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Scheme of lines on a family of 2-dimensional quadrics: geometry and derived category

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Abstract Given a generic family Q of 2-dimensional quadrics over a smooth 3-dimensional base Y we consider the relative Fano scheme M of lines of it. The scheme M has a structure of a generically conic bundle $M \to X$ over a double covering $X \to Y$ ramified in the degeneration locus of $Q \to Y$. The double covering $X \to Y$ is singular in a finite number of points (corresponding to the points $y \in Y$ such that the quadric Q_y degenerates to a union of two planes), the fibers of M over such points are unions of two planes intersecting in a point. The main result of the paper is a construction of a semiorthogonal decomposition for the derived category of coherent sheaves on M. This decomposition has three components, the first is the derived category of a small resolution X^+ of singularities of the double covering $X \to Y$, the second is a twisted resolution of singularities of X (given by the sheaf of even parts of Clifford algebras on Y), and the third is generated by a completely orthogonal exceptional collection.

1 Introduction

The subject of this note is a description of the structure of the derived category of coherent sheaves on the relative scheme of lines for a family of 2-dimensional quadrics. We had two motivations for the investigation of this category. First of all it has an interesting structure and exhibits some interesting features. For example, it combines the minimal resolution of

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singularities and the twisted resolution of singularities of the double covering of the base naturally associated with the family.

The second, and the most important motivation, comes from investigation of the derived categories of so-called nodal Enrique surfaces. These surfaces can be associated with families of 2-dimensional quadrics parameterized by $\mathbb{P}^3$ (webs of quadrics). The relative scheme of lines in this case can be identified with the blowup of the Grassmannian $\mathsf{Gr}(2, 4)$ in the corresponding Enriques surface. Consequently, its derived category contains the derived category of the Enriques surface as a semiorthogonal component. So, another description of the derived category of the relative scheme of lines, provided by the present paper, gives a link between the derived categories of nodal Enriques surfaces and of the associated double coverings of $\mathbb{P}^3$, which in this case are nothing but the Artin–Mumford quartic double solids, appeared in the famous paper [2] as examples of unirational but nonrational threefolds. See the companion paper [6] for more details.

The precise formulation of the main result of the paper is the following. Consider a family of 2-dimensional quadrics $q : Q \to Y$. This means that we are given a projectivization of a rank 4 vector bundle $\mathscr{V}$ on Y and a divisor $Q \subset \mathbb{P}_Y(\mathscr{V})$ of relative degree 2 which is flat over Y. Such divisor is given by a line subbundle $\mathcal{L} \subset S^2\mathscr{V}^\vee$.

Given this we consider *the relative Fano scheme of lines of Q over Y*. By definition this is the zero locus on the relative Grassmannian $\mathsf{Gr}_Y(2, \mathscr{V})$ of the global section

$$s \in \Gamma(\mathsf{Gr}_Y(2, \mathscr{V}), \mathcal{L}^\vee \otimes S^2\mathscr{U}^\vee),$$

where $\mathscr{U} \subset \mathscr{V}$ is the tautological subbundle on the Grassmannian. We denote this relative Fano scheme by M. The fibers M_y of the projection $\rho : M \to Y$ are the Fano schemes of lines on quadrics Q_y, and so they have the following structure

- M_y is a disjoint union of two smooth conics, if the quadric Q_y is smooth;
- M_y is a single smooth conic (with a nonreduced scheme structure), if the quadric Q_y has corank 1;
- M_y is a union of two planes intersecting in a point, if the quadric Q_y has corank 2;
- M_y is a single plane (with a nonreduced structure), if the quadric Q_y has corank 3.

The main result of this paper is a description of the derived category of M when $\dim Y = 3$ under the following genericity assumptions: we assume that

- the generic fiber of Q over Y is smooth, and
- the codimension of the locus $D_r \subset Y$ of quadrics of corank r equals $r(r+1)/2$.

As $\dim Y = 3$, the second assumtion implies that $D_3 = \emptyset$ and D_2 consists of a finite number N of isolated points $y_1, \dots, y_N$. Additionally we assume that

- D_1 has an ordinary double point (an ODP or a node for short) in each of y_i.

