Math. Z. (2014) 277:181–193
DOI 10.1007/s00209-013-1249-3

Mathematische Zeitschrift

On angles in Teichmüller spaces

Yun Hu · Yuliang Shen

Received: 25 September 2012 / Accepted: 4 October 2013 / Published online: 27 November 2013

Abstract We discuss the existence of the angle between two curves in Teichmüller spaces and show that, in any infinite dimensional Teichmüller space, there exist infinitely many geodesic triangles each of which has the same three vertices and satisfies the property that its three sides have the same and arbitrarily given length while its three angles are equal to any given three possibly different numbers from 0 to π. This implies that the sum of three angles of a geodesic triangle may be equal to any given number from 0 to 3π in an infinite dimensional Teichmüller space.

Keywords Teichmüller space · Geodesic segment · Angle · Beltrami coefficient

Mathematics Subject Classification (2000) Primary 32G15; Secondary 30C62 · 30F60

1 Introduction

The notion of geodesic segment plays an important role in the study of the geometry of a metric space. Recall that a geodesic segment in a metric space is a continuous curve such that for any subarc its length is equal to the distance between its two endpoints. It is well known that there always exists a geodesic segment between two points in any Teichmüller space (see [4]). However, there are some essential differences of the geodesic geometry between the finite and infinite dimensional Teichmüller spaces (see [1,2,8–11,18,19]). By Teichmüller's theorem, there exists precisely one geodesic segment between two points in finite dimensional Teichmüller spaces, while there exist infinitely many geodesic segments joining certain pair

Research supported by the National Natural Science Foundation of China and the Specialized Research Fund for the Doctoral Program of Higher Education of China.

Y. Hu · Y. Shen (✉)
Department of Mathematics, Soochow University, Suzhou 215006, People's Republic of China
e-mail: ylshen@suda.edu.cn

Y. Hu
e-mail: huyun_80@163.com

of points in any infinite dimensional Teichmüller space. The primary purpose of the paper is to explore the further geodesic property of infinite dimensional Teichmüller spaces.

It is known that the inner product on the tangent space to a Riemann manifold permits well-defined angle between two geodesic segments. Since the Teichmüller distance is induced by a Finsler structure (see [4,15,17]), it is not very clear how to define the angle between two geodesic segments in a Teichmüller space. Recently, following an idea of Professor Li, Yao [20] gave an approach to define the angle between two geodesic segments in a Teichmüller space, and showed that such an angle really exists in a finite dimensional Teichmüller space. Later, Li and Qi [12] gave a somewhat complicated condition under which there exists the angle between two geodesic segments (of special form) in an infinite dimensional Teichmüller space.

In this paper, we will continue to discuss the existence of the angle between two geodesic segments in (infinite dimensional) Teichmüller spaces. We first establish a variation formula for the Teichmüller distance, from which it is proved that the angle between two smooth geodesic segments exists in general. We then study the geometry of Teichmüller spaces from the point of angle. We show that in any infinite dimensional Teichmüller space, there exist infinitely many geodesic triangles each of which has the same three vertices and satisfies the property that its three sides have the same and arbitrarily given length while its three angles are equal to any given three possibly different numbers from 0 to π. This implies that the sum of three angles of a geodesic triangle may be equal to any given number from 0 to 3π in an infinite dimensional Teichmüller space. During the proof, we also find out that in any infinite dimensional Teichmüller space there do exist infinitely many pairs of geodesic segments between each pair of which the angle does not exist. Consequently, in the view of angle, the geometry of an infinite dimensional Teichmüller space is largely different from the standard Euclidean or hyperbolic geometry.

2 Preliminaries

In this section, we will recall some basic definitions and notations from Teichmüller theory. For more details see the books [3–5].

In what follows, R will always denote a hyperbolic Riemann surface covered by the unit disk in the complex plane. We denote by $M(R)$ the unit ball of the space $L^\infty(R)$ of all essentially bounded Beltrami differentials on R. We also denote by $SQ(R)$ the unit sphere of the space $Q(R)$ of all integrable holomorphic quadratic differentials on R.

For a given $\mu \in M(R)$, denote by f^μ the quasiconformal mapping with domain R and Beltrami coefficient μ, which is uniquely determined up to a conformal mapping on $R^\mu = f^\mu(R)$. Two elements μ and ν in $M(R)$ are equivalent, which is denoted by $\mu \sim \nu$, if f^μ and f^ν are Teichmüller equivalent, meaning as usual that there exists a conformal mapping g from R^μ onto R^ν such that f^ν and $g \circ f^\mu$ are homotopic (mod ∂R). Then $T(R) = M(R)/\sim$ is the Teichmüller space of R. Let $\Phi = \Phi_R : M(R) \to T(R)$ denote the canonical projection from $M(R)$ to $T(R)$ so that $\Phi(\mu)$ is the equivalence $[\mu]$. $\Phi(0) = [0]$ is called the base point of $T(R)$. It is known that $T(R)$ is finite dimensional precisely when R is of finite type, namely, R is a compact Riemann surface with possibly finitely many points removed. It is also known that $T(R)$ has a unique complex manifold structure such that Φ is a holomorphic split submersion.

For any Beltrami coefficient $\mu \in M(R)$, define

$$k_0(\mu) = \inf\{\|\nu\|_\infty : \nu \sim \mu\} \tag{2.1}$$

and set

$$K_0(\mu) = \frac{1 + k_0(\mu)}{1 - k_0(\mu)}. \tag{2.2}$$

Then the Teichmüller distance $\tau = \tau_R$ between points $\Phi(\mu_1)$ and $\Phi(\mu_2)$ is defined as

$$\tau(\Phi(\mu_1), \Phi(\mu_2)) = \frac{1}{2} \log K_0(\mu), \tag{2.3}$$

where μ is the Beltrami coefficient of the mapping $f^{\mu_1} \circ (f^{\mu_2})^{-1}$. It is known that the Teichmüller distance is compatible with the complex structure on $T(R)$, namely, it is the Kobayashi metric on $T(R)$. We will need an important fact about the Teichmüller distance: it is preserved under a so-called allowable map. Recall that a Beltrami coefficient μ in $M(R)$ induces an allowable map A_μ which maps $T(R)$ biholomorphically onto $T(R^\mu)$ and sends $[\mu]$ to the base point in $T(R^\mu)$.

