Math. Z. (2014) 276:799–827
DOI 10.1007/s00209-013-1223-0

Mathematische Zeitschrift

Diophantine approximation of the orbit of 1 in the dynamical system of beta expansions

Bing Li · Tomas Persson · Baowei Wang · Jun Wu

Received: 12 April 2013 / Accepted: 22 August 2013 / Published online: 20 September 2013

Abstract We consider the distribution of the orbits of the number 1 under the β-transformations T_β as β varies. Mainly, the size of the set of $\beta > 1$ for which a given point can be well approximated by the orbit of 1 is measured by its Hausdorff dimension. The dimension of the following set

$$E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \Big\{\beta > 1 : |T_\beta^n 1 - x_0| < \beta^{-\ell_n},\ \text{for infinitely many},\ n \in \mathbb{N}\Big\}$$

is determined, where x_0 is a given point in $[0, 1]$ and $\{\ell_n\}_{n\geq 1}$ is a sequence of integers tending to infinity as $n \to \infty$. For the proof of this result, the notion of the recurrence time of a word in symbolic space is introduced to characterise the lengths and the distribution of cylinders (the set of β with a common prefix in the expansion of 1) in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$.

Keywords β-expansion · Diophantine approximation · Hausdorff dimension

B. Li
Department of Mathematics, South China University of Technology,
Guangzhou 510640, People's Republic of China
e-mail: libing0826@gmail.com

B. Li
Department of Mathematical Sciences, University of Oulu,
P.O. Box 3000, 90014 Oulu, Finland

T. Persson
Centre for Mathematical Sciences, Lund University,
Box 118, 22100 Lund, Sweden
e-mail: tomasp@maths.lth.se

B. Wang (✉) · J. Wu
School of Mathematics and Statistics, Huazhong University of Science and Technology,
Wuhan 430074, People's Republic of China
e-mail: bwei_wang@yahoo.com.cn

J. Wu
e-mail: jun.wu@mail.hust.edu.cn

Mathematics Subject Classification (2010) 11K55 · 28A80

1 Introduction

The study of Diophantine properties of the orbits in a dynamical system has recently received much attention. This study contributes to a better understanding of the distribution of the orbits in a dynamical system. Let $(X, \mathcal{B}, \mu, T)$ be a measure-preserving dynamical system with a consistent metric d. If T is ergodic with respect to the measure μ, then Birkhoff's ergodic theorem yields the following hitting property, namely, for any $x_0 \in X$ and μ-almost all $x \in X$,

$$\liminf_{n\to\infty} d(T^n(x), x_0) = 0. \tag{1.1}$$

One can then ask, what are the quantitative properties of the convergence speed in (1.1)? More precisely, for a given sequence of balls $B(x_0, r_n)$ with center $x_0 \in X$ and shrinking radius $\{r_n\}$, what are the metric properties of the set

$$F(x_0, \{r_n\}) := \Big\{x \in X : d(T^n x, x_0) < r_n \text{ for infinitely many } n \in \mathbb{N}\Big\}$$

in the sense of measure and in the sense of dimension? More generally, let $\{B_n\}_{n\geq 1}$ be a sequence of measurable sets with $\mu(B_n)$ decreasing to 0 as $n \to \infty$. The study of the metric properties of the set

$$\Big\{x \in X : T^n x \in B_n \text{ for infinitely many } n \in \mathbb{N}\Big\} \tag{1.2}$$

is called the dynamical Borel–Cantelli Lemma [6] or the shrinking target problem [12].

In this paper, we consider a modified shrinking target problem. Let us begin with an example to illustrate the motivation. Let $R_\alpha : x \mapsto x + \alpha$ be a rotation map on the unit circle. Then the set studied in classical inhomogeneous Diophantine approximation can be written as

$$\Big\{\alpha \in \mathbb{Q}^c : |R_\alpha^n 0 - x_0| < r_n, \text{ for infinitely many } n \in \mathbb{N}\Big\}, \tag{1.3}$$

where $|x - y|$ means the distance between $x, y \in \mathbb{R}$. The size of the set (1.3) in the sense of Hausdorff measure and Hausdorff dimension was studied by Bugeaud [3], Levesley [15], Bugeaud and Chevallier [4] etc. Compared with the shrinking target problem (1.2), instead of considering the Diophantine properties in *one* given system, the set (1.3) concerns the properties of the orbit of some given point (the orbit of 0) in *a family* of dynamical systems. It is the set of parameters α such that R_α share some common properties.

Following this idea, in this paper, we consider the same setting as (1.3) in the dynamical systems $([0, 1], T_\beta)$ of β-transformations with β varying in the *parameter space* $\{\beta \in \mathbb{R} : \beta > 1\}$.

It is well-known that β-transformations are typical examples of one-dimensional expanding systems, whose properties are reflected by the orbit of some critical point. In the case of β-transformations, this critical point is the unit 1. This is because the β-expansion of 1 (or the orbit of 1 under T_β) can completely characterise all admissible sequences in the β-shift space (see [17]), the lengths and the distribution of cylinders induced by T_β [8], etc. Upon this, in this current work, we study the Diophantine properties of $\{T_\beta^n 1\}_{n\geq 1}$, the orbit of 1, as β varies in the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$.

Blanchard [1] gave a kind of classification of the parameters in the space $\{\beta \in \mathbb{R} : \beta > 1\}$ according to the distribution of $\mathcal{O}_\beta := \{T_\beta^n 1\}_{n\geq 1}$: (i) ultimately zero; (ii) ultimately non-zero periodic; (iii) 0 is not an accumulation point of $\mathcal{O}_\beta$ (exclude those β in classes (i,ii)); (iv) non-dense in [0, 1] (exclude β's in classes (i,ii,iii)); and (v) dense in [0, 1]. It was shown by Schmeling [21] that the class (v) is of full Lebesgue measure (the results in [21] give more, that for almost all β, all allowed words appear in the expansion of 1 with regular frequencies). This dense property of $\mathcal{O}_\beta$ for almost all β gives us a type of hitting property, i.e., for any $x_0 \in [0, 1]$,

$$\liminf_{n\to\infty} |T_\beta^n 1 - x_0| = 0, \quad \text{for } \mathcal{L}\text{-a.e. } \beta > 1, \tag{1.4}$$

where $\mathcal{L}$ is the Lebesgue measure on $\mathbb{R}$. Similarly as for (1.1), we would like to investigate the possible convergence speed in (1.4).

Fix a point $x_0 \in [0, 1]$ and a sequence of positive integers $\{\ell_n\}_{n\geq 1}$. Consider the set of $\beta > 1$ for which x_0 can be well approximated by the orbit of 1 under the β-transformations with given shrinking speed, namely the set

$$E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \Big\{\beta > 1 : |T_\beta^n 1 - x_0| < \beta^{-\ell_n}, \text{ for infinitely many } n \in \mathbb{N}\Big\}. \tag{1.5}$$

This can be viewed as a kind of shrinking target problem in the parameter space.

When $x_0 = 0$ and $\ell_n = \alpha n$ ($\alpha > 0$), Persson and Schmeling [18] proved that

$$\dim_{\mathsf{H}} E(\{\alpha n\}_{n\geq 1}, 0) = \frac{1}{1+\alpha},$$

where $\dim_{\mathsf{H}}$ denotes the Hausdorff dimension. For a general $x_0 \in [0, 1]$ and a sequence $\{\ell_n\}$, we have the following.

Theorem 1.1 *Let $x_0 \in [0, 1]$ and let $\{\ell_n\}_{n\geq 1}$ be a sequence of positive integers such that $\ell_n \to \infty$ as $n \to \infty$. Then*

$$\dim_{\mathsf{H}} E\big(\{\ell_n\}_{n\geq 1}, x_0\big) = \frac{1}{1+\alpha}, \quad \text{where } \alpha = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

In other words, the set in (1.5) consists of the points in the parameter space $\{\beta > 1 : \beta \in \mathbb{R}\}$ for which the orbit $\{T_\beta^n 1 : n \geq 1\}$ is close to the same point $x(\beta) = x_0$ for infinitely many moments in time. What can be said if the point $x(\beta)$ is also allowed to vary continuously with $\beta > 1$? Let $x = x(\beta)$ be a function on $(1, +\infty)$, taking values on [0, 1]. The setting (1.5) changes to

$$\widetilde{E}\big(\{\ell_n\}_{n\geq 1}, x\big) = \Big\{\beta > 1 : |T_\beta^n 1 - x(\beta)| < \beta^{-\ell_n}, \text{ for infinitely many } n \in \mathbb{N}\Big\}. \tag{1.6}$$

As will become apparent, the proof of Theorem 1.1 also works for this general case $x = x(\beta)$ after some minor adjustments, and we can therefore state the following theorem.

Theorem 1.2 *Let $x = x(\beta) : (1, +\infty) \to [0, 1]$ be a Lipschitz continuous function and $\{\ell_n\}_{n\geq 1}$ be a sequence of positive integers such that $\ell_n \to \infty$ as $n \to \infty$. Then*

$$\dim_{\mathsf{H}} \widetilde{E}\big(\{\ell_n\}_{n\geq 1}, x\big) = \frac{1}{1+\alpha}, \quad \text{where } \alpha = \liminf_{n\to\infty} \frac{\ell_n}{n}.$$

Theorems 1.1 (as well as Theorem 1.2) can be viewed as a generalisation of the result of Persson and Schmeling [18]. But there are essential differences between the three cases when the target $x_0 = 0$, $x_0 \in (0, 1)$ and $x_0 = 1$. The following three remarks serve as an outline of the differences.

Remark 1 The generality of $\{\ell_n\}_{n\geq 1}$ gives no extra difficulty compared with the special sequence $\{\ell_n = \alpha n\}_{n\geq 1}$. However, there are some essential difficulties when generalizing x_0 from zero to non-zero. The idea used in [18], to construct a suitable Cantor subset of $E\big(\{\ell_n\}_{n\geq 1}, x_0\big)$ to get the lower bound of $\dim_{\mathsf{H}} E(\{\ell_n\}_{n\geq 1}, x_0)$, is not applicable for $x_0 \neq 0$. For any $\beta > 1$, let

$$\varepsilon_1(x,\beta), \varepsilon_2(x,\beta), \ldots$$

be the digit sequence of the β-expansion of x. To guarantee that the two points $T_\beta^n 1$ and x_0 are close enough, a natural idea is to require that

$$\varepsilon_{n+1}(1,\beta) = \varepsilon_1(x_0,\beta),\ \ldots,\ \varepsilon_{n+\ell}(1,\beta) = \varepsilon_\ell(x_0,\beta) \tag{1.7}$$

for some $\ell \in \mathbb{N}$ sufficiently large. When $x_0 = 0$, the β-expansions of x_0 are the same (all digits are 0) no matter what β is. Thus to fulfill (1.7), one needs only to consider those β for which a long string of zeros follows $\varepsilon_n(1,\beta)$ in the β-expansion of 1. But when $x_0 \neq 0$, the β-expansions of x_0 under different β are different. Furthermore, the expansion of x_0 is not known to us, since β has not been determined yet. This difference constitutes a main difficulty in constructing points β fulfilling the conditions in the definition of $E\big(\{\ell_n\}_{n\geq 1}, x_0\big)$.

To overcome this difficulty, a better understanding of the parameter space seems necessary. In Sect. 3, we analyse the length and the distribution of a cylinder in the parameter space which relies heavily on a new notion that we call *the recurrence time of a word*.

Remark 2 When $x_0 \neq 1$, the set $E(\{\ell_n\}_{n\geq 1}, x_0)$ can be regarded as a type of shrinking target problem with fixed target. While when $x_0 = 1$, the set $E(\{\ell_n\}_{n\geq 1}, x_0)$ is the set of β for which the orbit of 1 returns to a shrinking neighbourhood of itself infinitely often. In this case, we have a so-called recurrence problem. There are some differences between these two cases. Therefore, their proofs for the lower bounds of $\dim_{\mathsf{H}} E(\{\ell_n\}_{n\geq 1}, x_0)$ are given separately in Sects. 5 and 6.

Remark 3 If $x(\beta)$, when developed in base β, is the same for all $\beta \in (\beta_0, \beta_1)$, then with an argument based on Theorem 15 in [18], one can give the dimension of $\widetilde{E}(\{\ell_n\}_{n\geq 1}, x(\beta))$. However as far as a general function $x(\beta)$ is concerned, the idea used in proving Theorem 1.1 can be applied to give a complete solution of the dimension of $\widetilde{E}(\{\ell_n\}_{n\geq 1}, x(\beta))$.

For more dimensional results related to the β-transformations, the readers are referred to [10,19,21,25,26] and references therein. For more dimensional results concerning the shrinking target problems, see [2,5,9,11–13,22–24,27] and references therein.

2 Preliminary

This section is devoted to recalling some basic properties of β-transformations and fixing some notation. For more information on β-transformations, see [1,14,17,20] and references therein.

The β-expansion of real numbers was first introduced by Rényi [20], which is given by the following algorithm. For any $\beta > 1$, let

$$T_\beta(0) := 0, \quad T_\beta(x) = \beta x - \lfloor \beta x \rfloor,\ x \in (0,1), \tag{2.1}$$

where $\lfloor \xi \rfloor$ is the integer part of $\xi \in \mathbb{R}$. By taking

$$\varepsilon_n(x,\beta) = \lfloor \beta T_\beta^{n-1} x \rfloor \in \mathbb{N}$$

recursively for each $n \geq 1$, every $x \in [0, 1)$ can be uniquely expanded into a finite or an infinite sum

$$x = \frac{\varepsilon_1(x, \beta)}{\beta} + \cdots + \frac{\varepsilon_n(x, \beta)}{\beta^n} + \cdots, \tag{2.2}$$

which is called the β-expansion of x and the sequence $\{\varepsilon_n(x, \beta)\}_{n\geq 1}$ is called the digit sequence of x. We also write (2.2) as $\varepsilon(x, \beta) = (\varepsilon_1(x, \beta), \ldots, \varepsilon_n(x, \beta), \ldots)$. The system $([0, 1), T_\beta)$ is called a *β-transformation*, *β-dynamical system* or a *β-system*.