The last assumption is equivalent to smoothness of M if Q is smooth (see Lemma 2.4 below).

To state the answer we need the following ingredients. First, consider the Stein factorization for the morphism $\rho : M \to Y$:

$$\begin{array}{ccc} M & \xrightarrow{\ \mu\ } & X \\ & {}_{\rho}\searrow \quad \swarrow_{f} & \\ & Y & \end{array}$$

where $f : X \to Y$ is the double covering ramified in the divisor D_1, and $\mu : M \to X$ is generically a conic bundle. Note that X is smooth away from N points $x_i = f^{-1}(y_i)$ which are isolated ordinary double points. So being 3-dimensional it has 2^N small resolutions of singularities in the category of Moishezon varieties. To fix one of these resolutions we should choose for each point $y_i \in D_2$ one of the planes in the corank 2 quadric Q_{y_i}, or equivalently one of the planes in the fiber M_{y_i} of M over Y. Let us pick one of these resolutions and denote it by $\sigma_+ : X^+ \to X$. Let us denote the planes in M_{y_i} corresponding to this choice by Σ_i^+, and the complementary planes by Σ_i^-, so that $M_{y_i} = \Sigma_i^+ \cup \Sigma_i^-$.

Second, consider the sheaf of even parts of Clifford algebras $\mathcal{B}_0$ on Y associated with the family of quadrics $Q \to Y$ (see Sect. 3 or [10] for details). If Q is smooth then the category $\mathcal{D}^b(Y, \mathcal{B}_0)$ is also smooth and can be thought of as a twisted noncommutative resolution of the double covering X.

The main result of this paper is the following

Theorem 1.1 *Assume that $Q \to Y$ is a family of quadrics, Y and Q are smooth, $\dim Y = 3$, and the degeneration locus D_1 has a finite number of ordinary double points $\{y_1, \dots, y_N\} = D_2$. Then the relative Fano scheme M of lines of Q over Y is smooth and there is a semiorthogonal decomposition*

$$\mathcal{D}^b(M) = \langle \mathcal{D}^b(X^+), \mathcal{D}^b(Y, \mathcal{B}_0), \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^N \rangle.$$

Here the third component is a completely orthogonal exceptional collection.

It is natural to ask, whether one can remove some of the assumptions and prove a similar result. It seems that indeed, there are some ways of a generalization of this decomposition. For example, if the dimension of Y is higher than 3 (but still Y and Q are smooth, D_1 has ordinary double points (in the transversal slice) along D_2, and $D_3 = \emptyset$), then probably one can construct a semiorthogonal decomposition with the first component being a categorical crepant resolution of X, the same second component, and the third component equal to the derived category of the stratum D_2. See Sect. 6 for a discussion of further perspectives of the question.

The proof of the Theorem goes as follows. In Sect. 2 we prove smoothness of M and investigate the local structure of M around the planes $\Sigma_i^\pm$. In particular, we check that the sheaves $\mathcal{O}_{\Sigma_i^+}$ form a completely orthogonal exceptional collection in $\mathcal{D}^b(M)$. In Sect. 3 we recall some facts about the sheaf of even parts of Clifford algebras $\mathcal{B}_0$ and construct a fully faithful embedding $\mathcal{D}^b(Y, \mathcal{B}_0) \to \mathcal{D}^b(M)$. In Sect. 4 we show that there is a birational transformation of M, known as a flip in N planes Σ_i^+, transforming it into a $\mathbb{P}^1$-fibration $\mu_+ : M^+ \to X^+$ over a small resolution $X^+ \to X$. This gives an identification of the orthogonal to the collection $\{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^N$ in $\mathcal{D}^b(M)$ with $\mathcal{D}^b(M^+)$. In Sect. 5 we construct a fully faithful embedding $\mathcal{D}^b(X^+) \to \mathcal{D}^b(M^+)$ and identify the complement with $\mathcal{D}^b(Y, \mathcal{B}_0)$. In the last Sect. 6 we discuss another way of proving Theorem 1.1 and suggest some further directions of investigation.