We say that $\mu \in M(R)$ is extremal if $\|\mu\|_\infty = k_0(\mu)$. Then we also say that f^μ is extremal. It is well known (see [6,7,16] or Chapter 6 in [4]) that μ is extremal if and only if μ satisfies the Hamilton-Krushkal condition, that is, there exists a sequence (ϕ_n) in $SQ(R)$ such that

$$\lim_{n\to\infty} \Re \iint_R \mu\phi_n dxdy = \|\mu\|_\infty. \tag{2.4}$$

Such a sequence (ϕ_n) is called a Hamilton sequence for μ. It is called degenerate if $\phi_n \to 0$ locally uniformly in R.

We also need a fundamental inequality of Reich-Strebel (see [16] or Chapter 6 in [4]). We first introduce some notations. For any $\mu \in M(R)$, set

$$I(\mu) = I_R(\mu) = \sup_{\phi\in SQ(R)} \left| \Re \iint_R \frac{\mu\phi}{1-|\mu|^2} dxdy \right|, \tag{2.5}$$

$$H(\mu) = H_R(\mu) = \sup_{\phi\in SQ(R)} \left| \Re \iint_R \mu\phi dxdy \right|, \tag{2.6}$$

$$J(\mu) = J_R(\mu) = \sup_{\phi\in SQ(R)} \iint_R \frac{|\mu|^2|\phi|}{1-|\mu|^2} dxdy. \tag{2.7}$$

Then, it holds that

$$\frac{k_0(\mu)}{1-k_0(\mu)} - J(\mu) \le I(\mu) \le \frac{k_0(\mu)}{1+k_0(\mu)} + J(\mu). \tag{2.8}$$

As stated in Sect. 1, the Teichmüller distance is induced by a Finsler structure (see [4,15,17]). For $\mu \in M(R)$ and $\nu \in L^\infty(R)$, the Finsler structure $F = F_R$ is

$$F(\Phi(\mu), \Phi'(\mu)\nu) = \inf \left\{ \left\| \frac{\tilde{\nu}}{1-|\mu|^2} \right\|_\infty : \quad \tilde{\nu} \in L^\infty(R) \text{ with } \Phi'(\mu)\tilde{\nu} = \Phi'(\mu)\nu \right\}. \tag{2.9}$$

From (2.8), it can be deduced that

$$F(\Phi(\mu), \Phi'(\mu)\nu) = H_{R^\mu}\left(\left(\frac{\nu}{1-|\mu|^2}\cdot\frac{\partial_z f^\mu}{\overline{\partial_z f^\mu}}\right)\circ (f^\mu)^{-1}\right)$$
$$= \sup_{\psi\in SQ(R^\mu)}\left|\Re\iint_{R^\mu}\psi\left[\left(\frac{\nu}{1-|\mu|^2}\cdot\frac{\partial_z f^\mu}{\overline{\partial_z f^\mu}}\right)\circ (f^\mu)^{-1}\right]dudv\right|. \quad (2.10)$$

In particular, $F(\Phi(0), \Phi'(0)\nu) = H(\nu)$. It is known that the Finsler structure F is continuous on the tangent bundle of the Teichmüller space $T(R)$.

3 A variation formula

Let $\mu(t)$ be a continuous curve from $[0, t_0]$ into $M(R)$. We say $\mu(t)$ is differentiable at 0 if there exist some $\mu \in L^\infty(R)$ such that $\mu(t) = \mu(0) + t\mu + o(t)$ as $t \to 0_+$, or more precisely,

$$\lim_{t\to 0_+}\left\|\frac{\mu(t)-\mu(0)}{t} - \mu\right\|_\infty = 0. \quad (3.1)$$

We call μ the derivative of $\mu(t)$ at 0, and denote it by $\mu'(0)$.

Theorem 3.1 *Let $\mu(t)$ and $\nu(t)$ be two continuous curves from $[0, t_0]$ into $M(R)$ which are differentiable at 0 and satisfy $\mu(0) = \nu(0)$. Then it holds that*

$$\tau(\Phi(\mu(t)), \Phi(\nu(t))) = tF(\Phi(\mu(0)), \Phi'(\mu(0))(\mu'(0) - \nu'(0))) + o(t), \quad t \to 0_+. \quad (3.2)$$

The following corollary is an immediate consequence of Theorem 3.1. When R is a compact Riemann surface, it was proved by Yao [20] by a lengthy computation.

Corollary 3.1 *For any two Beltrami differentials μ and ν in $L^\infty(R)$, it holds that*

$$\tau(\Phi(t\mu), \Phi(t\nu)) = tH(\mu - \nu) + o(t), \quad t \to 0_+.$$

To prove Theorem 3.1, we need the following lemma, which is a direct consequence of the fundamental inequality (2.8).

Lemma 3.1 *Suppose R_t is a Riemann surface which may depend on $t \in [0, t_0]$. If $\eta(t) \in M(R_t)$ satisfies $\eta(t) = t\delta(t) + o(t)$ as $t \to 0_+$, where $\delta(t) \in L^\infty(R_t)$ satisfies $\delta(t) = O(1)$ as $t \to 0_+$, then it holds that*

$$\tau_{R_t}(\Phi_{R_t}(0), \Phi_{R_t}(\eta(t))) = tH_{R_t}(\delta(t)) + o(t), \quad t \to 0_+. \quad (3.3)$$

Proof By definition, $\tau_{R_t}(\Phi_{R_t}(0), \Phi_{R_t}(\eta(t))) = (1 + o(1))k_0(\eta(t))$ as $t \to 0_+$. Now we replace μ by $\eta(t)$ in the inequality (2.8) on the Riemann surface R_t. Clearly, $I_{R_t}(\eta(t))$ differs from $H_{R_t}(\eta(t))$ by a term of order t^2, both $k_0(\eta(t))/(1-k_0(\eta(t)))$ and $k_0(\eta(t))/(1+k_0(\eta(t)))$ differ from $k_0(\eta(t))$ also by a term of order t^2, while $J_{R_t}(\eta(t))$ is a term of ordered t^2. We conclude that $k_0(\eta(t)) = H_{R_t}(\eta(t)) + o(t)$ as $t \to 0_+$ and (3.3) follows.

