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Cuspidal systems for affine Khovanov–Lauda–Rouquier algebras

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Abstract A cuspidal system for an affine Khovanov–Lauda–Rouquier algebra R_α yields a theory of standard modules. This allows us to classify the irreducible modules over R_α up to the so-called imaginary modules. We describe minuscule imaginary modules, laying the groundwork for future study of imaginary Schur–Weyl duality. We introduce colored imaginary tensor spaces and reduce a classification of imaginary modules to one color. We study the characters of cuspidal modules. We show that under the Khovanov–Lauda–Rouquier categorification, cuspidal modules correspond to dual root vectors.

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1 Introduction

Khovanov–Lauda–Rouquier (KLR) algebras were defined in [13,14,24]. Their representation theory is of interest for the theory of canonical bases, modular representation theory, cluster theory, knot theory, etc. Let F be an arbitrary ground field. The KLR algebra $R_\alpha = R_\alpha(\mathtt{C}, F)$ is a graded unital associative F-algebra depending on a Lie type $\mathtt{C}$ and an element α of the non-negative part Q_+ of the corresponding root lattice.

A natural approach to representation theory of R_α is provided by a theory of standard modules. For KLR algebras of *finite* Lie type such a theory was first described in [17], see also [4,9,23]. Key features of this theory are as follows. There is a natural induction functor $\mathrm{Ind}_{\alpha,\beta}$, which associates to an R_α-module M and an R_β-module N the $R_{\alpha+\beta}$-module

$$M \circ N := \mathrm{Ind}_{\alpha,\beta} M \boxtimes N$$

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for $\alpha, \beta \in Q_+$. We refer to this operation as the *induction product*. The functor $\mathrm{Ind}_{\alpha,\beta}$ has an obvious right adjoint $\mathrm{Res}_{\alpha,\beta}$.

To every positive root $\beta \in \Phi_+$ of the corresponding root system Φ, one associates a *cuspidal module* L_β. We point out a remarkable property of cuspidal modules which turns out to be key for building the theory of standard modules: the induction product powers $L_\beta^{\circ n}$ are irreducible for all $n > 0$, see [17, Lemma 6.6]. We make a special choice of a total order on Φ_+, and let $\beta_1 > \cdots > \beta_N$ be the positive roots taken in this order. A *root partition* of $\alpha \in Q_+$ is a tuple $\pi = (m_1, \dots, m_N)$ of nonnegative integers such that $\alpha = \sum_{n=1}^N m_n\beta_n$. The set of root partitions of α is denoted by $\Pi(\alpha)$.

Given $\pi = (m_1, \dots, m_N) \in \Pi(\alpha)$ we define the corresponding standard module $\Delta(\pi)$ as the induction product

$$\Delta(\pi) = L_{\beta_1}^{\circ m_1} \circ \cdots \circ L_{\beta_N}^{\circ m_N}\langle \mathtt{sh}(\pi)\rangle,$$

where $\langle \mathtt{sh}(\pi)\rangle$ means that grading is shifted by an explicit integer $\mathtt{sh}(\pi)$. Then the head of $\Delta(\pi)$ is proved to be irreducible, and, denoting this head by $L(\pi)$, we get a complete irredundant system

$$\{L(\pi) \mid \pi \in \Pi(\alpha)\}$$

of irreducible R_α-modules. Moreover, the decomposition matrix

$$([\Delta(\pi) : L(\sigma)])_{\pi,\sigma\in\Pi(\alpha)}$$

is unitriangular if we order its rows and columns according to the natural lexicographic order on root partitions.

We now comment on the order on Φ_+. In [17], the so-called Lyndon order is used, cf. [20]. This is determined by a choice of a total order on the set I of simple roots. Once such a choice has been made, we have a lexicographic order on the set $\langle I\rangle_\alpha$ of words of content α. These words play the role of weights in representation theory of R_α. In particular, each R_α-module has its highest word, and the highest word of an irreducible module determines the irreducible module uniquely up to an isomorphism. This leads to the natural notion of dominant words, namely the ones which occur as highest words in R_α-modules (called good words in [17]). The dominant words of cuspidal modules are characterized among all dominant words by the property that they are Lyndon words. It turns out that the dominant Lyndon words are in one-to-one correspondence with positive roots, and now we can compare positive roots by comparing the corresponding dominant Lyndon words lexicographically. This gives a total order on Φ_+ called a Lyndon order. We point out that the cuspidal modules themselves depend on the choice of a Lyndon order on Φ_+.

It is well-known that each Lyndon order is convex. However, there are in general more convex orders on Φ_+ than Lyndon orders. Recently McNamara [23] has found a remarkable generalization of the standard module theory which works for any convex order on Φ_+. In this generalization the cuspidal modules are defined via their restriction properties, which seems to be not quite as explicit as the definition via highest words. However, all the other important features of the theory, including the simplicity of induction powers of cuspidal modules, as well as the unitriangularity of decomposition matrices, remain the same.

In this paper, we begin to extend the results described above from finite to affine root systems. To describe the results in more detail we need some notation. Let the Lie type $\mathtt{C}$ be of arbitrary *untwisted affine type*. In particular, the simple roots are labeled by the elements of $I = \{0, 1, \dots, l\}$. We have an (affine) root system Φ and the subset $\Phi_+ \subset \Phi$ of *positive roots*. It is known that $\Phi_+ = \Phi_+^{\mathrm{re}} \sqcup \Phi_+^{\mathrm{im}}$, where Φ_+^{re} are the *real roots*, and $\Phi_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$, for the *null-root* δ, are the *imaginary roots*.

Following [1], we define a *convex preorder* on Φ_+ as a preorder $\preceq$ such that the following three conditions hold for all $\beta, \gamma \in \Phi_+$:

$$\beta \preceq \gamma \quad \text{or} \quad \gamma \preceq \beta; \tag{1.1}$$

$$\text{if } \beta \preceq \gamma \quad \text{and} \quad \beta + \gamma \in \Phi_+, \text{ then } \beta \preceq \beta + \gamma \preceq \gamma; \tag{1.2}$$

$$\beta \preceq \gamma \quad \text{and} \quad \gamma \preceq \beta \text{ if and only if } \beta \text{ and } \gamma \text{ are proportional.} \tag{1.3}$$

Convex preorders are known to exist. From (1.3) we have that $\beta \preceq \gamma$ and $\gamma \preceq \beta$ happens for $\beta \neq \gamma$ if and only if both β and γ are imaginary. We write $\beta \prec \gamma$ if $\beta \preceq \gamma$ but $\gamma \npreceq \beta$. The following set is totally ordered with respect to $\preceq$:

$$\Psi := \Phi_+^{\mathrm{re}} \cup \{\delta\}. \tag{1.4}$$

It is easy to see that the set of real roots splits into two disjoint infinite sets

$$\Phi_{\succ}^{\mathrm{re}} := \{\beta \in \Phi_+^{\mathrm{re}} \mid \beta \succ \delta\} \quad \text{and} \quad \Phi_{\prec}^{\mathrm{re}} := \{\beta \in \Phi_+^{\mathrm{re}} \mid \beta \prec \delta\}.$$

Root partitions are defined similarly to the case of finite root systems, except that now we need to take care of imaginary roots. We do this as follows. Consider the set $\mathscr{P}$ of l-multipartitions $\underline{\mu} = (\mu^{(1)}, \dots, \mu^{(l)})$, where each $\mu^{(i)}$ is a usual partition. We write $|\underline{\mu}| := |\mu^{(1)}| + \cdots + |\mu^{(l)}|$ and say that $\underline{\mu}$ is an l-multipartition of $|\underline{\mu}|$. Let $\alpha \in Q_+$. A *root partition of* α is a pair $(M, \underline{\mu})$, where M is a tuple $(m_\rho)_{\rho \in \Psi}$ of non-negative integers such that $\sum_{\rho \in \Psi} m_\rho \rho = \alpha$, and $\underline{\mu}$ is an l-multipartition of m_δ. It is clear that all but finitely many integers m_ρ are zero, so we can always choose a finite subset

$$\rho_1 > \cdots > \rho_s > \delta > \rho_{-t} > \cdots > \rho_{-1}$$

of Ψ such that $m_\rho = 0$ for ρ outside of this subset. Then, denoting $m_u := m_{\rho_u}$, we can write any root partition of α in the form

$$(\rho_1^{m_1}, \dots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \dots, \rho_{-1}^{m_{-1}}),$$

where all $m_u \in \mathbb{Z}_{\geq 0}$, $\underline{\mu} \in \mathscr{P}$, and

$$\sum_{u=1}^{s} m_u \rho_u + |\underline{\mu}|\delta + \sum_{u=1}^{t} m_{-u} \rho_{-u} = \alpha.$$

Denote by $\Pi(\alpha)$ the set of all root partitions of α. There is a natural partial order '$\leq$' on $\Pi(\alpha)$, which is a version of McNamara's bilexicographic order [23], see (3.3). In the following definition and throughout the paper, we always choose degree shifts of irreducible modules which make them graded-self-dual, see Sect. 2.4 for details.

A *cuspidal system* (for a fixed convex preorder) is the following data:

(Cus1) An irreducible R_ρ-module L_ρ assigned to every $\rho \in \Phi_+^{\mathrm{re}}$, with the following property: if $\beta, \gamma \in Q_+$ are non-zero elements such that $\rho = \beta + \gamma$ and $\mathrm{Res}_{\beta,\gamma} L_\rho \neq 0$, then β is a sum of positive roots less than ρ and γ is a sum of positive roots greater than ρ.

(Cus2) An irreducible $R_{n\delta}$-module $L(\underline{\mu})$ assigned to every l-multipartition $\underline{\mu}$ of n for every $n \in \mathbb{Z}_{\geq 0}$, with the following property: if $\beta, \gamma \in Q_+ \setminus \Phi_+^{\mathrm{im}}$ are non-zero elements such that $n\delta = \beta + \gamma$ and $\mathrm{Res}_{\beta,\gamma} L(\underline{\mu}) \neq 0$, then β is a sum of positive real roots less than δ and γ is a sum of positive real roots greater than δ. It is required that $L(\underline{\lambda}) \not\cong L(\underline{\mu})$ unless $\underline{\lambda} = \underline{\mu}$.

We call the irreducible modules L_ρ from (Cus1) *cuspidal modules*, and the irreducible modules $L(\underline{\mu})$ from (Cus2) *(irreducible) imaginary modules*.

It will be proved that cuspidal systems exist for all convex preorders, and cuspidal modules (for a fixed preorder) are determined uniquely up to an isomorphism. However, it is clearly not the case for imaginary modules: they are defined up to a permutation of multipartitions $\underline{\mu}$ of n. We give more comments on this after the Main Theorem.

Now, given a root partition

$$\pi = (\rho_1^{m_1}, \dots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \dots, \rho_{-1}^{m_{-1}}) \in \Pi(\alpha)$$

as above, we define the corresponding *standard module*

$$\Delta(\pi) := L_{\rho_1}^{\circ m_1} \circ \cdots \circ L_{\rho_s}^{\circ m_s} \circ L(\underline{\mu}) \circ L_{\rho_{-t}}^{\circ m_{-t}} \circ \cdots \circ L_{\rho_{-1}}^{\circ m_{-1}} \langle \mathtt{sh}(\pi) \rangle,$$

where $\mathtt{sh}(\pi)$ is an explicit integer defined in (3.5).

Main Theorem *For any convex preorder there exists a cuspidal system* $\{L_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}\}$. *Moreover:*

(i) *For every root partition* π, *the standard module* $\Delta(\pi)$ *has irreducible head; denote this irreducible module* $L(\pi)$.
(ii) $\{L(\pi) \mid \pi \in \Pi(\alpha)\}$ *is a complete and irredundant system of irreducible* R_α*-modules up to isomorphism and degree shift.*
(iii) $L(\pi)^{\circledast} \simeq L(\pi)$.
(iv) $[\Delta(\pi) : L(\pi)]_q = 1$, *and* $[\Delta(\pi) : L(\sigma)]_q \neq 0$ *implies* $\sigma \leq \pi$.
(v) $L_\rho^{\circ n}$ *is irreducible for every* $\rho \in \Phi_+^{\mathrm{re}}$ *and every* $n \in \mathbb{Z}_{>0}$.

This theorem, proved in Sect. 4, gives a 'rough classification' of irreducible R_α-modules. The main problem is that we did not give a canonical definition of individual irreducible imaginary modules $L(\underline{\mu})$. We just know that the amount of such modules for $R_{n\delta}$ is equal to the number of l-multipartitions of n, and so we have labeled them by such multipartitions in an arbitrary way. In fact, there is a solution to this problem. It turns out that there is a beautiful rich theory of imaginary representations of KLR algebras of affine type, which relies on the so-called imaginary Schur–Weyl duality. This theory in particular allows us to construct an equivalence between an appropriate category of imaginary representations of KLR algebras and the category of representations of the classical Schur algebras. We will address these matters in the forthcoming work [16].

In Sect. 5, we make some first steps in the study of imaginary representations and describe explicitly the *minuscule* imaginary representations—the ones which correspond to the l-multipartitions of 1. We introduce colored imaginary tensor spaces and reduce a classification of irreducible imaginary modules to one color. Minuscule imaginary representations are also used in Sects. 6.2 and 6.3 to describe explicitly the cuspidal modules corresponding to the roots of the form $n\delta \pm \alpha_i$. In Sect. 6 we also explain how the characters of other cuspidal modules can be computed by induction using the idea of minimal pairs which was suggested in [23]. In Sect. 4.8, we show that under the Khovanov–Lauda–Rouquier categorification, cuspidal modules correspond to dual root vectors of a dual PBW basis.

We mention that the methods of this paper can be used to simplify some of the proofs in [23], in particular, the identification of the characters of the cuspidal modules with dual PBW elements.

Immediately after the first version of this paper has been posted, the paper [27] has also been released on the arXiv. That paper suggests a different approach to standard module theory for affine KLR algebras, which is based on the theory of Mirkovic-Vilonen polytopes.

2 Preliminaries

Throughout the paper, F is a field of arbitrary characteristic $p \geq 0$. Denote the ring of Laurent polynomials in the indeterminate q by $\mathscr{A} := \mathbb{Z}[q, q^{-1}]$. We use quantum integers $[n]_q := (q^n - q^{-n})/(q - q^{-1}) \in \mathscr{A}$ for $n \in \mathbb{Z}$, and the quantum factorials $[n]_q^! := [1]_q[2]_q \dots [n]_q$. We have a bar-involution on $\mathscr{A}$ and on $\mathbb{Q}(q) \supset \mathscr{A}$ with $\overline{q} = q^{-1}$.

2.1 Lie theoretic notation

Throughout the paper $\mathtt{C} = (\mathtt{c}_{ij})_{i,j \in I}$ is a *Cartan matrix* of *untwisted affine type*, see [10, §4, Table Aff 1]. We have

$$I = \{0, 1, \dots, l\},$$

where 0 is the affine vertex. Following [10, §1.1], let $(\mathfrak{h}, \Pi, \Pi^\vee)$ be a realization of the Cartan matrix $\mathtt{C}$, so we have simple roots $\{\alpha_i \mid i \in I\}$, simple coroots $\{\alpha_i^\vee \mid i \in I\}$, and a bilinear form $(\cdot, \cdot)$ on $\mathfrak{h}^*$ such that

$$\mathtt{c}_{ij} = 2(\alpha_i, \alpha_j)/(\alpha_i, \alpha_i)$$

for all $i, j \in I$. We normalize $(\cdot, \cdot)$ so that $(\alpha_i, \alpha_i) = 2$ if α_i is a short simple root.

The fundamental dominant weights $\{\Lambda_i \mid i \in I\}$ have the property that $\langle \Lambda_i, \alpha_j^\vee \rangle = \delta_{i,j}$, where $\langle \cdot, \cdot \rangle$ is the natural pairing between $\mathfrak{h}^*$ and $\mathfrak{h}$. We have the integral weight lattice $P = \oplus_{i \in I} \mathbb{Z} \cdot \Lambda_i$ and the set of dominant weights $P_+ = \sum_{i \in I} \mathbb{Z}_{\geq 0} \cdot \Lambda_i$. For $i \in I$ we define

$$[n]_i := [n]_{q^{(\alpha_i, \alpha_i)/2}}, \qquad [n]_i^! := [1]_i [2]_i \dots [n]_i.$$

Denote $Q_+ := \bigoplus_{i \in I} \mathbb{Z}_{\geq 0} \alpha_i$. For $\alpha \in Q_+$, we write $\mathrm{ht}(\alpha)$ for the sum of its coefficients when expanded in terms of the α_i's.

Let $\mathfrak{g}' = \mathfrak{g}(\mathtt{C}')$ be the finite dimensional simple Lie algebra whose Cartan matrix $\mathtt{C}'$ corresponds to the subset of vertices $I' := I \setminus \{0\}$. The affine Lie algebra $\mathfrak{g} = \mathfrak{g}(\mathtt{C})$ is then obtained from $\mathfrak{g}'$ by a procedure described in [10, Section 7]. We denote by W (resp. W') the corresponding *affine Weyl group* (resp. *finite Weyl group*). It is a Coxeter group with standard generators $\{r_i \mid i \in I\}$ (resp. $\{r_i \mid i \in I'\}$), see [10, Proposition 3.13].

Let Φ' and Φ be the root systems of $\mathfrak{g}'$ and $\mathfrak{g}$, respectively. Denote by Φ'_+ and Φ_+ the set of *positive* roots in Φ' and Φ, respectively, cf. [10, §1.3]. Denote by δ the null-root. Let

$$\delta = a_0\alpha_0 + a_1\alpha_1 + \dots + a_l\alpha_l. \tag{2.1}$$

By [10, Table Aff 1], we always have

$$a_0 = 1. \tag{2.2}$$

We have

$$\delta - \alpha_0 = \theta, \tag{2.3}$$

where θ is the highest root in the finite root system Φ'. Finally,

$$\Phi_+ = \Phi_+^{\mathrm{im}} \sqcup \Phi_+^{\mathrm{re}},$$

where

$$\Phi_+^{\mathrm{im}} = \{n\delta \mid n \in \mathbb{Z}_{>0}\}$$

and

$$\Phi_+^{\mathrm{re}} = \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\}. \tag{2.4}$$

2.2 Words

Sequences of elements of I will be called *words*. The set of all words is denoted $\langle I \rangle$. If $\boldsymbol{i} = i_1 \dots i_d$ is a word, we denote $|\boldsymbol{i}| := \alpha_{i_1} + \dots + \alpha_{i_d} \in Q_+$. We refer to $|\boldsymbol{i}|$ as the *content* of the word $\boldsymbol{i}$. For any $\alpha \in Q_+$ we denote

$$\langle I \rangle_\alpha := \{\boldsymbol{i} \in \langle I \rangle \mid |\boldsymbol{i}| = \alpha\}.$$

If α is of height d, then the symmetric group $\mathfrak{S}_d$ with simple permutations $s_1, \dots, s_{d-1}$ acts on $\langle I \rangle_\alpha$ from the left by place permutations.

Let $\boldsymbol{i} = i_1 \dots i_d$ and $\boldsymbol{j} = i_{d+1} \dots i_{d+f}$ be two elements of $\langle I \rangle$. Define the *quantum shuffle product*:

$$\boldsymbol{i} \circ \boldsymbol{j} := \sum q^{-e(\sigma)} i_{\sigma(1)} \dots i_{\sigma(d+f)} \in \mathscr{A}\langle I \rangle,$$

where the sum is over all $\sigma \in S_{d+f}$ such that $\sigma^{-1}(1) < \dots < \sigma^{-1}(d)$ and $\sigma^{-1}(d+1) < \dots < \sigma^{-1}(d+f)$, and $e(\sigma) := \sum_{k \leq d < m,\ \sigma^{-1}(k) > \sigma^{-1}(m)} \mathtt{c}_{i_{\sigma(k)}, i_{\sigma(m)}}$. This defines an $\mathscr{A}$-algebra structure on the $\mathscr{A}$-module $\mathscr{A}\langle I \rangle$, which consists of all finite formal $\mathscr{A}$-linear combinations of elements $\boldsymbol{i} \in \langle I \rangle$.