Definition 2.1 A finite or an infinite sequence $(w_1, w_2, \ldots)$ is said to be *admissible* (with respect to the base β), if there exists an $x \in [0, 1)$ such that the digit sequence (in the β-expansion) of x begins with $(w_1, w_2, \ldots)$.

Denote by Σ_β^n the collection of all β-admissible sequences of length n and by Σ_β that of all infinite admissible sequences. Write $\mathcal{A} = \{0, 1, \ldots, \beta - 1\}$ when β is an integer and otherwise, $\mathcal{A} = \{0, 1, \ldots, \lfloor\beta\rfloor\}$. Let S_β be the closure of Σ_β under the product topology on $\mathcal{A}^{\mathbb{N}}$. Then $(S_\beta, \sigma|_{S_\beta})$ is a subshift of the symbolic space $(\mathcal{A}^{\mathbb{N}}, \sigma)$, where σ is the shift map on $\mathcal{A}^{\mathbb{N}}$.

Let us now turn to the *infinite β-expansion of 1*, which plays an important role in the study of β-expansions. At first, apply the algorithm (2.1) to the number $x = 1$. Then the number 1 can also be expanded into a series, denoted by

$$1 = \frac{\varepsilon_1(1, \beta)}{\beta} + \cdots + \frac{\varepsilon_n(1, \beta)}{\beta^n} + \cdots.$$

If the above series is finite, i.e. there exists $m \geq 1$ such that $\varepsilon_m(1, \beta) \neq 0$ but $\varepsilon_n(1, \beta) = 0$ for all $n > m$, then β is called a simple Parry number. In this case, the digit sequence of 1 is defined by

$$\varepsilon^*(1, \beta) := (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots) = (\varepsilon_1(1, \beta), \ldots, \varepsilon_{m-1}(1, \beta), \varepsilon_m(1, \beta) - 1)^\infty,$$

where $(w)^\infty$ denotes the periodic sequence $(w, w, w, \ldots)$. If β is not a simple Parry number, the digit sequence of 1 is defined by

$$\varepsilon^*(1, \beta) := (\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots) = (\varepsilon_1(1, \beta), \varepsilon_2(1, \beta), \ldots).$$

In both cases, the sequence $(\varepsilon_1^*(\beta), \varepsilon_2^*(\beta), \ldots)$ is called *the infinite β-expansion of 1* and we always have that

$$1 = \frac{\varepsilon_1^*(\beta)}{\beta} + \cdots + \frac{\varepsilon_n^*(\beta)}{\beta^n} + \cdots. \tag{2.3}$$

The lexicographical order $\prec$ between two infinite sequences is defined as follows:

$$w = (w_1, w_2, \ldots, w_n, \ldots) \prec w' = (w_1', w_2', \ldots, w_n', \ldots)$$

if there exists $k \geq 1$ such that $w_j = w_j'$ for $1 \leq j < k$, while $w_k < w_k'$. The notation $w \preceq w'$ means that $w \prec w'$ or $w = w'$. This ordering can be extended to finite blocks by identifying a finite block $(w_1, \ldots, w_n)$ with the sequence $(w_1, \ldots, w_n, 0, 0, \ldots)$.

The following result due to Parry [17] is a criterion for the admissibility of a sequence which relies heavily on the *infiniteβ-expansion of 1*.

Theorem 2.2 (Parry [17])

(1) Let $\beta > 1$. For each $n \geq 1$, a block of non-negative integers $w = (w_1, \ldots, w_n)$ belongs to Σ_β^n if and only if

$$\sigma^i w \preceq \varepsilon_1^*(1, \beta), \ldots, \varepsilon_{n-i}^*(1, \beta) \quad \textit{for all } 0 \leq i < n.$$

(2) The function $\beta \mapsto \varepsilon^*(1, \beta)$ is increasing with respect to the variable $\beta > 1$. Therefore, if $1 < \beta_1 < \beta_2$, then

$$\Sigma_{\beta_1} \subset \Sigma_{\beta_2}, \quad \Sigma_{\beta_1}^n \subset \Sigma_{\beta_2}^n \quad (\textit{for all } n \geq 1).$$

At the same time, Parry also presented a characterisation of when a sequence of integers is the infinite expansion of 1 for some $\beta > 1$. First, we introduce the notion of a *self-admissible* word.

Definition 2.3 A word $w = (\varepsilon_1, \ldots, \varepsilon_n)$ is called self-admissible if for all $1 \leq i < n$

$$\sigma^i(\varepsilon_1, \ldots, \varepsilon_n) \preceq \varepsilon_1, \ldots, \varepsilon_{n-i}.$$

An infinite digit sequence $w = (\varepsilon_1, \varepsilon_2, \ldots)$ is said to be self-admissible if for all $i \geq 1$, $\sigma^i w \preceq w$.

Theorem 2.4 (Parry [17]) *A digit sequence $(\varepsilon_1, \varepsilon_2, \ldots)$ with $\varepsilon_1 \geq 1$ is the infinite expansion of 1 for some $\beta > 1$ if and only if it is self-admissible.*

The following result of Rényi implies that the dynamical system $([0, 1), T_\beta)$ admits $\log \beta$ as its topological entropy. Here and hereafter $\sharp$ denotes the cardinality of a finite set.

Theorem 2.5 (Rényi [20]) *Let $\beta > 1$. For any $n \geq 1$,*

$$\beta^n \leq \sharp\Sigma_\beta^n \leq \beta^{n+1}/(\beta - 1).$$

3 Distribution of regular cylinders in parameter space

From this section on, we turn to the parameter space $\{\beta \in \mathbb{R} : \beta > 1\}$, instead of considering a fixed $\beta > 1$. We will address the length of a cylinder in the parameter space, which is closely related to the notion of *recurrence time*.

Definition 3.1 Let $(\varepsilon_1, \ldots, \varepsilon_n)$ be self-admissible. A cylinder in the parameter space is defined as

$$I_n^P(\varepsilon_1, \ldots, \varepsilon_n) = \Big\{ \beta > 1 : \varepsilon_1(1, \beta) = \varepsilon_1, \ldots, \varepsilon_n(1, \beta) = \varepsilon_n \Big\},$$

i.e., the set of β for which the β-expansion of 1 begins with the common prefix $\varepsilon_1, \ldots, \varepsilon_n$. Denote by C_n^P the collection of cylinders of order n in the parameter space.

When $(\varepsilon_1, \ldots, \varepsilon_n)$ is a self-admissible word, we will sometimes talk about "the cylinder $(\varepsilon_1, \ldots, \varepsilon_n)$". When we do so, we mean the cylinder $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$.

3.1 Recurrence time of words

Definition 3.2 Let $w = (\varepsilon_1, \ldots, \varepsilon_n)$ be a word of length n. The recurrence time $\tau(w)$ of w is defined as

$$\tau(w) := \inf\left\{k \geq 1 : \sigma^k(\varepsilon_1, \ldots, \varepsilon_n) = (\varepsilon_1, \ldots, \varepsilon_{n-k})\right\}.$$

If such an integer k does not exist, then $\tau(w)$ is defined to be n and w is said to be a non-recurrent word.

From the definition of the recurrence time $\tau(\cdot)$, it is clear that if $w = (\varepsilon_1, \ldots, \varepsilon_n)$ is recurrent with $\tau(w) = k < n$, then

$$(\varepsilon_1, \ldots, \varepsilon_n) = (\varepsilon_1, \ldots, \varepsilon_k)^{\lfloor n/k \rfloor} \varepsilon_1, \ldots, \varepsilon_{n-k\lfloor n/k \rfloor},$$

where $\lfloor \xi \rfloor$ denotes the integer part of ξ.

Applying the definition of recurrence time and the criterion of self-admissibility of a sequence, we obtain the following.

Lemma 3.3 *Let $w = (\varepsilon_1, \ldots, \varepsilon_n)$ be self-admissible with the recurrence time $\tau(w) = k$. Then for each $1 \leq i < k$,*

$$\varepsilon_{i+1}, \ldots, \varepsilon_k \prec \varepsilon_1, \ldots, \varepsilon_{k-i}. \tag{3.1}$$

Proof The self-admissibility of w ensures that

$$\varepsilon_{i+1}, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_n \preceq \varepsilon_1, \ldots, \varepsilon_{k-i}, \varepsilon_{k-i+1}, \ldots, \varepsilon_{n-i}.$$

The recurrence time $\tau(w) = k$ of w implies that for $1 \leq i < k$,

$$\varepsilon_{i+1}, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_n \neq \varepsilon_1, \ldots, \varepsilon_{k-i}, \varepsilon_{k-i+1}, \ldots, \varepsilon_{n-i}.$$

Combining the above two facts, we arrive at

$$\varepsilon_{i+1}, \ldots, \varepsilon_k, \varepsilon_{k+1}, \ldots, \varepsilon_n \prec \varepsilon_1, \ldots, \varepsilon_{k-i}, \varepsilon_{k-i+1}, \ldots, \varepsilon_{n-i}. \tag{3.2}$$

When $k = n$, $(\varepsilon_{k+1}, \ldots, \varepsilon_n)$ is an empty word. Then the result follows directly by (3.2). Now we assume $k < n$ and compare the suffixes of the two words in (3.2). By the definition of $\tau(w)$, the left one ends with

$$\varepsilon_{k+1}, \ldots, \varepsilon_n = \varepsilon_1, \ldots, \varepsilon_{n-k},$$

while the right one ends with

$$\varepsilon_{k-i+1}, \ldots, \varepsilon_{n-i}.$$

By the self-admissibility of $\varepsilon_1, \ldots, \varepsilon_n$, we get

$$\varepsilon_{k+1}, \ldots, \varepsilon_n = \varepsilon_1, \ldots, \varepsilon_{n-k} \succeq \varepsilon_{k-i+1}, \ldots, \varepsilon_{n-i}. \tag{3.3}$$

Then the formulae (3.2) and (3.3) enable us to conclude the result. □

We give a sufficient condition to ensure that a word is non-recurrent.

Lemma 3.4 *Assume that $(\varepsilon_1, \ldots, \varepsilon_{m-1}, \varepsilon_m)$ and $(\varepsilon_1, \ldots, \varepsilon_{m-1}, \overline{\varepsilon}_m)$ are both self-admissible and $0 \leq \varepsilon_m < \overline{\varepsilon}_m$. Then*

$$\tau(\varepsilon_1, \ldots, \varepsilon_m) = m.$$

Proof Let $\tau(\varepsilon_1, \ldots, \varepsilon_m) = k$. Suppose that $k < m$. We show that this will lead to a contradiction. Write $m = tk + i$ with $0 < i \le k$. By the definition of the recurrence time τ, we have

$$\sigma^{tk}(\varepsilon_1, \ldots, \varepsilon_m) = (\varepsilon_{tk+1}, \ldots, \varepsilon_m) = (\varepsilon_1, \ldots, \varepsilon_i). \tag{3.4}$$

From the self-admissibility of the other sequence $(\varepsilon_1, \ldots, \varepsilon_{m-1}, \overline{\varepsilon}_m)$, we know

$$\sigma^{tk}(\varepsilon_1, \ldots, \varepsilon_{m-1}, \overline{\varepsilon}_m) = (\varepsilon_{tk+1}, \ldots, \overline{\varepsilon}_m) \preceq (\varepsilon_1, \ldots, \varepsilon_i). \tag{3.5}$$

The assumption $\varepsilon_m < \overline{\varepsilon}_m$ implies that

$$(\varepsilon_{tk+1}, \ldots, \varepsilon_m) \prec (\varepsilon_{tk+1}, \ldots, \overline{\varepsilon}_m).$$

Combining this with (3.4) and (3.5), we arrive at the contradiction $(\varepsilon_1, \ldots, \varepsilon_i) \prec (\varepsilon_1, \ldots, \varepsilon_i)$. □

3.2 Maximal admissible sequences in parameter space

Now we recall a result of Schmeling [21] concerning the length of $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$.

Lemma 3.5 ([21]) *The cylinder $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$ is a half-open interval $[\beta_0, \beta_1)$. The left endpoint β_0 is given as the only solution in $(1, \infty)$ of the equation*

$$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n}.$$

The right endpoint β_1 is the limit of the unique solutions $\{\beta_N\}_{N \ge n}$ in $(1, \infty)$ of the equations

$$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n} + \frac{\varepsilon_{n+1}}{\beta^{n+1}} + \cdots + \frac{\varepsilon_N}{\beta^N}, \quad N \ge n$$

where $(\varepsilon_1, \ldots, \varepsilon_n, \varepsilon_{n+1}, \ldots, \varepsilon_N)$ is the maximal self-admissible sequence of length $n + N$ beginning with $\varepsilon_1, \ldots, \varepsilon_n$ in the lexicographical order. Moreover,

$$\left|I_n^P(\varepsilon_1, \ldots, \varepsilon_n)\right| \le \beta_1^{-n+1}.$$

Therefore, to give an accurate estimate on the length of $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$, we are led to determine the maximal self-admissible sequences beginning with a given self-admissible word $\varepsilon_1, \ldots, \varepsilon_n$.