Proof of Theorem 3.1 Let $\eta(t)$ be the Beltrami coefficient of $f^{\mu(t)} \circ (f^{\nu(t)})^{-1}$, namely,

$$\eta(t) = \left(\frac{\mu(t) - \nu(t)}{1 - \overline{\nu(t)}\mu(t)} \cdot \frac{\partial_z f^{\nu(t)}}{\overline{\partial_z f^{\nu(t)}}} \right) \circ (f^{\nu(t)})^{-1}.$$

By the differentiability of $\mu(t)$ and $\nu(t)$, we obtain

$$\frac{\mu(t) - \nu(t)}{1 - \overline{\nu(t)}\mu(t)} = \frac{t(\mu'(0) - \nu'(0))}{1 - |\nu(t)|^2} + o(t), \quad t \to 0_+.$$

Clearly, with $R_t = R^{\nu(t)}$, $\eta(t) \in M(R_t)$ satisfies the assumption of Lemma 3.1 with

$$\delta(t) = \left(\frac{(\mu'(0) - \nu'(0))}{1 - |\nu(t)|^2} \cdot \frac{\partial_z f^{\nu(t)}}{\overline{\partial_z f^{\nu(t)}}} \right) \circ (f^{\nu(t)})^{-1}.$$

By Lemma 3.1,

$$\tau_{R_t}(\Phi_{R_t}(0), \Phi_{R_t}(\eta(t))) = t H_{R_t}(\delta(t)) + o(t), \quad t \to 0_+.$$

But by (2.10) and the continuity of the Finsler structure F, as $t \to 0_+$ it holds that

$$\begin{aligned} H_{R_t}(\delta(t)) &= F(\Phi(\nu(t)), \Phi'(\nu(t))(\mu'(0) - \nu'(0))) \\ &\to F(\Phi(\mu(0)), \Phi'(\mu(0))(\mu'(0) - \nu'(0))). \end{aligned}$$

Thus,

$$\tau_{R_t}(\Phi_{R_t}(0), \Phi_{R_t}(\eta(t))) = t F(\Phi(\mu(0)), \Phi'(\mu(0))(\mu'(0) - \nu'(0))) + o(t), \quad t \to 0_+.$$

Finally, $\tau(\Phi(\mu(t)), \Phi(\nu(t))) = \tau_{R_t}(\Phi_{R_t}(0), \Phi_{R_t}(\eta(t)))$ by the distance-preserving property of the allowable map $A_{\nu(t)} : T(R) \to T(R_t)$ and (3.4) follows.

4 Existence of angle

We first introduce the notion of the angle between two joint curves in a general metric space (X, d). Let α and β be two continuous curves in X with one common endpoint p. For any $r > 0$, we choose $x(r) \in \alpha$ and $y(r) \in \beta$ such that the length of the sub-curve of α between p and $x(r)$ is the same as that of the sub-curve of β between p and $y(r)$ and equal to r. Then the angle at p between α and β, denoted by $\langle \alpha, \beta \rangle_p$, is defined as the number $\theta \in [0, \pi]$ by the equation

$$2 \sin \frac{\theta}{2} = \lim_{r \to 0} \frac{d(x(r), y(r))}{r}, \tag{4.1}$$

if the limit exists. Notice that when both α and β are geodesic segments in a Teichmüller space, the notion of the angle is reduced to the one introduced by Yao [20] and Li and Qi [12]. A trivial case is when $\alpha \equiv \beta$ in a neighborhood of p, then the angle at p between α and β exists and equals 0. Another trivial case is when $\alpha \cup \beta$ is geodesic at p, namely, there exists some closed neighborhood $U(p)$ of p such that $(\alpha \cup \beta) \cap U(p)$ is a geodesic segment. Then, it is clear that the angle at p between α and β exists and equals π. In what follows we always assume that $\alpha \neq \beta$ in a punctured neighborhood of p, and $\alpha \cup \beta$ is not geodesic at p, and call these two curves are distinct. As will be seen in the next section, the angle between two distinct geodesic segments in an infinite dimensional Teichmüller space may still be equal to 0 or π, however.

The following result, which follows directly from Theorem 3.1, gives a general condition under which there exists the angle between two geodesic segments in a Teichmüller space.

Theorem 4.1 *Let α and β be two geodesic segments in $T(R)$ given by the equations $\alpha = \Phi(\mu(t))$ and $\beta = \Phi(\nu(t))$, $t \in [0, t_0]$ $(t_0 < 1)$, respectively. Suppose both $\mu(t)$ and $\nu(t)$ are continuous from $[0, t_0]$ into $M(R)$, differentiable at 0 with $\mu(0) = \nu(0)$, and*

$$\tau(\Phi(\mu(0)), \Phi(\mu(t))) = \tau(\Phi(\nu(0)), \Phi(\nu(t))) = \frac{1}{2}\log\frac{1+t}{1-t}, \quad t \in [0, t_0]. \tag{4.2}$$

Then the angle at $\Phi(\mu(0))$ between α and β exists, and

$$2\sin\frac{\langle\alpha, \beta\rangle_{\Phi(\mu(0))}}{2} = F(\Phi(\mu(0)), \Phi'(\mu(0))(\mu'(0) - \nu'(0))). \tag{4.3}$$