2 Geometry of *M*

For each quadric Q_y in the family $Q \to Y$ denote by $K_y \subset \mathcal{V}_y$ the kernel of the corresponding quadratic form (thus $\mathbb{P}(K_y)$ is the singular locus of the quadric Q_y). Note that the differential of the section $s \in \Gamma(Y, \mathcal{L}^{-1} \otimes S^2\mathcal{V}^\vee)$ at y gives a linear map $T_yY \to S^2\mathcal{V}_y^\vee$ (depending on a choice of trivializations on the bundles $\mathcal{V}$ and $\mathcal{L}$ near y). Composing it with the natural projection $S^2\mathcal{V}_y^\vee \to S^2K_y^\vee$ we obtain a map

$$\kappa_y : T_y Y \to S^2 K_y^\vee$$

which does not depend on the choices of the trivializations.

In terms of these maps one can check the smoothness of Q and M.

Proposition 2.1 *Assume that Y is smooth. Then*

1. *Q is smooth if and only if for any $y \in Y$ and any subspace $K \subset K_y$ with* $\dim K \le 1$, *the composition $T_y Y \longrightarrow S^2 K_y^\vee \longrightarrow S^2 K^\vee$ is surjective;*
2. *M is smooth if and only if for any $y \in Y$ and any embedding $K \to K_y$ with* $\dim K \le 2$, *the composition $T_y Y \longrightarrow S^2 K_y^\vee \longrightarrow S^2 K^\vee$ is surjective.*

Proof The question is local in Y so we can assume that $\mathscr{V}$ is a trivial bundle, $\mathscr{V} \cong V \otimes \mathcal{O}_Y$, and so $\mathbb{P}_Y(\mathscr{V}) = Y \times \mathbb{P}(V)$. Let (y, K) be an arbitrary point of Q, where $K \subset V$ is a 1-dimensional subspace of V. Then the smoothness of Q at the point (y, K) is equivalent to the surjectivity of the differential $ds : T_y Y \oplus \mathsf{Hom}(K, V/K) \to S^2 K^\vee$. If $K \not\subset K_y$ then the second summand maps surjectively. If, on a contrary, $K \subset K_y$ then the second summand maps trivially, while the map of the first summand is the map in the statement of the Lemma. This proves the first part.

For the second part, let (y, K) be an arbitrary point of M, where $K \subset V$ is a 2-dimensional subspace of V. Then the smoothness of M at the point (y, K) is equivalent to the surjectivity of the differential $ds : T_y Y \oplus \mathsf{Hom}(K, V/K) \to S^2 K^\vee$. If $K \cap K_y = 0$ then the second summand maps surjectively. Further, if the space $K_1 := K \cap K_y$ is 1-dimensional, then the cokernel of the map of the second summand is equal to $S^2 K_1^\vee$ and the map of the first summand is the map in the statement of the Lemma for K_1. Finally, if $K \subset K_y$, then the second summand maps trivially, while the map of the first summand is the map in the statement of the Lemma. This proves the second part. □

Remark 2.2 This result generalizes to arbitrary relative isotropic Grassmannians of families of quadrics of arbitrary dimension. The smoothness of the Grassmannian of k-dimensional subspaces is equivalent to the surjectivity of the corresponding map for all K with $\dim K \le k$.

Corollary 2.3 *Assume that Y is smooth and $D_3 = \emptyset$. Then M is smooth if and only if Q is smooth and for any $y \in D_2$ the map $\kappa_y : T_y Y \to S^2 K_y^\vee$ is surjective.*

Another consequence of the smoothness of Q is the smoothness of $D_1 \setminus D_2$. On the other hand, the points of D_2 are always singular on D_1. In fact they are ordinary double points if M is smooth.