2.3 KLR algebras

Define the polynomials in the variables u, v

$$\{Q_{ij}(u, v) \in F[u, v] \mid i, j \in I\}$$

as follows. For the case where the Cartan matrix $\mathtt{C} \neq \mathtt{A}_1^{(1)}$, choose signs ε_{ij} for all $i, j \in I$ with $\mathtt{c}_{ij} < 0$ so that $\varepsilon_{ij}\varepsilon_{ji} = -1$. Then set:

$$Q_{ij}(u, v) := \begin{cases} 0 & \text{if } i = j; \\ 1 & \text{if } \mathtt{c}_{ij} = 0; \\ \varepsilon_{ij}(u^{-\mathtt{c}_{ij}} - v^{-\mathtt{c}_{ji}}) & \text{if } \mathtt{c}_{ij} < 0. \end{cases} \tag{2.5}$$

For type $A_1^{(1)}$ we define

$$Q_{ij}(u, v) := \begin{cases} 0 & \text{if } i = j; \\ (u - v)(v - u) & \text{if } i \neq j. \end{cases} \tag{2.6}$$

Fix $\alpha \in Q_+$ of height d. The *KLR-algebra* R_α is an associative graded unital F-algebra, given by the generators

$$\{1_{\boldsymbol{i}} \mid \boldsymbol{i} \in \langle I \rangle_\alpha\} \cup \{y_1, \dots, y_d\} \cup \{\psi_1, \dots, \psi_{d-1}\} \tag{2.7}$$

and the following relations for all $\boldsymbol{i},\boldsymbol{j}\in\langle I\rangle_\alpha$ and all admissible r,t:

$$1_{\boldsymbol{i}}1_{\boldsymbol{j}}=\delta_{\boldsymbol{i},\boldsymbol{j}}1_{\boldsymbol{i}},\quad \textstyle\sum_{\boldsymbol{i}\in\langle I\rangle_\alpha}1_{\boldsymbol{i}}=1; \tag{2.8}$$

$$y_r1_{\boldsymbol{i}}=1_{\boldsymbol{i}}y_r;\quad y_ry_t=y_ty_r; \tag{2.9}$$

$$\psi_r1_{\boldsymbol{i}}=1_{s_r\boldsymbol{i}}\psi_r; \tag{2.10}$$

$$(y_t\psi_r-\psi_ry_{s_r(t)})1_{\boldsymbol{i}}=\delta_{i_r,i_{r+1}}(\delta_{t,r+1}-\delta_{t,r})1_{\boldsymbol{i}}; \tag{2.11}$$

$$\psi_r^21_{\boldsymbol{i}}=Q_{i_r,i_{r+1}}(y_r,y_{r+1})1_{\boldsymbol{i}} \tag{2.12}$$

$$\psi_r\psi_t=\psi_t\psi_r\qquad(|r-t|>1); \tag{2.13}$$

$$(\psi_{r+1}\psi_r\psi_{r+1}-\psi_r\psi_{r+1}\psi_r)1_{\boldsymbol{i}}=\delta_{i_r,i_{r+2}}\frac{Q_{i_r,i_{r+1}}(y_{r+2},y_{r+1})-Q_{i_r,i_{r+1}}(y_r,y_{r+1})}{y_{r+2}-y_r}1_{\boldsymbol{i}}. \tag{2.14}$$

The *grading* on R_α is defined by setting:

$$\deg(1_{\boldsymbol{i}})=0,\quad \deg(y_r1_{\boldsymbol{i}})=(\alpha_{i_r},\alpha_{i_r}),\quad \deg(\psi_r1_{\boldsymbol{i}})=-(\alpha_{i_r},\alpha_{i_{r+1}}).$$

It is pointed out in [14] and [24, §3.2.4] that up to isomorphism the graded F-algebra R_α depends only on the Cartan matrix and α.

Fix in addition a dominant weight $\Lambda\in P_+$. The corresponding *cyclotomic KLR algebra* R_α^Λ is the quotient of R_α by the following ideal:

$$J_\alpha^\Lambda:=(y_1^{\langle\Lambda,\alpha_{i_1}^\vee\rangle}1_{\boldsymbol{i}}\mid \boldsymbol{i}=(i_1,\dots,i_d)\in\langle I\rangle_\alpha). \tag{2.15}$$

For each element $w\in S_d$ fix a reduced expression $w=s_{r_1}\dots s_{r_m}$ and set

$$\psi_w:=\psi_{r_1}\dots\psi_{r_m}.$$

In general, ψ_w depends on the choice of the reduced expression of w.

Theorem 2.1 [13, Theorem 2.5], [24, Theorem 3.7] *The elements*

$$\{\psi_wy_1^{m_1}\dots y_d^{m_d}1_{\boldsymbol{i}}\mid w\in S_d,\ m_1,\dots,m_d\in\mathbb{Z}_{\geq 0},\ \boldsymbol{i}\in\langle I\rangle_\alpha\}$$

form an F-basis of R_α.

There exists a homogeneous algebra anti-involution

$$\tau:R_\alpha\longrightarrow R_\alpha,\quad 1_{\boldsymbol{i}}\mapsto 1_{\boldsymbol{i}},\quad y_r\mapsto y_r,\quad \psi_s\mapsto\psi_s \tag{2.16}$$

for all $\boldsymbol{i}\in\langle I\rangle_\alpha$, $1\leq r\leq d$, and $1\leq s<d$. If $M=\bigoplus_{d\in\mathbb{Z}}M_d$ is a finite dimensional graded R_α-module, then the *graded dual* $M^{\circledast}$ is the graded R_α-module such that $(M^{\circledast})_n:=\mathrm{Hom}_F(M_{-n},F)$, for all $n\in\mathbb{Z}$, and the R_α-action is given by $(xf)(m)=f(\tau(x)m)$, for all $f\in M^{\circledast}$, $m\in M$, $x\in R_\alpha$.

2.4 Basic representation theory of R_α

For any ($\mathbb{Z}$-)graded F-algebra H, we denote by H-mod the abelian subcategory of all *finite dimensional* graded H-modules, with morphisms being *degree-preserving* module homomorphisms, and $[H\text{-mod}]$ denotes the corresponding Grothendieck group. Then $[H\text{-mod}]$ is an $\mathscr{A}$-module via $q^m[M]:=[M\langle m\rangle]$, where $M\langle m\rangle$ denotes the module obtained by shifting the grading up by m, i.e. $M\langle m\rangle_n := M_{n-m}$. We denote by $\hom_H(M, N)$ the space of morphism in H-mod. For $n \in \mathbb{Z}$, let $\mathrm{Hom}_H(M, N)_n := \hom_H(M\langle n\rangle, N)$ denote the space of all homomorphisms that are homogeneous of degree n. Set

$$\mathrm{Hom}_H(M, N) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}_H(M, N)_n.$$

For graded H-modules M and N we write $M \simeq N$ to mean that M and N are isomorphic as graded modules and $M \cong N$ to mean that they are isomorphic as H-modules after we forget the gradings. For a finite dimensional graded vector space $V = \oplus_{n\in\mathbb{Z}} V_n$, its *graded dimension* is $\dim_q\ V := \sum_{n\in\mathbb{Z}}(\dim V_n)q^n \in \mathscr{A}$. Given $M, L \in H$-mod with L irreducible, we write $[M : L]_q$ for the corresponding *graded composition multiplicity*, i.e. $[M : L]_q := \sum_{n\in\mathbb{Z}} a_n q^n$, where a_n is the multiplicity of $L\langle n\rangle$ in a graded composition series of M.

Going back to the algebras R_α, every irreducible graded R_α-module is finite dimensional [13, Proposition 2.12], and there are finitely many irreducible modules in R_α-mod up to isomorphism and grading shift [13, §2.5]. A prime field is a splitting field for R_α [13, Corollary 3.19], so working with irreducible R_α-modules we do not need to assume that F is algebraically closed. Finally, for every irreducible module L, there is a unique choice of the grading shift so that we have $L^{\circledast} \simeq L$ [13, Section 3.2]. When speaking of irreducible R_α-modules we often assume by fiat that the shift has been chosen in this way.

For $\boldsymbol{i} \in \langle I\rangle_\alpha$ and $M \in R_\alpha$-mod, the $\boldsymbol{i}$-*word space* of M is $M_{\boldsymbol{i}} := 1_{\boldsymbol{i}} M$. We have

$$M = \bigoplus_{\boldsymbol{i}\in\langle I\rangle_\alpha} M_{\boldsymbol{i}}.$$

We say that $\boldsymbol{i}$ is a *word of* M if $M_{\boldsymbol{i}} \neq 0$. Note from the relations that $\psi_r M_{\boldsymbol{i}} \subset M_{s_r\boldsymbol{i}}$. Define the *(graded formal) character* of M as follows:

$$\mathrm{ch}_q\ M := \sum_{\boldsymbol{i}\in\langle I\rangle_\alpha} (\dim_q\ M_{\boldsymbol{i}})\boldsymbol{i} \in \mathscr{A}\langle I\rangle_\alpha.$$

The character map $\mathrm{ch}_q\ : R_\alpha\text{-mod} \to \mathscr{A}\langle I\rangle_\alpha$ factors through to give an *injective* $\mathscr{A}$-linear map $\mathrm{ch}_q\ : [R_\alpha\text{-mod}] \to \mathscr{A}\langle I\rangle_\alpha$, see [13, Theorem 3.17].

2.5 Induction, coinduction, and duality

Given $\alpha, \beta \in Q_+$, we set $R_{\alpha,\beta} := R_\alpha \otimes R_\beta$. Let $M \boxtimes N$ be the outer tensor product of the R_α-module M and the R_β-module N. There is an injective homogeneous non-unital algebra homomorphism $R_{\alpha,\beta} \hookrightarrow R_{\alpha+\beta},\ 1_{\boldsymbol{i}} \otimes 1_{\boldsymbol{j}} \mapsto 1_{\boldsymbol{ij}}$, where $\boldsymbol{ij}$ is the concatenation of $\boldsymbol{i}$ and $\boldsymbol{j}$. The image of the identity element of $R_{\alpha,\beta}$ under this map is

$$1_{\alpha,\beta} := \sum_{\boldsymbol{i}\in\langle I\rangle_\alpha,\boldsymbol{j}\in\langle I\rangle_\beta} 1_{\boldsymbol{ij}}.$$

Let $\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta}$ and $\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta}$ be the induction and restriction functors:

$$\mathrm{Ind}_{\alpha,\beta}^{\alpha+\beta} := R_{\alpha+\beta}1_{\alpha,\beta}\otimes_{R_{\alpha,\beta}} ? : R_{\alpha,\beta}\text{-mod} \to R_{\alpha+\beta}\text{-mod},$$
$$\mathrm{Res}_{\alpha,\beta}^{\alpha+\beta} := 1_{\alpha,\beta}R_{\alpha+\beta}\otimes_{R_{\alpha+\beta}} ? : R_{\alpha+\beta}\text{-mod} \to R_{\alpha,\beta}\text{-mod}\,.$$

We often omit upper indices and write simply $\mathrm{Ind}_{\alpha,\beta}$ and $\mathrm{Res}_{\alpha,\beta}$. These functors have obvious generalizations to $n \geq 2$ factors:

$$\begin{aligned}\mathrm{Ind}_{\gamma_1,\dots,\gamma_n} &: R_{\gamma_1,\dots,\gamma_n}\text{-mod} \to R_{\gamma_1+\dots+\gamma_n}\text{-mod},\\ \mathrm{Res}_{\gamma_1,\dots,\gamma_n} &: R_{\gamma_1+\dots+\gamma_n}\text{-mod} \to R_{\gamma_1,\dots,\gamma_n}\text{-mod}.\end{aligned}$$

The functor $\mathrm{Ind}_{\gamma_1,\dots,\gamma_n}$ is left adjoint to $\mathrm{Res}_{\gamma_1,\dots,\gamma_n}$. If $M_a \in R_{\gamma_a}$-Mod, for $a = 1, \dots, n$, we define

$$M_1 \circ \dots \circ M_n := \mathrm{Ind}_{\gamma_1,\dots,\gamma_n} M_1 \boxtimes \dots \boxtimes M_n. \tag{2.17}$$

In view of [13, Lemma 2.20], we have

$$\mathrm{ch}_q\,(M_1 \circ \dots \circ M_n) = \mathrm{ch}_q\,(M_1) \circ \dots \circ \mathrm{ch}_q\,(M_n). \tag{2.18}$$

The functors of induction and restriction have obvious parabolic analogues. Given a family $(\alpha_b^a)_{1\leq a\leq n,\ 1\leq b\leq m}$ of elements of Q_+, set $\sum_{a=1}^n \alpha_b^a =: \beta_b$ for all $1 \leq b \leq m$. Then we have functors

$$\mathrm{Ind}^{\beta_1\,;\,\dots\,;\,\beta_m}_{\alpha_1^1,\dots,\alpha_1^n\,;\,\dots\,;\,\alpha_m^1,\dots,\alpha_m^n} \qquad \text{and} \qquad \mathrm{Res}^{\beta_1\,;\,\dots\,;\,\beta_m}_{\alpha_1^1,\dots,\alpha_1^n\,;\,\dots\,;\,\alpha_m^1,\dots,\alpha_m^n}$$

The right adjoint to the functor $\mathrm{Ind}_{\gamma_1,\dots,\gamma_n}$ is given by the coinduction:

$$\mathrm{Coind}_{\gamma_1,\dots,\gamma_n} := \mathrm{Hom}_{R_{\gamma_1,\dots,\gamma_n}}(1_{\gamma_1,\dots,\gamma_n} R_{\gamma_1+\dots+\gamma_n},\ ?)$$

Induction and coinduction are related as follows:

Lemma 2.2 [19, Theorem 2.2] *Let* $\underline{\gamma} := (\gamma_1, \dots, \gamma_n) \in Q_+^n$, *and* $V_m \in R_{\gamma_m}$-mod *for* $m = 1, \dots, n$. *Denote* $d(\underline{\gamma}) = \sum_{1\leq m<k\leq n}(\gamma_m, \gamma_k)$. *Then*

$$(\mathrm{Coind}_{\gamma_n,\dots,\gamma_1} V_n \boxtimes \dots \boxtimes V_1) \simeq \mathrm{Ind}_{\gamma_1,\dots,\gamma_n} V_1 \boxtimes \dots \boxtimes V_n\langle d(\underline{\gamma})\rangle.$$

Lemma 2.3 *Let* $\underline{\gamma} := (\gamma_1, \dots, \gamma_n) \in Q_+^n$, *and* $V_m \in R_{\gamma_m}$-mod *for* $m = 1, \dots, n$. *Denote* $d(\underline{\gamma}) = \sum_{1\leq m<k\leq n}(\gamma_m, \gamma_k)$. *Then*

$$(V_1 \circ \dots \circ V_n)^{\circledast} \simeq (V_n^{\circledast} \circ \dots \circ V_1^{\circledast})\langle d(\underline{\gamma})\rangle.$$

Proof Follows from Lemma 2.2 by uniqueness of adjoint functors as in the proof of [15, Corollary 3.7.4] □

2.6 Mackey theorem

We state a slight generalization of the Mackey Theorem of Khovanov and Lauda [13, Proposition 2.18]. Given $x \in \mathfrak{S}_n$ and $\underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in Q_+^n$, we denote

$$x\underline{\gamma} := (\gamma_{x^{-1}(1)}, \dots, \gamma_{x^{-1}(n)}).$$

Correspondingly, define the integer

$$s(x, \underline{\gamma}) := -\sum_{1\leq m<k\leq n,\ x(m)>x(k)} (\gamma_m, \gamma_k).$$

Writing $R_{\underline{\gamma}}$ for $R_{\gamma_1,\dots,\gamma_n}$, there is an obvious natural algebra isomorphism

$$\varphi^x : R_{x\underline{\gamma}} \to R_{\underline{\gamma}}$$

permuting the components. Composing with this isomorphism, we get a functor

$$R_{\underline{\gamma}}\text{-mod} \to R_{x\underline{\gamma}}\text{-mod}, \quad M \mapsto {}^{\varphi^x}M.$$

Making in addition a degree shift, we get a functor

$$R_{\underline{\gamma}}\text{-mod} \to R_{x\underline{\gamma}}\text{-mod}, \quad M \mapsto {}^{x}M := {}^{\varphi^x}M\langle s(x, \underline{\gamma})\rangle. \tag{2.19}$$

Theorem 2.4 *Let* $\underline{\gamma} = (\gamma_1, \dots, \gamma_n) \in Q_+^n$ *and* $\underline{\beta} = (\beta_1, \dots, \beta_m) \in Q_+^m$ *with*

$$\gamma_1 + \dots + \gamma_n = \beta_1 + \dots + \beta_m =: \alpha.$$

Then for any $M \in R_{\underline{\gamma}}$-mod *we have that* $\mathrm{Res}_{\underline{\beta}}\,\mathrm{Ind}_{\underline{\gamma}} M$ *has filtration with factors of the form*

$$\mathrm{Ind}^{\beta_1;\dots;\beta_m}_{\alpha_1^1,\dots,\alpha_1^n;\dots;\alpha_m^1,\dots,\alpha_m^n} {}^{x(\underline{\alpha})}\left(\mathrm{Res}^{\gamma_1;\dots;\gamma_n}_{\alpha_1^1,\dots,\alpha_m^1;\dots;\alpha_1^n,\dots,\alpha_m^n} M\right)$$

with $\underline{\alpha} = (\alpha_b^a)_{1\le a\le n,\ 1\le b\le m}$ *running over all tuples of elements of* Q_+ *such that* $\sum_{b=1}^m \alpha_b^a = \gamma_a$ *for all* $1 \le a \le n$ *and* $\sum_{a=1}^n \alpha_b^a = \beta_b$ *for all* $1 \le b \le m$*, and* $x(\underline{\alpha})$ *is the permutation of* mn *which maps*

$$(\alpha_1^1, \dots, \alpha_m^1; \alpha_1^2, \dots, \alpha_m^2; \dots; \alpha_1^n, \dots, \alpha_m^n)$$

to

$$(\alpha_1^1, \dots, \alpha_1^n; \alpha_2^1, \dots, \alpha_2^n; \dots; \alpha_m^1, \dots, \alpha_m^n).$$

Proof For $m = n = 2$ this follows from [13, Proposition 2.18]. The general case can be proved by the same argument or deduced from the case $m = n = 2$ by induction. □

2.7 Crystal operators

The theory of crystal operators has been developed in [13,19] and [11] following ideas of Grojnowski [8], see also [15]. We review necessary facts for reader's convenience.

Let $\alpha \in Q_+$ and $i \in I$. It is known that $R_{n\alpha_i}$ is a nil-Hecke algebra with unique (up to a degree shift) irreducible module, which we denote by $L(i^n)$. Moreover, $\dim_q L(i^n) = [n]_i^!$. We have functors

$$e_i : R_\alpha\text{-mod} \to R_{\alpha-\alpha_i}\text{-mod}, \quad M \mapsto \mathrm{Res}^{R_{\alpha-\alpha_i,\alpha_i}}_{R_{\alpha-\alpha_i}} \circ \mathrm{Res}_{\alpha-\alpha_i,\alpha_i} M,$$
$$f_i : R_\alpha\text{-mod} \to R_{\alpha+\alpha_i}\text{-mod}, \quad M \mapsto \mathrm{Ind}_{\alpha,\alpha_i} M \boxtimes L(i).$$

If $L \in R_\alpha$-mod is irreducible, we define

$$\tilde{f}_i L := \mathrm{head}(f_i L), \quad \tilde{e}_i L := \mathrm{soc}(e_i L).$$

A fundamental fact is that $\tilde{f}_i L$ is again irreducible and $\tilde{e}_i L$ is irreducible or zero. We refer to $\tilde{e}_i$ and $\tilde{f}_i$ as the crystal operators. These are operators on $B \cup \{0\}$, where B is the set of isomorphism classes of irreducible R_α-modules for all $\alpha \in Q_+$. Define $\mathrm{wt} : B \to P,\ [L] \mapsto -\alpha$ if $L \in R_\alpha$-mod.