Lemma 3.6 *Let $w = (\varepsilon_1, \ldots, \varepsilon_n)$ be self-admissible with $\tau(w) = k$. Then for each $m \ge 1$ and $0 \le \ell < k$ with $km + \ell \ge n$, the periodic sequence*

$$(\varepsilon_1, \ldots, \varepsilon_k)^m \varepsilon_1, \ldots, \varepsilon_\ell,$$

is the maximal self-admissible sequence of length $km + \ell$ beginning with $\varepsilon_1, \ldots, \varepsilon_n$. Consequently, if we denote by β_1 the right endpoint of $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$, then the β_1-expansion of 1 and the infinite β_1-expansion of 1 are given respectively as

$$\varepsilon(1, \beta_1) = (\varepsilon_1, \ldots, \varepsilon_k + 1), \quad \varepsilon^*(1, \beta_1) = (\varepsilon_1, \ldots, \varepsilon_k)^\infty.$$

Proof By Lemma 3.3, we get for all $1 \le i < k$

$$\varepsilon_{i+1}, \ldots, \varepsilon_k \prec \varepsilon_1, \ldots, \varepsilon_{k-i}. \tag{3.6}$$

For each $m \in \mathbb{N}$ and $0 \le \ell < k$ with $km + \ell \ge n$, we check that

$$w_0 = (\varepsilon_1, \ldots, \varepsilon_k)^m \varepsilon_1, \ldots, \varepsilon_\ell$$

is the maximal self-admissible sequence beginning with $\varepsilon_1, \ldots, \varepsilon_n$ of length $mk + \ell$.

The admissibility of w_0 follows directly from (3.6). Now we show that w_0 is maximal. Let

$$w = (\varepsilon_1, \ldots, \varepsilon_k)^t w_1, \ldots, w_k, \ldots, w_{(m-t-1)k+1}, \ldots, w_{(m-t)k}, w_{(m-t)k+1}, \ldots, w_{(m-t)k+\ell}$$

be a self-admissible word different from w_0, where $t \geq 1$ is the maximal integer such that w begins with $(\varepsilon_1, \ldots, \varepsilon_k)^t$. We distinguish two cases according to $t < m$ or $t = m$. We consider only the case $t < m$, since the other case can be treated similarly.

If $t < m$, then

$$w_1, \ldots, w_k \neq \varepsilon_1, \ldots, \varepsilon_k.$$

The self-admissibility of w ensures that

$$w_1, \ldots, w_k \preceq \varepsilon_1, \ldots, \varepsilon_k.$$

Hence, we arrive at

$$w_1, \ldots, w_k \prec \varepsilon_1, \ldots, \varepsilon_k. \tag{3.7}$$

This shows $w \prec w_0$. □

The following fact is just the self-admissibility of w_0 proven in Lemma 3.6. We state it as a corollary for later use.

Corollary 3.7 *Assume that* $(\varepsilon_1, \ldots, \varepsilon_k)$ *is a non-recurrent word. Then for any integer* $m \geq 1$ *and* $0 \leq \ell < k$*, the word*

$$(\varepsilon_1, \ldots, \varepsilon_k)^m, \varepsilon_1, \ldots, \varepsilon_\ell$$

is self-admissible.

The following simple calculation will be used several times in the sequel, so we state it in advance.

Lemma 3.8 *Let* $1 < \beta_0 < \beta_1$ *and* $0 \leq \varepsilon_k < \beta_0$ *for all* $k \geq 1$*. Then for every* $n \geq 1$*,*

$$\left(\frac{\varepsilon_1}{\beta_0} + \cdots + \frac{\varepsilon_n}{\beta_0^n}\right) - \left(\frac{\varepsilon_1}{\beta_1} + \cdots + \frac{\varepsilon_n}{\beta_1^n}\right) \leq \frac{\beta_0}{(\beta_0 - 1)^2}(\beta_1 - \beta_0).$$

Now we apply Lemma 3.6 to give a lower bound on the length of $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$.

Theorem 3.9 *Let* $w = (\varepsilon_1, \ldots, \varepsilon_n)$ *be self-admissible with* $\tau(w) = k$*. Let* β_0 *and* β_1 *be the left and right endpoints of* $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$*. Then we have*

$$\left|I_n^P(\varepsilon_1, \ldots, \varepsilon_n)\right| \geq \begin{cases} C\beta_1^{-n}, & \text{when } k = n; \\ C\frac{1}{\beta_1^n}\left(\frac{\varepsilon_{t+1}}{\beta_1} + \cdots + \frac{\varepsilon_k + 1}{\beta_1^{k-t}}\right), & \text{otherwise}, \end{cases} \tag{3.8}$$

where $C := (\beta_0 - 1)^2/\beta_0$ *is a constant depending on* β_0*; the integers* t *and* ℓ *are given by* $\ell k < n \leq (\ell + 1)k$ *and* $t = n - \ell k$*.*

Proof When $k = n$, the endpoints β_0 and β_1 of $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$ are given respectively as the solutions to

$$1 = \frac{\varepsilon_1}{\beta_0} + \cdots + \frac{\varepsilon_n}{\beta_0^n}, \quad \text{and} \quad 1 = \frac{\varepsilon_1}{\beta_1} + \cdots + \frac{\varepsilon_n + 1}{\beta_1^n}. \tag{3.9}$$

Thus,

$$\frac{1}{\beta_1^n} = \left(\frac{\varepsilon_1}{\beta_0} + \cdots + \frac{\varepsilon_n}{\beta_0^n}\right) - \left(\frac{\varepsilon_1}{\beta_1} + \cdots + \frac{\varepsilon_n}{\beta_1^n}\right) \le C^{-1}(\beta_1 - \beta_0).$$

Then $|I_n^P(\varepsilon_1, \ldots, \varepsilon_n)| = \beta_1 - \beta_0 \ge C\beta_1^{-n}$.

When $k < n$, the endpoints β_0 and β_1 of $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$ are given respectively as the solutions to

$$1 = \frac{\varepsilon_1}{\beta_0} + \cdots + \frac{\varepsilon_n}{\beta_0^n}, \quad \text{and} \quad 1 = \frac{\varepsilon_1}{\beta_1} + \cdots + \frac{\varepsilon_n}{\beta_1^n} + \frac{\varepsilon_{t+1}}{\beta_1^{n+1}} + \cdots + \frac{\varepsilon_k + 1}{\beta_1^{(\ell+1)k}}.$$

Thus,

$$\frac{\varepsilon_{t+1}}{\beta_1^{n+1}} + \cdots + \frac{\varepsilon_k + 1}{\beta_1^{(\ell+1)k}} = \left(\frac{\varepsilon_1}{\beta_0} + \cdots + \frac{\varepsilon_n}{\beta_0^n}\right) - \left(\frac{\varepsilon_1}{\beta_1} + \cdots + \frac{\varepsilon_n}{\beta_1^n}\right) \le C^{-1}(\beta_1 - \beta_0),$$

and we obtain the desired result. □

Combining Lemma 3.5 and Theorem 3.9, we know that when $(\varepsilon_1, \ldots, \varepsilon_n)$ is a non-recurrent word, the length of $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$ satisfies

$$C\beta_1^{-n} \le |I_n^P(\varepsilon_1, \ldots, \varepsilon_n)| \le \beta_1^{-n}.$$

In this case, $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$ is called a regular cylinder.

The following corollary of Theorem 3.9 indicates that if the digit 1 appears regularly in a self-admissible sequence w, then we can have a good lower bound for the length of the cylinder generated by w. This will be applied in constructing a Cantor subset of $E(\{\ell_n\}_{n\ge1}, x_0)$.

Corollary 3.10 *Let $w = (\varepsilon_1, \ldots, \varepsilon_n)$ be self-admissible and d an integer such that for every $0 \le i \le n - d$, the word $(w_{i+1}, \ldots, w_{i+d})$ is nonzero. Then we have*

$$|I_n(w)| \ge C\beta_1^{-n-d},$$

where C and β_1 are as those in Theorem 3.9.

Proof Let $\tau(w) = k$. When n is a multiple of k, the maximal self-admissible sequence beginning with w is just the periodic sequence $(w, w, w, \ldots)$. Then the desired result follows with the same argument as that for the first inequality in (3.8).

When n is not a multiple of k, we argue as follows. Keep the notation as in Theorem 3.9. If $k - t \ge d$, then $(\varepsilon_{t+1}, \ldots, \varepsilon_{t+d})$ is nonzero. Thus by the second inequality in (3.8), we have $|I_n(w)| \ge C\beta_1^{-(n+d)}$. If $k - t < d$, then still by the second inequality in (3.8), we have

$$|I_n(w)| \ge C\beta_1^{-n} \cdot \beta_1^{-(k-t)} \ge C\beta_1^{-(n+d)}.$$

□

3.3 Distribution of regular cylinders

The following result presents a relationship between the recurrence time of two consecutive cylinders in the parameter space.

Proposition 3.11 *Let w_1, w_2 be two self-admissible words of length n. Assume that $w_2 \prec w_1$ and w_2 is next to w_1 in the lexicographic order. If $\tau(w_1) < n$, then*

$$\tau(w_2) > \tau(w_1).$$

Proof Since $\tau(w_1) := k_1 < n,\ w_1$ can be written as

$$w_1 = (\varepsilon_1, \ldots, \varepsilon_{k_1})^t, \varepsilon_1, \ldots, \varepsilon_\ell, \quad \text{for some integers } t \geq 1 \text{ and } 1 \leq \ell \leq k_1.$$

It is clear that $\varepsilon_1 \geq 1$ which ensures the self-admissibility of the sequence

$$w = (\varepsilon_1, \ldots, \varepsilon_{k_1})^t, \underbrace{0, \ldots, 0}_{\ell}.$$

Since w_2 is less than w_1 and is next to w_1, we have

$$w \preceq w_2 \prec w_1.$$

This implies that w_1 and w_2 have common prefixes up to at least $k_1 \cdot t$ terms. Then w_2 can be expressed as

$$w_2 = (\varepsilon_1, \ldots, \varepsilon_{k_1})^t, \varepsilon_1', \ldots, \varepsilon_\ell'.$$

First, we claim that $\tau(w_2) := k_2 \neq k_1$. Otherwise, by the definition of $\tau(w_2)$, we obtain

$$\varepsilon_1', \ldots, \varepsilon_\ell' = \varepsilon_1, \ldots, \varepsilon_\ell,$$

which indicates that $w_1 = w_2$.

Second, we show that k_2 cannot be strictly smaller than k_1. Otherwise, consider the prefix $\varepsilon_1, \ldots, \varepsilon_{k_1}$ which is also the prefix of w_1. If $k_2 < k_1$, we have

$$\varepsilon_{k_2+1}, \ldots, \varepsilon_{k_1} = \varepsilon_1, \ldots, \varepsilon_{k_1-k_2},$$

which contradicts Lemma 3.3 by applying to w_1.

Therefore, $\tau(w_2) > \tau(w_1)$ holds. □

The following corollary indicates that cylinders with regular length (equivalent with β_1^{-n}) are well distributed among the parameter space. This result was found for the first time by Persson and Schmeling [18].

Corollary 3.12 *Among any n consecutive cylinders in C_n^P, there is at least one with regular length.*

Proof Let $w_1 \succ w_2 \succ \cdots \succ w_n$ be n consecutive cylinders in C_n^P. By Theorem 3.9, it suffices to show that there is at least one word w whose recurrence time is equal to n. If this is not the case, then by Proposition 3.11, we have

$$1 \leq \tau(w_1) < \tau(w_2) < \cdots < \tau(w_n) < n,$$

i.e. there would be n different integers in $\{1, 2, \ldots, n-1\}$. This is impossible. This completes the proof. □

4 Proof of Theorem 1.1: upper bound

The proof of the upper bound of $\dim_{\mathsf{H}} E(\{\ell_n\}_{n\ge1}, x_0)$ is given in a unified way no matter whether $x_0 = 1$ or not. Before providing an upper bound of $\dim_{\mathsf{H}} E(\{\ell_n\}_{n\ge1}, x_0)$, we begin with a lemma.

Lemma 4.1 *Let* $(\varepsilon_1, \ldots, \varepsilon_n)$ *be self-admissible. Then the set*

$$\left\{ T_\beta^n 1 : \beta \in I_n^P(\varepsilon_1, \ldots, \varepsilon_n) \right\} \tag{4.1}$$

is a half-open interval $[0, a)$ *for some* $a \le 1$. *Moreover,* $T_\beta^n 1$ *is continuous and increasing on* $\beta \in I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$.