We consider a special case of Theorem 4.1. For μ_0, μ in $M(R)$, we consider the curve

$$\alpha_{\mu_0,\mu}(t) = \Phi\left(\frac{\delta(\mu_0, \mu)\mu_0(1 - \bar{\mu}_0\mu) + t(\mu - \mu_0)}{\delta(\mu_0, \mu)(1 - \bar{\mu}_0\mu) + t\bar{\mu}_0(\mu - \mu_0)}\right), \quad t \in [0, \delta(\mu_0, \mu)], \tag{4.4}$$

where

$$\delta(\mu_0, \mu) = \left\|\frac{\mu - \mu_0}{1 - \bar{\mu}_0\mu}\right\|_\infty. \tag{4.5}$$

We set $\alpha_\mu = \alpha_{0,\mu}$ for simplicity. When $f^\mu \circ (f^{\mu_0})^{-1}$ is extremal, $\alpha_{\mu_0,\mu}$ is a geodesic segment, and

$$\tau([\mu_0], [\alpha_{\mu_0,\mu}(t)]) = \frac{1}{2}\log\frac{1+t}{1-t}.$$

We call it a standard geodesic segment joining $[\mu_0]$ to $[\mu]$. By the well-known theorem of Teichmüller, a geodesic segment α beginning at the base point in a finite dimensional Teichmüller space must be a standard, actually, a Teichmüller geodesic segment, that is, $\alpha = \alpha_\mu$ for a so-called Teichmüller differential $\mu = k|\phi|/\phi$ with $0 < k < 1$, $\phi \in SQ(R)$. While in an infinite dimensional Teichmüller space, a geodesic segment need not be standard.

The following corollary follows immediately from Theorem 4.1. It provides an affirmative answer to Problem A posed by Li-Qi [12].

Corollary 4.1 *Let μ_0, μ_1 and μ_2 be three Beltrami coefficients in $M(R)$ such that $f^{\mu_1} \circ (f^{\mu_0})^{-1}$ and $f^{\mu_2} \circ (f^{\mu_0})^{-1}$ are extremal. Then there exists the angle at the point $[\mu_0]$ between the two standard geodesic segments α_{μ_0,μ_1} and α_{μ_0,μ_2}, and*

$$\begin{aligned} &2\sin\frac{\langle\alpha_{\mu_0,\mu_1}, \alpha_{\mu_0,\mu_2}\rangle_{[\mu_0]}}{2} \\ &= F\left(\Phi(\mu_0), \Phi'(\mu_0)\left(\frac{(\mu_1 - \mu_0)(1 - |\mu_0|^2)}{\delta(\mu_0, \mu_1)(1 - \bar{\mu}_0\mu_1)} - \frac{(\mu_2 - \mu_0)(1 - |\mu_0|^2)}{\delta(\mu_0, \mu_2)(1 - \bar{\mu}_0\mu_2)}\right)\right). \end{aligned} \tag{4.6}$$

In particular, when $\mu_0 = 0$,

$$2\sin\frac{\langle\alpha_{\mu_1}, \alpha_{\mu_2}\rangle_{[0]}}{2} = H\left(\frac{\mu_1}{\|\mu_1\|_\infty} - \frac{\mu_2}{\|\mu_2\|_\infty}\right). \tag{4.7}$$

In the next section, we will see that in any infinite dimensional Teichmüller space there do exist infinitely many pairs of geodesic segments (one of which even may be a Teichmüller geodesic segment) between each pair of which the angle does not exist.

5 An example

In Sect. 4, we have introduced the notion of the angle between two curves in Teichmüller spaces and show that such defined angle exists in a much general situation. A natural question is to determine whether so-defined angle behaves like that under the standard Euclidean or hyperbolic geometry. Recall that a geodesic triangle Δ in a general metric space consists of three distinct geodesic segments, called the sides of Δ, any two of which have precisely one common endpoint. In this section, we will prove the following result.

Theorem 5.1 *Let R be a Riemann surface of infinite type so that $T(R)$ is infinite dimensional. Given any four numbers l and $\theta_1, \theta_2, \theta_3$ with $0 < l < \infty$ and $0 \le \theta_j \le \pi$ for $j = 1, 2, 3$, there exist infinitely many geodesic triangles in $T(R)$ each of which has the same three vertices and a common side and satisfies the property that its three sides have the same length l while its three angles are equal to $\theta_1, \theta_2, \theta_3$ respectively.*

Theorem 5.1 implies that the sum of three angles of a geodesic triangle may be equal to any given number from 0 to 3π in an infinite dimensional Teichmüller space. Thus, the geometry of an infinite dimensional Teichmüller space is largely different from the standard Euclidean or hyperbolic geometry in the view of angle. This also provides a negative answer to Problem B posed by Li-Qi [12] in the infinite dimensional case. During the proof of Theorem 5.1, we will find out that there do not exist the angles between infinitely many pairs of geodesic segments (one of which even may be a Teichmüller geodesic segment) in any infinite dimensional Teichmüller space, as stated at the end of Sect. 4 (see Lemma 5.1 below).

Proof of Theorem 5.1 Let R be a given Riemann surface of infinite type so that $T(R)$ is infinite dimensional. Choose an extremal Beltrami coefficient μ in $M(R)$ which satisfies $|\mu| \equiv k = \frac{e^{2l}-1}{e^{2l}+1} < 1$ and possess a degenerating Hamilton sequence (ϕ_n). It is known there even exist infinitely many Teichmüller differentials each of which possess a degenerating Hamilton sequence (see [13]).