Lemma 2.4 *Assume that Q is smooth and* $\dim Y = 3$. *Then M is smooth if and only if D_2 is a finite number of points and any point of D_2 is an ordinary double point of D_1.*

Proof Take any $y \in D_2$, so that $\dim K_y = 2$. The map $T_y Y \to S^2 K_y^\vee$ can be thought of as a net of quadrics on K_y parameterized by $T_y Y$. Its degeneration locus in $\mathbb{P}(T_y Y)$ is a conic, either nondegenerate (if the map $T_y Y \to S^2 K_y^\vee$ is surjective) or singular (since the kernel of the map lies in the singular locus). But on the other hand, this degeneration locus is the base of the tangent cone to D_1 at y_i. So, if y_i is an ODP of D_1, the conic should be nondegenerate, hence the map should be surjective. □

Remark 2.5 Note also that if the map κ_y for $y \in D_2$ is surjective, then it is an isomorphism (since $\dim T_y Y = \dim S^2 K_y^\vee = 3$).

For each point $y_i \in D_2$ we have $Q_{y_i} = \mathbb{P}(W_i^+) \cup \mathbb{P}(W_i^-)$, both W_i^+ and W_i^- being 3-dimensional subspaces in $\mathscr{V}_{y_i}$, the fiber of $\mathscr{V}$ over y_i. The planes $\mathbb{P}(W_i^+)$ and $\mathbb{P}(W_i^-)$

intersect along the line $\mathbb{P}(K_{y_i})$. In these terms the fiber M_{y_i} of M over the point y_i can be written as

$$M_{y_i} = \mathsf{Gr}(2, W_i^+) \cup \mathsf{Gr}(2, W_i^-) = \Sigma_i^+ \cup \Sigma_i^-,$$

where both Σ_i^+ and Σ_i^- are isomorphic to projective planes, intersecting transversally in the point $P_i = \mathsf{Gr}(2, K_{y_i})$.

Choose an arbitrary point $y = y_i \in D_2$ and one of the planes $\Sigma = \Sigma_i^\pm$. The following Proposition computes its normal bundle in M.

Proposition 2.6 *If a point $y \in D_2$ is an ODP of D_1 then $\mathcal{N}_{\Sigma/M} \cong \mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(-1)$.*

Proof Choosing a local trivialization of the bundles $\mathcal{V}$ and $\mathcal{L}$ we obtain an isomorphism

$$\mathcal{N}_{\Sigma/\mathsf{Gr}_Y(2,\mathcal{V})} \cong \mathcal{N}_{\Sigma/\mathsf{Gr}(2,\mathcal{V}_y)} \oplus T_yY \otimes \mathcal{O}_\Sigma \cong \mathcal{U}^\vee_{|\Sigma} \oplus T_yY \otimes \mathcal{O}_\Sigma.$$

On the other hand, $\mathcal{N}_{M/\mathsf{Gr}_Y(2,\mathcal{V})} \cong S^2\mathcal{U}^\vee$. Hence the standard exact sequence

$$0 \to \mathcal{N}_{\Sigma/M} \to \mathcal{N}_{\Sigma/\mathsf{Gr}_Y(2,\mathcal{V})} \to (\mathcal{N}_{M/\mathsf{Gr}_Y(2,\mathcal{V})})_{|\Sigma} \to 0$$

gives

$$0 \to \mathcal{N}_{\Sigma/M} \to \mathcal{U}^\vee_{|\Sigma} \oplus T_yY \otimes \mathcal{O}_\Sigma \to S^2\mathcal{U}^\vee_{|\Sigma} \to 0.$$

Since $\Sigma = \mathsf{Gr}(2, W)$, $W \subset \mathcal{V}_y$, the cohomology exact sequence looks like

$$0 \to H^0(\Sigma, \mathcal{N}_{\Sigma/M}) \to W^* \oplus T_yY \to S^2W^* \to H^1(\Sigma, \mathcal{N}_{\Sigma/M}) \to 0.$$

Consider the map $W^* \oplus T_yY \to S^2W^*$. Its first component is the multiplication by the equation of the line $K_y \subset W$. Hence the sequence can be rewritten as

$$0 \to H^0(\Sigma, \mathcal{N}_{\Sigma/M}) \to T_yY \xrightarrow{\kappa_y} S^2K_y^\vee \to H^1(\Sigma, \mathcal{N}_{\Sigma/M}) \to 0.$$

The middle map here is just the map κ_y, hence by Remark 2.5 it is an isomorphism. Thus the bundle $\mathcal{N}_{\Sigma/M}$ is acyclic. Moreover, it is easy to see that the bundle $\mathcal{N}_{\Sigma/M}(-1)$ is acyclic as well. It follows that $\mathcal{N}_{\Sigma/M}$ is isomorphic to $\mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(-1)$. □

From now on we assume that every point $y_i \in D_2$ is an ODP of D_1.