Proof Note that for any $\beta \in I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$, we have

$$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n + T_\beta^n 1}{\beta^n}.$$

Thus

$$T_\beta^n 1 = \beta^n - \beta^n \left(\frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n} \right).$$

Denote

$$f(\beta) = \beta^n - \left(\varepsilon_1 \beta^{n-1} + \varepsilon_2 \beta^{n-2} + \cdots + \varepsilon_n \right). \tag{4.2}$$

Then the set in (4.1) is just the set

$$\{ f(\beta) : \beta \in I_n^P(\varepsilon_1, \ldots, \varepsilon_n) \}.$$

To show the monotonicity of $\beta \mapsto T_\beta^n 1$, it suffices to show that the derivative $f'(\beta)$ is positive. In fact,

$$\begin{aligned} f'(\beta) &= n\beta^{n-1} - \left((n-1)\varepsilon_1 \beta^{n-2} + (n-2)\varepsilon_2 \beta^{n-3} + \cdots + \varepsilon_{n-1} \right) \\ &\ge n\beta^{n-1} - (n-1)\beta^{n-1} \left(\frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_{n-1}}{\beta^{n-1}} \right) \\ &\ge n\beta^{n-1} - (n-1)\beta^{n-1} > 0. \end{aligned}$$

Since f is continuous and $I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$ is an interval with the left endpoint β_0 given as the solution to the equation

$$1 = \frac{\varepsilon_1}{\beta} + \cdots + \frac{\varepsilon_n}{\beta^n},$$

the set (4.1) is an interval with 0 being its left endpoint and some right endpoint $a \le 1$. □

Now we give an upper bound of $\dim_{\mathsf{H}} E\big(\{\ell_n\}_{n\ge1}, x_0\big)$. For any $1 < \beta_0 < \beta_1$, denote

$$E(\beta_0, \beta_1) = \left\{ \beta_0 < \beta \le \beta_1 : |T_\beta^n 1 - x_0| < \beta^{-\ell_n}, \text{ i.o. } n \in \mathbb{N} \right\}.$$

For any $\delta > 0$, we partition the parameter space $(1, \infty)$ into $\{(a_i, a_{i+1}] : i \ge 1\}$ with $\frac{\log a_{i+1}}{\log a_i} < 1 + \delta$ for all $i \ge 1$. Then

$$E\big(\{\ell_n\}_{n\ge1}, x_0\big) = \cup_{i=1}^{\infty} E(a_i, a_{i+1}).$$

By the σ-stability of the Hausdorff dimension, it suffices to give an upper bound on $\dim_{\mathsf{H}} E(\beta_0, \beta_1)$ for any $1 < \beta_0 < \beta_1$ with $\frac{\log \beta_1}{\log \beta_0} < 1 + \delta$.

Proposition 4.2 *For any* $1 < \beta_0 < \beta_1$, *we have*

$$\dim_H E(\beta_0, \beta_1) \le \frac{1}{1+\alpha} \frac{\log \beta_1}{\log \beta_0}. \tag{4.3}$$

Proof Let $B(x, r)$ be a ball with center $x \in [0, 1]$ and radius r. By using the simple inclusion $B(x_0, \beta^{-\ell_n}) \subset B(x_0, \beta_0^{-\ell_n})$ for any $\beta > \beta_0$, we have

$$\begin{aligned} E(\beta_0, \beta_1) &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \beta \in (\beta_0, \beta_1] : T_\beta^n 1 \in B(x_0, \beta^{-\ell_n}) \right\} \\ &\subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \left\{ \beta \in (\beta_0, \beta_1] : T_\beta^n 1 \in B(x_0, \beta_0^{-\ell_n}) \right\} \\ &= \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(i_1, \ldots, i_n) \in \Sigma_{\beta_0, \beta_1}^{P,n}} I_n^P(i_1, \ldots, i_n; \beta_0^{-\ell_n}), \end{aligned}$$

where $\Sigma_{\beta_0,\beta_1}^{P,n}$ denotes the set of self-admissible words of length n between $(\varepsilon_1^*(\beta_0), \ldots, \varepsilon_n^*(\beta_0))$ and $(\varepsilon_1^*(\beta_1), \ldots, \varepsilon_n^*(\beta_1))$ in the lexicographic order, and

$$I_n^P(i_1, \ldots, i_n; \beta_0^{-\ell_n}) := \{ \beta \in (\beta_0, \beta_1] : \beta \in I_n^P(i_1, \ldots, i_n), \ T_\beta^n 1 \in B(x_0, \beta_0^{-\ell_n}) \}.$$

By Lemma 4.1, we know that the set $I_n^P(i_1, \ldots, i_n; \beta_0^{-\ell_n})$ is an interval. In case it is non-empty we denote its left and right endpoints by β_0' and β_1' respectively. Thus

$$\beta_1' \le i_1 + \frac{i_2}{\beta_1'} + \cdots + \frac{i_n}{\beta_1'^{n-1}} + \frac{x_0 + \beta_0^{-\ell_n}}{\beta_1'^{n-1}}$$

and

$$\beta_0' \ge i_1 + \frac{i_2}{\beta_0'} + \cdots + \frac{i_n}{\beta_0'^{n-1}} + \frac{x_0 - \beta_0^{-\ell_n}}{\beta_0'^{n-1}} \ge i_1 + \frac{i_2}{\beta_1'} + \cdots + \frac{i_n}{\beta_1'^{n-1}} + \frac{x_0 - \beta_0^{-\ell_n}}{\beta_1'^{n-1}}.$$

Therefore,

$$\begin{aligned} &\beta_1' - \beta_0' \\ &\le \left(i_1 + \frac{i_2}{\beta_1'} + \cdots + \frac{i_n}{\beta_1'^{n-1}} + \frac{x_0 + \beta_0^{-\ell_n}}{\beta_1'^{n-1}} \right) - \left(i_1 + \frac{i_2}{\beta_1'} + \cdots + \frac{i_n}{\beta_1'^{n-1}} + \frac{x_0 - \beta_0^{-\ell_n}}{\beta_1'^{n-1}} \right) \\ &= \frac{2\beta_0^{-\ell_n}}{\beta_1'^{n-1}} \le \frac{2\beta_0^{-\ell_n}}{\beta_0^{n-1}} = 2\beta_0^{-(\ell_n+n-1)}. \end{aligned}$$

By the monotonicity of $\varepsilon(1, \beta)$ with respect to β (Theorem 2.2 (2)), we have $\varepsilon(1, \beta) \in \Sigma_{\beta_1}$ for any $\beta < \beta_1$. Therefore

$$\sharp \Sigma_{\beta_0,\beta_1}^{P,n} \le \sharp \Sigma_{\beta_1}^n \le \frac{\beta_1^{n+1}}{\beta_1 - 1},$$

where the last inequality follows from Theorem 2.5. It is clear that the family

$$\left\{ I_n^P(i_1, \ldots, i_n, \beta_0^{-\ell_n}) : (i_1, \ldots, i_n) \in \Sigma_{\beta_0,\beta_1}^{P,n}, n \ge N \right\}$$

is a cover of the set $E(\beta_0, \beta_1)$. Recall that $\alpha = \liminf_{n\to\infty} \ell_n/n$. Thus for any $s > \frac{1}{1+\alpha}\frac{\log\beta_1}{\log\beta_0}$, we have

$$\mathcal{H}^s(E(\beta_0,\beta_1)) \le \liminf_{N\to\infty}\sum_{n\ge N}\sum_{(i_1,\dots,i_n)\in\Sigma^{P,n}_{\beta_0,\beta_1}} \left|I_n^P(i_1,\dots,i_n,\beta_0^{-\ell_n})\right|^s$$
$$\le \liminf_{N\to\infty}\sum_{n\ge N}\frac{\beta_1^{n+1}}{\beta_1-1}\cdot 2^s\cdot\beta_0^{-(\ell_n+n-1)s} < \infty.$$

This gives the estimate (4.3). □

5 Lower bound of $E(\{\ell_n\}_{n\ge1}, x_0) : x_0 \ne 1$

The proof of the lower bound of $\dim_{\mathrm{H}} E(\{\ell_n\}_{n\ge1}, x_0)$, when $x_0 \ne 1$, is done by using a classic method: first construct a Cantor subset $\mathcal{F}$, then define a measure μ supported on $\mathcal{F}$, and estimate the Hölder exponent of the measure μ. At last, conclude the result by applying the following mass distribution principle [7, Proposition 4.4].

Proposition 5.1 (Falconer [7]) *Let E be a Borel subset of $\mathbb{R}^d$ and μ be a Borel measure with $\mu(E) > 0$. Assume that, for any $x \in E$*

$$\liminf_{r\to0}\frac{\log\mu(B(x,r))}{\log r} \ge s.$$

Then $\dim_H E \ge s$.

Instead of dealing with $E(\{\ell_n\}_{n\ge1}, x_0)$ directly, we give some technical modifications by considering the following set

$$E = \left\{\beta > 1 : |T_\beta^n 1 - x_0| < 4(n+\ell_n)\beta^{-\ell_n},\ \text{i.o.}\, n\in\mathbb{N}\right\}.$$

It is clear that if we replace $\beta^{-\ell_n}$ by $\beta^{-(\ell_n+n\epsilon)}$ for any $\epsilon > 0$ in defining E above, the set E will be a subset of $E(\{\ell_n\}_{n\ge1}, x_0)$. Therefore, once we show the dimension of E is bounded from below by $1/(1+\alpha)$, so is $E(\{\ell_n\}_{n\ge1}, x_0)$. We always assume in the following that $\alpha > 0$, if not, just replace ℓ_n by $\ell_n + n\epsilon$. In the remaining part of this section, we are going to prove that

$$\dim_{\mathrm{H}} E \ge \frac{1}{1+\alpha}, \quad \text{with} \quad \alpha > 0.$$

5.1 Cantor subset

Let x_0 be a real number in $[0, 1)$. Let $\beta_0 > 1$ be such that its expansion $\varepsilon(1, \beta_0)$ of 1 is infinite, i.e. there are infinitely many nonzero terms in $\varepsilon(1, \beta_0)$. The infinity of the digit sequence $\varepsilon(1, \beta_0)$ implies that for each $n \ge 1$, the number β_0 is not the right endpoint of the cylinder $I_n^P(\beta_0)$ containing β_0 by Lemma 3.6. Hence we can choose another $\beta_1 > \beta_0$ such that the β_1-expansion $\varepsilon(1, \beta_1)$ of 1 is infinite and has a sufficiently long common prefix with $\varepsilon(1, \beta_0)$ so that

$$\frac{\beta_1(\beta_1-\beta_0)}{(\beta_0-1)^2} \le \frac{1-x_0}{2}. \tag{5.1}$$

Let

$$M = \min\{n \geq 1 : \varepsilon_n(1, \beta_0) \neq \varepsilon_n(1, \beta_1)\},$$

that is, $\varepsilon_i(1, \beta_0) = \varepsilon_i(1, \beta_1)$ for all $1 \leq i < M$ and $\varepsilon_M(1, \beta_0) \neq \varepsilon_M(1, \beta_1)$. Let β_2 be the maximal element beginning with $w(\beta_0) := (\varepsilon_1(1, \beta_0), \ldots, \varepsilon_M(1, \beta_0))$ in its infinite expansion of 1, that is, β_2 is the right endpoint of $I_M^P(w(\beta_0))$. Then it follows that $\beta_0 < \beta_2 < \beta_1$. Note that the word

$$(\varepsilon_1(1, \beta_0), \ldots, \varepsilon_{M-1}(1, \beta_0), \varepsilon_M(1, \beta_1)) = (\varepsilon_1(1, \beta_1), \ldots, \varepsilon_{M-1}(1, \beta_1), \varepsilon_M(1, \beta_1))$$

is self-admissible and $\varepsilon_M(1, \beta_0) < \varepsilon_M(1, \beta_1)$. So by Lemma 3.4, we know that $\tau(w(\beta_0)) = M$. As a result, Lemma 3.6 compels that the infinite β_2-expansion of 1 is

$$\varepsilon^*(1, \beta_2) = (\varepsilon_1(1, \beta_0), \ldots, \varepsilon_M(1, \beta_0))^\infty. \tag{5.2}$$

Since the following fact will be used frequently, we highlight it here:

$$\varepsilon_1^*(1, \beta_2), \ldots, \varepsilon_M^*(1, \beta_2) \prec \varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1). \tag{5.3}$$

Lemma 5.2 *For any $w \in S_{\beta_2}$, the sequence*

$$\varepsilon = \varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1), 0^M, w$$

is self-admissible.

Proof This will be checked by using properties of the recurrence time and the fact (5.3). Denote $\tau(\varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1)) = k$. Then $\varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1)$ is periodic with a period k. Thus ε can be rewritten as

$$(\varepsilon_1, \ldots, \varepsilon_k)^{t_0} \varepsilon_1, \ldots, \varepsilon_s, 0^M, w$$

for some $t_0 \in \mathbb{N}$ and $0 \leq s < k$. We will compare $\sigma^i \varepsilon$ and ε for all $i \geq 1$. The proof is divided into three steps according to $i \leq M$, $M < i < 2M$ or $i \geq 2M$.

(1) $i \leq M$. When $i = tk$ for some $t \in \mathbb{N}$, then $\sigma^i(\varepsilon)$ and ε have a common prefix up to the $(M - tk)$th digit. Following this prefix, the next k digits in $\sigma^i(\varepsilon)$ are 0^k, while they are $(\varepsilon_{s+1}, \ldots, \varepsilon_k, \varepsilon_1, \ldots, \varepsilon_s)$ in ε, which implies that $\sigma^i \varepsilon \prec \varepsilon$.
When $i = tk + \ell$ for some $0 < \ell < k$, then $\sigma^i(\varepsilon)$ begins with $(\varepsilon_{\ell+1}, \ldots, \varepsilon_s, 0^{k-s})$ if $t = t_0$ and begins with $(\varepsilon_{\ell+1}, \ldots, \varepsilon_k)$ if $t < t_0$. By Lemma 3.3, we know that

$$\varepsilon_{\ell+1}, \ldots, \varepsilon_s, 0^{k-s} \preceq \varepsilon_{\ell+1}, \ldots, \varepsilon_k \prec \varepsilon_1, \ldots, \varepsilon_{k-\ell}.$$

Thus $\sigma^i(\varepsilon) \prec \varepsilon$.

(2) $M < i < 2M$. For this case, it is trivial because $\sigma^i \varepsilon$ begins with 0.