Since (ϕ_n) is degenerating, we can always choose a sequence of compact subsets D_n of R, and a subsequence of (ϕ_n) which we still denote by (ϕ_n), such that

$$D_{n-1} \subset D_n, \quad R = \cup_{n=1}^{\infty} D_n, \tag{5.1}$$

and

$$\iint\limits_{D_n \setminus D_{n-1}} |\phi_n| dxdy = 1 + o(1), \quad n \to \infty. \tag{5.2}$$

We need to consider the inverse map $(f^\mu)^{-1}$, and denote by μ^* its Beltrami coefficient. Since μ is extremal with a degenerating Hamilton sequence, μ^* is also extremal, and has a degenerating Hamilton sequence, which we denote by (ϕ_n^*). Set $D_n^* = f^\mu(D_n)$. Then (D_n^*) is a sequence of compact subsets of R^μ. We now choose subsequences of (ϕ_n^*) and (D_n^*), which we still denote by (ϕ_n^*) and (D_n^*), such that

$$D_{n-1}^* \subset D_n^*, \quad R^\mu = \cup_{n=1}^{\infty} D_n^*, \tag{5.3}$$

and

$$\iint\limits_{D_n^* \setminus D_{n-1}^*} |\phi_n^*| dxdy = 1 + o(1), \quad n \to \infty. \tag{5.4}$$

We list two basic properties of these constructions. Set $R_1 = \cup_{n=0}^{\infty}(D_{2n+1} \setminus D_{2n})$ ($D_0 = \varnothing$), $R_2 = R \setminus R_1$, $R_1^{\mu} = \cup_{n=0}^{\infty}(D_{2n+1}^* \setminus D_{2n}^*)$ ($D_0^* = \varnothing$), $R_2^{\mu} = R^{\mu} \setminus R_1^{\mu}$. Let χ denote the characteristic function of a set. Then for any two numbers c_1 and c_2, we have the following two statements:

$$H_R((c_1\chi_{R_1} + c_2\chi_{R_2})\mu) = \|(c_1\chi_{R_1} + c_2\chi_{R_2})\mu\|_\infty = k\max(|c_1|, |c_2|). \tag{P_1}$$

$$H_{R^\mu}((c_1\chi_{R_1^\mu} + c_2\chi_{R_2^\mu})\mu^*) = \|(c_1\chi_{R_1^\mu} + c_2\chi_{R_2^\mu})\mu^*\|_\infty = k\max(|c_1|, |c_2|). \tag{P_2}$$

Step 1 Constructing a geodesic segment between $\Phi(\mu)$ and $\Phi(\chi_{R_1}\mu)$

For simplicity, set $\mu_1 = \chi_{R_1}\mu$. (P_1) implies that μ_1 is extremal, and $\|\mu_1\|_\infty = k$. To construct a geodesic segment between $[\mu]$ and $[\mu_1]$, we adapt some discussion from Li [10] and the second-named author [18]. Define μ_t in $M(R)$ joining μ to μ_1 as follows:

$$\mu_t = \left(\sigma(t)\chi_{R_1} + \frac{k-t}{k(1-kt)}\chi_{R_2}\right)\mu, \tag{5.5}$$

where $\sigma(t)$ is a continuous function of t in $[0, k]$ satisfying the following condition:

$$\max\left\{\frac{k-t}{k(1-kt)}, \frac{t}{k}\right\} \le \sigma(t) \le \min\left\{\frac{k+t}{k(1+kt)}, \frac{2k-(1+k^2)t}{k(1+k^2-2kt)}\right\}. \tag{5.6}$$

Clearly, $\|\mu_t\|_\infty = \sigma(t)k$. Using ($P_1$) again, we see that each μ_t is extremal. We first prove that $\Phi(\mu_t)$, $t \in [0, k]$, is a geodesic segment between $[\mu]$ and $[\mu_1]$.

In fact, if we set $f_t = f^{\mu_t}$, then the complex dilatation $\tilde{\mu}_t$ of $f_t \circ f_0^{-1}$ is

$$\tilde{\mu}_t = \left(\frac{\mu_t - \mu}{1-\bar{\mu}\mu_t} \cdot \frac{\partial_z f_0}{\overline{\partial_z f_0}}\right) \circ f_0^{-1} = \left(\frac{1-\sigma(t)}{1-k^2\sigma(t)}\chi_{R_1^\mu} + \frac{t}{k}\chi_{R_2^\mu}\right)\mu^*. \tag{5.7}$$

A direct but tedious computation from (5.6) yields $\|\tilde{\mu}_t\|_\infty = t$. On the other hand, by (P_2) we conclude that $\tilde{\mu}_t$ is extremal. So we get

$$\tau([\mu], [\mu_t]) = \frac{1}{2}\log\frac{1+t}{1-t}. \tag{5.8}$$

Now the Beltrami coefficient $\tilde{\nu}_t$ of $f_t \circ f_1^{-1}$ is

$$\tilde{\nu}_t = \left(\frac{\mu_t - \mu_1}{1-\bar{\mu}_1\mu_t} \cdot \frac{\partial_z f_1}{\overline{\partial_z f_1}}\right) \circ f_1^{-1}. \tag{5.9}$$

By the definition of μ_t and the inequality (5.6) we get

$$\|\tilde{\nu}_t\|_\infty = \frac{k-t}{1-kt}. \tag{5.10}$$

On the other hand, by (5.8) and (5.10),

$$\tau([\mu_1], [\mu_t]) \ge \tau([\mu], [\mu_1]) - \tau([\mu], [\mu_t]) = \frac{1}{2}\log\left(\frac{1+k}{1-k} \cdot \frac{1-t}{1+t}\right) = \frac{1}{2}\log\frac{1+\|\tilde{\nu}_t\|_\infty}{1-\|\tilde{\nu}_t\|_\infty}.$$

Therefore, $\tilde{\nu}_t$ is extremal, and

$$\tau([\mu_1], [\mu_t]) = \frac{1}{2}\log\frac{1+\|\tilde{\nu}_t\|_\infty}{1-\|\tilde{\nu}_t\|_\infty}. \tag{5.11}$$

Consequently,

$$\tau([\mu],[\mu_t])+\tau([\mu_1],[\mu_t])=\tau([\mu],[\mu_1]),\quad t\in[0,k].$$

This implies that $\Phi(\mu_t)$, $t\in[0,k]$, is a geodesic segment between $[\mu]$ and $[\mu_1]$.