Theorem 2.5 [19] *The set B with the operators* $\tilde{e}_i$, $\tilde{f}_i$ *and the function* wt *is the crystal graph of the negative part* $U_q(\mathfrak{n}_-)$ *of the quantized enveloping algebra of* $\mathfrak{g}$.

For any $M \in R_\alpha$-mod, we define

$$\varepsilon_i(M) := \max\{k \geq 0 \mid e_i^k(M) \neq 0\}.$$

Then $\varepsilon_i(M)$ is also the length of the longest 'i-tail' of words of M, i.e. the maximum of $k \geq 0$ such that $j_{d-k+1} = \cdots = j_d = i$ for some word $\boldsymbol{j} = (j_1, \dots, j_d)$ of M. Define also

$$\varepsilon_i^*(M) := \max\{k \geq 0 \mid j_1 = \cdots = j_k = i \quad \text{for a word} \quad \boldsymbol{j} = (j_1, \dots, j_d) \text{of} M\}$$

to be the length of the longest 'i-head' of words of M.

Proposition 2.6 [13,19] *Let L be an irreducible R_α-module, $i \in I$, and $\varepsilon = \varepsilon_i(L)$.*

(i) $\tilde{e}_i \tilde{f}_i L \cong L$ *and if* $\tilde{e}_i L \neq 0$ *then* $\tilde{f}_i \tilde{e}_i L \cong L$;
(ii) $\varepsilon = \max\{k \geq 0 \mid \tilde{e}_i^k(L) \neq 0\}$;
(iii) $\mathrm{Res}_{\alpha - \varepsilon\alpha_i, \varepsilon\alpha_i} L \cong \tilde{e}_i^\varepsilon L \boxtimes L(i^\varepsilon)$.

Recall from (2.15) the cyclotomic ideal J_α^Λ. We have an obvious functor of inflation $\mathrm{infl}^\Lambda : R_\alpha^\Lambda\text{-mod} \to R_\alpha\text{-mod}$ and its left adjoint

$$\mathrm{pr}^\Lambda : R_\alpha\text{-mod} \to R_\alpha^\Lambda\text{-mod}, \ M \mapsto M/J_\alpha^\Lambda M.$$

Lemma 2.7 [19, Proposition 2.4] *Let L be an irreducible R_α-module. Then* $\mathrm{pr}^\Lambda L \neq 0$ *if and only if* $\varepsilon_i^*(L) \leq \langle \Lambda, \alpha_i^\vee \rangle$ *for all* $i \in I$.

2.8 Extremal words and multiplicity one results

Let $i \in I$. Consider the map $\theta_i^* : \langle I \rangle \to \langle I \rangle$ such that for $\boldsymbol{j} = (j_1, \dots, j_d) \in \langle I \rangle$, we have

$$\theta_i^*(\boldsymbol{j}) = \begin{cases} (j_1, \dots, j_{d-1}) & \text{if } j_d = i; \\ 0 & \text{otherwise.} \end{cases} \tag{2.20}$$

We extend θ_i^* by linearity to a map $\theta_i^* : \mathscr{A}\langle I \rangle \to \mathscr{A}\langle I \rangle$.

Let x be an element of $\mathscr{A}\langle I \rangle$. Define

$$\varepsilon_i(x) := \max\{k \geq 0 \mid (\theta_i^*)^k(x) \neq 0\}.$$

A word $i_1^{a_1} \dots i_b^{a_b} \in \langle I \rangle$, with $a_1, \dots, a_b \in \mathbb{Z}_{\geq 0}$, is called *extremal* for x if

$$a_b = \varepsilon_{i_b}(x), \ a_{b-1} = \varepsilon_{i_{b-1}}((\theta_{i_b}^*)^{a_b}(x)) \ , \ \dots \ , \ a_1 = \varepsilon_{i_1}\big((\theta_{i_2}^*)^{a_2} \dots (\theta_{i_b}^*)^{a_b}(x)\big).$$

A word $i_1^{a_1} \dots i_b^{a_b} \in \langle I \rangle_\alpha$ is called *extremal* for $M \in R_\alpha$-mod if it is an extremal word for $\mathrm{ch}_q M \in \mathscr{A}\langle I \rangle$, in other words, if

$$a_b = \varepsilon_{i_b}(M), \ a_{b-1} = \varepsilon_{i_{b-1}}(\tilde{e}_{i_b}^{a_b} M) \ , \ \dots \ , \ a_1 = \varepsilon_{i_1}(\tilde{e}_{i_2}^{a_2} \dots \tilde{e}_{i_b}^{a_b} M).$$

The following useful result, which is a version of [5, Corollary 2.17], describes the multiplicities of extremal word spaces in irreducible modules. We denote by 1_F the trivial module F over the trivial algebra $R_0 \simeq F$.

Lemma 2.8 *Let L be an irreducible R_α-module, and $\boldsymbol{i} = i_1^{a_1} \dots i_b^{a_b} \in \langle I \rangle_\alpha$ be an extremal word for L. Then* $\dim_q L_{\boldsymbol{i}} = [a_1]^!_{i_1} \dots [a_b]^!_{i_b}$, *and*

$$L \cong \tilde{f}_{i_b}^{a_b} \tilde{f}_{i_{b-1}}^{a_{b-1}} \dots \tilde{f}_{i_1}^{a_1} 1_F.$$

Moreover, $\boldsymbol{i}$ is not an extremal word for any irreducible module $L' \not\cong L$.

Proof Follows easily from Proposition 2.6, cf. [5, Theorem 2.16]. □

Corollary 2.9 *Let* $M \in R_\alpha$-mod, *and* $\boldsymbol{i} = i_1^{a_1} \dots i_b^{a_b} \in \langle I\rangle_\alpha$ *be an extremal word for* M. *Then we can write* $\dim_q M_{\boldsymbol{i}} = m[a_1]^!_{i_1} \dots [a_b]^!_{i_b}$ *for some* $m \in \mathscr{A}$. *Moreover, if* $L \cong \tilde f_{i_b}^{a_b} \tilde f_{i_{b-1}}^{a_{b-1}} \dots \tilde f_{i_1}^{a_1} 1_F$ *and* $L^{\circledast} \simeq L$, *then we have* $[M : L]_q = m$.

Proof Apply Lemma 2.8, cf. [5, Corollary 2.17]. □

Now we establish some useful 'multiplicity-one results'. The first one shows that in every irreducible module there is a word space with a one dimensional graded component:

Lemma 2.10 *Let* L *be an irreducible* R_α*-module, and* $\boldsymbol{i} = i_1^{a_1} \dots i_b^{a_b} \in \langle I\rangle_\alpha$ *be an extremal word for* L. *Set* $N := \sum_{m=1}^b a_m(a_m - 1)(\alpha_{i_m}, \alpha_{i_m})/4$. *Then* $\dim 1_{\boldsymbol{i}} L_N = \dim 1_{\boldsymbol{i}} L_{-N} = 1$.

Proof This follows immediately from the equality $\dim_q 1_{\boldsymbol{i}} L = [a_1]^!_{i_1} \dots [a_b]^!_{i_b}$, which comes from Lemma 2.8. □

The following result shows that any induction product of irreducible modules always has a multiplicity one composition factor.

Proposition 2.11 *Suppose that* $n \in \mathbb{Z}_{>0}$ *and for* $r = 1, \dots, n$, *we have* $\alpha^{(r)} \in Q_+$, *an irreducible* $R_{\alpha^{(r)}}$*-module* $L^{(r)}$, *and* $\boldsymbol{i}^{(r)} := i_1^{a_1^{(r)}} \dots i_k^{a_k^{(r)}} \in \langle I\rangle_{\alpha^{(r)}}$ *is an extremal word for* $L^{(r)}$. *Denote* $a_m := \sum_{r=1}^n a_m^{(r)}$ *for all* $1 \le m \le k$. *Then* $\boldsymbol{j} := i_1^{a_1} \dots i_k^{a_k}$ *is an extremal word for* $L^{(1)} \circ \dots \circ L^{(n)}$, *and the graded multiplicity of the* $\circledast$*-self-dual irreducible module*

$$N \cong \tilde f_{i_k}^{a_k} \tilde f_{i_{k-1}}^{a_{k-1}} \dots \tilde f_{i_1}^{a_1} 1_F$$

in $L^{(1)} \circ \dots \circ L^{(n)}$ *is* q^m, *where*

$$m := -\sum_{1 \le t < u \le n} \Big(\sum_{1 \le r < s \le k} a_r^{(u)} a_s^{(t)} (\alpha_{i_r}, \alpha_{i_s}) + \tfrac{1}{2} \sum_{r=1}^k a_r^{(t)} a_r^{(u)} (\alpha_{i_r}, \alpha_{i_r})\Big).$$

In particular, the ungraded multiplicity of N *in* $L^{(1)} \circ \dots \circ L^{(n)}$ *is one.*

Proof By Lemma 2.8, the multiplicity of $\boldsymbol{i}^{(r)}$ in $\mathrm{ch}_q\, L^{(r)}$ is $[a_1^{(r)}]^!_{i_1} \dots [a_k^{(r)}]^!_{i_k}$. By (2.18), we have

$$\mathrm{ch}_q\, (L^{(1)} \circ \dots \circ L^{(n)}) = \mathrm{ch}_q\, (L^{(1)}) \circ \dots \circ \mathrm{ch}_q\, (L^{(n)}).$$

It is easy to see that the word $\boldsymbol{j}$ is an extremal word for $L^{(1)} \circ \dots \circ L^{(n)}$, and that $\boldsymbol{j}$ can be obtained only from the shuffle product $\boldsymbol{i}^{(1)} \circ \dots \circ \boldsymbol{i}^{(n)}$. An elementary computation shows that $\boldsymbol{j}$ appears in $\boldsymbol{i}^{(1)} \circ \dots \circ \boldsymbol{i}^{(n)}$ with multiplicity $q^m [a_1]^!_{i_1} \dots [a_k]^!_{i_k}$. Now apply Corollary 2.9. □

Corollary 2.12 *Let* L *be an irreducible* R_α*-module and* $n \in \mathbb{Z}_{>0}$. *Then there is an irreducible* $R_{n\alpha}$*-module* N *which appears in* $L^{\circ n}$ *with graded multiplicity* $q^{-(\alpha,\alpha)n(n-1)/4}$. *In particular, the ungraded multiplicity of* N *is one.*

Proof Apply Proposition 2.11 with $L^{(1)} = \dots = L^{(n)} = L$. □

2.9 Khovanov–Lauda–Rouquier categorification

We recall the Khovanov–Lauda–Rouquier categorification of the quantized enveloping algebra $\mathbf{f}$ obtained in [13,14,24]. We follow the presentation of [6,17]. Let $\mathbf{f}_{\mathscr{A}} \subset \mathbf{f}$ be the $\mathscr{A}$-form of the Lusztig's quantum group $\mathbf{f}$ corresponding to the Cartan matrix $\mathtt{C}$. This $\mathscr{A}$-algebra is generated by the divided powers $\theta_i^{(n)} = \theta_i^n/[n]_i^!$ of the standard generators. The algebra $\mathbf{f}_{\mathscr{A}}$ has a Q_+-grading $\mathbf{f}_{\mathscr{A}} = \oplus_{\alpha \in Q_+}(\mathbf{f}_{\mathscr{A}})_\alpha$ determined by the condition that each θ_i is in degree α_i.

There is a bilinear form $(\cdot,\cdot)$ on $\mathbf{f}$ defined in [21, §1.2.5, §33.1.2]. Let $\mathbf{f}_{\mathscr{A}}^* = \{y \in \mathbf{f} \mid (x, y) \in \mathscr{A} \text{ for all } x \in \mathbf{f}_{\mathscr{A}}\}$. Let $(\theta_i^*)^{(n)}$ be the map dual to the map $\mathbf{f}_{\mathscr{A}} \to \mathbf{f}_{\mathscr{A}}$, $x \mapsto x\theta_i^{(n)}$. Finally, there is a coproduct r on $\mathbf{f}$ such that $\mathbf{f}$ is a twisted unital and counital bialgebra. Moreover, for all $x, y, z \in \mathbf{f}$ we have

$$(xy, z) = (x \otimes y, r(z)). \tag{2.21}$$

The field $\mathbb{Q}(q)$ possesses a unique automorphism called the *bar involution* such that $\overline{q} = q^{-1}$. With respect to this involution, let $\mathtt{b} : \mathbf{f} \to \mathbf{f}$ be the anti-linear algebra automorphism such that $\mathtt{b}(\theta_i) = \theta_i$ for all $i \in I$. Also let $\mathtt{b}^* : \mathbf{f} \to \mathbf{f}$ be the adjoint anti-linear map to $\mathtt{b}$ with respect to Lusztig's form, so $(x, \mathtt{b}^*(y)) = \overline{(\mathtt{b}(x), y)}$ for all $x, y \in \mathbf{f}$. The maps $\mathtt{b}$ and $\mathtt{b}^*$ preserve $\mathbf{f}_{\mathscr{A}}$ and $\mathbf{f}_{\mathscr{A}}^*$, respectively.

Let $[R\text{-mod}] = \bigoplus_{\alpha \in Q_+}[R_\alpha\text{-mod}]$ denote the Grothendieck ring, which is an $\mathscr{A}$-algebra via induction product and $q^n[V] = [V\langle n\rangle]$. Similarly the functors of restriction define a coproduct r on $[R\text{-mod}]$. This product and coproduct make $[R\text{-mod}]$ into a twisted unital and counital bialgebra [13, Proposition 3.2].

In [13,14] an explicit $\mathscr{A}$-bialgebra isomorphisms $\gamma^* : [R\text{-mod}] \xrightarrow{\sim} \mathbf{f}_{\mathscr{A}}^*$ is constructed; in fact [13] establishes a dual isomorphism, see [17, Theorem 4.4] for details on this. Moreover, $\gamma^*([V^{\circledast}]) = \mathtt{b}^*(\gamma^*([V]))$, and we have a commutative triangle

$$\begin{array}{ccc} & \mathscr{A}\langle I\rangle & \\ {\scriptstyle \mathrm{ch}_q}\nearrow & & \nwarrow{\scriptstyle \iota} \\ [R\text{-mod}] & \xrightarrow{\quad\gamma^*\quad} & \mathbf{f}_{\mathscr{A}}^* \end{array}, \tag{2.22}$$

where the map ι is defined as follows:

$$\iota(x) = \sum_{\boldsymbol{i}=(i_1,\dots,i_d)\in\langle I\rangle} (x, \theta_{i_1}\dots\theta_{i_d})\boldsymbol{i} \qquad (x \in \mathbf{f}_{\mathscr{A}}^*).$$

Lemma 2.13 *Let v^* be a dual canonical basis element of $\mathbf{f}$, and $\boldsymbol{i} = i_1^{a_1}\dots i_k^{a_k}$ be an extremal word of $\iota(v^*)$ in the sence of Sect. 2.8. Then $\boldsymbol{i}$ appears in $\iota(v^*)$ with coefficient $[a_1]_{i_1}^!\dots[a_k]_{i_k}^!$.*

Proof Apply induction on $a_1 + \dots + a_k$. The induction base is $a_1 + \dots + a_k = 0$, in which case $v^* = 1 \in \mathbf{f}_{\mathscr{A}}^*$ and $\iota(1)$ is the empty word. Recall the map $\theta_i^* : \mathscr{A}\langle I\rangle \to \mathscr{A}\langle I\rangle$ from (2.20). For all $x \in \mathbf{f}_{\mathscr{A}}^*$ we have $\iota((\theta_i^*)^{(n)}(x)) = (\theta_i^*)^{(n)}(\iota(x))$, where in the right hand side $(\theta_i^*)^{(n)} = (\theta_i^*)^n/[n]_i^!$. By [12, Proposition 5.3.1], $(\theta_{i_k}^*)^{(a_{i_k})}(v^*)$ is again a dual canonical basis element, and by induction, the word $i_1^{a_1}\dots i_{k-1}^{a_{k-1}}$ appears in $\iota((\theta_{i_k}^*)^{(a_{i_k})}(v^*))$ with coefficient $[a_1]_{i_1}^!\dots[a_{k-1}]_{i_{k-1}}^!$. The result follows. □

3 Cuspidal systems and standard modules

3.1 Convex preorders on Φ_+

Recall the notion of a convex preorder on Φ_+ from (1.1)–(1.3). Convex preorders exist, see e.g. [1, Example 2.11(ii)].

Lemma 3.1 *For any positive root β, the convex cones spanned by $\Phi_+(\beta) := \{\gamma \in \Phi_+ \mid \gamma \succeq \beta\}$ and $\Phi_+ \backslash \Phi_+(\beta)$ intersect only at the origin.*

Proof The set $\{\gamma \in \Phi_+ \mid \gamma \succeq \beta\}$ is a terminal section for the preorder $\preceq$ in the sense of [1, Section 2.4]. By [1, Lemma 2.9], this set is biconvex, which is equivalent to the statement about the cones by [1, Remark 2.3]. □

Lemma 3.1 immediately implies the following properties:

(Con1) Let $\rho \in \Phi_+^{\mathrm{re}}$, $m \in \mathbb{Z}_{>0}$, and $m\rho = \sum_{a=1}^{b} \gamma_a$ for some positive roots γ_a. Assume that either $\gamma_a \preceq \rho$ for all $a = 1, \dots, b$ or $\gamma_a \succeq \rho$ for all $a = 1, \dots, b$. Then $b = m$ and $\gamma_a = \rho$ for all $a = 1, \dots, b$.
(Con2) Let β, κ be two positive roots, not both imaginary. If $\beta + \kappa = \sum_{a=1}^{b} \gamma_a$ for some positive roots $\gamma_a \preceq \beta$, then $\beta \succeq \kappa$.
(Con3) Let $\rho \in \Phi_+^{\mathrm{im}}$, and $\rho = \sum_{a=1}^{b} \gamma_a$ for some positive roots γ_a. If either $\gamma_a \preceq \rho$ for all $a = 1, \dots, b$ or $\gamma_a \succeq \rho$ for all $a = 1, \dots, b$, then all γ_a are imaginary.

Indeed, for (Con1), we may assume that all $\gamma_a \prec \rho$, and apply the lemma with $\beta = \rho$. For (Con2), taking into account (Con1), we may assume that all $\gamma_a \prec \beta$, and apply the lemma. For (Con3), we may assume that all γ_a are real and apply the lemma with $\beta = \rho$.

The Main Theorem from the introduction will be proved for an arbitrary convex preorder, but later results which rely on the theory of imaginary representations, beginning from Sect. 5, require an additional assumption. Recall from (2.4) that

$$\Phi_+^{\mathrm{re}} = \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\} \sqcup \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\}.$$

A convex preorder $\preceq$ will be called *balanced* if

$$\Phi_{\succ}^{\mathrm{re}} = \{\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{\geq 0}\}. \tag{3.1}$$

Then of course we also have $\Phi_{\prec}^{\mathrm{re}} = \{-\beta + n\delta \mid \beta \in \Phi'_+,\ n \in \mathbb{Z}_{>0}\}$. A convex preorder $\preceq$ is balanced if and only if $\alpha_i \succ \delta \succ \alpha_0$ for all $i \in I'$. Balanced convex preorders exist, see for example [3].