Corollary 2.7 *We have $(\omega_M)_{|\Sigma_i^\pm} \cong \mathcal{O}_{\Sigma_i^\pm}(-1)$.*

Proof By adjunction formula $\mathcal{O}_\Sigma(-3) \cong \omega_\Sigma \cong \omega_{M|\Sigma} \otimes \det \mathcal{N}_{\Sigma/M} \cong \omega_{M|\Sigma} \otimes \mathcal{O}_\Sigma(-2)$, hence the claim. □

The most important corollary is the following

Corollary 2.8 *The structure sheaf $\mathcal{O}_\Sigma \in \mathcal{D}^b(M)$ is exceptional.*

Proof We have an isomorphism $\mathcal{E}xt^t(\mathcal{O}_\Sigma, \mathcal{O}_\Sigma) \cong \Lambda^t\mathcal{N}_{\Sigma/M} \cong \Lambda^t(\mathcal{O}_\Sigma(-1) \oplus \mathcal{O}_\Sigma(-1))$. Note that for $t = 1$, and $t = 2$ this sheaf on $\Sigma = \mathbb{P}^2$ is acyclic. Hence $\mathsf{Ext}^\bullet(\mathcal{O}_\Sigma, \mathcal{O}_\Sigma) \cong H^\bullet(\Sigma, \mathcal{H}om(\mathcal{O}_\Sigma, \mathcal{O}_\Sigma)) \cong H^\bullet(\Sigma, \mathcal{O}_\Sigma)$ implies exceptionality of $\mathcal{O}_\Sigma$. □

Another simple observation is that $\Sigma_i^\pm$ with different i are completely orthogonal.

Lemma 2.9 *If $i \neq j$ then $\mathsf{Ext}^\bullet(\mathcal{O}_{\Sigma_i^\pm}, \mathcal{O}_{\Sigma_j^\pm}) = 0$.*

Proof The planes $\Sigma_i^\pm$ and $\Sigma_j^\pm$ are contained in fibers of M over different points $y_i, y_j \in Y$, hence there is no local Ext's between their structure sheaves. Hence global Ext's also vanish. □

Thus choosing one plane for each y_i we obtain a completely orthogonal exceptional collection.

Corollary 2.10 *The collection* $\{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^N$ *is a completely orthogonal exceptional collection in* $\mathcal{D}^b(M)$.

3 The Clifford algebra

For the precise definition and basic results about the sheaves of even parts of Clifford algebras, see [10]. Here we remind some of their properties. Recall that the Clifford multiplication is the composition

$$\Lambda^p\mathcal{V}\otimes\Lambda^q\mathcal{V}\to\bigoplus_{i=0}^{\min(p,q)}\Lambda^{p-i}\mathcal{V}\otimes\underbrace{\mathcal{V}\otimes\cdots\otimes\mathcal{V}}_{i}\otimes\underbrace{\mathcal{V}\otimes\cdots\otimes\mathcal{V}}_{i}\otimes\Lambda^{q-i}\mathcal{V}$$
$$\xrightarrow{s^{\otimes i}}\bigoplus_{i=0}^{\min(p,q)}\Lambda^{p-i}\mathcal{V}\otimes\mathcal{L}^{-i}\otimes\Lambda^{q-i}\mathcal{V}\to\bigoplus_{i=0}^{\min(p,q)}\Lambda^{p+q-2i}\mathcal{V}\otimes\mathcal{L}^{-i},$$

where the first map is the partial polarization; the second map is the iterated application of the map $s:\mathcal{V}\otimes\mathcal{V}\to\mathcal{L}^{-1}$, first to the two copies of $\mathcal{V}$ in the middle, then to the next two, and so on; and the third map is the wedge product. The Clifford multiplication provides the sheaf