(3) $i = 2M + \ell$ for some $\ell \geq 0$. Then the sequence $\sigma^i(\varepsilon)$ begins with the subword $(w_{\ell+1}, \ldots, w_{\ell+M})$ of w. Since $w \in S_{\beta_2}$, we have

$$w_{\ell+1}, \ldots, w_{\ell+M} \preceq \varepsilon_1^*(1, \beta_2), \ldots, \varepsilon_M^*(1, \beta_2) \prec \varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1).$$

where the last inequality follows from (5.3). Therefore, $\sigma^i(\varepsilon) \prec \varepsilon$.

□

Now we use Lemmas 4.1, 5.2 and a suitable choice of the self-admissible sequence to show that the interval defined in (4.1) can be large enough. Fix $q \geq M$ such that

$$0^q \prec \varepsilon_{M+1}(1, \beta_1), \ldots, \varepsilon_{M+q}(1, \beta_1),$$

that is, find a position $M+q$ in $\varepsilon(1,\beta_1)$ with nonzero element $\varepsilon_{M+q}(1,\beta_1)$. The choice of the integer q guarantees that the cylinder $I^P_{M+q}(\varepsilon_1(1,\beta_1),\ldots,\varepsilon_M(1,\beta_1),0^q)$ lies on the left hand side of β_1.

Lemma 5.3 *Suppose β_0 and β_1 are close enough such that (5.1) holds. For any $w\in\Sigma^{n-M-q}_{\beta_2}$ ending with M zeros, the interval*

$$\Gamma_n=\left\{T^n_\beta 1:\beta\in I^P_n(\varepsilon_1(1,\beta_1),\ldots,\varepsilon_M(1,\beta_1),0^q,w)\right\}$$

contains $(x_0+1)/2$.

Proof Recall $\varepsilon^*(1,\beta_2)=(\varepsilon_1(1,\beta_0),\ldots,\varepsilon_M(1,\beta_0))^\infty:=(e_1,\ldots,e_M)^\infty$. Since w ends with M zeros, the sequence $(w,(e_1,\ldots,e_M)^\infty)$ is in S_{β_2}. Thus, the number β^* for which

$$\varepsilon^*(1,\beta^*)=\varepsilon_1(1,\beta_1),\ldots,\varepsilon_M(1,\beta_1),0^q,w,e_1,e_2,\ldots e_M,e_1,e_2,\ldots$$

belongs to the closure of $I^P_n(\varepsilon_1(1,\beta_1),\ldots,\varepsilon_M(1,\beta_1),0^q,w)$ by Lemma 5.2. Note that $\beta^*\le\beta_1$ by the choice of q. For such a number β^*,

$$T^n_{\beta^*}1=\frac{e_1}{\beta^*}+\frac{e_2}{\beta^{*2}}+\cdots\ge\frac{e_1}{\beta_1}+\frac{e_2}{\beta_1^2}+\cdots.$$

Note also that

$$1=\frac{e_1}{\beta_2}+\frac{e_2}{\beta_2^2}+\cdots.$$

Thus

$$1-T^n_{\beta^*}1\le\left(\frac{e_1}{\beta_2}+\frac{e_2}{\beta_2^2}+\cdots\right)-\left(\frac{e_1}{\beta_1}+\frac{e_2}{\beta_1^2}+\cdots\right)\le\frac{\beta_1(\beta_1-\beta_0)}{(\beta_0-1)^2}.$$

Hence $T^n_{\beta^*}1>\frac{x_0+1}{2}$ by (5.1). Then we obtain the statement of Lemma 5.3. □

Now we are in the position to construct a Cantor subset $\mathcal{F}$ of E. Let $\mathfrak{N}$ be a subsequence of integers such that

$$\liminf_{n\to\infty}\frac{\ell_n}{n}=\lim_{n\in\mathfrak{N},\,n\to\infty}\frac{\ell_n}{n}=\alpha>0.$$

5.1.1 Generation 0 of the Cantor set

Write

$$\varepsilon^{(0)}=(\varepsilon_1(1,\beta_1),\ldots,\varepsilon_M(1,\beta_1),0^q),\quad \text{and}\ \mathbb{F}_0=\{\varepsilon^{(0)}\}.$$

Then the 0th generation of the Cantor set is defined as

$$\mathcal{F}_0=\left\{I^P_{M+q}(\varepsilon^{(0)}):\varepsilon^{(0)}\in\mathbb{F}_0\right\}.$$

5.1.2 Generation 1 of the Cantor set

Recall that M is the integer defined for β_2 in the beginning of this subsection. Let $N \gg M$. Denote by U_ℓ a collection of words in S_{β_2}:

$$U_\ell = \Big\{ u = (0^M, 1, 0^M, a_1, \ldots, a_N, 0^M, 1, 0^M) : (a_1, \ldots, a_N) \in S_{\beta_2} \Big\}, \tag{5.4}$$

where $\ell = 4M + 2 + N$ is the length of the words in U_ℓ. Without causing any confusion, in the sequel, the family $\mathbb{F}_0$ of words is also called the 0th generation of the Cantor set $\mathcal{F}$.

Remark 4 We give a remark on the way that the family U_ℓ is constructed.

(1) The first M-zeros guarantee that for any β_2-admissible word v and any element $u \in U_\ell$, the concatenation (v, u) is still β_2-admissible.
(2) With the same reason as for (1), the other three blocks of 0^M guarantee that $U_\ell \subset \Sigma_{\beta_2}^\ell$.
(3) The two digits 1 are added so that the digit 1 appears regularly in $u \in U_\ell$ (recall Corollary 3.10).

Let $m_0 = M + q$ be the length of $\varepsilon^{(0)} \in \mathbb{F}_0$. Choose an integer $n_1 \in \mathfrak{N}$ such that $n_1 \gg m_0$, $\beta_0^{-n_1} \le 2(\beta_0 - 1)^2/\beta_1$ and $4(n_1 + \ell_{n_1})\beta^{-\ell_{n_1}} < \frac{1-x_0}{2}$ by noting $\alpha > 0$. Write

$$n_1 - m_0 = t_1\ell + i_1, \quad \text{for some} \quad t_1 \in \mathbb{N},\ 0 \le i_1 < \ell.$$

First, we collect a family of self-admissible sequences beginning with $\varepsilon^{(0)}$:

$$\mathfrak{M}(\varepsilon^{(0)}) = \Big\{ (\varepsilon^{(0)}, u_1, \ldots, u_{t_1-1}, u_{t_1}, 0^{i_1}) : u_1, \ldots, u_{t_1} \in U_\ell \Big\}.$$

Here the self-admissibility of the elements in $\mathfrak{M}(\varepsilon^{(0)})$ follows from Lemma 5.2.

Second, for each $w \in \mathfrak{M}(\varepsilon^{(0)})$, we will extract an element belonging to $\mathbb{F}_1$ (the first generation of $\mathcal{F}$). Let $\Gamma_{n_1}(w) := \{T_\beta^{n_1} 1 : \beta \in I_{n_1}^P(w)\}$. By Lemma 5.3, we have that

$$\Gamma_{n_1} = \Gamma_{n_1}(w) \supset B(x_0, 4(n_1 + \ell_{n_1})\beta_0^{-\ell_{n_1}}). \tag{5.5}$$

Now we consider the set of all possible self-admissible sequences of order $n_1 + \ell_{n_1}$ beginning with w, denoted by

$$\mathbb{A}(w) := \big\{ (w, \eta_1, \ldots, \eta_{\ell_{n_1}}) : (w, \eta_1, \ldots, \eta_{\ell_{n_1}}) \text{ is self-admissible} \big\}.$$

Then

$$\Gamma_{n_1}(w) = \bigcup_{\varepsilon \in \mathbb{A}(w)} \big\{ T_\beta^{n_1} 1 : \beta \in I_{n_1+\ell_{n_1}}^P(\varepsilon) \big\}. \tag{5.6}$$

We show that for each $\varepsilon \in \mathbb{A}(w)$,

$$\big|\big\{ T_\beta^{n_1} 1 : \beta \in I_{n_1+\ell_{n_1}}^P(\varepsilon) \big\}\big| \le 4\beta_0^{-\ell_{n_1}}. \tag{5.7}$$

In fact, for each pair $\beta, \beta' \in I_{n_1+\ell_{n_1}}^P(\varepsilon)$, we have

$$T_\beta^{n_1} 1 = \frac{\eta_1}{\beta} + \cdots + \frac{\eta_{\ell_{n_1}} + y}{\beta^{\ell_{n_1}}}, \qquad T_{\beta'}^{n_1} 1 = \frac{\eta_1}{\beta'} + \cdots + \frac{\eta_{\ell_{n_1}} + y'}{\beta'^{\ell_{n_1}}}$$

for some $0 \le y, y' \le 1$. Then

$$\left|T_\beta^{n_1}1 - T_{\beta'}^{n_1}1\right| \le \sum_{k=1}^{\ell_{n_1}} \left|\frac{\eta_k}{\beta^k} - \frac{\eta_k}{\beta'^k}\right| + \frac{1}{\beta^{\ell_{n_1}}} + \frac{1}{\beta'^{\ell_{n_1}}}$$
$$\le \frac{\beta_1}{(\beta_0-1)^2}\beta_0^{-n_1-\ell_{n_1}} + \frac{1}{\beta^{\ell_{n_1}}} + \frac{1}{\beta'^{\ell_{n_1}}} \le 4\beta_0^{-\ell_{n_1}}.$$

Now Lemma 4.1, together with the estimate (5.7), enables us to conclude the following simple facts:

- for each $\varepsilon \in \mathbb{A}(w)$, $\left\{T_\beta^{n_1}1 : \beta \in I^P_{n_1+\ell_{n_1}}(\varepsilon)\right\}$ is an interval, since $I^P_{n_1+\ell_{n_1}}(\varepsilon)$ is an interval;
- for every pair $\varepsilon, \varepsilon' \in \mathbb{A}(w)$, if $\varepsilon \prec \varepsilon'$, then by the monotonicity of $T_\beta^{n_1}1$ with respect to β we have that $\left\{T_\beta^{n_1}1 : \beta \in I^P_{n_1+\ell_{n_1}}(\varepsilon)\right\}$ lies on the left hand side of $\left\{T_\beta^{n_1}1 : \beta \in I^P_{n_1+\ell_{n_1}}(\varepsilon')\right\}$. Therefore, the intervals in the union in (5.6) are arranged in [0, 1] consecutively;
- moreover, there are no gaps between adjoint intervals in the union in (5.6), since $\Gamma_{n_1}(w)$ is an interval;
- the length of the interval $\left\{T_\beta^{n_1}1 : \beta \in I^P_{n_1+\ell_{n_1}}(\varepsilon)\right\}$ is less than $4\beta_0^{-\ell_{n_1}}$.

By these four facts, we conclude that there are at least $(n_1 + \ell_{n_1})$ consecutive cylinders $I^P_{n_1+\ell_{n_1}}(\varepsilon)$ with $\varepsilon \in \mathbb{A}(w)$ such that $\left\{T_\beta^{n_1}1 : \beta \in I^P_{n_1+\ell_{n_1}}(\varepsilon)\right\}$ is contained in the ball $B(x_0, 4(n_1+\ell_{n_1})\beta_0^{-\ell_{n_1}})$. Thus by Corollary 3.12, there exists a cylinder, denoted by

$$I^P_{n_1+\ell_{n_1}}(w, w_1^{(1)}, \ldots, w_{\ell_{n_1}}^{(1)})$$

satisfying that

- The word $(w, w_1^{(1)}, \ldots, w_{\ell_{n_1}}^{(1)})$ is non-recurrent;
- The set $\left\{T_\beta^{n_1}1 : \beta \in I^P_{n_1+\ell_{n_1}}(w, w_1^{(1)}, \ldots, w_{\ell_{n_1}}^{(1)})\right\}$ is contained in the ball $B(x_0, 4(n_1 + \ell_{n_1})\beta_0^{-\ell_{n_1}})$. Thus, for any $\beta \in I^P_{n_1+\ell_{n_1}}(w, w_1^{(1)}, \ldots, w_{\ell_{n_1}}^{(1)})$,

$$\left|T_\beta^{n_1}1 - x_0\right| < 4(n_1+\ell_{n_1})\beta_0^{-\ell_{n_1}}. \tag{5.8}$$

This is the cylinder corresponding to $w \in \mathfrak{M}(\varepsilon^{(0)})$ that we are looking for in composing the first generation of the Cantor set.

Finally the first generation of the Cantor set is defined as

$$\mathbb{F}_1 = \left\{\varepsilon^{(1)} = (w, w_1^{(1)}, \ldots, w_{\ell_{n_1}}^{(1)}) : w \in \mathfrak{M}(\varepsilon^{(0)})\right\}, \qquad \mathcal{F}_1 = \bigcup_{\varepsilon^{(1)}\in\mathbb{F}_1} I^P_{n_1+\ell_{n_1}}(\varepsilon^{(1)}),$$

where $w_1^{(1)}, \ldots, w_{\ell_{n_1}}^{(1)}$ depend on $w \in \mathfrak{M}(\varepsilon^{(0)})$, but for simplicity we do not emphasize this dependence in our notation. Let $m_1 = n_1 + \ell_{n_1}$.

5.1.3 From generation $k-1$ to generation k of the Cantor set $\mathcal{F}$

Assume that the $(k-1)$th generation $\mathbb{F}_{k-1}$ has been well defined, and is composed by a collection of non-recurrent words.

To repeat the process of the construction of the Cantor set, we present similar results as Lemmas 5.2 and 5.3.

Lemma 5.4 *Let* $\varepsilon^{(k-1)} \in \mathbb{F}_{k-1}$. *Then for any* $u \in S_{\beta_2}$ *ending with M zeros, the sequence*

$$(\varepsilon^{(k-1)}, u)$$

is self-admissible.