3.2 Root partitions

Recall that $I' = \{1, \dots, l\}$. We will consider the set $\mathscr{P}$ of l-multipartitions $\underline{\lambda} = (\lambda^{(i)})_{i \in I'}$, where each $\lambda^{(i)} = (\lambda_1^{(i)}, \lambda_2^{(i)}, \dots)$ is a usual partition. For all $i \in I'$, we denote $|\lambda^{(i)}| := \lambda_1^{(i)} + \lambda_2^{(i)} + \dots$, and set $|\underline{\lambda}| := \sum_{i \in I'} |\lambda^{(i)}|$. For $m \in \mathbb{Z}_{\geq 0}$, denote

$$\mathscr{P}_m := \{\underline{\lambda} \in \mathscr{P} \mid |\underline{\lambda}| = m\}.$$

We work with a fixed convex preorder $\preceq$ on Φ_+. Recall the totally ordered set Ψ from (1.4). Denote by $\mathtt{T}$ the set of all finitary (i.e. with almost all terms zero) tuples $M = (m_\rho)_{\rho \in \Psi} \in \mathbb{Z}_{\geq 0}^{\Psi}$ of non-negative integers. The left lexicographic order on $\mathtt{T}$ is denoted $\leq_l$ and the right lexicographic order on $\mathtt{T}$ is denoted $\leq_r$. We will use the following *bilexicographic* partial order on $\mathtt{T}$:

$$M \leq N \qquad \text{if and only if} \qquad M \leq_l N \text{ and } M \geq_r N.$$

Recall from the introduction that a *root partition* is a pair $(M, \underline{\mu})$ with $M \in \mathtt{T}$, $\underline{\mu} \in \mathscr{P}_{m_\delta}$, and that, for $\alpha \in Q_+$, a root partition $\pi \in \Pi(\alpha)$ can be written in the form

$$\pi = (\rho_1^{m_1}, \ldots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \ldots, \rho_{-1}^{m_{-1}}), \tag{3.2}$$

where $\rho_1 > \cdots > \rho_s > \delta > \rho_{-t} > \cdots > \rho_{-1}$, all $m_u \in \mathbb{Z}_{\geq 0}$, $\underline{\mu} \in \mathscr{P}$ and $\sum_{u=1}^{s} m_u\rho_u + |\underline{\mu}|\delta + \sum_{u=1}^{t} m_{-u}\rho_{-u} = \alpha$. For a root partition $\pi = (M, \underline{\mu})$ and $\rho \in \Psi$, we define $M_\rho := m_\rho \rho$, and consider a tuple $|M| = (M_\rho)_{\rho\in\Psi} \in Q_+^{\Psi}$. If π is written in the form (3.2), we also write (ignoring trivial terms)

$$|M| = (m_1\rho_1, \ldots, m_s\rho_s, m_\delta\delta, m_{-t}\rho_{-t}, \ldots, m_{-1}\rho_{-1}).$$

Then we have a parabolic subalgebra

$$R_{|M|} = R_{m_1\rho_1,\ldots,m_s\rho_s,m_\delta\delta,m_{-t}\rho_{-t},\ldots,m_{-1}\rho_{-1}} \subseteq R_\alpha.$$

We will use the following partial order on $\Pi(\alpha)$:

$$(M, \underline{\mu}) \leq (N, \underline{\nu}) \quad \text{if and only if} \quad M \leq N \quad \text{and if} \quad M = N \text{ then } \underline{\mu} = \underline{\nu}. \tag{3.3}$$

The positive subalgebra $\mathfrak{n}_+ \subset \mathfrak{g}$ has a basis consisting of *root vectors*

$$\{E_\rho,\ E_{n\delta,i} \mid \rho \in \Phi_+^{\mathrm{re}},\ n \in \mathbb{Z}_{>0},\ i \in I'\}.$$

For $i \in I'$, assign to a partition $\mu^{(i)} = (\mu_1^{(i)}, \mu_2^{(i)}, \ldots)$ a PBW monomial $E_{\mu^{(i)}} := E_{\mu_1^{(i)}\delta,i} E_{\mu_2^{(i)}\delta,i} \ldots$. Now, to a root partition π as in (3.2), we assign a PBW monomial

$$E_\pi := E_{\rho_1}^{m_1} \ldots E_{\rho_s}^{m_s}\ E_{\mu^{(1)}} E_{\mu^{(2)}}\ \ldots\ E_{\mu^{(l)}}\ E_{\rho_{-t}}^{m_{-t}} \ldots E_{\rho_{-1}}^{m_{-1}}.$$

Then $\{E_\pi \mid \pi \in \Pi(\alpha)\}$ is a basis of the weight space $U(\mathfrak{n}_+)_\alpha$. In particular, $|\Pi(\alpha)| = \dim U(\mathfrak{n}_+)_\alpha$ is the *Kostant partition function* of α. In view of the isomorphism γ^* from (2.22), we conclude:

Lemma 3.2 *The number of irreducible R_α-modules (up to isomorphism) is $|\Pi(\alpha)|$.*

Given a root partition $\pi = (M, \underline{\mu})$ and $\rho \in \Psi$, denote by $\pi'_\rho = (M, \underline{\mu})'_\rho$ the root partition obtained from π by 'annihilating' its ρth component; to be more precise, $(M, \underline{\mu})'_\rho = (M', \underline{\mu}')$, where

$$m'_\beta = \begin{cases} 0 & \text{if } \beta = \rho \\ m_\beta & \text{if } \beta \neq \rho \end{cases} \qquad \text{and} \qquad \underline{\mu}' = \begin{cases} \emptyset & \text{if } \rho = \delta \\ \underline{\mu} & \text{otherwise.} \end{cases} \tag{3.4}$$

3.3 Standard modules

We continue to work with a fixed convex preorder $\preceq$ on Φ_+. Recall from the introduction the definition of the corresponding cuspidal system. It consists of certain cuspidal modules L_ρ for $\rho \in \Phi_+^{\mathrm{re}}$ and irreducible imaginary modules $L(\underline{\mu})$ for $\underline{\mu} \in \mathscr{P}$ satisfying the properties (Cus1) and (Cus2). For every $\alpha \in Q_+$ and a root partition $\pi = (M, \underline{\mu}) \in \Pi(\alpha)$, written in the form (3.2), we define an integer

$$\mathtt{sh}(\pi) = \mathtt{sh}(M, \underline{\mu}) := \sum_{\rho\in\Phi_+^{\mathrm{re}}} (\rho, \rho)m_\rho(m_\rho - 1)/4. \tag{3.5}$$

Next, we define the $R_{|M|}$-module

$$L_\pi = L_{M,\underline{\mu}} := L_{\rho_1}^{\circ m_1} \boxtimes \cdots \boxtimes L_{\rho_s}^{\circ m_s} \boxtimes L(\underline{\mu}) \boxtimes L_{\rho_{-t}}^{\circ m_{-t}} \boxtimes \cdots \boxtimes L_{\rho_{-1}}^{\circ m_{-1}} \langle \mathtt{sh}(\pi) \rangle, \quad (3.6)$$

and we define the *standard module*

$$\Delta(\pi) = \Delta(M, \underline{\mu}) := L_{\rho_1}^{\circ m_1} \circ \cdots \circ L_{\rho_s}^{\circ m_s} \circ L(\underline{\mu}) \circ L_{\rho_{-t}}^{\circ m_{-t}} \circ \cdots \circ L_{\rho_{-1}}^{\circ m_{-1}} \langle \mathtt{sh}(\pi) \rangle. \quad (3.7)$$

Note that $\Delta(M, \underline{\mu}) = \mathrm{Ind}_{|M|} L_{M,\underline{\mu}} \in R_\alpha$-mod.

Lemma 3.3 *Let $\rho \in \Phi_+^{\mathrm{re}}$, L_ρ be the corresponding cuspidal module, and $n \in \mathbb{Z}_{>0}$. Then*

$$(L_\rho^{\circ n})^{\circledast} \simeq L_\rho^{\circ n} \langle (\rho, \rho) n(n-1)/2 \rangle.$$

In particular, the module $L_\rho^{\circ n} \langle (\rho, \rho) n(n-1)/4 \rangle$ is $\circledast$-self-dual.

Proof Recall that our standard choice of shifts of irreducible modules is so that $L_\rho^{\circledast} \simeq L_\rho$. Now the result follows from Lemma 2.3. □

Lemma 3.4 *We have $L_\pi^{\circledast} \simeq L_\pi$*

Proof Follows from Lemma 3.3. □

3.4 Restrictions of standard modules

The proof of the following proposition is similar to [23, Lemma 3.3].

Proposition 3.5 *Let $(M, \underline{\mu}), (N, \underline{\nu}) \in \Pi(\alpha)$. Then:*

(i) $\mathrm{Res}_{|N|} \Delta(N, \underline{\nu}) \simeq L_{N,\underline{\nu}}$.
(ii) $\mathrm{Res}_{|M|} \Delta(N, \underline{\nu}) \neq 0$ *implies* $M \leq N$.

Proof We may write the root partitions $(M, \underline{\mu})$ and $(N, \underline{\nu})$ in the form (3.2):

$$(M, \underline{\mu}) = (\rho_1^{m_1}, \ldots, \rho_s^{m_s}, \underline{\mu}, \rho_{-t}^{m_{-t}}, \ldots, \rho_{-1}^{m_{-1}}),$$
$$(N, \underline{\nu}) = (\rho_1^{n_1}, \ldots, \rho_s^{n_s}, \underline{\nu}, \rho_{-t}^{n_{-t}}, \ldots, \rho_{-1}^{n_{-1}})$$

with $m_u, n_u \geq 0$.

Let $\mathrm{Res}_{|M|} \Delta(N, \underline{\nu}) \neq 0$. It suffices to prove that $M \geq_l N$ or $M \leq_r N$ implies that $M = N$ and $\mathrm{Res}_{|M|} \Delta(N, \underline{\nu}) \cong L_{N,\underline{\nu}}$. We may assume that $M \geq_l N$, the case $M \leq_r N$ being similar. We apply induction on $\mathrm{ht}(\alpha)$ and consider three cases.

Case 1: $m_\rho > 0$ for some $\rho > \delta$. Pick the maximal such ρ, and let $(M', \underline{\mu}') = (M, \underline{\mu})'_\rho$ and $(N', \underline{\nu}') = (N, \underline{\nu})'_\rho$, see (3.4). By the Mackey Theorem 2.4, $\mathrm{Res}_{|M|} \Delta(N, \underline{\nu})$ has filtration with factors of the form

$$\mathrm{Ind}_{\kappa_1,\ldots,\kappa_c;\underline{\gamma}}^{m_\rho \rho;|M'|} V,$$

where $m_\rho \rho = \kappa_1 + \cdots + \kappa_c$, with $\kappa_1, \ldots, \kappa_c \in Q_+ \backslash \{0\}$, and $\underline{\gamma}$ is a refinement of $|M'|$. Moreover, the module V is obtained by twisting and degree shifting as in (2.19) of a module obtained by restriction of

$$L_{\rho_1}^{\boxtimes n_1} \boxtimes \cdots \boxtimes L_{\rho_s}^{\boxtimes n_s} \boxtimes L(\underline{\nu}) \boxtimes L_{\rho_{-t}}^{\boxtimes n_{-t}} \boxtimes \cdots \boxtimes L_{\rho_{-1}}^{\boxtimes n_{-1}}$$

to a parabolic which has $\kappa_1, \ldots, \kappa_c$ in the beginnings of the corresponding blocks. In particular, if $V \neq 0$, then for each $b = 1, \ldots, c$ we have that $\mathrm{Res}_{\kappa_b, \rho_k - \kappa_b} L_{\rho_k} \neq 0$ for some $k = k(b)$ with $n_k \neq 0$ or $\mathrm{Res}_{\kappa_b, n_\delta \delta - \kappa_b} L(\underline{\nu}) \neq 0$.

If $\mathrm{Res}_{\kappa_b,\rho_k-\kappa_b} L_{\rho_k} \neq 0$, then by (Cus1), κ_b is a sum of roots $\preceq \rho_k$. Moreover, since $M \geq_l N$ and $n_k \neq 0$, we have that $\rho_k \preceq \rho$. Thus κ_b is a sum of roots $\preceq \rho_a$. On the other hand, if $\mathrm{Res}_{\kappa_b,n_\delta\delta-\kappa_b} L(\underline{\nu}) \neq 0$, then by (Cus2), either κ_b is an imaginary root or it is a sum of real roots less than $n_\delta\delta$. In either case we conclude again that κ_b is a sum of roots $\preceq \rho$. Using (Con1), we can now conclude that $c = m_\rho$, and $\kappa_b = \rho = \rho_{k(b)}$ for all $b = 1, \dots, c$. Hence $n_\rho \geq m_\rho$. Since $M \geq_l N$, we conclude that $n_\rho = m_\rho$, and

$$\mathrm{Res}_{|M|} \Delta(N, \underline{\nu}) \cong L_\rho^{\circ m_\rho} \boxtimes \mathrm{Res}^{\alpha - m_\rho\rho}_{|M'|} \Delta(N', \underline{\nu}').$$

Now, since $\mathrm{ht}(\alpha - m_\rho\rho) < \mathrm{ht}(\alpha)$, we can apply the inductive hypothesis.

Case 2: $m_\rho = 0$ for all $\rho > \delta$, but $m_\delta \neq 0$. Since $N \leq_l M$, we also have that $n_\rho = 0$ for all $\rho > \delta$. Let $(M', \underline{\mu}') = (M, \underline{\mu})'_\delta$, $(N', \underline{\nu}') = (N, \underline{\nu})'_\delta$. By the Mackey Theorem 2.4, $\mathrm{Res}_{|M|} \Delta(N, \underline{\nu})$ has filtration with factors of the form

$$\mathrm{Ind}^{m_\delta\delta;|M'|}_{\kappa_1,\dots,\kappa_c;\underline{\gamma}} V,$$

where $m_\delta\delta = \kappa_1 + \cdots + \kappa_c$, with $\kappa_1, \dots, \kappa_c \in Q_+ \backslash \{0\}$, and $\underline{\gamma}$ is a refinement of $|M'|$. Moreover, the module V is obtained by twisting and degree shifting of a module obtained by parabolic restriction of the module $L(\underline{\nu}) \boxtimes L_{\rho_{-t}}^{\boxtimes n_{-t}} \boxtimes \cdots \boxtimes L_{\rho_{-1}}^{\boxtimes n_{-1}}$ to a parabolic which has $\kappa_1, \dots, \kappa_c$ in the beginnings of the corresponding blocks. In particular, if $V \neq 0$, then either

(1) $\mathrm{Res}_{\kappa_1,n_\delta\delta-\kappa_1} L(\underline{\nu}) \neq 0$ and for $b = 2, \dots, c$, there is $k = k(b) < 0$ such that $\mathrm{Res}_{\kappa_b,\rho_k-\kappa_b} L_{\rho_k} \neq 0$, or
(2) for $b = 1, \dots, c$ there is $k = k(b) < 0$ such that $\mathrm{Res}_{\kappa_b,\rho_k-\kappa_b} L_{\rho_k} \neq 0$.

By (Cus1) and (Con3), only (1) is possible, and in that case, using also (Cus2), we must have $c = 1$ and $\kappa_1 = m_\delta\delta$. Since $M \geq_l N$, we conclude that $n_\delta = m_\delta$, and

$$\mathrm{Res}_{|M|} \Delta(N, \underline{\nu}) \cong L(\underline{\nu}) \boxtimes \mathrm{Res}^{\alpha - m_\delta\delta}_{|M'|} \Delta(N', \underline{\nu}).$$

Now, since $\mathrm{ht}(\alpha - m_\delta\delta) < \mathrm{ht}(\alpha)$, we can apply the inductive hypothesis.

Case 3: $m_\rho = 0$ for all $\rho \geq \delta$. This case is similar to Case 1. □

4 Rough classification of irreducible modules

We continue to work with a fixed convex preorder $\preceq$ on Φ_+. In this section we prove the Main Theorem from the introduction.

4.1 Statement and the structure of the proof

We will prove the following result, which contains slightly more information than the Main Theorem:

Theorem 4.1 *For a given convex preorder, there exists a corresponding cuspidal system* $\{L_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}\}$. *Moreover:*

(i) *For every root partition* $(M, \underline{\mu})$, *the standard module* $\Delta(M, \underline{\mu})$ *has an irreducible head; denote this irreducible module* $L(M, \underline{\mu})$.
(ii) $\{L(M, \underline{\mu}) \mid (M, \underline{\mu}) \in \Pi(\alpha)\}$ *is a complete and irredundant system of irreducible* R_α-*modules up to isomorphism.*
(iii) $L(M, \underline{\mu})^{\circledast} \simeq L(M, \underline{\mu})$.
(iv) $[\Delta(M, \underline{\mu}) : L(M, \underline{\mu})]_q = 1$, *and* $[\Delta(M, \underline{\mu}) : L(N, \underline{\nu})]_q \neq 0$ *implies* $(N, \underline{\nu}) \leq (M, \underline{\mu})$.

(v) $\mathrm{Res}_{|M|}L(M,\underline{\mu}) \simeq L_{M,\underline{\mu}}$ *and* $\mathrm{Res}_{|N|}L(M,\underline{\mu}) \neq 0$ *implies* $N \leq M$.
(vi) $L_\rho^{\circ n}$ *is irreducible for all* $\rho \in \Phi_+^{\mathrm{re}}$ *and all* $n \in \mathbb{Z}_{>0}$.

The rest of Sect. 4 is devoted to the proof of Theorem 4.1, which goes by induction on $\mathrm{ht}(\alpha)$. To be more precise, we prove the following statements for all $\alpha \in Q_+$ by induction on $\mathrm{ht}(\alpha)$:

(1) For each $\rho \in \Phi_+^{\mathrm{re}}$ with $\mathrm{ht}(\rho) \leq \mathrm{ht}(\alpha)$ there exists a unique up to isomorphism irreducible R_ρ-module L_ρ which satisfies the property (Cus1). Moreover, L_ρ then also satisfies the property (vi) of Theorem 4.1 if $\mathrm{ht}(n\rho) \leq \mathrm{ht}(\alpha)$.
(2) For each $n \in \mathbb{Z}_{\geq 0}$ with $\mathrm{ht}(n\delta) \leq \mathrm{ht}(\alpha)$ there exist irreducible $R_{n\delta}$-modules $\{L(\underline{\mu}) \mid \underline{\mu} \in \mathscr{P}_n\}$ which satisfy the property (Cus2).
(3) The standard modules $\Delta(M,\underline{\mu})$ for all $(M,\underline{\mu}) \in \Pi(\alpha)$, defined as in (3.7) using the modules from (1) and (2), satisfy the properties (i)–(v) of Theorem 4.1.

The induction starts with $\mathrm{ht}(\alpha) = 0$, and for $\mathrm{ht}(\alpha) = 1$ the theorem is also clear since R_{α_i} is a polynomial algebra, which has only the trivial representation L_{α_i}. The inductive assumption will stay valid throughout Sect. 4.

4.2 Irreducible heads

In the following proposition, we exclude the cases where the standard module is either of the form $L_\rho^{\circ n}$ for a real root ρ, or is imaginary of the form $L(\underline{\lambda})$. The excluded cases will be dealt with in this Sects. 4.3, 4.4 and 4.5.

Proposition 4.2 *Let* $(M,\underline{\mu}) \in \Pi(\alpha)$*, and suppose that there are elements* $\rho \neq \beta$ *of* Ψ *such that* $m_\rho \neq 0$ *and* $m_\beta \neq 0$.

(i) $\Delta(M,\underline{\mu})$ *has an irreducible head; denote this irreducible module* $L(M,\underline{\mu})$.
(ii) *If* $(M,\underline{\mu}) \neq (N,\underline{\nu})$*, then* $L(M,\underline{\mu}) \not\cong L(N,\underline{\nu})$.
(iii) $L(M,\underline{\mu})^{\circledast} \simeq L(M,\underline{\mu})$.
(iv) $[\Delta(M,\underline{\mu}) : L(M,\underline{\mu})]_q = 1$*, and* $[\Delta(M,\underline{\mu}) : L(N,\underline{\nu})]_q \neq 0$ *implies* $(N,\underline{\nu}) \leq (M,\underline{\mu})$.
(v) $\mathrm{Res}_{|M|}L(M,\underline{\mu}) \simeq L_{M,\underline{\mu}}$ *and* $\mathrm{Res}_{|N|}L(M,\underline{\mu}) \neq 0$ *implies* $N \leq M$.

Proof (i) and (v) If L is an irreducible quotient of $\Delta(M,\underline{\mu}) = \mathrm{Ind}_{|M|}L_{M,\underline{\mu}}$, then by adjointness of $\mathrm{Ind}_{|M|}$ and $\mathrm{Res}_{|M|}$ and the irreducibility of the $R_{|M|}$-module $L_{M,\underline{\mu}}$, which holds by the inductive assumption, we conclude that $L_{M,\underline{\mu}}$ is a submodule of $\mathrm{Res}_{|M|}L$. On the other hand, by Proposition 3.5(i) the multiplicity of $L_{M,\underline{\mu}}$ in $\mathrm{Res}_{|M|}\Delta(M,\underline{\mu})$ is one, so (i) follows. Note that we have also proved the first statement in (v), while the second statement in (v) follows from Proposition 3.5(ii) and the exactness of the functor $\mathrm{Res}_{|M|}$.