$$\mathcal{B}_0=\mathcal{O}_Y\oplus\Lambda^2\mathcal{V}\otimes\mathcal{L}\oplus\Lambda^4\mathcal{V}\otimes\mathcal{L}^2$$

with a structure of a sheaf of $\mathcal{O}_Y$-algebras, called the even part of the Clifford algebra, and the sheaf

$$\mathcal{B}_1=\mathcal{V}\oplus\Lambda^3\mathcal{V}\otimes\mathcal{L}$$

(the odd part of the Clifford algebra) with a structure of a $\mathcal{B}_0$-bimodule. It is convenient to extend this pair of sheaves to a sequence defined by

$$\mathcal{B}_{2k}=\mathcal{B}_0\otimes\mathcal{L}^{-k},\qquad\mathcal{B}_{2k+1}=\mathcal{B}_1\otimes\mathcal{L}^{-k}.$$

This sequence can be thought of as a sequence of powers of a line bundle. In particular, by [10] the functors $-\otimes_{\mathcal{B}_0}\mathcal{B}_l$ and $\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_l,-)$ are exact and we have

$$\mathcal{B}_k\otimes_{\mathcal{B}_0}\mathcal{B}_l\cong\mathcal{B}_{k+l},\qquad\mathsf{R}\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_k,\mathcal{B}_l)\cong\mathcal{B}_{l-k}.\tag{1}$$

Remark 3.1 It follows that for any coherent sheaves $\mathcal{F}$ and $\mathcal{F}'$ on Y there is an isomorphism between the space of maps of $\mathcal{B}_0$-modules $\mathcal{F}\otimes\mathcal{B}_k\to\mathcal{F}'\otimes\mathcal{B}_l$ and the space of maps of $\mathcal{O}$-modules $\mathcal{F}\to\mathcal{F}'\otimes\mathcal{B}_{l-k}$. In fact the corresponding map of $\mathcal{B}_0$-modules is given by the composition $\mathcal{F}\otimes\mathcal{B}_k\to\mathcal{F}'\otimes\mathcal{B}_{l-k}\otimes\mathcal{B}_k\to\mathcal{F}'\otimes\mathcal{B}_l$, where the second map is the Clifford multiplication. We will use frequently this observation to define and compare such maps.

Let α denote the embedding $M\to\mathsf{Gr}_Y(2,\mathcal{V})$. Let

$$g:=c_1(\mathcal{U}^\vee)\in\mathsf{Pic}(\mathsf{Gr}_Y(2,\mathcal{V})/Y)$$

be the positive generator of the relative Picard group. Since the scheme M is the zero locus of the section $s \in \Gamma(\mathsf{Gr}_Y(2, V), \mathcal{L}^\vee \otimes S^2\mathcal{U}^\vee)$ we have the Koszul resolution for its structure sheaf

$$0 \to \mathcal{L}^3(-3g) \to \mathcal{L}^2 \otimes S^2\mathcal{U}(-g) \to \mathcal{L} \otimes S^2\mathcal{U} \to \mathcal{O}_{\mathsf{Gr}_Y(2,\mathcal{V})} \to \alpha_*\mathcal{O}_M \to 0. \quad (2)$$

Now we will show that M also comes with a sequence of naturally defined $\mathcal{B}_0$-modules. To unburden the notation we denote the pullbacks of the sheaves $\mathcal{B}_k$ to $\mathsf{Gr}_Y(2, \mathcal{V})$ by the same letters. For each $k \in \mathbb{Z}$ consider the morphism $\mathcal{U} \otimes \mathcal{B}_{k-1} \to \mathcal{B}_k$ of sheaves of $\mathcal{B}_0$-modules on $\mathsf{Gr}_Y(2, \mathcal{V})$ induced by the embedding $\mathcal{U} \subset \mathcal{V} \subset \mathcal{B}_1$ (see Remark 3.1).