Proof Let $1 \le i < m_{k-1}$, where m_{k-1} is the order of $\varepsilon^{(k-1)}$. Since $\varepsilon^{(k-1)}$ is non-recurrent, an application of Lemma 3.3 yields that

$$\sigma^i(\varepsilon^{(k-1)}, u) \prec \varepsilon^{(k-1)}.$$

Moreover, combining the assumption of $u \in S_{\beta_2}$ and (5.3), we obtain that any block of M consecutive digits in u is strictly less than the prefix of $\varepsilon^{(k-1)}$. In other words, when $m_{k-1} \le i \le m_{k-1} + |u| - M$, we have $\sigma^i(\varepsilon^{(k-1)}, u) \prec \varepsilon^{(k-1)}$.

At last, since u ends with M zeros, clearly when $i \ge m_{k-1} + |u| - M$, we have $\sigma^i(\varepsilon^{(k-1)}, u) \prec \varepsilon^{(k-1)}$. □

Lemma 5.5 *For any* $\varepsilon^{(k-1)} \in \mathbb{F}_{k-1}$ *and* $u \in S_{\beta_2}$ *ending with M zeros, write* $n = |\varepsilon^{(k-1)}| + |u|$. *Then*

$$\Gamma_n = \left\{ T_\beta^n 1 : \beta \in I_n^P(\varepsilon^{(k-1)}, u) \right\}$$

contains $(x_0 + 1)/2$.

Proof With the same argument as Lemma 5.4, we can prove that the sequence $(\varepsilon^{(k-1)}, u, (e_1, \ldots, e_M)^\infty)$ is self-admissible. Then with the same argument as that in Lemma 5.3, we can conclude the assertion. □

Let $\varepsilon^{(k-1)} \in \mathbb{F}_{k-1}$ be a word of length m_{k-1}. Choose an integer $n_k \in \mathfrak{N}$ such that $n_k \gg m_{k-1}$. Write

$$n_k - m_{k-1} = t_k \ell + i_k, \quad \text{for some} \quad 0 \le i_k < \ell.$$

We collect a family of self-admissible sequences beginning with $\varepsilon^{(k-1)}$:

$$\mathfrak{M}(\varepsilon^{(k-1)}) = \left\{ \varepsilon^{(k-1)}, u_1, \ldots, u_{t_k - 1}, u_{t_k}, 0^{i_k} : u_1, \ldots, u_{t_k} \in U_\ell \right\}. \tag{5.9}$$

Here the self-admissibility of the elements in $\mathfrak{M}(\varepsilon^{(k-1)})$ follows from Lemma 5.4.

Then in the light of Lemma 5.5, the remaining argument for the construction of $\mathbb{F}_k$ (the kth generation of $\mathcal{F}$) is absolutely the same as that for $\mathbb{F}_1$.

For each $w \in \mathfrak{M}(\varepsilon^{(k-1)})$, we can extract a non-recurrent word of length $n_k + \ell_{n_k}$ belonging to $\mathbb{F}_k$, denoted by

$$(w, w_1^{(k)}, \ldots, w_{\ell_{n_k}}^{(k)}).$$

Then the kth generation $\mathbb{F}_k$ is defined as

$$\mathbb{F}_k = \left\{ \varepsilon^{(k)} = (w, w_1^{(k)}, \ldots, w_{\ell_{n_k}}^{(k)}) : w \in \mathfrak{M}(\varepsilon^{(k-1)}),\ \varepsilon^{(k-1)} \in \mathbb{F}_{k-1} \right\}, \tag{5.10}$$

and

$$\mathcal{F}_k = \bigcup_{\varepsilon^{(k)} \in \mathbb{F}_k} I_{n_k + \ell_{n_k}}^P(\varepsilon^{(k)}).$$

Note also that $w_1^{(k)}, \ldots, w_{\ell_{n_k}}^{(k)}$ depend on w for each $w \in \mathfrak{M}(\varepsilon^{(k-1)})$.

Continuing this procedure, we get a nested sequence $\{\mathcal{F}_k\}_{k\ge1}$ consisting of cylinders. Finally, the desired Cantor set is defined as

$$\mathcal{F} = \bigcap_{k=1}^{\infty} \bigcup_{\varepsilon^{(k)}\in\mathbb{F}_k} I^P_{|\varepsilon^{(k)}|}(\varepsilon^{(k)}) = \bigcap_{k=1}^{\infty} \bigcup_{\varepsilon^{(k)}\in\mathbb{F}_k} I^P_{n_k+\ell_{n_k}}(\varepsilon^{(k)}).$$

Lemma 5.6 $\mathcal{F} \subset E$.

Proof This is clear by (5.8). □

5.2 Measure supported on $\mathcal{F}$

Though $\mathcal{F}$ can only be viewed as a *locally* homogeneous Cantor set, we define a measure *uniformly* distributed on $\mathcal{F}$. This measure is defined along the cylinders with non-empty intersection with $\mathcal{F}$. For any $\beta \in \mathcal{F}$, let $\{I^P_n(\beta)\}_{n\ge1}$ be the cylinders containing β and write

$$\varepsilon(1,\beta) = (\varepsilon^{(k-1)}, u_1, \ldots, u_{t_k}, 0^{i_k}, w_1^{(k)}, \ldots, w_{\ell_{n_k}}^{(k)}, \ldots).$$

To simplify the notation, we still use u_{t_k} to denote $(u_{t_k}, 0^{i_k})$ in the above formula. Then the β-expansion of 1 will be read as

$$\varepsilon(1,\beta) = (\varepsilon^{(k-1)}, u_1, \ldots, u_{t_k}, w_1^{(k)}, \ldots, w_{\ell_{n_k}}^{(k)}, \ldots).$$

Note that the order of $\varepsilon^{(k-1)}$ is $n_{k-1} + \ell_{n_{k-1}}$.

Now define

$$\mu\big(I^P_{M+q}(\varepsilon^{(0)})\big) = 1,$$

and let

$$\mu\big(I^P_{n_1}(\varepsilon^{(0)}, u_1, \ldots, u_{t_1})\big) = \left(\frac{1}{\sharp\Sigma^N_{\beta_2}}\right)^{t_1}.$$

In other words, the measure is uniformly distributed among the offsprings of the cylinder $I^P_{M+q}(\varepsilon^{(0)})$ with nonempty intersection with $\mathcal{F}$.

Next for each $n_1 < n \le n_1 + \ell_{n_1}$, let

$$\mu\big(I^P_n(\beta)\big) = \mu\big(I^P_{n_1}(\beta)\big).$$

Assume that $\mu\big(I^P_{n_{k-1}+\ell_{n_{k-1}}}(\beta)\big)$, i.e. $\mu\big(I^P_{n_{k-1}+\ell_{n_{k-1}}}(\varepsilon^{(k-1)})\big)$ has been defined.

(1) Define

$$\mu\big(I^P_{n_k}(\varepsilon^{(k-1)}, u_1, \ldots, u_{t_k})\big) := \left(\frac{1}{\sharp\Sigma^N_{\beta_2}}\right)^{t_k} \mu\big(I^P_{|\varepsilon^{(k-1)}|}(\varepsilon^{(k-1)})\big) = \left(\prod_{j=1}^{k} \big(\sharp\Sigma^N_{\beta_2}\big)^{t_j}\right)^{-1}. \tag{5.11}$$

(2) When $n_{k-1} + \ell_{n_{k-1}} < n < n_k$, let

$$\mu\big(I^P_n(\beta)\big) = \sum_{I^P_{n_k}(w)\in\mathcal{F}_k : I^P_{n_k}(w)\cap I^P_n(\beta)\ne\emptyset} \mu\big(I^P_{n_k}(w)\big).$$

More precisely, when $n = n_{k-1} + \ell_{n_{k-1}} + t\ell$,

$$\mu\big(I_n^P(\beta)\big) = \prod_{j=1}^{k-1}\left(\frac{1}{\sharp\Sigma_{\beta_2}^N}\right)^{t_j}\cdot\left(\frac{1}{\sharp\Sigma_{\beta_2}^N}\right)^{t}, \tag{5.12}$$

and when $n = n_{k-1} + \ell_{n_{k-1}} + t\ell + i$ for some $i \neq 0$, we have

$$\mu\big(I^P_{n_{k-1}+\ell_{n_{k-1}}+t\ell}(\beta)\big) \geq \mu\big(I_n^P(\beta)\big) \geq \max\Big\{\mu\big(I^P_{n_{k-1}+\ell_{n_{k-1}}+(t+1)\ell}(\beta)\big),\ \mu\big(I^P_{n_k}(\beta)\big)\Big\}. \tag{5.13}$$

(3) When $n_k < n \leq n_k + \ell_{n_k}$, take

$$\mu\big(I_n^P(\beta)\big) = \mu\big(I^P_{n_k}(\beta)\big). \tag{5.14}$$

5.3 Lengths of cylinders

Now we estimate the lengths of cylinders with non-empty intersection with $\mathcal{F}$.

Let $(\varepsilon_1, \ldots, \varepsilon_n)$ be self-admissible such that $I_n^P := I_n^P(\varepsilon_1, \ldots, \varepsilon_n)$ has non-empty intersection with $\mathcal{F}$. Thus there exists $\beta \in \mathcal{F}$ such that I_n^P is just the cylinder containing β. Let $n_k \leq n < n_{k+1}$ for some $k \geq 1$. The estimate of the length of I_n^P is divided into two cases according to the range of n.

(1) When $n_k \leq n < n_k + \ell_{n_k}$. The length of I_n^P is bounded from below by the length of the cylinder containing β with order $n_k + \ell_{n_k} + M$.
By the construction of $\mathcal{F}_k$, we know that $\varepsilon(1, \beta)$ can be expressed as

$$\varepsilon(1,\beta) = (\varepsilon^{(k)}, 0^M, 1, \ldots),$$

which implies the self-admissibility of $(\varepsilon^{(k)}, 0^M, 1)$. Then clearly $(\varepsilon^{(k)}, 0^M, 0)$ is self-admissible as well. Then by Lemma 3.4, we know that $(\varepsilon^{(k)}, 0^M, 0)$ is non-recurrent. Thus,

$$\begin{aligned}\big|I_n^P\big| &\geq \big|I^P_{n_k+\ell_{n_k}+M}(\beta)\big| \geq \big|I^P_{n_k+\ell_{n_k}+M+1}(\varepsilon^{(k)}, 0^M, 0)\big| \\ &\geq C\beta_1^{-(n_k+\ell_{n_k}+M+1)} := C_1\beta_1^{-(n_k+\ell_{n_k})}.\end{aligned} \tag{5.15}$$

(2) When $n_k + \ell_{n_k} \leq n < n_{k+1}$. Let $t = n - n_k - \ell_{n_k}$. Write $\varepsilon(1, \beta)$ as

$$\varepsilon(1,\beta) = (\varepsilon^{(k)}, \eta_1, \ldots, \eta_t, \ldots)$$

for some $(\eta_1, \ldots, \eta_t) \in \Sigma^t_{\beta_2}$. Lemma 5.4 tells us that

$$(\varepsilon^{(k)}, \eta_1, \ldots, \eta_t, 0^M, 1, 0^M)$$

is self-admissible. Then with the same argument as case (1), we obtain

$$\big|I_n^P\big| \geq \big|I^P_{n+M+1}(\varepsilon^{(k)}, \eta_1, \ldots, \eta_t, 0^M, 0)\big| \geq C\beta_1^{-(n+M+1)} := C_1\beta_1^{-n}. \tag{5.16}$$

5.4 Measure of balls

We estimate the measure of arbitrary balls $B(\beta, r)$ with $\beta \in \mathcal{F}$ and r small enough.

Together with the μ-measure and the lengths of cylinders with non-empty intersection with $\mathcal{F}$ given in the last two subsections, it follows directly that

Corollary 5.7 *For any $\beta \in \mathbb{F}$,*

$$\liminf_{n\to\infty} \frac{\log \mu\big(I_n^P(\beta)\big)}{\log |I_{n+1}^P(\beta)|} \geq \frac{1}{1+\alpha} \frac{\log \beta_2}{\log \beta_1} \frac{N}{\ell}, \tag{5.17}$$

where N and ℓ are the integers in the definition of U_ℓ (see (5.4)).

Now we refine the cylinders containing some $\beta \in \mathcal{F}$ as follows. For each $\beta \in \mathcal{F}$ and $n \geq 1$, define

$$J_n(\beta) = \begin{cases} I^P_{n_k+\ell_{n_k}}(\beta), & \text{when } n_k \leq n < n_k + \ell_{n_k} \text{ for some } k \geq 1; \\ I_n^P(\beta), & \text{when } n_k + \ell_{n_k} \leq n < n_{k+1} \text{ for some } k \geq 1. \end{cases} \tag{5.18}$$

and call $J_n(\beta)$ the *basic interval* of order n containing β.