(iv) By (v), $\mathrm{Res}_{|N|}L(N,\underline{\nu}) \cong L_{N,\underline{\nu}} \neq 0$. Therefore, if $L(N,\underline{\nu})$ is a composition factor of $\Delta(M,\underline{\mu})$, then $\mathrm{Res}_{|N|}\Delta(M,\underline{\mu}) \neq 0$ by exactness of $\mathrm{Res}_{|N|}$. By Proposition 3.5, we then have $N \leq M$ and the first equality in (iv). If $N < M$, then $(N,\underline{\nu}) < (M,\underline{\mu})$. If $N = M$, and $\underline{\nu} \neq \underline{\mu}$, then we get a contribution of $L_{N,\underline{\nu}}$ into $\mathrm{Res}_{|M|}\Delta(M,\underline{\mu})$, which contradicts (v).

(ii) If $L(M,\underline{\mu}) \cong L(N,\underline{\nu})$, then we deduce from (iv) that $(M,\underline{\mu}) \leq (N,\underline{\nu})$ and $(N,\underline{\nu}) \leq (M,\underline{\mu})$, whence $(M,\underline{\mu}) = (N,\underline{\nu})$.

(iii) follows from (v) and Lemma 3.4. □

4.3 Imaginary modules

In this subsection we assume that $\alpha = n\delta$ for some $n \in \mathbb{Z}_{\geq 0}$. Then Proposition 4.2, yields $|\Pi(\alpha)| - |\mathscr{P}_n|$ (pairwise non-isomorphic) irreducible modules, namely the modules $L(M,\underline{\mu})$

corresponding to the root partitions $(M, \underline{\mu})$ such that $m_\rho \neq 0$ for some $\rho \in \Phi_+^{\mathrm{re}}$. Let us label the remaining $|\mathscr{P}_n|$ irreducible $R_{n\delta}$-modules by the elements of $\mathscr{P}_n$ in *some* way, cf. Lemma 3.2. So we get irreducible $R_{n\delta}$-modules $\{L(\underline{\mu}) \mid \underline{\mu} \in \mathscr{P}_n\}$, and then $\{L(M, \underline{\mu}) \mid (M, \underline{\mu}) \in \Pi(\alpha)\}$ is a complete and irredundant system of irreducible R_α-modules up to isomorphism. Our next goal is Lemma 4.3 which proves that the modules $\{L(\underline{\mu}) \mid \underline{\mu} \in \mathscr{P}_n\}$ are imaginary in the sense of (Cus2).

We need some terminology. Let $(M, \underline{\mu})$ be a root partition. We say that $\rho \in \Psi$ *appears in the support of* M if $m_\rho > 0$. Let κ be the largest root appearing in the support of M, and $\beta \in \Phi_+$ satisfies $\beta \succeq \kappa$. Note that if β is real then $L_\beta \circ \Delta(M, \underline{\mu})$ is, up to a degree shift, a standard module again. If $\beta = n\delta$ is imaginary, $\underline{\nu} \in \mathscr{P}_n$, and κ is real, then $L(\underline{\nu}) \circ \Delta(M, \underline{\mu})$ is again a standard module.

Lemma 4.3 *Let $\underline{\lambda} \in \mathscr{P}_n$. Suppose that $\beta, \gamma \in Q_+ \backslash \Phi_+^{\mathrm{im}}$ are non-zero elements such that $n\delta = \beta + \gamma$ and $\mathrm{Res}_{\beta,\gamma} L(\underline{\lambda}) \neq 0$. Then β is a sum of real roots less than δ and γ is a sum of real roots greater than δ.*

Proof We prove that β is a sum of real roots less than δ, the proof that γ is a sum of real roots greater than δ being similar. Let $L(M, \underline{\mu}) \boxtimes L(N, \underline{\nu})$ be an irreducible submodule of $\mathrm{Res}_{\beta,\gamma} L(\underline{\lambda}) \neq 0$, so that $(M, \underline{\mu}) \in \Pi(\beta)$ and $(N, \underline{\nu}) \in \Pi(\gamma)$. Note that $\mathrm{ht}(\beta), \mathrm{ht}(\gamma) < \mathrm{ht}(\alpha)$, so the modules $L(M, \underline{\mu}), L(N, \underline{\nu})$ are defined by induction.

Let χ be the largest root appearing in the support of M. If $\chi \preceq \delta$, then, since β is not an imaginary root, we conclude that β is a sum of real roots less than δ. So we may assume that $\chi \succ \delta$. Moreover, $\mathrm{Res}_{\chi,\beta-\chi} L(M, \underline{\mu}) \neq 0$, and hence $\mathrm{Res}_{\chi,\gamma+\beta-\chi} L(\underline{\lambda}) \neq 0$. So we may assume from the beginning that $\beta \in \Phi_\succ^{\mathrm{re}}$ and $L(M, \underline{\mu}) \simeq L_\beta$. Moreover, we may assume that β is the largest possible real root for which $\mathrm{Res}_{\beta,\gamma} L(\underline{\lambda}) \neq 0$.

Now, let κ be the largest root appearing in the support of N. If κ is a real root, we have the cuspidal module L_κ. If κ is imaginary, then let us denote by L_κ the module $L(\underline{\nu})$. Then we have a non-zero map $L_\beta \boxtimes L_\kappa \boxtimes V \to \mathrm{Res}_{\beta,\kappa,\gamma-\kappa} L(\underline{\lambda})$, for some non-zero $R_{\gamma-\kappa}$-module V. By adjunction, this yields a non-zero map

$$f : (\mathrm{Ind}_{\beta,\kappa} L_\beta \boxtimes L_\kappa) \boxtimes V \to \mathrm{Res}_{\beta+\kappa,\gamma-\kappa} L(\underline{\lambda})$$

If $\kappa = \gamma$ note that $\beta \neq \gamma$, since it has been assumed that $\beta, \gamma \notin \Phi_+^{\mathrm{im}}$. Now we conclude that $\beta \prec \gamma$, for otherwise $L(\underline{\lambda})$ is a quotient of the standard module $L_\beta \circ L_\gamma$, which contradicts the definition of the irreducible imaginary module $L(\underline{\lambda})$. Now, since $n\delta = \beta + \kappa$, we have by (Con3) that $\beta \prec \delta \prec \gamma$, as desired.

Next, let $\kappa \neq \gamma$, and pick a composition factor $L(M', \underline{\mu}')$ of $\mathrm{Ind}_{\beta,\kappa} L_\beta \boxtimes L_\kappa$, which is not in the kernel of f. By the assumption on the maximality of β, every root κ' in the support of M' satisfies $\kappa' \preceq \beta$. Thus $\beta + \kappa$ is a sum of roots $\preceq \beta$. Now (Con2) implies that $\kappa \preceq \beta$, and so by adjointness, $L(\underline{\lambda})$ is a quotient of the standard module $L_\beta \circ \Delta(N, \underline{\nu})$, which is a contradiction. □

We now establish a useful property of imaginary modules:

Lemma 4.4 *Let $\underline{\mu} \in \mathscr{P}_r$ and $\underline{\nu} \in \mathscr{P}_s$ with $r + s = n$. Then all composition factors of $L(\underline{\mu}) \circ L(\underline{\nu})$ are of the form $L(\underline{\kappa})$ for $\underline{\kappa} \in \mathscr{P}_n$.*

Proof Let $L(K, \underline{\kappa})$ be a composition factor of $L(\underline{\mu}) \circ L(\underline{\nu})$. We need to prove that $k_\rho = 0$ for all $\rho \in \Phi_+^{\mathrm{re}}$, i.e. $L(K, \underline{\kappa}) = L(\underline{\kappa})$. If this is not the case, there is $\rho > \delta$ with $k_\rho > 0$. Pick the largest such ρ, and set $(K', \underline{\kappa}') := (K, \underline{\kappa})'_\rho$, see (3.4). By Proposition 4.2(v), we have that $\mathrm{Res}_{|K|}L(K,\underline{\kappa}) \neq 0$, so $\mathrm{Res}_{|K|}(L(\underline{\mu}) \circ L(\underline{\nu})) \neq 0$. We apply the Mackey Theorem to conclude that the last module has a filtration with factors of the form

$$\mathrm{Ind}_{\lambda_1,\lambda_2;\underline{\gamma}}^{k_\rho \rho;|K'|} V,$$

where $k_\rho \rho = \lambda_1 + \lambda_2$, $\underline{\gamma}$ is a refinement of $|K'|$, and

$$\mathrm{Res}_{\lambda_1, r\delta-\lambda_1} L(\underline{\mu}) \neq 0 \neq \mathrm{Res}_{\lambda_2, s\delta-\lambda_2} L(\underline{\nu}).$$

By the inductive assumption, we know that $L(\underline{\mu})$ and $L(\underline{\nu})$ satisfy (Cus2), i.e. λ_1 and λ_2 are either imaginary roots or a sum of roots less than δ. In either case, λ_1 and λ_2 are sums of roots less than ρ, and then so is $k_\rho \rho$. This contradicts (Con1). □

4.4 Cuspidal modules

Throughout this subsection we assume that $\alpha = \rho \in \Phi_+^{\mathrm{re}}$. Let $(M, \underline{\mu}) \in \Pi(\alpha)$ be a root partition of α. There is a *trivial* root partition (α). Proposition 4.2 yields $|\Pi(\alpha)| - 1$ irreducible R_α-modules, namely the ones which correspond to the *non-trivial* root partitions $(M, \underline{\mu})$. We define the cuspidal module L_α to be the missing irreducible R_α-module, cf. Lemma 3.2. Then, of course, we have that $\{L(M, \underline{\mu}) \mid (M, \underline{\mu}) \in \Pi(\alpha)\}$ is a complete and irredundant system of irreducible R_α-modules up to isomorphism. We now prove that L_α satisfies the property (Cus1) and is uniquely determined by it. To be more precise:

Lemma 4.5 *If $\beta, \gamma \in Q_+$ are non-zero elements such that $\alpha = \beta + \gamma$ and $\mathrm{Res}_{\beta,\gamma} L_\alpha \neq 0$, then β is a sum of roots less than α and γ is a sum of roots greater than α. Moreover, this property characterizes L_α among the irreducible R_α-modules uniquely up to isomorphism and degree shift.*

Proof We prove that β is a sum of roots less than α, the proof that γ is a sum of roots greater than α being similar. Let $L(M, \underline{\mu}) \boxtimes L(N, \underline{\nu})$ be an irreducible submodule of $\mathrm{Res}_{\beta,\gamma} L_\alpha$, so that $(M, \underline{\mu}) \in \Pi(\beta)$ and $(N, \underline{\nu}) \in \Pi(\gamma)$. Let χ be the largest root appearing in the support of M. Then $\mathrm{Res}_{\chi,\beta-\chi} L(M, \underline{\mu}) \neq 0$, and hence $\mathrm{Res}_{\chi,\gamma+\beta-\chi} L_\alpha \neq 0$. If we can prove that χ is a sum of roots less than α, then by (Con1), (Con3), χ is a root less than α, whence, by the maximality of χ, we have that β is a sum of roots less than α. So we may assume from the beginning that β is a root and $L(M, \underline{\mu}) = L_\beta$ (if β is imaginary, L_β is interpreted as $L(\underline{\mu})$). Moreover, we may assume that β is the largest possible root for which $\mathrm{Res}_{\beta,\gamma} L_\alpha \neq 0$.

Now, let κ be the largest root appearing in the support of N. If κ is a real root, we have the cuspidal module L_κ. If κ is imaginary, then we interpret L_κ as $L(\underline{\nu})$. Then we have a non-zero map

$$L_\beta \boxtimes L_\kappa \boxtimes V \to \mathrm{Res}_{\beta,\kappa,\gamma-\kappa} L_\alpha,$$

for some $0 \neq V \in R_{\gamma-\kappa}$-mod. By adjunction, this yields a non-zero map

$$f : (\mathrm{Ind}_{\beta,\kappa} L_\beta \boxtimes L_\kappa) \boxtimes V \to \mathrm{Res}_{\beta+\kappa,\gamma-\kappa} L_\alpha.$$

If $\kappa = \gamma$, then we must have $\beta \prec \gamma$, for otherwise L_α is a quotient of the standard module $L_\beta \circ L_\gamma$, which contradicts the definition of the cuspidal module L_α. Now, since $\alpha = \beta + \kappa$, we have by (Con1) that $\beta \prec \alpha \prec \gamma$, in particular $\beta \prec \alpha$ as desired.

Next, let $\kappa \neq \gamma$, and pick a composition factor $L(M', \underline{\mu}')$ of $\mathrm{Ind}_{\beta,\kappa} L_\beta \boxtimes L_\kappa$, which is not in the kernel of f. By the assumption on the maximality of β, every root κ' in the support of M' satisfies $\kappa' \preceq \beta$. Thus $\beta + \kappa$ is a sum of roots $\preceq \beta$. If β and κ are not both imaginary, then

(Con2) implies that $\kappa \preceq \beta$, and so by adjointness, L_α is a quotient of the standard module $L_\beta \circ \Delta(N, \underline{\nu})$, which is a contradiction.

If β and κ are both imaginary, then $\Delta(N, \underline{\nu}) = L(\underline{\nu}) \circ \Delta(N', \emptyset)$ for N' such that a maximal root appearing in the support of N' is of the form $\psi < \delta$. In this case, we have by adjunction that L_α is a quotient of $L(\underline{\mu}) \circ L(\underline{\nu}) \circ L(N', \emptyset)$. It now follows from Lemma 4.4 that L_α is a quotient of the standard module of the form $L(\underline{\lambda}) \circ L(N', \emptyset)$ for some composition factor $L(\underline{\lambda})$ of $L(\underline{\mu}) \circ L(\underline{\nu})$, so we get a contradiction again, since L_α is cuspidal.

The second statement of the lemma is clear since, in view of Proposition 4.2(v) and (Con1), the irreducible modules $L(M, \underline{\mu})$, corresponding to non-trivial root partitions $(M, \underline{\mu}) \in \Pi(\alpha)$, do not satisfy the property (Cus1). □

4.5 Powers of cuspidal modules

Assume finally that $\alpha = n\rho$ for some $\rho \in \Phi_+^{\mathrm{re}}$ and $n \in \mathbb{Z}_{>1}$.

Lemma 4.6 *The induced module $L_\rho^{\circ n}$ is irreducible.*

Proof In view of Proposition 4.2, we have the irreducible modules $L(M, \underline{\mu})$ for all root partitions $(M, \underline{\mu}) \in \Pi(\alpha)$, except for $(N, \underline{\nu}) = (\rho^n)$ for which $\Delta(N, \underline{\nu}) = L_\rho^{\circ n}$. By (Con1), we have that $N \leq M$ for all $(M, \underline{\mu}) \in \Pi(\alpha)$, and if $M = N$, then $(M, \underline{\mu}) = (N, \underline{\nu})$. By Proposition 4.2(v), we conclude that $L_\rho^{\circ n}$ has only one composition factor L appearing with certain multiplicity $c(q) \in \mathscr{A}$, and such that $L \ncong L(M, \underline{\mu})$ for all $(M, \underline{\mu}) \in \Pi(\alpha) \backslash \{(N, \underline{\nu})\}$. Finally, by Corollary 2.12, we conclude that $L_\rho^{\circ n} \cong L$. □

The proof of Theorem 4.1 is now complete.

4.6 Another version of the Main Theorem

We now formulate and prove a slightly stronger version of the Main Theorem. For each n, fix an arbitrary partial order $\trianglelefteq$ on the set of multipartitions $\mathscr{P}_n$. Let $\alpha \in Q_+$. Define a partial order $\leq'$ on $\Pi(\alpha)$ as follows: $(M, \underline{\mu}) \leq' (N, \underline{\nu})$ if and only if the following two conditions hold: (1) $M \leq N$, (2) if $m_\rho = n_\rho$ for all $\rho \geq \delta$ or for all $\rho \leq \delta$, then $\underline{\mu} \trianglelefteq \underline{\nu}$.

Now we modify the data (Cus2) of a cuspidal system as follows:

(Cus2′) An $R_{n\delta}$-module $\Delta(\underline{\mu})$ is assigned to every $\underline{\mu} \in \mathscr{P}_n$ for all $n \in \mathbb{Z}_{\geq 0}$ with the following properties:

- (a) each $\Delta(\underline{\mu})$ has an irreducible head; denote this head by $L(\underline{\mu})$;
- (b) $L(\underline{\mu})^{\circledast} \simeq L(\underline{\mu})$;
- (c) $[\Delta(\underline{\mu}) : L(\underline{\mu})]_q = 1$ and $[\Delta(\underline{\mu}) : L(\underline{\nu})]_q \neq 0$ implies $\underline{\nu} \trianglelefteq \underline{\mu}$;
- (d) $L(\underline{\lambda}) \ncong L(\underline{\mu})$ unless $\underline{\lambda} = \underline{\mu}$;
- (e) if $\beta, \gamma \in Q_+ \backslash \Phi_+^{\mathrm{im}}$ are non-zero elements such that $n\delta = \beta + \gamma$ and $\mathrm{Res}_{\beta,\gamma} L(\underline{\mu}) \neq 0$, then β is a sum of positive real roots less than δ and γ is a sum of positive real roots greater than δ

A *weak cuspidal system* (for a fixed convex preorder) is the data of (Cus1) and (Cus2′).

Given a weak cuspidal system, for every $\alpha \in Q_+$ and $\pi = (M, \underline{\mu}) \in \Pi(\alpha)$ in the form (3.2), we define

$$\Delta'(\pi) = \Delta'(M, \underline{\mu}) := L_{\rho_1}^{\circ m_1} \circ \cdots \circ L_{\rho_s}^{\circ m_s} \circ \Delta(\underline{\mu}) \circ L_{\rho_{-t}}^{\circ m_{-t}} \circ \cdots \circ L_{\rho_{-1}}^{\circ m_{-1}} \langle \mathtt{sh}(\pi) \rangle.$$

The following is a version of the Main Theorem for weak cuspidal systems:

Theorem 4.7 *For a weak cuspidal system* $\{L_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{\Delta(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}\}$*, we have:*

(i) *For every root partition* π*, the standard module* $\Delta'(\pi)$ *has an irreducible head; denote this irreducible module* $L(\pi)$*.*
(ii) $\{L(\pi) \mid \pi \in \Pi(\alpha)\}$ *is a complete and irredundant system of irreducible* R_α*-modules up to isomorphism.*
(iii) $L(\pi)^{\circledast} \simeq L(\pi)$*.*
(iv) $[\Delta'(\pi) : L(\pi)]_q = 1$*, and* $[\Delta'(\pi) : L(\sigma)]_q \neq 0$ *implies* $\sigma \leq' \pi$*.*

Proof Since $\{L_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}\}$ is a weak cuspidal system, it is also a cuspidal system. So we only need to prove (i) and (iv). To see (i), we observe using Proposition 3.5 and dimensions that

$$\mathrm{Res}_{|M|}\Delta'(M,\underline{\mu}) \simeq L_{\rho_1}^{\circ m_1} \boxtimes \cdots \boxtimes L_{\rho_s}^{\circ m_s} \boxtimes \Delta(\underline{\mu}) \boxtimes L_{\rho_{-t}}^{\circ m_{-t}} \boxtimes \cdots \boxtimes L_{\rho_{-1}}^{\circ m_{-1}} \langle \mathtt{sh}(\pi)\rangle.$$

Now (i) follows by the adjointness of Ind and Res. Finally, (iv) is proved using a variation of Proposition 3.5. □

4.7 Reduction modulo p

In this section we work with two fields: F of characteristic $p > 0$ and K of characteristic 0. We use the corresponding indices to distinguish between the two situations. Given an irreducible $R_\alpha(K)$-module L_K for a root partition $\pi \in \Pi(\alpha)$ we can pick a (graded) $R_\alpha(\mathbb{Z})$-invariant lattice $L_{\mathbb{Z}}$ as follows: pick a homogeneous 'word vector' $v \in L_K$ and set $L_{\mathbb{Z}} := R_\alpha(\mathbb{Z})v$. The lattice $L_{\mathbb{Z}}$ can be used to *reduce modulo* p:

$$\bar{L} := L_{\mathbb{Z}} \otimes_{\mathbb{Z}} F.$$

In general, the $R_\alpha(F)$-module $\bar{L}$ depends on the choice of the lattice $L_{\mathbb{Z}}$. However, we have $\mathrm{ch}_q\, \bar{L} = \mathrm{ch}_q\, L_K$, so by linear independence of characters of irreducible $R_\alpha(F)$-modules, composition multiplicities of irreducible $R_\alpha(F)$-modules in $\bar{L}$ are well-defined. In particular, we have well-defined *decomposition numbers*

$$d_{\pi,\sigma} := [\bar{L}(\pi) : L_F(\sigma)]_q \qquad (\pi, \sigma \in \Pi(\alpha)),$$

which depend only on the characteristic p of F, since prime fields are splitting fields for irreducible modules over KLR algebras.