Proposition 3.2 *There are isomorphisms* $\mathsf{Coker}(\mathcal{U}\otimes\mathcal{B}_{k-1} \to \mathcal{B}_k) \cong \alpha_*\mathcal{S}_k$, *where*

$$\mathcal{S}_{2k+1} = (\mathcal{V}/\mathcal{U}) \otimes \mathcal{L}^{-k}, \quad (3)$$

and there is an exact sequence

$$0 \to \mathcal{L}^{-k} \to \mathcal{S}_{2k} \to \det \mathcal{V} \otimes \mathcal{L}^{1-k}(g) \to 0. \quad (4)$$

Moreover, the sheaves $\mathcal{S}_k$ *have a structure of right* $\mathcal{B}_0$*-modules such that*

$$\mathcal{S}_k\otimes_{\mathcal{B}_0}\mathcal{B}_l \cong \mathcal{S}_{k+l}, \qquad \mathsf{R}\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_l, \mathcal{S}_k) \cong \mathcal{S}_{k-l}. \quad (5)$$

Finally, for each k *there is an exact sequence of right* $\mathcal{B}_0$*-modules on* $\mathsf{Gr}_Y(2, \mathcal{V})$

$$0 \to \mathcal{O}(-2g) \otimes \mathcal{B}_{k-4} \to \mathcal{U}(-g) \otimes \mathcal{B}_{k-3} \to \mathcal{U} \otimes \mathcal{B}_{k-1} \to \mathcal{B}_k \to \alpha_*\mathcal{S}_k \to 0. \quad (6)$$

Proof First let us check that the cokernels are supported on M scheme-theoretically. For this we note that the composition of the maps $S^2\mathcal{U} \otimes \mathcal{B}_{k-2} \to \mathcal{U} \otimes \mathcal{B}_{k-1} \to \mathcal{B}_k$ (both of which are induced by the Clifford multiplication) coincides with the map $S^2\mathcal{U} \otimes \mathcal{B}_{k-2} \cong S^2\mathcal{U} \otimes \mathcal{B}_k \otimes \mathcal{L} \to \mathcal{B}_k$ induced by the section s defining the family Q. Indeed, by Remark 3.1 it is enough to compare the corresponding maps $S^2\mathcal{U} \to \mathcal{B}_2$. The first map is the composition

$$S^2\mathcal{U} \to \mathcal{U} \otimes \mathcal{U} \to \mathcal{V} \otimes \mathcal{V} \to \Lambda^2\mathcal{V} \oplus \mathcal{L}^{-1}$$

of the natural embeddings and of the Clifford multiplication. The first component of the composition is zero since the wedge product vanishes on symmetric tensors, and the second component coincides with the second map.

Thus, the cokernel of the map $\mathcal{U} \otimes \mathcal{B}_{k-1} \to \mathcal{B}_k$ is a quotient of $\alpha_*\alpha^*\mathcal{B}_k$, hence it can be written as $\alpha_*\mathcal{S}_k$, where $\mathcal{S}_k$ is a sheaf of $\mathcal{B}_0$-modules on M. Note also that the formulas (5) follow from the definition of $\mathcal{S}_k$ combined with Eqs. (1) and exactness of functors $-\otimes_{\mathcal{B}_0}\mathcal{B}_l$ and $\mathcal{H}om_{\mathcal{B}_0}(\mathcal{B}_l, -)$. So, it remains to verify (3), (4) and (6).

For this let us construct the first two maps in (6) using Remark 3.1. The first map is induced by the embedding

$$\mathcal{O}(-2g) = \Lambda^2\mathcal{U} \otimes \Lambda^2\mathcal{U} \to \Lambda^2\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{V} \subset \Lambda^2\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{B}_1,$$

and the second is induced by the composition

$$\mathcal{U}(-g) = \Lambda^2\mathcal{U} \otimes \mathcal{U} \subset \mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U} \xrightarrow{(\wedge_{23}, -s_{23})} \mathcal{U} \otimes \Lambda^2\mathcal{V} \oplus \mathcal{U} \otimes \mathcal{L}^{-1} \subset \mathcal{U} \otimes \mathcal{B}_2$$

(where the map $\wedge_{23}$ is the wedge product of the second and the third factors, and the map s_{23} is the map s again applied to the second and the third factors of the tensor product $\mathcal{U} \otimes \mathcal{U} \otimes \mathcal{U}$).