Fix a ball $B(\beta, r)$ with $\beta \in \mathcal{F}$ and r small. Let n be the integer such that

$$\big|J_{n+1}(\beta)\big| \leq r < \big|J_n(\beta)\big|.$$

Let k be the integer such that $n_k \leq n < n_{k+1}$. The difference of the lengths of $J_{n+1}(\beta)$ and $J_n(\beta)$ (i.e., $|J_{n+1}(\beta)| < |J_n(\beta)|$) yields that

$$n_k + \ell_{n_k} \leq n < n_{k+1}.$$

Recall the definition of μ. It should be noticed that

$$\mu\big(J_n(\beta)\big) = \mu\big(I_n^P(\beta)\big), \quad \text{for all } n \in \mathbb{N}.$$

Then all *basic intervals* J with the same order are of equal μ-measure. So, to bound the measure of the ball $B(\beta, r)$ from above, it suffices to estimate the number of basic intervals with non-empty intersection with the ball $B(\beta, r)$. We denote this number by $\mathcal{N}$. Note that by (5.16) and (5.18), when $n_k + \ell_{n_k} \leq n < n_{k+1}$, all basic intervals of order n are of length no less than $C_1\beta_1^{-n}$. Since $r \leq |J_n(\beta)| \leq \beta_0^{-n}$, we have

$$\mathcal{N} \leq 2r/(C_1\beta_1^{-n}) + 2 \leq 2\beta_0^{-n}/(C_1\beta_1^{-n}) + 2 \leq C_2\beta_0^{-n}\beta_1^n.$$

It follows that

$$\mu\big(B(\beta, r)\big) \leq C_2\beta_0^{-n}\beta_1^n \cdot \mu\big(I_n^P(\beta)\big). \tag{5.19}$$

Now we give a lower bound for r. When $n < n_{k+1} - 1$, we have

$$r \geq \big|J_{n+1}(\beta)\big| = \big|I_{n+1}^P(\beta)\big| \geq C_1\beta_1^{-n-1}. \tag{5.20}$$

When $n = n_{k+1} - 1$, we have

$$r \geq \big|J_{n+1}(\beta)\big| \geq C_1\beta_1^{-n_{k+1}-\ell_{n_{k+1}}} \tag{5.21}$$

Thus, by the formulae (5.19) (5.20) (5.21) and Corollary 5.7, we have

$$\liminf_{r\to 0} \frac{\log \mu(B(\beta, r))}{\log r} \geq \left(\frac{\log \beta_0 - \log \beta_1}{\log \beta_1} + \frac{\log \beta_2}{\log \beta_1}\frac{N}{\ell}\right)\frac{1}{1+\alpha}.$$

Applying the mass distribution principle (Proposition 5.1), we obtain

$$\dim_{\mathrm{H}} E \geq \left(\frac{\log \beta_0 - \log \beta_1}{\log \beta_1} + \frac{\log \beta_2}{\log \beta_1}\frac{N}{\ell}\right)\frac{1}{1+\alpha}.$$

Letting $N \to \infty$ and then $\beta_1 \to \beta_0$, we arrive at

$$\dim_{\mathsf{H}} E \geq \frac{1}{1+\alpha}.$$

6 Lower bound of $E(\{\ell_n\}_{n\geq 1}, x_0) : x_0 = 1$

We still use the classic strategy to estimate the dimension of $E(\{\ell_n\}_{n\geq 1}, 1)$ from below. In fact, we will show a little stronger result: for any $\beta_0 < \beta_1$, the Hausdorff dimension of the set $E(\{\ell_n\}_{n\geq 1}, 1) \cap (\beta_0, \beta_1)$ is $1/(1+\alpha)$.

The first step is devoted to constructing a Cantor subset $\mathcal{F}$ of $E(\{\ell_n\}_{n\geq 1}, 1)$. We begin with some notation.

As in the beginning of Sect. 5.1, we can require that β_0 and β_1 are sufficiently close such that the common prefix

$$(\varepsilon_1(1, \beta_1), \ldots, \varepsilon_{M-1}(1, \beta_1))$$

of $\varepsilon(1, \beta_0)$ and $\varepsilon(1, \beta_1)$ contains at least four nonzero terms. Assume that $\varepsilon(1, \beta_1)$ begins with the word $o = (a_1, 0^{r_1-1}, a_2, 0^{r_2-1}, a_3, 0^{r_3-1}, a_4)$ with $a_i \neq 0$. Let

$$\overline{o} = (0^{r_1}, 1, 0^{r_2}, 1, 0^{r_3}), \quad \overline{O} = (0^{r_1}, 1, 0^{r_2+1}).$$

By the self-admissibility of o, it follows that if $a_1 = 1$, then $\min\{r_2, r_3\} \geq r_1$. So it is direct to check that for any $i \geq 0$, we have

$$\sigma^i(\overline{o}) \prec \varepsilon_1(1, \beta_1), \ldots, \varepsilon_{(r_1+r_2+r_3+2)-i}(1, \beta_1). \tag{6.1}$$

Recall that β_2 is given in (5.2). Fix an integer $\ell \gg M$. Define the collection

$$U_\ell = \big\{u = (\overline{o}, \varepsilon_1, \ldots, \varepsilon_{\ell-r_1-r_2-r_3-2-M}, 0^M) \in \Sigma_{\beta_2}^{\ell}\big\}.$$

Following the same argument as the case (3) in proving Lemma 5.2 and then by (5.3), we have for any $u \in U_\ell$ and $i \geq r_1 + r_2 + r_3 + 2$,

$$\sigma^i(u) \prec (\varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1)). \tag{6.2}$$

Combining (6.1) and (6.2), we get for any $u \in U_\ell$ and $i \geq 0$,

$$\sigma^i(u) \prec (\varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1)). \tag{6.3}$$

Recall that q is the integer such that

$$(\varepsilon_{M+1}(1, \beta_1), \ldots, \varepsilon_{M+q}(1, \beta_1)) \neq 0^q.$$

With the help of (6.3), we present a result with the same role as that of Lemma 5.2.

Lemma 6.1 *Let $k \in \mathbb{N}$. For any $u_1, \ldots, u_k \in U_\ell$, the word*

$$\varepsilon = (\varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1), 0^q, u_1, u_2, \ldots, u_k)$$

is non-recurrent.

Proof We check that $\sigma^i(\varepsilon) \prec \varepsilon$ for all $i \geq 1$. When $i < M + q$, the argument is absolutely the same as that for $i < M + q$ in Lemma 5.2. When $i \geq M + q$, it follows by (6.3). □

6.1 Construction of the Cantor subset

Now we return to the set

$$E_0 := \left\{ \beta_0 < \beta < \beta_1 : \left|T_\beta^n 1 - 1\right| < \beta^{-\ell_n},\ \text{i.o.}\, n \in \mathbb{N} \right\}.$$

We will use the following strategy to construct a Cantor subset of E_0.

- STRATEGY: *If the β-expansion of 1 has a long periodic prefix with period n, then $T_\beta^n 1$ and 1 will be close enough.*

Let $\{n_k\}_{k\geq 1}$ be a subsequence of integers such that

$$\lim_{k\to\infty} \frac{\ell_{n_k}}{n_k} = \liminf_{n\to\infty} \frac{\ell_n}{n} = \alpha, \quad \text{and} \quad n_{k+1} \gg n_k, \text{ for all } k \geq 1.$$

6.1.1 First generation $\mathcal{F}_1$ of the Cantor set $\mathcal{F}$

Let $\varepsilon^{(0)} = (\varepsilon_1(1, \beta_1), \ldots, \varepsilon_M(1, \beta_1), 0^q)$ and $m_0 = M + q$. Write $n_1 = m_0 + t_1\ell + i_1$ for some $t_1 \in \mathbb{N}$ and $0 \leq i_1 < \ell$. Now consider the collection of self-admissible words of length n_1

$$\mathfrak{M}(\varepsilon^{(0)}) = \left\{ (\varepsilon^{(0)}, u_1, \ldots, u_{t_1}, 0^{i_1}) : u_1, \ldots, u_{t_1} \in U_\ell \right\}.$$

Lemma 6.1 says that all the elements in $\mathfrak{M}(\varepsilon^{(0)})$ are non-recurrent words.

Enlarging ℓ_{n_1} by at most $m_0 + \ell$ if necessary, the number ℓ_{n_1} can be written as

$$\ell_{n_1} = z_1 n_1 + m_0 + j_1\ell, \quad \text{with}\, z_1 \in \mathbb{N},\ 0 \leq j_1 < t_1. \tag{6.4}$$

Corollary 3.7 convinces us that for any $(\varepsilon_1, \ldots, \varepsilon_{n_1}) \in \mathfrak{M}(\varepsilon^{(0)})$, the word

$$\varepsilon := \Big((\varepsilon_1, \ldots, \varepsilon_{n_1}), (\varepsilon_1, \ldots, \varepsilon_{n_1})^{z_1}, (\varepsilon^{(0)}, u_1, \ldots, u_{j_1}) \Big) \tag{6.5}$$

is self-admissible. In other words, ε is a periodic self-admissible word with length $n_1 + \ell_{n_1}$. We remark that the suffix $(\varepsilon^{(0)}, u_1, \ldots, u_{j_1})$ is the prefix of $(\varepsilon_1, \ldots, \varepsilon_{n_1})$ but not chosen freely.

Now consider the cylinder

$$I^P_{n_1+\ell_{n_1}} := I^P_{n_1+\ell_{n_1}} \Big((\varepsilon_1, \ldots, \varepsilon_{n_1})^{z_1+1}, (\varepsilon^{(0)}, u_1, \ldots, u_{j_1}) \Big).$$

It is clear that for each $\beta \in I^P_{n_1+\ell_{n_1}}$, the β-expansion of $T_\beta^{n_1} 1$ and that of 1 coincide for the first ℓ_{n_1} terms. So, we conclude that for any $\beta \in I^P_{n_1+\ell_{n_1}}$,

$$\left|T_\beta^{n_1} 1 - 1\right| < \beta^{-\ell_{n_1}}. \tag{6.6}$$

Now we prolong the word in (6.5) to a non-recurrent word. Still by Corollary 3.7, we know that (ε, u_{j_1+1}) is self-admissible, which implies the admissibility of the word

$$(\varepsilon, 0^{r_1}, 1, 0^{r_2}, 1).$$

So, by Lemma 3.4, we obtain that the word $(\varepsilon, \overline{O})$ is non-recurrent. Then finally, the first generation $\mathcal{F}_1$ of the Cantor set $\mathcal{F}$ is defined as

$$\mathcal{F}_1 = \Big\{ I^P_{(n_1+\ell_{n_1}+r_1+r_2+2)} \Big((\varepsilon_1, \ldots, \varepsilon_{n_1})^{z_1+1}, (\varepsilon^{(0)}, u_1, \ldots, u_{j_1}, \overline{O}) \Big) : (\varepsilon_1, \ldots, \varepsilon_{n_1}) \in \mathfrak{M}(\varepsilon^{(0)}) \Big\}.$$

6.1.2 Second generation $\mathcal{F}_2$ of the Cantor set $\mathcal{F}$

Let $m_1 = n_1 + \ell_{n_1} + r_1 + r_2 + 2$ and write

$$n_2 = m_1 + t_2\ell + i_2 \quad \text{for some} \quad t_2 \in \mathbb{N}, \quad 0 \le i_2 < \ell.$$

For each $\varepsilon^{(1)} \in \mathcal{F}_1$, consider the collection of self-admissible words of length n_2

$$\mathfrak{M}(\varepsilon^{(1)}) = \left\{ (\varepsilon^{(1)}, u_1, \ldots, u_{t_2}, 0^{i_2}) : u_1, \ldots, u_{t_2} \in U_\ell \right\}.$$

By noting that $\varepsilon^{(1)}$ is non-recurrent and by the formula (6.3), we know that all elements in $\mathfrak{M}(\varepsilon^{(1)})$ are non-recurrent words.

Similar to the modification on ℓ_{n_1}, by enlarging ℓ_{n_2} by at most $m_1 + \ell$ if necessary, the number ℓ_{n_2} can be written as

$$\ell_{n_2} = z_2 n_2 + m_1 + j_2\ell, \quad \text{with} \quad z_2 \in \mathbb{N}, \quad 0 \le j_2 < t_2. \tag{6.7}$$

Then, following the same line as for the construction of the first generation, we get the second generation $\mathcal{F}_2$, defined by

$$\mathcal{F}_2 = \Big\{ I^P_{(n_2+\ell_{n_2}+r_1+r_2+1)}\Big(\big(\varepsilon_1, \ldots, \varepsilon_{n_2}\big)^{z_2+1}, \big(\varepsilon^{(1)}, u_1, \ldots, u_{j_2}, \overline{O}\big)\Big) : (\varepsilon_1, \ldots, \varepsilon_{n_2}) \\ \in \mathfrak{M}(\varepsilon^{(1)}) \Big\}.$$

We remark that the suffix $\big(\varepsilon^{(1)}, u_1, \ldots, u_{j_2}\big)$ is the prefix of $(\varepsilon_1, \ldots, \varepsilon_{n_2})$ but not chosen freely. Then let $m_2 = n_2 + \ell_{n_2} + r_1 + r_2 + 2$.

Then, proceeding along the same line, we get a nested sequence $\mathcal{F}_k$ consisting of a family of cylinders. The desired Cantor set is defined as

$$\mathcal{F} = \bigcap_{k \ge 1} \mathcal{F}_k.$$

Noting (6.6), we know that $\mathcal{F} \subset E_0$.

6.2 Estimate on the supported measure

The remaining argument for the dimension of $\mathcal{F}$ is almost the same as what we did in Sect. 5: constructing an evenly distributed measure supported on $\mathcal{F}$ and then applying the mass distribution principle. Thus, we will not repeat it here.

7 Proof of Theorem 1.2

The proof of Theorem 1.2 can be established with almost the same argument as that for Theorem 1.1. Therefore only differences of the proof are marked below.