Lemma 4.8 *Let* L_K *be an irreducible* $R_\alpha(K)$*-module and let* $\boldsymbol{i} = i_1^{a_1} \dots i_b^{a_b}$ *be an extremal word for* L_K*. Let* N *be the irreducible* $\circledast$*-selfdual* $R_\alpha(F)$*-module defined by* $N := \tilde{f}_{i_k}^{a_k} \dots \tilde{f}_{i_1}^{a_1} 1_F$*. Then* $[\bar{L} : N]_q = 1$*.*

Proof Reduction modulo p preserves formal characters, so the result follows from Corollary 2.9. □

Proposition 4.9 *Let* $(M,\underline{\mu}), (N,\underline{\nu}) \in \Pi(\alpha)$*. Then* $d_{(M,\underline{\mu}),(N,\underline{\nu})} \neq 0$ *implies* $N \leq M$*. In particular, reduction modulo* p *of any cuspidal module is an irreducible cuspidal module again:* $\bar{L}_\rho \simeq L_{\rho,F}$*.*

Proof By Theorem 4.1(v), which holds over any field, we conclude that any composition factor of $\bar{L}_\rho$ is isomorphic to $L_{\rho,F}$ up to a degree shift. Now use Lemma 4.8. □

4.8 Cuspidal modules and dual PBW bases

Recall the Q_+-graded $\mathscr{A}$-algebras $\mathbf{f}^*_{\mathscr{A}}$ and $\mathbf{f}_{\mathscr{A}}$ and $\mathbb{Q}(q)$-algebras $\mathbf{f}^*$ and $\mathbf{f}$. Suppose that we are given elements

$$\{E^*_\rho \in (\mathbf{f}^*_{\mathscr{A}})_\rho \mid \rho \in \Phi^{\mathrm{re}}_+\} \cup \{E^*_{\underline{\lambda}} \in (\mathbf{f}_{\mathscr{A}})_{|\underline{\lambda}|\delta} \mid \underline{\lambda} \in \mathscr{P}\}. \tag{4.1}$$

If $\pi = (M, \underline{\mu})$ is a root partition written in the form (3.2), define the corresponding *dual PBW monomial*

$$E^*_\pi = E^*_{M,\underline{\mu}} := (E^*_{\rho_1})^{m_1} \dots (E^*_{\rho_s})^{m_s} E^*_{\underline{\mu}} (E^*_{\rho_{-t}})^{m_{-t}} \dots (E^*_{\rho_{-1}})^{m_{-1}} \in \mathbf{f}^*_{\mathscr{A}}.$$

We say that (4.1) is a *dual PBW family* if the following properties are satisfied:

(i) ('convexity') if $\beta \succ \gamma$ are positive roots then $E^*_\gamma E^*_\beta - q^{-(\beta,\gamma)} E^*_\beta E^*_\gamma$ is an $\mathscr{A}$-linear combination of elements E^*_π with $\pi < (\beta, \gamma) \in \Pi(\beta+\gamma)$; here if $\beta = n\delta$ is imaginary, then E^*_β is interpreted as $E^*_{\underline{\mu}}$ and (β, γ) is interpreted as $(\underline{\mu}, \gamma) \in \Pi(\beta + \gamma)$ for an arbitrary $\underline{\mu} \in \mathscr{P}_n$, and similarly for γ (both β and γ cannot be imaginary since then $\beta \not\succ \gamma$);
(ii) ('basis') $\{E^*_\pi \mid \pi \in \Pi(\alpha)\}$ is an $\mathscr{A}$-basis of $(\mathbf{f}^*_{\mathscr{A}})_\alpha$ for all $\alpha \in Q_+$;
(iii) ('orthogonality')

$$(E^*_{M,\underline{\mu}}, E^*_{N,\underline{\nu}}) = \delta_{M,N} (E^*_{\underline{\mu}}, E^*_{\underline{\mu}}) \prod_{\rho \in \Phi^{\mathrm{re}}_+} ((E^*_\rho)^{m_\rho}, (E^*_\rho)^{m_\rho});$$

(iv) ('bar-triangularity') $\mathrm{b}^*(E^*_\pi) = E^*_\pi +$ an $\mathscr{A}$-linear combination of dual PBW monomials E^*_σ for $\sigma < \pi$.

The following result shows in particular that the elements E^*_ρ of the dual PBW family are determined uniquely up to signs (for a fixed preorder $\preceq$):

Lemma 4.10 *Assume that (4.1) is a dual PBW family. Then:*

(i) *The elements of (4.1) are* b^**-invariant.*
(ii) *Suppose that we are given another family* $\{{}'E^*_\rho \in (\mathbf{f}^*_{\mathscr{A}})_\rho \mid \rho \in \Phi^{\mathrm{re}}_+\} \cup \{{}'E^*_{\underline{\lambda}} \in (\mathbf{f}_{\mathscr{A}})_{|\underline{\lambda}|\delta} \mid \underline{\lambda} \in \mathscr{P}\}$ *of* b^**-invariant elements which satisfies the basis and orthogonality properties. Then* $E^*_\rho = \pm {}'E^*_\rho$ *for all* $\rho \in \Phi^{\mathrm{re}}_+$*, and for any* $\underline{\mu} \in \mathscr{P}_n$*, we have that* $E^*_{\underline{\mu}}$ *is an* $\mathscr{A}$*-linear combination of elements* ${}'E^*_{\underline{\nu}}$ *with* $\underline{\nu} \in \mathscr{P}_n$.

Proof (i) The convexity of $\preceq$ implies that for $\rho \in \Phi^{\mathrm{re}}_+$ the root partition $(\rho) \in \Pi(\rho)$ is a minimal element of $\Pi(\rho)$ and for $\underline{\mu} \in \mathscr{P}_n$ the root partition $(\underline{\mu}) \in \Pi(n\delta)$ is a minimal element of $\Pi(n\delta)$. So the bar-triangularity property (iv) implies that the elements of a dual PBW family are b^*-invariant.

Part (ii) has two statements, one for E^*_ρ with $\rho \in \Phi^{\mathrm{re}}_+$ and another for $E^*_{\underline{\mu}}$ with $\underline{\mu} \in \mathscr{P}_n$. Let $\alpha := \rho$ in the first statement and $\alpha := n\delta$ in the second. We prove (ii) by induction on $\mathrm{ht}(\alpha)$, the induction base being clear. For the first statement, by the basis property of dual PBW families, we can write

$${}'E^*_\rho = cE^*_\rho + \sum_{\pi \in \Pi(\rho)\setminus\{(\rho)\}} c_\pi E^*_\pi \qquad (c, c_\pi \in \mathscr{A}). \tag{4.2}$$

Fix for a moment a root partition $\pi = (M, \underline{\mu}) \in \Pi(\rho)\setminus\{(\rho)\}$. By the orthogonality property of dual PBW families and non-degeneracy of the form $(\cdot,\cdot)$, there is a $\mathbb{Q}(q)$-linear combination X_π of elements $E^*_{M,\underline{\nu}}$ with $\underline{\nu} \in \mathscr{P}_{|\underline{\mu}|}$ such that $(E^*_\sigma, X_\pi) = \delta_{\sigma,\pi}$ for all $\sigma \in$

$\Pi(\rho)$. So pairing the right hand side of (4.2) with X_π yields c_π. On the other hand, by the inductive assumption, $E^*_{M,\underline{\nu}}$ for each $\underline{\nu}$ is a linear combination of elements of the form ${}'E^*_{M,\underline{\lambda}}$. So using the orthogonality property for the primed family in (ii), we must have $({}'E^*_\rho, X_\pi) = 0$ for all $\pi \in \Pi(\rho) \setminus \{(\rho)\}$. So $c_\pi = 0$. Thus ${}'E^*_\rho = cE^*_\rho$. Furthermore, the elements ${}'E^*_\rho$ and E^*_ρ belong to the algebra $\mathbf{f}^*_{\mathscr{A}}$ and are parts of its $\mathscr{A}$-bases, whence ${}'E^*_\rho = \pm q^n E^*_\rho$. Since both ${}'E^*_\rho$ and E^*_ρ are $\mathtt{b}^*$-invariant, we conclude that $n = 0$.

Now, we prove the second statement in (ii). We can write $E^*_{\underline{\mu}}$ as

$$'E^*_{\underline{\mu}} = \sum_{\underline{\lambda}\in\mathscr{P}_n} c_{\underline{\lambda}}\, E^*_{\underline{\lambda}} + \sum_{(N,\underline{\nu})\in\Pi(n\delta)\text{ with }|\underline{\nu}|<n} c_{N,\underline{\nu}}\, E^*_{N,\underline{\nu}} \qquad (c_{\underline{\lambda}}, c_{N,\underline{\nu}} \in \mathscr{A}).$$

Now one shows that all $c_{N,\underline{\nu}} = 0$ by an argument using orthogonality and the inductive assumption as in the previous two paragraphs. □

We now show that under the Khovanov–Lauda–Rouquier categorification (see Sect. 2.9), cuspidal systems yield dual PBW families.

Proposition 4.11 *The following set of elements in* $\mathbf{f}^*_{\mathscr{A}}$

$$\{E^*_\rho := \gamma^*([L_\rho]) \mid \rho \in \Phi^{\mathrm{re}}_+\} \cup \{E^*_{\underline{\mu}} := \gamma^*([L(\underline{\mu})]) \mid \underline{\lambda} \in \mathscr{P}\} \tag{4.3}$$

is a dual PBW family.

Proof Under the categorification map γ^*, the graded duality $\circledast$ corresponds to $\mathtt{b}^*$, so $\gamma^*([L])$ is $\mathtt{b}^*$-invariant for any $\circledast$-self-dual R_α-module L. Moreover, under γ^*, the induction product corresponds to the product in $\mathbf{f}^*_{\mathscr{A}}$, so the convexity condition (i) follows from Theorem 4.1(iv) and Lemma 2.3. Now, note that $E^*_\pi = \gamma^*([\Delta(\pi)])$, so the conditions (ii) and (iv) follow from Theorem 4.1(iv) again. It remains to establish the orthogonality property (iii). Let $(M, \underline{\mu})$ be written in the form (3.2). Under γ^*, the coproduct r corresponds to the map on the Grothendieck group induces by Res. So using (2.21), we get

$$(E^*_{M,\underline{\mu}}, E^*_{N,\underline{\nu}}) = \big((E^*_{\rho_1})^{m_1} \otimes \cdots \otimes E^*_{\underline{\mu}} \otimes \cdots \otimes (E^*_{\rho_{-1}})^{m_{-1}}, \gamma^*([\mathrm{Res}_{|M|}\Delta(N, \underline{\nu})])\big).$$

By Proposition 3.5, $\mathrm{Res}_{|M|}\Delta(N, \underline{\nu}) = 0$ unless $M = N$, and for $M = N$ we have

$$\mathrm{Res}_{|M|}\Delta(N, \underline{\nu}) = L_{\rho_1}^{\circ m_1} \boxtimes \cdots \boxtimes L(\underline{\nu}) \boxtimes \cdots \boxtimes L_{\rho_{-1}}^{\circ m_{-1}}.$$

Since the form $(\cdot, \cdot)$ is symmetric, the orthogonality follows from the preceding remarks. □

Remark 4.12 Let $\preceq$ be an arbitrary convex order,

$$\{L_\rho \mid \rho \in \Phi^{\mathrm{re}}_+\} \cup \{L(\underline{\mu}) \mid \underline{\lambda} \in \mathscr{P}\}$$

be the corresponding cuspidal system, and set again $E^*_\rho := \gamma^*([L_\rho])$ for all ρ and $E^*_{\underline{\mu}} := \gamma^*([L(\underline{\mu})])$ for all $\underline{\mu}$.

(i) We claim that each E^*_ρ is a dual canonical basis element. Indeed, for symmetric Cartan matrices, this is true by the main result of [28] and Proposition 4.9. We now sketch an argument, which works in general. This will not be used elsewhere in the paper. Let $\rho \in \Phi^{\mathrm{re}}_+$. To prove that E^*_ρ is a dual canonical basis element, it suffices to prove the following
Claim. There exists a dual canonical basis element v^* such that $E^*_\rho = \pm v^*$.

Indeed, then in view of the commutativity of the triangle (2.22), to show that $E_\rho^* = v^*$, it suffices to know that for an arbitrary element w^* of the dual canonical basis, there at least one word $\boldsymbol{i} \in \langle I \rangle$ such that the coefficient of $\boldsymbol{i}$ in $\iota(w^*)$ evaluated at $q = 1$ is positive. But this follows from Lemma 2.13.

We now sketch the proof of the Claim. Fix ρ and write $\rho = \sum_{i\in I} c_i\alpha_i$. Set

$$\Phi(\rho)' = \left\{ \beta = \sum_{i\in I} b_i\alpha_i \in \Phi_+^{\mathrm{re}} \mid b_i \le c_i \text{ for all } i \in I \right\}.$$

Let $\Phi(\rho)'_\succ = \{\beta \in \Phi(\rho)' \mid \beta \succ \delta\}$ and $\Phi(\rho)'_\prec = \{\beta \in \Phi(\rho)' \mid \beta \prec \delta\}$. Next, let $\Phi(\rho)_\succ$ (resp. $\Phi(\rho)_\prec$) be the set of all positive roots which can be written as $\mathbb{Z}_{\ge 0}$-linear combinations of roots in $\Phi(\rho)'_\succ$ (resp. $\Phi(\rho)'_\prec$). Finally, put $\Phi(\rho) = \Phi(\rho)_\succ \sqcup \Phi(\rho)_\prec$. Note that the sets $\Phi(\rho)_\succ$ and $\Phi(\rho)_\prec$ are finite and compatible in the sense of [7, Definition on p. 213]. By [7, Proposition 3.2 and Remark (1) on p. 214], there exist reduced words $r_{i_1} \dots r_{i_m}$ and $r_{j_1} \dots r_{j_n}$ such that

$$\Phi(\rho)_\succ = \{\alpha_{i_1} \succ r_{i_1}\alpha_{i_2} \succ \dots \succ r_{i_1} \dots r_{i_{m-1}}\alpha_{i_m}\}$$

and

$$\Phi(\rho)_\prec = \{\alpha_{j_1} \succ r_{j_1}\alpha_{j_2} \succ \dots \succ r_{j_1} \dots r_{j_{n-1}}\alpha_{j_n}\}.$$

We now use 'partial PBW basis' from [22, Proposition 8.2] and [21, Section 40.2] (up to dualizing). To be more precise, Lusztig uses a braid group action to define bar-invariant dual PBW-elements $\{'E_\beta^* \mid \beta \in \Phi(\rho)\}$ which lie in the dual canonical basis by [22, Proposition 8.2], and satisfy the defining properties (ii) and (iii) of a dual PBW family for the weight space ρ and all smaller weight spaces of $\mathbf{f}_{\mathscr{A}}^*$. Now the argument as in the proof of Lemma 4.10(ii) shows that $'E_\rho^* = \pm E_\rho^*$, i.e. up to a sign E_ρ^* is a dual canonical basis element.

(ii) For certain special convex preorders, which we refer to as *Beck preorders*, (dual) PBW families have been constructed in [2,3]. Fix a Beck preorder and denote by $\{'E_\rho^* \in (\mathbf{f}_{\mathscr{A}}^*)_\rho \mid \rho \in \Phi_+^{\mathrm{re}}\} \cup \{'E_{\underline{\lambda}}^* \in (\mathbf{f}_{\mathscr{A}})_{|\underline{\lambda}|\delta} \mid \underline{\lambda} \in \mathscr{P}\}$ the corresponding dual PBW family from [2,3]. By Lemma 4.10(ii), $'E_\rho^* = \pm E_\rho^*$ for all $\rho \in \Phi_+^{\mathrm{re}}$. In fact, $'E_\rho^* = E_\rho^*$ for all $\rho \in \Phi_+^{\mathrm{re}}$ by an argument in (ii) since the real dual root elements $'E_\rho^*$ of Beck-Chari-Pressley basis are known to belong to the dual canonical basis.

(iii) By the main result of [28], each $E_{\underline{\mu}}^*$ is a dual canonical basis element provided $\mathtt{C}$ is symmetric and char $F = 0$. This is certainly false if char $F \ne 0$. On the other hand, we conjecture that for not necessarily symmetric $\mathtt{C}$ we still have that each $E_{\underline{\mu}}^*$ is a dual canonical basis element provided char $F = 0$. An argument similar to the one sketched in part (i) would apply, provided the Claim in (i) holds with $E_{\underline{\mu}}^*$ in place of E_ρ^*. But we do not know how to prove such a claim for non-symmetric $\mathtt{C}$.

5 Minuscule representations and imaginary tensor spaces

In this section we study the 'smallest' imaginary representations, namely the imaginary representations of R_δ. Then we consider induction powers of these minuscule representations, which turn out to play a role of tensor spaces. Denote

$$e := \mathrm{ht}(\delta).$$

Throughout the section we assume that our convex preorder $\preceq$ is balanced, as defined in (3.1), so that $\alpha_i \succ n\delta \succ \alpha_0$ for all $i \in I'$ and $n \in \mathbb{Z}_{>0}$. So for any irreducible imaginary representation L of $R_{n\delta}$, we conclude using (Cus2) that $\mathrm{Res}_{\alpha_i, n\delta-\alpha_i} L = 0$ for all $i \in I'$, i.e. all words $\boldsymbol{i} = (i_1, \dots, i_d)$ of L have the property that $i_1 = 0$.

5.1 Minuscule representations

Note that $|\mathscr{P}_1| = l$, so there are exactly l irreducible imaginary representations of R_δ. We call these representations *minuscule*. The following lemma shows that a description of minuscule imaginary modules is equivalent to a description of the irreducible $R_\delta^{\Lambda_0}$-modules.

Lemma 5.1 *Let L be an irreducible R_δ-module. The following are equivalent:*

(i) *L is minuscule imaginary;*
(ii) *L factors through to the cyclotomic quotient $R_\delta^{\Lambda_0}$;*
(iii) *we have $i_1 = 0$ for any word $\boldsymbol{i} = (i_1, \dots, i_e)$ of L.*

Proof By (2.2), there is exactly one 0 among the entries $i_1, \dots, i_e$ of an arbitrary word $\boldsymbol{i} \in \langle I \rangle_\delta$. Now (ii) and (iii) are equivalent by Lemma 2.7. The implication (i) $\implies$ (iii) follows from the remarks in the beginning of Sect. 5. Finally, let $L(M, \underline{\mu})$ be an irreducible R_δ-module, which is not imaginary, i.e. there is $\rho \in \Phi_+^{\mathrm{re}}$ with $m_\rho \neq 0$. Then, since $\sum_{\beta\in\Psi} M_\beta = \delta$, we conclude that there is $\rho > \delta$ with $m_\rho \neq 0$. Let ρ be the largest such. Then $\rho \in \Phi'_+$, in particular, $j_1 \neq 0$ for all words $\boldsymbol{j} = (j_1, \dots)$ of L_ρ. In view of Theorem 4.1(v), we have $L_{M,\underline{\mu}} \subseteq \mathrm{Res}_{|M|} L(M, \underline{\mu})$. In particular, there is a word $\boldsymbol{i} = (i_1, \dots)$ of $L(M, \underline{\mu})$ with $i_1 \neq 0$. □

We always consider $R_\alpha^{\Lambda_0}$-modules as R_α-modules via $\mathrm{infl}^{\Lambda_0}$.