Now we consider the standard geodesic segments $\alpha_\mu=\Phi(t/k\mu)$, $\alpha_{\mu_1}=\Phi(t/k\mu_1)$ and the above-constructed geodesic segment $\beta_\sigma=\Phi(\mu_t)$, $t\in[0,k]$. Then they have the same length $\frac{1}{2}\log\frac{1+k}{1-k}=l$. Clearly, $\alpha_\mu\cap\beta_\sigma=\{[\mu]\}$, $\alpha_{\mu_1}\cap\beta_\sigma=\{[\mu_1]\}$. Before we discuss the existence of the angles at $[\mu]$ and $[\mu_1]$, we point out when $\alpha_\mu\cup\beta_\sigma$ is (not) geodesic at $[\mu]$, and $\alpha_{\mu_1}\cup\beta_\sigma$ is (not) geodesic at $[\mu_1]$. Since both α_μ and β_σ are geodesic segments, $\alpha_\mu\cup\beta_\sigma$ is not geodesic at $[\mu]$ if and only if, as $t\to0_+$,

$$\tau([0],[\mu])+\tau([\mu],[\mu_t])>\tau([0],[\mu_t]),$$

which implies by (5.8) that

$$\sigma(t)<\frac{k+t}{k(1+kt)},\quad t\to0_+. \tag{5.12}$$

Similarly, $\alpha_{\mu_1}\cup\beta_\sigma$ is not geodesic at $[\mu_1]$ if and only if

$$\sigma(t)<\frac{2k-(1+k^2)t}{k(1+k^2-2kt)},\quad t\to k_-. \tag{5.13}$$

Under these two conditions, α_μ, α_{μ_1} and β_σ are distinct geodesic segments. In the following, we assume that σ satisfies (5.6), (5.12) and (5.13). Corollary (4.1) and (P_1) imply that

$$\langle\alpha_\mu,\alpha_{\mu_1}\rangle_{[0]}=2\arcsin\frac{H(\mu-\mu_1)}{2k}=2\arcsin\frac{H(\chi_{R_2}\mu)}{2k}=\frac{\pi}{3}.$$

We end this step by pointing out that $\sigma(t)\equiv1$ meets all the conditions (5.6), (5.12) and (5.13). In this case β_σ is the standard geodesic segment α_{μ,μ_1}. This will be essential in our final step to construct the desired geodesic triangle. We also point out that there are infinitely many continuous functions σ satisfying the conditions (5.6), (5.12) and (5.13), and two different such functions determine two different geodesic segments between $[\mu]$ and $[\mu_1]$.

Step 2 On the existence of the angle at $[\mu]$ between α_μ and β_σ

Lemma 5.1 *There exists the angle at $[\mu]$ between α_μ and β_σ if and only if σ is differentiable at* 0. *Furthermore, we can choose σ so that the angle at $[\mu]$ between α_μ and β_σ attains any given number from* 0 *to* π.

Proof We first assume σ is differentiable at 0. (5.6) implies that $|\sigma'(0)|\le(1-k^2)/k$. Then μ_t is differentiable with $\mu_0=\mu$, and

$$\mu'(0)=\lim_{t\to0_+}\frac{\mu_t-\mu_0}{t}=\left(\sigma'(0)\chi_{R_1}+\frac{k^2-1}{k}\chi_{R_2}\right)\mu. \tag{5.14}$$

Interchanging the endpoints, $\alpha_\mu=\Phi(\nu_t)$, with

$$\nu_t=\frac{k-t}{k(1-kt)}\mu,\quad t\in[0,k].$$

Then ν_t is differentiable with $\nu_0=\mu$, and

$$\nu'(0)=\lim_{t\to0_+}\frac{\nu_t-\mu}{t}=\frac{k^2-1}{k}\mu. \tag{5.15}$$

It is easy to see that

$$\tau([\mu],[\nu_t]) = \frac{1}{2}\log\frac{1+t}{1-t}. \tag{5.16}$$

By (5.8), (5.14)–(5.16), we find out that μ_t and ν_t satisfy the assumption in Theorem 4.1. Consequently, there exists the angle at $[\mu]$ between α_μ and β_σ, and

$$2\sin\frac{\langle\alpha_\mu,\beta_\sigma\rangle_{[\mu]}}{2} = F(\Phi(\mu),\Phi'(\mu)(\mu'(0)-\nu'(0))).$$

Noting that

$$\mu'(0)-\nu'(0) = \left(\frac{1-k^2}{k}+\sigma'(0)\right)\chi_{R_1}\mu,$$

we obtain

$$\begin{aligned} F(\Phi(\mu),\Phi'(\mu)(\mu'(0)-\nu'(0))) &= \left(\frac{1-k^2}{k}+\sigma'(0)\right)F(\Phi(\mu),\Phi'(\mu)(\chi_{R_1}\mu)) \\ &= \left(\frac{1}{k}+\frac{\sigma'(0)}{1-k^2}\right)H_{R^\mu}\left(\left(\chi_{R_1}\mu\cdot\frac{\partial_z f_0}{\overline{\partial_z f_0}}\right)\circ f_0^{-1}\right) \\ &= \left(\frac{1}{k}+\frac{\sigma'(0)}{1-k^2}\right)H_{R^\mu}(\chi_{R_1^\mu}\mu^*) = 1+\frac{k\sigma'(0)}{1-k^2}. \end{aligned}$$

Consequently,

$$2\sin\frac{\langle\alpha_\mu,\beta_\sigma\rangle_{[\mu]}}{2} = 1+\frac{k\sigma'(0)}{1-k^2}. \tag{5.17}$$

Since $|\sigma'(0)|\le(1-k^2)/k$, we see that $\langle\alpha_\mu,\beta_\sigma\rangle_{[\mu]}\in[0,\pi]$.

