_{n-1})$ with $Y_i = \{\mathbf{p} \in \mathrm{Spec}(R) \mid \mathbf{p}$ has height $> i\}$ for any $0 \leq i < n$.

We infer from [7, Theorem 4.2] that a module $M$ is Gorenstein-injective if and only if it satisfies $\mathrm{Tor}_i^R(R/\mathbf{p}, M) = 0$ for all $i < n$ and all $\mathbf{p} \in Y_i$, that is, for all prime ideals of height $> i$.

Finally, by Corollary 3.5 a module $M$ is Gorenstein-flat if and only if $M$ is in $\mathcal{U}^o \cap \mathrm{Mod}\text{-}R$, where $(\mathcal{U}, \mathcal{U}^o[1])$ is the compactly generated t-structure associated to $(Y_0, \dots, Y_{n-1})$. By Lemma 2.10, this means that

$$\mathrm{Ass}\, E_i(M) \subseteq \mathrm{Spec}(R) \setminus Y_i = \{\mathbf{p} \in \mathrm{Spec}(R)\colon \mathrm{height}(\mathbf{p}) \leq i\}$$

or in other words, $E_i(M)$ is a direct sum of indecomposable modules of the form $E(R/\mathbf{p})$ where $\mathbf{p}$ is a prime ideal of height $\leq i$. □

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