Lemma 5.2 *Let $\beta \in \Phi'_+$. The cuspidal module $L_{\delta-\beta}$ factors through $R_{\delta-\beta}^{\Lambda_0}$ and it is the only irreducible $R_{\delta-\beta}^{\Lambda_0}$-module.*

Proof Let $\pi \in \Pi(\delta-\beta)$. In view of Lemma 2.7, it suffices to prove that if $\pi \neq (\delta-\beta)$ then $i_1 \neq 0$ for some word $\boldsymbol{i} = (i_1, \dots)$ of $L(\pi)$. But if $\pi = (M, \underline{\mu})$ is non-trivial, then there is $\rho > \delta$ with $m_\rho \neq 0$. Take the largest such ρ. Then $\rho \in \Phi'_+$, so $j_1 \neq 0$ for all words $\boldsymbol{j} = (j_1, \dots)$ of L_ρ. By Theorem 4.1(v), we have $L_{M,\underline{\mu}} \subseteq \mathrm{Res}_{|M|} L(M, \underline{\mu})$. In particular, there is a word $\boldsymbol{i} = (i_1, \dots)$ of $L(M, \underline{\mu})$ with $i_1 \neq 0$. □

Corollary 5.3 *The minuscule imaginary modules are exactly*

$$\{L_{\delta,i} := \tilde{f}_i L_{\delta-\alpha_i} \mid i \in I'\}.$$

Moreover, $e_j L_{\delta,i} = 0$ for all $j \in I \backslash \{i\}$. Thus, for each $i \in I'$, the minuscule imaginary module $L_{\delta,i}$ can be characterized uniquely up to isomorphism as the irreducible $R_\delta^{\Lambda_0}$-module such that $i_e = i$ for all words $\boldsymbol{i} = (i_1, \dots, i_e)$ of $L_{\delta,i}$.

Proof If L and L' are two minuscule imaginary modules, with $e_i L \neq 0$ and $e_i L' \neq 0$, then by Lemmas 5.1 and 5.2, we have that $\tilde{e}_i L \cong \tilde{e}_i L'$, whence $L \cong L'$ by Proposition 2.6(i). It follows by a counting argument that for each minuscule imaginary module L there exists exactly one i with $e_i L \neq 0$, and then, by Lemma 5.2, we must have $\tilde{e}_i L \cong L_{\delta-\alpha_i}$ and $L \cong \tilde{f}_i L_{\delta-\alpha_i}$. □

For each $i \in I'$, we refer to the minuscule module $L_{\delta,i}$ described in Corollary 5.3 as the minuscule module of *color* i. Let

$$\underline{\mu}(i) := (\emptyset, \dots, \emptyset, (1), \emptyset, \dots, \emptyset) \in \mathscr{P}_1 \qquad (i \in I') \tag{5.1}$$

be the l-multipartition of 1 with the partition (1) in the ith component. We associate to it the minuscule module $L_{\delta,i}$:

$$L(\underline{\mu}(i)) := L_{\delta,i} \qquad (i \in I'). \tag{5.2}$$

Lemma 5.4 *Let $i \in I'$. Then $\varepsilon_i(L_{\delta,i}) = 1$.*

Proof Otherwise $e_i^2(L_{\delta,i}) \neq 0$, whence $\Lambda_0 - \delta + 2\alpha_i$ is a weight of $V(\Lambda_0)$, which is a contradiction. □

Remark 5.5 The minuscule modules are defined over $\mathbb{Z}$. To be more precise, for each $i \in I'$, there exists an $R_\delta(\mathbb{Z})$-module $L_{\delta,i,\mathbb{Z}}$ which is free finite rank over $\mathbb{Z}$ and such that $L_{\delta,i,\mathbb{Z}} \otimes F$ is the minuscule imaginary module $L_{\delta,i,F}$ over $R_\delta(F)$ for any ground field F. To construct $L_{\delta,i,\mathbb{Z}}$, recall that a prime field is a splitting field for R_α. Now, start with the minuscule module $L_{\delta,i,\mathbb{Q}}$ over $\mathbb{Q}$, pick any word vector v and consider the lattice $L_{\delta,i,\mathbb{Q}} := R_\delta(\mathbb{Z})v$. Then $L_{\delta,i,\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q} \cong L_{\delta,i,\mathbb{Q}}$. To see that $L_{\delta,i,\mathbb{Z}} \otimes_{\mathbb{Z}} F$ is the minuscule module $L_{\delta,i,F}$ over any filed F, it suffices to prove that $L_{\delta,i,\mathbb{Z}} \otimes_{\mathbb{Z}} F$ is irreducible. If $L(M, \underline{\mu})$ is a composition factor of $L_{\delta,i,\mathbb{Z}} \otimes_{\mathbb{Z}} F$ with $m_\rho \neq 0$ for some $\rho \in \Phi_+^{\mathrm{re}}$, then we get a contradiction with the definition of an imaginary module. So, taking into account the character information, all composition factors of $L_{\delta,i,\mathbb{Z}} \otimes_{\mathbb{Z}} F$ are of the form $L_{\delta,i,F}$. Now, in fact we must have $L_{\delta,i,\mathbb{Z}} \otimes_{\mathbb{Z}} F \simeq L_{\delta,i,F}$ using the multiplicity one result from Lemma 4.8.

5.2 Imaginary tensor spaces

The *imaginary tensor space of color* i is the $R_{n\delta}$-module

$$M_{n,i} := L_{\delta,i}^{\circ n}.$$

In this definition we allow n to be zero, in which case $M_{0,i}$ is interpreted as the trivial module over the trivial algebra R_0.

Lemma 5.6 $M_n^{\circledast} \simeq M_n$.

Proof This comes from Lemma 2.3 using $(\delta, \delta) = 0$. □

A composition factor of $M_{n,i}$ is called an irreducible imaginary module of *color* i. We remark that by Lemma 4.4 such composition factor is an irreducible imaginary module in the sense of (Cus2). Another application of Lemma 4.4 now gives:

Lemma 5.7 *All composition factors of $M_{n_1,1} \circ \cdots \circ M_{n_l,l}$ are imaginary.*

We next observe that if an irreducible $R_{n\delta}$-module L (with $n > 0$) is imaginary of color $i \in I'$, then L cannot be imaginary of color $j \in I'$, i.e. the color is well defined. Indeed, if L is imaginary of color i, then by (2.18) we have that $\varepsilon_i(L) > 0$ while $\varepsilon_j(L) = 0$ for any $j \neq i$.

Lemma 5.8 *Let $i \in I'$ and $n_1, \dots, n_a \in \mathbb{Z}_{>0}$. Set $n := n_1 + \cdots + n_a$. Then all composition factors of* $\mathrm{Res}_{n_1\delta,\dots,n_a\delta} M_{n,i}$ *are of the form $L^1 \boxtimes \cdots \boxtimes L^a$ where $L^1, \dots, L^a$ are imaginary of color i.*

Proof By the Mackey Theorem, $\mathrm{Res}_{n_1\delta,\dots,n_a\delta} M_{n,i}$ has filtration with factors of the form

$$\mathrm{Ind}^{n_1\delta\,;\,\dots\,;\,n_a\delta}_{\nu_{11},\dots,\nu_{n1}\,;\,\dots\,;\,\nu_{1a},\dots,\nu_{na}} V,$$

where $\sum_{m=1}^{n} \nu_{mb} = n_b\delta$ for all $b = 1, \dots, a$, $\sum_{b=1}^{a} \nu_{mb} = \delta$ for all $m = 1, \dots, n$, and V is obtained by an appropriate twisting of the module

$$(\mathrm{Res}_{\nu_{11},\dots,\nu_{1a}} L_{\delta,i}) \boxtimes \cdots \boxtimes (\mathrm{Res}_{\nu_{n1},\dots,\nu_{na}} L_{\delta,i}).$$

If $\nu_{m1} \neq 0$ and $\nu_{m1} \neq \delta$ for some m, then by Lemma 4.3, we have that ν_{m1} is a sum of real roots less than δ, which leads to a contradiction with $\sum_{m=1}^{n} \nu_{m1} = n_1\delta$. So we deduce that $\nu_{m1} = \delta$ for n_1 different values of m, and $\nu_{m1} = 0$ for all other values of m. Then $L^1 \boxtimes L^2 \boxtimes \cdots \boxtimes L^a$ is a composition factor of

$$M_{n_1,i} \boxtimes \mathrm{Res}_{n_2\delta,\dots,n_a\delta} M_{n-n_1,i},$$

and the lemma follows by induction. □

Corollary 5.9 *Let $i \in I'$ and $n_1, \dots, n_a \in \mathbb{Z}_{\geq 0}$. Set $n := n_1 + \cdots + n_a$. If L is an imaginary irreducible $R_{n\delta}$-module of color i, then all composition factors of $\mathrm{Res}_{n_1\delta,\dots,n_a\delta} L$ are of the form $L^1 \boxtimes \cdots \boxtimes L^a$ where $L^1, \dots, L^a$ are imaginary of color i.*

Proof Follows from Lemma 5.8, since by definition L is a composition factor of $M_{n,i}$. □

5.3 Reduction to one color

The goal of this section is to prove:

Theorem 5.10 *Suppose that for each $n \in \mathbb{Z}_{\geq 0}$ and $i \in I'$, we have an irredundant family $\{L_i(\lambda) \mid \lambda \vdash n\}$ of irreducible imaginary $R_{n\delta}$-modules of color i. For a multipartition $\underline{\lambda} = (\lambda^{(1)}, \dots, \lambda^{(l)}) \in \mathscr{P}_n$, define*

$$L(\underline{\lambda}) := L_1(\lambda^{(1)}) \circ \cdots \circ L_l(\lambda^{(l)}).$$

Then $\{L(\underline{\lambda}) \mid \underline{\lambda} \in \mathscr{P}_n\}$ is a complete and irredundant system of imaginary irreducible $R_{n\delta}$-modules. In particular, the given modules $\{L_i(\lambda) \mid \lambda \vdash n\}$ give all the irreducible imaginary modules of color i up to isomorphism.

We prove the theorem by induction on n. The induction base is clear. Throughout this section we work under the induction hypothesis.

Lemma 5.11 *Let $\underline{\lambda}, \underline{\mu} \in \mathscr{P}_n$ with $\lambda^{(i)} \vdash n_i$ for $i = 1, \dots, l$. If the irreducible $R_{n_1\delta,\dots,n_l\delta}$-module $L_1(\lambda^{(1)}) \boxtimes \cdots \boxtimes L_l(\lambda^{(l)})$ appears as a composition factor in*

$$\mathrm{Res}_{n_1\delta,\dots,n_l\delta} L(\underline{\mu}), \tag{5.3}$$

then $\underline{\lambda} = \underline{\mu}$, and the multiplicity of this composition factor is one.

Proof Let $\mu^{(i)} \vdash m_i$ for $i = 1, \dots, l$. By the Mackey Theorem, the module in (5.3) has filtration with factors of the form

$$\mathrm{Ind}^{n_1\delta\,;\,\dots\,;\,n_l\delta}_{\nu_{11},\dots,\nu_{l1}\,;\,\dots\,;\,\nu_{1l},\dots,\nu_{ll}} V, \tag{5.4}$$

where $\sum_{i=1}^{l} \nu_{ij} = n_j\delta$ for all $j \in I'$, $\sum_{j=1}^{l} \nu_{ij} = m_i\delta$ for all $i \in I'$, and V is obtained by an appropriate twisting of the module

$$(\mathrm{Res}_{\nu_{11},\dots,\nu_{1l}} L_1(\mu^{(1)})) \boxtimes \cdots \boxtimes (\mathrm{Res}_{\nu_{l1},\dots,\nu_{ll}} L_l(\mu^{(l)})). \tag{5.5}$$

Assume that the module in (5.4) is non-zero.

Since each $L_i(\mu^{(i)})$ is imaginary and $\mathrm{Res}_{\nu_{i1},\dots,\nu_{il}} L_i(\mu^{(i)}) \neq 0$, it follows by Lemma 4.3 that either $\nu_{i1} = n_{i1}\delta$ for some $n_{i,1} \in \mathbb{Z}_{\geq 0}$, or ν_{i1} a sum of real roots less than $m_i\delta$. Since $\sum_{i=1}^{l} \nu_{i1} = n_1\delta$, we conclude that the second option is impossible. Next, we claim that also each $\nu_{i2} = n_{i2}\delta$ for some $n_{i2} \in \mathbb{Z}_{\geq 0}$. Indeed, since $\mathrm{Res}_{\nu_{i1},\dots,\nu_{il}} L_i(\mu^{(i)}) \neq 0$, we have that $\mathrm{Res}_{\nu_{i1}+\nu_{i2}, m_i\delta-\nu_{i1}-\nu_{i2}} L_i(\mu^{(i)}) \neq 0$. By Lemma 4.3, either $\nu_{i1} + \nu_{i2}$ is an imaginary root, or it is a sum of real roots less than $m_i\delta$. Since we already know that the $\nu_{i,1}$ are imaginary roots (or zero), the equality $\sum_{i=1}^{l} \nu_{i2} = n_2\delta$ implies that $\nu_{i2} = n_{i2}\delta$ for some $n_{i2} \in \mathbb{Z}_{\geq 0}$. Continuing this way, we establish that all ν_{ij} are of the form $n_{ij}\delta$.

Now, by Corollary 5.9, all composition factors of $\mathrm{Res}_{\nu_{i1},\dots,\nu_{il}} L_i(\mu^{(i)})$ are of the form $L_i(\mu^{(i1)}) \boxtimes \cdots \boxtimes L_i(\mu^{(il)})$. Then the module in (5.3) has filtration with factors of the form

$$\big(L_1(\mu^{(11)}) \circ \cdots \circ L_l(\mu^{(l1)})\big) \boxtimes \cdots \boxtimes \big(L_1(\mu^{(1l)}) \circ \cdots \circ L_l(\mu^{(ll)})\big).$$

By the inductive hypothesis, each $L_1(\mu^{(1j)}) \circ \cdots \circ L_l(\mu^{(lj)})$ is irreducible, and

$$L_1(\mu^{(1j)}) \circ \cdots \circ L_l(\mu^{(lj)}) \cong L_j(\lambda^{(j)})$$

if and only if $\mu^{(jj)} = \lambda^{(j)}$ and $\mu^{(ij)} = \emptyset$ for all $i \neq j$. Thus $\nu_{jj} = n_j\delta$, $\nu_{ij} = 0$ for all $i \neq j$. We conclude that $m_j = n_j$ and $\mu^{(j)} = \lambda^{(j)}$ for all j. □

Corollary 5.12 *The module $L(\underline{\lambda})$ has simple head; denote it by $L^{\underline{\lambda}}$. The multiplicity of $L^{\underline{\lambda}}$ in $L(\underline{\lambda})$ is one.*

Proof If an irreducible module L is in the head of $L(\underline{\lambda})$, then by the adjunction of Ind and Res, we have that $L_1(\lambda^{(1)}) \boxtimes \cdots \boxtimes L_l(\lambda^{(l)}) \subseteq \mathrm{Res}_{n_1\delta,\dots,n_l\delta} L$. Now the result follows from Lemma 5.11 with $\underline{\lambda} = \underline{\mu}$. □

Corollary 5.13 *If $\underline{\lambda} \neq \underline{\mu}$, then $L^{\underline{\lambda}} \not\cong L^{\underline{\mu}}$.*

Proof Assume that $L^{\underline{\lambda}} \cong L^{\underline{\mu}}$. Then $L^{\underline{\mu}}$ is a quotient of $L(\underline{\lambda})$. By the adjunction of Ind and Res, we have that $L_1(\lambda^{(1)}) \boxtimes \cdots \boxtimes L_l(\lambda^{(l)}) \subseteq \mathrm{Res}_{n_1\delta,\dots,n_l\delta} L^{\underline{\mu}}$. In particular, $L_1(\lambda^{(1)}) \boxtimes \cdots \boxtimes L_l(\lambda^{(l)})$ is a composition factor of $\mathrm{Res}_{n_1\delta,\dots,n_l\delta} L(\underline{\mu})$. Now, by Lemma 5.11, we have $\underline{\lambda} = \underline{\mu}$. □

Now we can finish the proof of Theorem 5.10. By counting using Theorem 4.1, Lemma 5.7, and Corollary 5.13, we see that $\{L^{\underline{\lambda}} \mid \underline{\lambda} \in \mathscr{P}_n\}$ is a complete and irredundant set of irreducible imaginary $R_{n\delta}$-modules. It remains to prove that $L(\underline{\mu})$ is irreducible, i.e. $L(\underline{\mu}) = L^{\underline{\mu}}$, for each $\underline{\mu}$. If $L(\underline{\mu})$ is not irreducible, let $L^{\underline{\lambda}} \not\cong L^{\underline{\mu}}$ be an irreducible submodule in the socle of $L(\underline{\mu})$, see Corollary 5.12. Then there is a nonzero homomorphism $L(\underline{\lambda}) \to L(\underline{\mu})$, whence by the adjunction of Ind and Res, we have that $L_1(\lambda^{(1)}) \boxtimes \cdots \boxtimes L_l(\lambda^{(l)}) \subseteq \mathrm{Res}_{n_1\delta,\dots,n_l\delta} L(\underline{\mu})$. Now, by Lemma 5.11, we have $\underline{\lambda} = \underline{\mu}$. Theorem 5.10 is proved.

5.4 Homogeneous modules

In the remainder of Sect. 5 we describe the minuscule imaginary modules more explicitly for symmetric (affine) Cartan matrices. This is done using the theory of homogeneous representations developed in [18], which we review next. Throughout this subsection we assume that the Cartan matrix C is symmetric. As usual, we work with an arbitrary fixed $\alpha \in Q_+$ of height d. A graded R_α-module is called *homogeneous* if it is concentrated in one degree.

Let $\boldsymbol{i} \in \langle I \rangle_\alpha$. We call $s_r \in S_d$ an *admissible transposition* for $\boldsymbol{i}$ if $\mathsf{c}_{i_r,i_{r+1}} = 0$. The *word graph* G_α is the graph with the set of vertices $\langle I \rangle_\alpha$, and with $\boldsymbol{i}, \boldsymbol{j} \in \langle I \rangle_\alpha$ connected by an edge if and only if $\boldsymbol{j} = s_r\boldsymbol{i}$ for some admissible transposition s_r for $\boldsymbol{i}$.

Recall from Sect. 2.1 the Weyl group $W = \langle r_i \mid i \in I \rangle$. Let C be a connected component of G_α, and $\boldsymbol{i} = (i_1, \ldots, i_d) \in C$. We set

$$w_C := r_{i_d} \ldots r_{i_1} \in W.$$

Clearly the element w_C depends only on C and not on $\boldsymbol{i} \in C$. An element $w \in W$ is called *fully commutative* if any reduced expression for w can be obtained from any other by using only the Coxeter relations that involve commuting generators, see e.g. [25]. For an integral weight $\Lambda \in P$, an element $w \in W$ is called Λ-*minuscule* if there is a reduced expression $w = r_{i_l} \ldots r_{i_1}$ such that

$$\langle r_{i_{k-1}} \ldots r_{i_1} \Lambda, \alpha_{i_k}^\vee \rangle = 1 \qquad (1 \leq k \leq l),$$

cf. [26, Section 2]. By [26, Proposition 2.1], if w is Λ-minuscule for some $\Lambda \in P$, then w is fully commutative.

A connected component C of G_α is called *homogeneous* (resp. *strongly homogeneous*) if for some (equivalently every) $\boldsymbol{i} = (i_1, \ldots, i_d) \in C$, we have that $r_{i_d} \ldots r_{i_1}$ is a reduced expression for a fully commutative (resp. *minuscule*) element $w_C \in W$, cf. [18, Sections 3.2, Definition 3.5, Proposition 3.7]. In that case, there is an obvious one-to-one correspondence between the elements $\boldsymbol{i} \in C$ and the reduced expressions of w_C.