To show that $\langle\alpha_\mu,\beta_\sigma\rangle_{[\mu]}$ may attain any number from 0 to π, it is sufficient to show that for any number δ with $|\delta|\le(1-k^2)/k$, there exists a continuous function $\sigma_\delta(t)$ in $[0,k]$ which satisfies the inequalities (5.6), (5.12) and is differentiable at zero with $\sigma_\delta'(0)=\delta$. Actually, by (5.12), we only need to find $\sigma_\delta(t)$ when $t\to 0_+$. When $\delta=(1-k^2)/k$, we choose

$$\sigma_\delta(t) = 1+\frac{1-k^2}{k}t-(1-k^2)t^2,\quad t\to 0_+,$$

when $-(1-k^2)/k\le\delta<(1-k^2)/k$, we choose

$$\sigma_\delta(t) = 1+\delta t,\quad t\to 0_+.$$

Conversely, suppose that there exists the angle at $[\mu]$ between α_μ and β_σ. By (5.8) and (5.16) we conclude that

$$\lim_{t\to 0_+}\frac{\tau([\mu_t],[\nu_t])}{t} = 2\sin\frac{\langle\alpha_\mu,\beta_\sigma\rangle_{[\mu]}}{2}.$$

Let μ_t^* denote the Beltrami coefficient of $f^{\mu_t}\circ(f^{\nu_t})^{-1}$. It is routine to show that μ_t^* is extremal, and

$$k_0(\mu_t^*) = \left\|\frac{k\sigma(t)-\frac{k-t}{1-kt}}{1-\frac{k-t}{1-kt}k\sigma(t)}\right\|_\infty. \tag{5.18}$$

Then,

$$\tau([\mu_t],[\nu_t]) = \frac{1}{2}\log\frac{1+k_0(\mu_t^*)}{1-k_0(\mu_t^*)} = k_0(\mu_t^*) + o(t) = \frac{(\sigma(t)-1)k}{1-k^2} + t + o(t), \quad t \to 0_+.$$

Consequently, σ is differentiable at 0, and

$$\sigma'(0) = \frac{1-k^2}{k}\left(2\sin\frac{\langle\alpha_\mu,\beta_\sigma\rangle_{[\mu]}}{2} - 1\right).$$

Finally, we need to find a continuous function σ which satisfies (5.6), (5.12) but is not differentiable at 0 so that the angle at $[\mu]$ between α_μ and β_σ does not exist. In fact, the following function works:

$$\sigma(t) = 1 + \frac{1-k^2}{2k}t\sin^2\frac{1}{t}, \quad t \to 0_+.$$

Step 3 On the existence of the angle at $[\mu_1]$ between α_{μ_1} and β_σ

Lemma 5.2 *There exists the angle at $[\mu_1]$ between α_{μ_1} and β_σ if and only if σ is differentiable at k. Furthermore, we can choose σ so that the angle at $[\mu_1]$ between α_{μ_1} and β_σ attains any given number from 0 to π.*

Proof By the same reasoning as in Step 2, we can prove that the angle at $[\mu_1]$ between α_{μ_1} and β_σ exists if and only if σ is differentiable at k. In this case, $|\sigma'(k)| \le 1/k$, and

$$2\sin\frac{\langle\alpha_{\mu_1},\beta_\sigma\rangle_{[\mu_1]}}{2} = 1 - k\sigma'(k). \tag{5.19}$$

We omit the details here. Consequently, $\langle\alpha_{\mu_1},\beta_\sigma\rangle_{[\mu_1]} \in [0,\pi]$, and it attains any number from 0 to π. As above, it is sufficient to show that for any number η with $|\eta| \le 1/k$, there exists a continuous function $\sigma_\eta(t)$ in $[0,k]$ which satisfies the inequalities (5.8), (5.13) and is differentiable at k with $\sigma_\eta'(k) = \eta$. This time, by (5.13), we only need to find $\sigma_\delta(t)$ when $t \to k_-$.When $\eta = -1/k$, we choose

$$\sigma_\eta(t) = 1 - \frac{t-k}{k} - \frac{2(t-k)^2}{1-k^2}, \quad t \to k_-,$$

when $-1/k < \eta \le 1/k$, we choose

$$\sigma_\eta(t) = 1 + \eta(t-k), \quad t \to k_-.$$

□

Step 4 Constructing the triangle

We have proved that $\alpha_\mu \cup \alpha_{\mu_1} \cup \beta_\sigma$ is a geodesic triangle such that its two angles at $[\mu]$ and $[\mu_1]$ can be equal to any two given numbers from 0 to π for an appropriate function σ, but the third angle at [0] is fixed and equal to $\pi/3$. We now show that α_{μ_1} can be modified in a neighborhood of [0] so that the new angle at [0] can be equal to any given number from 0 to π.

Recall that the existence and the value of the angle between two geodesic segments are preserved under an allowable map. By (5.7) we see that $\tilde{\mu}_1 = \chi_{R_2^\mu}\mu^*$. Replacing μ, μ_1 and R_1 by μ^*, $\tilde{\mu}_1$ and R_2^μ respectively in the three steps above, we obtain a geodesic segment $\gamma_{\tilde{\sigma}}$ in $T(R^\mu)$ joining $[\mu^*]$ to $[\tilde{\mu}_1]$ such that the angle at $[\mu^*]$ between $\gamma_{\tilde{\sigma}}$ and α_{μ^*} exists and equals any given number from 0 to π, and $\gamma_{\tilde{\sigma}}$ coincides with the standard geodesic segment

$\alpha_{\mu^*,\tilde{\mu}_1}$ in a neighborhood of $[\tilde{\mu}_1]$ (as remarked at the end of Step 1). By the allowable map $A_\mu : T(R) \to T(R^\mu)$, $\gamma_{\tilde{\sigma}}$ becomes a geodesic segment $\tilde{\gamma}_{\tilde{\sigma}}$ in $T(R)$ joining [0] to $[\mu_1]$ such that the angle at [0] between $\tilde{\gamma}_{\tilde{\sigma}}$ and α_μ exists and equals any given number from 0 to π, and $\tilde{\gamma}_{\tilde{\sigma}}$ coincides with the standard geodesic segment α_{μ_1} in a neighborhood of $[\mu_1]$. Now let Δ_0 denote the geodesic triangle $\alpha_\mu \cup \beta_\sigma \cup \tilde{\gamma}_{\tilde{\sigma}}$ with vertices [0], $[\mu]$ and $[\mu_1]$. Then the angles at these three vertices exist and equal any three given numbers from 0 to π. This finishes our construction and completes the proof of Theorem 5.1.