7.1 Proof of the upper bound

For each self-admissible sequence $(i_1, \ldots, i_n)$, denote

$$J_n(i_1, \ldots, i_n) := \left\{ \beta \in I^P_n(i_1, \ldots, i_n) : |T^n_\beta 1 - x(\beta)| < \beta_0^{-\ell_n} \right\}.$$

These sets correspond to the sets $I_n^P(i_1, \ldots, i_n; \beta_0^{-\ell_n})$ studied in the proof of Proposition 4.2, where the upper bound for the case of constant x_0 was obtained. We have that

$$\big(\widetilde{E}\big(\{\ell_n\}_{n\ge1}, x\big) \cap (\beta_0, \beta_1)\big) \subset \bigcap_{N=1}^{\infty} \bigcup_{n=N}^{\infty} \bigcup_{(i_1,\ldots,i_n)\text{ self-admissible}} J_n(i_1, \ldots, i_n).$$

What remains is to estimate the diameter of $J_n(i_1, \ldots, i_n)$ for any self-admissible sequence $(i_1, \ldots, i_n)$. If we can get a good estimate of the diameter, then we can do as in the proof of Proposition 4.2 to get an upper bound of the dimension of $\widetilde{E}\big(\{\ell_n\}_{n\ge1}, x\big) \cap (\beta_0, \beta_1)$.

Suppose J_n is non-empty, and let $\beta_2 < \beta_3$ denote the infimum and supremum of J_n. Let L be such that $\beta \mapsto x(\beta)$ is Lipschitz continuous, with constant L. Denote by ψ the map $\beta \mapsto T_\beta^n(1)$, and note that ψ satisfies

$$|\psi(\beta_3) - \psi(\beta_2)| \ge \beta_0^n \cdot |\beta_3 - \beta_2|.$$

Clearly, β_2 and β_3 must satisfy

$$|\psi(\beta_3) - \psi(\beta_2)| - |x(\beta_3) - x(\beta_2)| < 2\beta_0^{-\ell_n},$$

and hence, we must have

$$\beta_0^n \cdot |\beta_3 - \beta_2| - L \cdot |\beta_3 - \beta_2| < 2\beta_0^{-\ell_n}. \tag{7.1}$$

Take $K > 2$. Then we must have $|\beta_3 - \beta_2| \le K\beta_0^{-\ell_n - n}$ for sufficiently large n, otherwise (7.1) will not be satisfied.

Thus, we have proved that $|J_n(i_1, \ldots, i_n)| \le K\beta_0^{-\ell_n - n}$ for some constant K. This is all what is needed to make the proof of Proposition 4.2 work also for the case of non-constant x_0.

7.2 Proof of the lower bound

Case 1. If $x(\beta) = 1$ for all $\beta \in [\beta_0, \beta_1]$, this falls into the proof of Theorem 1.1.
Case 2. Otherwise, we can find a subinterval of (β_0, β_1) such that the supremum of $x(\beta)$ on this subinterval is strictly less than 1. We denote by $0 \le x_0 < 1$ the supremum of $x(\beta)$ on this subinterval. We note that with this definition of x_0, Lemma 5.3 still holds.

Now that we have Lemma 5.3, we can get a lower bound in the same way as in Sect. 5, i.e. we construct a Cantor set with desired properties. The proof is more or less unchanged, but some minor changes are nessesary, as we will describe below.

The sets $\mathbb{F}_0$ and $\mathfrak{M}(\varepsilon^{(0)})$ are defined as before, and we consider a $w \in \mathfrak{M}(\varepsilon^{(0)})$. On the interval $I_{n_1}^P(w)$ we define $\psi\colon \beta \mapsto T_\beta^{n_1}(1)$, and we observe that there are constants c_1 and c_2 such that

$$c_1\beta_0^{n_1} \le \psi'(\beta) \le c_2\beta_0^{n_1},$$

holds for all $\beta \in I_{n_1}^P(w)$. As in the proof of the upper bound, we let L denote the Lipschitz constant of the function $\beta \mapsto x(\beta)$.

We need to estimate the size of the set

$$J = \{\, \beta \in I_{n_1}^P(w) : \psi(\beta) \in B(x_0(\beta), C(n_1 + \ell_{n_1})\beta_0^{-\ell_{n_1}})\,\}.$$

The constant C appearing in the definition of J above, was equal to 4 in Sect. 5. We remark that the value of C has no influence on the result of the proof, so we may choose it more freely, as will be done here.

Lemma 5.3 implies that there is a $\beta_a \in J$ such that $\psi(\beta_a) = x(\beta_a)$. Suppose $\beta_b \in I^P_{n_1}(w)$ is such that $|\beta_a - \beta_b| < 4(n_1 + \ell_{n_1})\beta_0^{-n_1-\ell_{n_1}}$. We can choose C so large that we have

$$|\psi(\beta_b) - x(\beta_b)| \leq |\psi(\beta_a) - \psi(\beta_b)| + |x(\beta_a) - x(\beta_b)|$$
$$\leq c_2 4(n_1 + \ell_{n_1})\beta_0^{-\ell_{n_1}} + L \cdot 4(n_1 + \ell_{n_1})\beta_0^{-n_1-\ell_{n_1}} < C(n_1 + \ell_{n_1})\beta_0^{-\ell_{n_1}}.$$

This proves that β_b is in J, and hence, J contains an interval of length at least $4(n_1 + \ell_{n_1})\beta_0^{-n_1-\ell_{n_1}}$.

Analogous to the estimate in (5.7), we have that $|I^P_{n_1+\ell_{n_1}}(\varepsilon)| \leq 4\beta_0^{-n_1-\ell_{n_1}}$. This implies that there are at least $(n_1 + \ell_{n_1})$ consequtive cylinders $I^P_{n_1+\ell_{n_1}}(\varepsilon)$ with the desired hitting property, where $\varepsilon \in \mathbb{A}(w)$.

With the changes indicated above, the proof then continues just as in Sect. 5.

8 Application

This section is devoted to an application of Theorem 1.1. For each $n \geq 1$, denote by $\ell_n(\beta)$ the length of the longest string of zeros just after the nth digit in the β-expansion of 1, namely,

$$\ell_n(\beta) := \max\{k \geq 0 : \varepsilon^*_{n+1}(\beta) = \cdots = \varepsilon^*_{n+k}(\beta) = 0\}.$$

Let

$$\ell(\beta) = \limsup_{n\to\infty} \frac{\ell_n(\beta)}{n}.$$

Li and Wu [16] gave a kind of classification of betas according to the growth of $\{\ell_n\}_{n\geq 1}$ as follows:

$$A_0 = \Big\{\beta > 1 : \{\ell_n(\beta)\} \text{ is bounded}\Big\};$$
$$A_1 = \Big\{\beta > 1 : \{\ell_n(\beta)\} \text{ is unbounded and } \quad \ell(\beta) = 0\Big\};$$
$$A_2 = \Big\{\beta > 1 : \ell(\beta) > 0\Big\}.$$

We will use the dimensional result of $E(\{\ell_n\}_{n\geq 1}, x_0)$ to determine the size of A_1, A_2 and A_3 in the sense of Lebesgue measure $\mathcal{L}$ and Hausdorff dimension. In the argument below only the dimension of $E(\{\ell_n\}_{n\geq 1}, x_0)$ when $x_0 = 0$ is used. In other words, the result in [18] by Persson and Schmeling is already sufficient for the following conclusions.

Proposition 8.1 *(Size of A_0)* $\mathcal{L}(A_0) = 0$ *and* $\dim_H(A_0) = 1$.

Proof The set A_0 is nothing but the collections of β with specification properties. Then this proposition is just Theorem A in [21]. □

Proposition 8.2 *(Size of A_2)* $\mathcal{L}(A_2) = 0$ *and* $\dim_H(A_2) = 1$.

Proof For any $\alpha > 0$, let

$$F(\alpha) = \{\,\beta > 1 : \ell(\beta) \geq \alpha\,\}.$$

Then $A_2 = \bigcup_{\alpha>0} F(\alpha)$. Since $F(\alpha)$ is increasing with respect to α, the above union can be expressed as a countable union. Now we show that for each $\alpha > 0$

$$\dim_{\mathsf{H}} F(\alpha) = \frac{1}{1+\alpha},$$

which is sufficient for the desired result.

Recall the algorithm of T_β. Since for each $\beta \in A_2$, the β-expansion of 1 is infinite, then for each $n \geq 1$, we have

$$T_\beta^n 1 = \frac{\varepsilon_{n+1}^*(\beta)}{\beta} + \frac{\varepsilon_{n+2}^*(\beta)}{\beta^2} + \cdots .$$

Then by the definition of $\ell_n(\beta)$, it follows that

$$\beta^{-(\ell_n(\beta)+1)} \leq T_\beta^n 1 \leq (\beta+1)\beta^{-(\ell_n(\beta)+1)}. \tag{8.1}$$

As a consequence, for any $\delta > 0$,

$$F(\alpha) \subset \{\beta > 1 : T_\beta^n 1 < (\beta+1)\beta^{-n(\alpha-\delta)-1} \text{ for infinitely many } n \in \mathbb{N}\}. \tag{8.2}$$

On the other hand, it is clear that

$$\{\beta > 1 : T_\beta^n 1 < \beta^{-n\alpha} \text{ for infinitely many } n \in \mathbb{N}\} \subset F(\alpha). \tag{8.3}$$

Applying Theorem 1.1 to (8.2) and (8.3), we get that

$$\dim_{\mathsf{H}} F(\alpha) = \frac{1}{1+\alpha}.$$

□

Since $A_1 = (1,\infty)\backslash(A_0 \cup A_2)$, it follows directly that

Proposition 8.3 *(Size of A_1) The set A_1 is of full Lebesgue measure.*

Acknowledgments This work was in part finished when some of the authors visited the Morning Center of Mathematics (Beijing). The authors are grateful for the host's warm hospitality. This work was partially supported by NSFC Nos. 11126071, 10901066, 11225101 and 11171123.

References

1. Blanchard, F.: β-expansion and symbolic dynamics. Theor. Comput. Sci. **65**, 131–141 (1989)
2. Bugeaud, Y.: A note on inhomogeneous Diophantine approximation. Glasgow Math. J. **45**, 105–110 (2003)
3. Bugeaud, Y.: An inhomogeneous Jarník theroem. J. Anal. Math. **92**, 327–349 (2004)
4. Bugeaud, Y., Chevallier, N.: On simultaneous inhomogeneous Diophantine approximation. Acta Arith. **123**, 97–123 (2006)
5. Bugeaud, Y., S. Harrap, S., Kristensen, S., Velani, S.: On shrinking targets for $\mathbb{Z}^m$ actions on tori. Mathematika **56**, 193–202 (2010)
6. Chernov, N., Kleinbock, D.: Dynamical Borel-Cantelli lemmas for Gibbs measures. Israel J. Math. **122**, 1–27 (2001)
7. Falconer, K.J.: Fractal Geometry: Mathematical Foundations and Application. Wiley, New York (1990)
8. Fan, A.H., Wang, B.W.: On the lengths of basic intervals in beta expansion. Nonlinearity **25**, 1329–1343 (2012)
9. Fan, A.H., Wu, J.: A note on inhomogeneous Diophantine approximation with a general error function. Glasg. Math. J. **48**, 187–191 (2006)

10. Färm, D., Persson, T., Schmeling, J.: Dimenion of countable intersections of some sets arising in expansions in non-integer bases. Fundamenta Math. **209**, 157–176 (2010)
11. Fernández, J.L., Melián, M.V., Pestana, D.: Quantitative recurrence properties in expanding systems. arXiv: math/0703222 (2007)
12. Hill, R., Velani, S.: The ergodic theory of shrinking targets. Invent. Math. **119**, 175–198 (1995)
13. Hill, R., Velani, S.: Metric Diophantine approximation in Julia sets of expanding rational maps. Inst. Hautes Études Sci. Publ. Math **85**, 193–216 (1997)
14. Hofbauer, F.: β-shifts have unique maximal measure. Monatsh. Math. **85**, 189–198 (1978)
15. Levesley, J.: A general inhomogeneous Jarník-Besicovitch theorem. J. Number Theory **71**, 65–80 (1998)
16. Li, B., Wu, J.: Beta-expansion and continued fraction expansion. J. Math. Anal. Appl. **339**, 1322–1331 (2008)
17. Parry, W.: On the β-expansions of real numbers. Acta Math. Acad. Sci. Hunger. **11**, 401–416 (1960)
18. Persson, T., Schmeling, J.: Dyadic Diophantine approximation and Katok's horseshoe approximation. Acta Arith. **132**, 205–230 (2008)
19. Pfister, C.-E., Sullivan, W.G.: Large deviations estimates for dynamical systems without the specification property. Applications to the β-shifts. Nonlinearity **18**, 237–261 (2005)
20. Rényi, A.: Representations for real numbers and their ergodic properties. Acta Math. Acad. Sci. Hungar. **8**, 477–493 (1957)
21. Schmeling, J.: Symbolic dynamics for β-shifts and self-normal numbers. Ergod. Theory Dyn. Syst. **17**, 675–694 (1997)
22. Schmeling, J., Troubetzkoy, S.: Inhomogeneous Diophantine approximation and angular recurrence properties of the billiard flow in certain polygons. Math. Sbornik **194**, 295–309 (2003)
23. Shen, L.M., Wang, B.W.: Shrinking target problems for beta-dynamical system. Sci. China Math. **56**, 91–104 (2013)
24. Stratmann, B., Urbański, M.: Jarník and Julia; a Diophantine analysis for parabolic rational maps. Math. Scand. **91**, 27–54 (2002)
25. Tan, B., Wang, B.W.: Quantitive recurrence properties of beta dynamical systems. Adv. Math. **228**, 2071–2097 (2011)
26. Thompson, D.: Irregular sets, the β-transformation and the almost specification property. Trans. Am. Math. Soc. **364**, 5395–5414 (2012)
27. Urbański, M.: Diophantine analysis of conformal iterated function systems. Monatsh. Math. **137**, 325–340 (2002)