Lemma 5.14 [18, Lemma 3.3] *A connected component C of G_α is homogeneous if and only if for every $\boldsymbol{i} = (i_1, \ldots, i_d) \in C$ the following conditions hold:*

$$\begin{aligned} &i_r \neq i_r + 1 \quad \textit{for all} \quad r = 1, 2 \ldots, d-1; \\ &\textit{if } i_r = i_{r+2} \quad \textit{for some} \quad 1 \leq r \leq d-2, \quad \textit{then } \mathtt{c}_{i_r, i_{r+1}} \neq -1. \end{aligned} \tag{5.6}$$

The main theorem on homogeneous representations is:

Theorem 5.15 [18, Theorems 3.6, 3.10, (3.3)]

(i) *Let C be a homogeneous connected component of G_α. Let $L(C)$ be the vector space concentrated in degree 0 with basis $\{v_{\boldsymbol{i}} \mid \boldsymbol{i} \in C\}$ labeled by the elements of C. The formulas*

$$\begin{aligned} 1_{\boldsymbol{j}} v_{\boldsymbol{i}} &= \delta_{\boldsymbol{i},\boldsymbol{j}} v_{\boldsymbol{i}} \qquad (\boldsymbol{j} \in \langle I \rangle_\alpha,\ \boldsymbol{i} \in C), \\ y_r v_{\boldsymbol{i}} &= 0 \qquad (1 \leq r \leq d,\ \boldsymbol{i} \in C), \\ \psi_r v_{\boldsymbol{i}} &= \begin{cases} v_{s_r \boldsymbol{i}} & \textit{if } s_r \boldsymbol{i} \in C, \\ 0 & \textit{otherwise;} \end{cases} \qquad (1 \leq r < d,\ \boldsymbol{i} \in C) \end{aligned}$$

define an action of R_α on $L(C)$, under which $L(C)$ is a homogeneous irreducible R_α-module.

(ii) *$L(C) \not\cong L(C')$ if $C \neq C'$, and every homogeneous irreducible R_α-module, up to a degree shift, is isomorphic to one of the modules $L(C)$.*

(iii) *If $\beta, \gamma \in Q_+$ with $\alpha = \beta + \gamma$, then* $\mathrm{Res}_{\beta,\gamma} L(C)$ *is either zero or irreducible.*

5.5 Minuscule representations for symmetric Cartan matrices

Throughout this subsection we assume that the Cartan matrix $\mathtt{C}$ is symmetric.

Lemma 5.16 *Let $i \in I'$. Then we can write $\Lambda_0 - \delta + \alpha_i = w(i)\Lambda_0$ for a unique Λ_0-minuscule element $w(i) \in W$.*

Proof Let θ be the highest root in the finite root system Φ'. Pick a (unique) minimal length element u of the finite Weyl group W' with $u\theta = \alpha_i$. Now, take $w(i) = ur_0$. Note that

$$\begin{aligned} w(i)(\Lambda_0) &= ur_0(\Lambda_0) = u(\Lambda_0 - \alpha_0) = u(\Lambda_0 - \alpha_0 - \theta + \theta) = u(\Lambda_0 - \delta + \theta) \\ &= \Lambda_0 - \delta + u(\theta) = \Lambda_0 - \delta + \alpha_i. \end{aligned}$$

Since the α-string through β has length 0 or 1 for any distinct roots $\alpha, \beta \in \Phi'$, we deduce that u is θ-minuscule, and the lemma follows. □

By the theory described in Sect. 5.4, the minuscule element $w(i)$ constructed in Lemma 5.16 is of the form $w_{C(i)}$ for some strongly homogeneous component $C(i)$ of $G_{\delta-\alpha_i}$.

Lemma 5.17 *Let $i \in I'$, $d := e - 1 = \mathrm{ht}(\delta - \alpha_i)$ and $\boldsymbol{j} = (j_1, \dots, j_d) \in C(i)$. Then:*

(i) $j_1 = 0$;
(ii) *j_d is connected to i in the Dynkin diagram, i.e. $\mathtt{c}_{j_d,i} < 0$;*
(iii) *if $j_b = i$ for some b, then there are at least three indices b_1, b_2, b_3 such that $b < b_1 < b_2 < b_3 \le d$ such that $\mathtt{c}_{i,b_1} = \mathtt{c}_{i,b_2} = \mathtt{c}_{i,b_3} = -1$.*

Proof (i) is clear from the construction of $w(i)$ which always has r_0 as the last simple reflection in its reduced decomposition.

(ii) Let $w(i) = r_{j_d} \dots r_{j_1}$ be a reduced decomposition. By definition of a minuscule element, we conclude that $\langle \Lambda_0 - \delta + \alpha_i, \alpha_{j_d}^\vee \rangle < 0$, so $\langle \alpha_i, \alpha_{j_d}^\vee \rangle < 0$.

(iii) If $j_b = i$, then, using the definition of a minuscule element and the equality $w(i)\Lambda_0 = r_{j_d} \dots r_{j_1} \Lambda_0 = \Lambda_0 - \delta + \alpha_i$, we see that

$$\langle r_{j_{b+1}} \dots r_{j_d}(\Lambda_0 - \delta + \alpha_i), \alpha_i^\vee \rangle = \langle r_{j_b} r_{j_{b-1}} \dots r_{j_1} \Lambda_0, \alpha_{j_b}^\vee \rangle = -1.$$

This implies (iii), since $\langle \Lambda_0 - \delta + \alpha_i, \alpha_i^\vee \rangle = 2$. □

Corollary 5.18 *Let $i \in I'$. Then the cuspidal module $L_{\delta-\alpha_i}$ is the homogeneous module $L(C(i))$.*

Proof By Lemmas 5.17(i) and 2.7, the module $L(C(i))$ factors through $H_{\delta-\alpha_i}^{\Lambda_0}$. So $L(C(i)) \cong L_{\delta-\alpha_i}$ by Lemma 5.2. □

Proposition 5.19 *Let $i \in I'$. The set of concatenations*

$$C_i := \{\boldsymbol{j}i \mid \boldsymbol{j} \in C(i)\}$$

is a homogeneous component of G_δ, and the corresponding homogeneous R_δ-module $L(C_i)$ is isomorphic to the minuscule imaginary module $L_{\delta,i}$.

Proof By Lemmas 5.14 and 5.17(ii),(iii), we have that C_i is a homogeneous connected component of G_δ. By Lemmas 5.17(i) and 2.7, the corresponding homogeneous representation $L(C_i)$ factors through to $R_\delta^{\Lambda_0}$, and so it must be one of the minuscule representations $L_{\delta,1}, \dots, L_{\delta,l}$, see Corollary 5.3. Finally, by the second statement in Corollary 5.3, we must have $L(C_i) \cong L_{\delta,i}$. □

Example 5.20 Let $\mathtt{C} = \mathtt{A}_l^{(1)}$ and $i \in I'$. Then $L_{\delta,i}$ is the homogeneous irreducible R_δ-module with character

$$\mathrm{ch}_q\, L_{\delta,i} = 0\big((12 \dots i-1) \circ (l, l-1, \dots, i+1)\big)i.$$

For example, $L_{\delta,1}$ and $L_{\delta,l}$ are 1-dimensional with characters

$$\mathrm{ch}_q\, L_{\delta,1} = (0, l, l-1, \dots, 1), \quad \mathrm{ch}_q\, L_{\delta,l} = (01\dots l),$$

while for $l \geq 3$, the module $L_{\delta,l-1}$ is $(l-2)$-dimensional with character

$$\mathrm{ch}_q\, L_{\delta,l-1} = \sum_{r=0}^{l-3} (0, 1, \dots, r, l, r+1, \dots, l-1).$$

6 More on cuspidal modules

In this section we first work again with an arbitrary convex preorder $\preceq$, and then in Sects. 6.2 and 6.3 we assume that the preorder is balanced.

6.1 Minimal pairs

Let $\rho \in \Phi_+^{\mathrm{re}}$. A pair of positive roots (β, γ) is called a *minimal pair* for ρ if

(i) $\beta + \gamma = \rho$ and $\beta \succ \gamma$;
(ii) for any other pair (β', γ') satisfying (i) we have $\beta' \succ \beta$ or $\gamma' \prec \gamma$.

In view of convexity, (β, γ) is a minimal pair for ρ if and only if (β, γ) is a minimal element of $\Pi(\rho)\backslash\{(\rho)\}$. A minimal pair (β, γ) is called *real* if both β and γ are real roots.

Lemma 6.1 *Let $\rho \in \Phi_+^{\mathrm{re}}$ and (β, γ) be a minimal pair for ρ. If L is a composition factor of the standard module $\Delta(\beta, \gamma) = L(\beta) \circ L(\gamma)$, then $L \cong L(\beta, \gamma)$ or $L \cong L_\rho$.*

Proof Use the minimality of (β, γ) in $\Pi(\rho)\backslash\{(\rho)\}$ and Theorem 4.1(iv). □

Let (β, γ) be a *real* minimal pair for $\rho \in \Phi_+^{\mathrm{re}}$. Denote

$$p_{\beta,\gamma} := \max\{n \in \mathbb{Z}_{\geq 0} \mid \beta - n\gamma \in \Phi_+\}.$$

Motivated by [6, Theorem 4.2] we conjecture:

Conjecture 6.2 Let $\rho \in \Phi_+^{\mathrm{re}}$, and (β, γ) be a real minimal pair for ρ. Then In the Grothendieck group we have:

$$[L_\gamma \circ L_\delta] - q^{-(\beta,\gamma)}[L_\beta \circ L_\gamma] = q^{-p_{\beta,\gamma}}(1 - q^{2(p_{\beta,\gamma}-(\beta,\gamma))})[L_\rho].$$

Remark 6.3 Although this goes beyond the scope of this paper, we remark that Conjecture 6.2 can be proved following the steps in the proof of [6, Theorem 4.2]. That proof uses dual root elements (in finite types) constructed using Lusztig's braid group action. Even though 'globally' all of our dual root elements E_ρ^* cannot in general be constructed like that, we have already observed in Remark 4.12(i) that this can be done locally, i.e. for all roots in $\Phi(\rho)$ for a fixed ρ. Note that by definition $\beta, \gamma \in \Phi(\rho)$.

Using Conjecture 6.2 one can compute the character of the cuspidal module L_ρ by induction on $\mathrm{ht}(\rho)$, provided ρ possesses a real minimal pair, cf. Lemma 6.6 below. Moreover, by Lemma 6.1, we can write in the Grothendieck group

$$[L_\beta \circ L_\gamma] = [L(\beta, \gamma)] + m(q)[L_\rho].$$

Now, by Lemma 2.3, we also have

$$[L_\gamma \circ L_\beta] = q^{-(\beta,\gamma)}[L(\beta,\gamma)] + q^{-(\beta,\gamma)} m(q^{-1})[L_\rho].$$

So Conjecture 6.2 implies

$$q^{-(\beta,\gamma)}(m(q^{-1}) - m(q)) = q^{-p_{\beta,\gamma}}(1 - q^{2(p_{\beta,\gamma}-(\beta,\gamma))}),$$

whence

$$m(q) - m(q^{-1}) = q^{p_{\beta,\gamma}-(\beta,\gamma)} - q^{(\beta,\gamma)-p_{\beta,\gamma}}.$$

Now, assume that the Cartan matric $\mathtt{C}$ is symmetric and char $F = 0$. Then by the main result of [28], we have that $m(q) \in q\mathbb{Z}[q]$, and so the last equality implies

$$m(q) = q^{p_{\beta,\gamma}-(\beta,\gamma)}, \tag{6.1}$$

i.e. there is a short exact sequence

$$0 \longrightarrow L_\rho\langle p_{\beta,\gamma} - (\beta,\gamma)\rangle \longrightarrow L_\beta \circ L_\gamma \longrightarrow L(\beta,\gamma) \longrightarrow 0. \tag{6.2}$$

(Note that for symmetric $\mathtt{C}$ we always have $p_{\beta,\gamma} = 0$ and $p_{\beta,\gamma} - (\beta,\gamma) = 1$.)

We conjecture that this also holds in all affine types for all fields (a similar result for all finite types is established in [6, Theorem 4.7]):

Conjecture 6.4 Let $\rho \in \Phi_+^{\mathrm{re}}$, and (β,γ) be a real minimal pair for ρ. Then there is a short exact sequence of the form (6.2).

Example 6.5 Let $n \in \mathbb{Z}_{>0}$ and $i \in I'$. Assume that the preorder is balanced.

(i) If $\rho = n\delta + \alpha_i$, then $(\alpha_i + (n-1)\delta, \delta)$ is a minimal pair for ρ.
(ii) If $n > 1$ and $\rho = n\delta - \alpha_i$, then $(\delta, (n-1)\delta - \alpha_i)$ is a minimal pair for ρ.

Lemma 6.6 *Assume that the preorder is balanced. Let ρ be a non-simple positive root. Then there exists a real minimal pair for ρ, unless ρ is of the form $n\delta \pm \alpha_i$.*

Proof If $\rho \in \Phi_{\succ}^{\mathrm{re}}$ is not of the form $n\delta + \alpha_i$, then we can always write ρ as a sum of two roots in $\Phi_{\succ}^{\mathrm{re}}$, and so there exists a real minimal pair for ρ.

If $\rho \in \Phi_{\prec}^{\mathrm{re}}$ is not of the form $n\delta - \alpha_i$ and $n \geq 2$, then we can write ρ as a sum of two roots in $\Phi_{\prec}^{\mathrm{re}}$, and so again there exists a real minimal pair for ρ. Finally, in the special case where ρ is a non-simple root of the form $\delta - \alpha$ for $\alpha \in \Phi'_+$, by an argument of [23, Lemma 2.1] we can write ρ as a sum of two real roots, which implies the result. □

In view of the lemma, the cuspidal modules corresponding to the roots of the form $n\delta \pm \alpha_i$ play a special role. In Sects. 6.2 and 6.3 we will investigate them in detail.

6.2 Cuspidal modules $L_{n\delta+\alpha_i}$

We continue to assume (until the end of the paper) that the convex preorder $\preceq$ is balanced. Fix $i \in I'$. In this section we consider the cuspidal modules corresponding to the real roots of the form $n\delta + \alpha_i$ for $i \in I'$. Fix also an extremal word

$$\boldsymbol{i} = i_1^{a_1} \dots i_k^{a_k} \tag{6.3}$$

of the minuscule imaginary module $L_{\delta,i}$, see Sect. 2.8. Recall from Corollary 5.3 and Lemma 5.4 that $i_k = i$ and $a_k = 1$. We will use the concatenations $\boldsymbol{i}^n \in \langle I\rangle_{n\delta}, \boldsymbol{i}^n i \in \langle I\rangle_{n\delta+\alpha_i}$ and also the special word

$$\boldsymbol{i}^{\{n\}} := i_1^{na_1} \dots i_{k-1}^{na_{k-1}} i^{n+1} \in \langle I\rangle_{n\delta+\alpha_i}.$$

Proposition 6.7 *Let $i \in I'$, $n \in \mathbb{Z}_{>0}$, $\alpha = n\delta + \alpha_i$, and $\beta = (n-1)\delta + \alpha_i$. Then:*

(i) *The standard module $\Delta(\beta, \delta^{(i)}) = L_\beta \circ L_{\delta,i}$ has composition series of length two with head $L(\beta, \delta^{(i)})$ and socle $L_\alpha\langle(\alpha_i, \alpha_i)/2\rangle$.*

(ii) *We have*

$$\mathrm{ch}_q\, L_\alpha = \frac{1}{q_i - q_i^{-1}}\big((\mathrm{ch}_q\, L_\beta) \circ (\mathrm{ch}_q\, L_{\delta,i}) - (\mathrm{ch}_q\, L_{\delta,i}) \circ (\mathrm{ch}_q\, L_\beta)\big).$$

(iii) *We have*

$$\mathrm{ch}_q\, L_\alpha = \frac{1}{q_i - q_i^{-1}} \sum_{m=0}^{n} (-1)^m (\mathrm{ch}_q\, L_{\delta,i})^{\circ m} \circ i \circ (\mathrm{ch}_q\, L_{\delta,i})^{\circ(n-m)}.$$

(iv) *The word $\boldsymbol{i}^{\{n\}}$ is an extremal word of L_α.*

Proof We apply induction on n. Consider the induced modules $W_1 := L_\beta \circ L_{\delta,i}$ and $W_2 := L_{\delta,i} \circ L_\beta$. When evaluated at $q = 1$, the formal characters of these two modules are the same. It follows from the linear independence of ungraded formal characters of irreducible R_α-modules that W_1 and W_2 have the same composition factors, but possibly with different degree shifts. We also know that the graded multiplicity of $L(\beta, \delta^{(i)})$ in $W_1 = \Delta(\beta, \delta^{(i)})$ is 1. By Lemma 2.3, we have that $W_1^{\circledast} \simeq W_2$, so the graded multiplicity of $L(\beta, \delta^{(i)})$ in W_2 is also 1. In view of Lemma 6.1 and Example 6.5(i), in the Grothendieck group $[R_\alpha\text{-mod}]$ we now have

$$[W_i] = [L(\beta, \delta^{(i)})] + c_i[L_\rho] \qquad (i = 1, 2)$$

for some graded multiplicities $c_i \in \mathscr{A}$ such that $\overline{}c_1 = c_2$.

To compute c_1 and c_2, we look at the multiplicity of the word $\boldsymbol{i}^{\{n\}}$ in W_1. By induction, $\boldsymbol{i}^{\{n-1\}}$ is extremal in L_β. Let N be a $\circledast$-selfdual irreducible R_α-module such that

$$N \cong \tilde{f}_i^{n+1} \tilde{f}^{na_{k-1}} \dots \tilde{f}_{i_1}^{na_1} 1_F.$$

By Proposition 2.11, $\boldsymbol{i}^{\{n\}}$ is an extremal word for W_1. An elementary computation using Proposition 2.11 also shows that N appears in W_1 with graded multiplicity q_i. So we must have $N \simeq L_\alpha$, and $c_1 = q_i$. We have proved (i) and (iv). Part (ii) easily follows from (i), and (ii) implies (iii) by induction on n. □

6.3 Cuspidal modules $L_{n\delta-\alpha_i}$

Fix $i \in I'$. In this section we consider the cuspidal modules corresponding to the real roots of the form $n\delta - \alpha_i$ for $i \in I'$. Recall that we have $i_k = i$ and $a_k = 1$ for the extremal word $\boldsymbol{i}$ of $L_{\delta,i}$ picked in (6.3). So in view of Corollary 5.3 and Lemma 5.4, the word

$$\boldsymbol{j} = i_1^{a_1} \dots i_{k-1}^{a_{k-1}}$$

is an extremal word of $L_{\delta-\alpha_i}$. We will use the notation

$$\boldsymbol{i}^{[n]} := i_1^n \dots i_{e-1}^n i_e^{n-1} \in \langle I \rangle_{n\delta-\alpha_i}.$$

Proposition 6.8 *Let $i \in I'$, $n \in \mathbb{Z}_{>1}$, and $\alpha = n\delta - \alpha_i$, $\beta = (n-1)\delta - \alpha_i$. Then:*

(i) *The standard module $\Delta(\delta^{(i)}, \beta) = L_{\delta,i} \circ L_\beta$ has composition series of length two with head $L(\delta^{(i)}, \beta)$ and socle $L_\alpha\langle(\alpha_i, \alpha_i)/2\rangle$.*

(ii) *We have*

$$\mathrm{ch}_q\, L_\alpha = \frac{1}{q_i - q_i^{-1}}\big((\mathrm{ch}_q\, L_{\delta,i}) \circ (\mathrm{ch}_q\, L_\beta) - (\mathrm{ch}_q\, L_\beta) \circ (\mathrm{ch}_q\, L_{\delta,i})\big).$$

(iii) *We have*

$$\mathrm{ch}_q\, L_\alpha = \frac{1}{q_i - q_i^{-1}} \sum_{m=0}^{n} (-1)^{n-m} (\mathrm{ch}_q\, L_{\delta,i})^{\circ m} \circ (\mathrm{ch}_q\, L_{\delta-\alpha_i}) \circ (\mathrm{ch}_q\, L_{\delta,i})^{\circ(n-m)}.$$

(iv) *The word* $\boldsymbol{i}^{[n]}$ *is an extremal word of* L_α.

Proof The proof is similar to that of Proposition 6.7. □

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