Remark 1 In a finite dimensional Teichmüller space, the angle between two distinct geodesic segments exists and is always positive and less than π. It is not known whether the sum of three angles of a geodesic triangle in a finite dimensional Teichmüller space is less than π. This seems to be a difficult problem.

Remark 2 Masur [14] proved that any Teichmüller space of finite dimension (≥ 2) does not have negative curvature. In general, a metric space (X, d) is said to have negative curvature if for any geodesic triangle Δ in X with vertices A, B and C, $d(B, C) > 2d(\tilde{B}, \tilde{C})$, where $\tilde{B}$ is the midpoint of the side $\overline{AB}$ between A and B, and $\tilde{C}$ is the midpoint of the side $\overline{AC}$ between A and C. In fact, by Masur's discussion, for any Teichmüller space $T(R)$ of finite dimension (≥ 2) and any number $\epsilon > 1$, there exists a geodesic triangle Δ in $T(R)$ with vertices A, B and C such that $d(B, C) \leq \epsilon d(\tilde{B}, \tilde{C})$. It is of interest to determine whether one can take $\epsilon = 1$ here. Our proof of Theorem 5.1 shows that this is the case when $T(R)$ is infinite dimensional. Thus, an infinite dimensional Teichmüller space is far from having negative curvature.

In fact, we may consider the triangle $\Delta = \alpha_\mu \cup \alpha_{\mu_1} \cup \beta_\sigma$. Here we assume that

$$\sigma(t_0) = \frac{2 + \sqrt{1-k^2}}{1 + k^2 + \sqrt{1-k^2}} \tag{5.20}$$

with

$$t_0 = \frac{k}{1 + \sqrt{1-k^2}}. \tag{5.21}$$

By (5.8), (5.16) and (5.20), we obtain

$$\tau([\mu], [\mu_{t_0}]) = \tau([\mu], [\nu_{t_0}]) = \frac{1}{4} \log \frac{1+k}{1-k} = \frac{l}{2}.$$

Thus, $[\mu_{t_0}]$ and $[\nu_{t_0}]$ are the midpoints of β_σ and α_μ, respectively. Now it follows from (5.18), (5.20) and (5.21) that

$$\tau([\mu_{t_0}], [\nu_{t_0}]) = \frac{1}{2} \log \frac{1 + k_0(\mu^*_{t_0})}{1 - k_0(\mu^*_{t_0})} = \frac{1}{2} \log \frac{1+k}{1-k} = l = \tau([\mu_1], [0]).$$

Acknowledgments The authors would like to thank the referee for useful advice.

References

1. Earle, C.J., Kra, I., Krushkal, S.L.: Holomorphic motions and Teichmüller spaces. Trans. Am. Math. Soc. **343**, 927–948 (1994)
2. Earle, C.J., Li, Z.: Isometrically embedded polydisks in infinite dimensional Teichmüller spaces. J. Geom. Anal. **9**, 51–71 (1999)

3. Fletcher, A., Markovic, V.: Quasiconformal Maps and Teichmüller Theory Oxford Graduate Texts in Mathematics, vol. 11. Oxford University Press, Oxford (2007)
4. Gardiner, F.P.: Teichmüller Theory and Quadratic Differentials. Wiley-Interscience, New York (1987)
5. Gardiner, F.P., Lakic, N.: Quasiconformal Teichmüller Theory. Math. Surveys Monogr., 76, Amer. Math. Soc., Providence, RI (2000)
6. Hamilton, R.S.: Extremal quasiconformal mappings with prescribed boundary values. Trans. Am. Math. Soc. **138**, 399–406 (1969)
7. Krushkal, S.L.: Extremal quasiconformal mappings. Siberian Math. J. **10**, 411–418 (1969)
8. Li, Z.: Non-uniqueness of geodesics in infinite dimensional Teichmüller spaces. Complex Var. Theory Appl. **16**, 261–272 (1991)
9. Li, Z.: Non-uniqueness of geodesics in infinite dimensional Teichmüller spaces (II). Ann. Acad. Sci. Fenn. Math. **18**, 355–367 (1993)
10. Li, Z.: A note on geodesics in infinite dimensional Teichmüller spaces. Ann. Acad. Sci. Fenn. Math. **20**, 301–313 (1995)
11. Li, Z.: Closed geodesics and non-differentiability of the Teichmüller metric in infinite dimensional Teichmüller spaces. Proc. Am. Math. Soc. **124**, 1459–1465 (1996)
12. Li, Z., Qi, Y.: Fundamental inequalities of Reich-Strebel and triangles in a Teichmüller space. Contemp. Math. **575**, 283–297 (2012)
13. Li, Z., Shen, Y.: A remark on the weak uniform convexity of the space of holomorphic quadratic differentials. Beijing Math. **1**, 187–193 (1995)
14. Masur, H.: On a class of geodesics in Teichmüller space. Ann. Math. **102**, 205–221 (1975)
15. O'Byrne, B.: On Finsler geometry and applications to Teichmüller spaces. Ann. Math. Stud. **66**, 317–328 (1971)
16. Reich, E., Strebel, K.: Extremal quasiconformal mappings with given boundary values. In: Contributions to Analysis, A collection of papers dedicated to Lipman Bers, pp. 375–391. Academic Press, New York (1974)
17. Royden, H.: Automorphisms and isometrics of Teichmüller space. Ann. Math. Stud. **66**, 369–383 (1971)
18. Shen, Y.: On the geometry of infinite dimensional Teichmüller spaces. Acta Math. Sinica **13**, 413–420 (1997)
19. Tanigawa, H.: Holomorphic families of geodesic disks in infinite dimensional Teichmüller spaces. Nagoya Math. J. **127**, 117–128 (1992)
20. Yao, G.: A binary infinitesimal form of Teichmüller metric, preprint, arXiv:0901.3822