Now let us check that the constructed sequence

$$0 \to \mathcal{O}(-2g) \otimes \mathcal{B}_{k-4} \to \mathcal{U}(-g) \otimes \mathcal{B}_{k-3} \to \mathcal{U} \otimes \mathcal{B}_{k-1} \to \mathcal{B}_k \to 0.$$

is a complex. Indeed, to check that the composition of the second and the third arrow is zero we use Remark 3.1. It says that it is enough to compute the corresponding map $\Lambda^2\mathcal{U}\otimes\mathcal{U}\to\mathcal{B}_3$. The map is the composition

$$\Lambda^2\mathcal{U}\otimes\mathcal{U}\to\mathcal{U}\otimes\mathcal{U}\otimes\mathcal{U}\xrightarrow{(\wedge_{23},-s_{23})}\mathcal{U}\otimes\Lambda^2\mathcal{V}\oplus\mathcal{U}\otimes\mathcal{L}^{-1}\xrightarrow{\left(\begin{smallmatrix}\wedge & 0\\ s_{12} & 1\end{smallmatrix}\right)}\Lambda^3\mathcal{V}\oplus\mathcal{V}\otimes\mathcal{L}^{-1}.$$

It takes a vector $(u_1\wedge u_2)\otimes u_3$ to $u_1\otimes u_2\otimes u_3-u_2\otimes u_1\otimes u_3$, then to

$$\Big(u_1\otimes(u_2\wedge u_3)-u_2\otimes(u_1\wedge u_3),\,-s(u_2,u_3)u_1+s(u_1,u_3)u_2\Big),$$

and then to

$$\begin{aligned}&(2u_1\wedge u_2\wedge u_3,\,s(u_1,u_2)u_3-s(u_1,u_3)u_2-s(u_2,u_1)u_3+s(u_2,u_3)u_1\\&\quad -s(u_2,u_3)u_1+s(u_1,u_3)u_2).\end{aligned}$$

The first component is zero since the rank of $\mathcal{U}$ is 2, and in the second component all summands cancel out. This proves that the composition of the second and the third arrows in (6) is zero. A similar computation shows that the composition of the first two arrows is zero as well.

Note that each term of the complex is naturally filtered. Consider the spectral sequence of the filtered complex in case $k=0$. The first term looks like

$$\begin{array}{llll}
\mathcal{O}(-2g)\otimes\Lambda^4\mathcal{V}\otimes\mathcal{L}^4 & & & \\
\mathcal{O}(-2g)\otimes\Lambda^2\mathcal{V}\otimes\mathcal{L}^3\longrightarrow & \mathcal{U}(-g)\otimes\Lambda^3\mathcal{V}\otimes\mathcal{L}^3 & & \\
\quad\mathcal{O}(-2g)\otimes\mathcal{L}^2\longrightarrow & \mathcal{U}(-g)\otimes\mathcal{V}\otimes\mathcal{L}^2\longrightarrow & \mathcal{U}\otimes\Lambda^3\mathcal{V}\otimes\mathcal{L}^2\longrightarrow & \Lambda^4\mathcal{V}\otimes\mathcal{L}^2\\
 & & \mathcal{U}\otimes\mathcal{V}\otimes\mathcal{L}\longrightarrow & \Lambda^2\mathcal{V}\otimes\mathcal{L}\\
 & & & \mathcal{O}
\end{array}$$

The rows are natural complexes with maps corresponding to the wedge multiplication. Their cohomology are easy to compute, so it is not difficult to see that the second term looks like

$$\begin{array}{lllll}
\det\mathcal{V}\otimes\mathcal{L}^4(-2g)\searrow & & & & \\
\mathcal{L}^3(-3g)\searrow & \det\mathcal{V}\otimes\mathcal{L}^3\otimes S^2\mathcal{U}\searrow & & & \\
 & \mathcal{L}^2\otimes S^2\mathcal{U}(-g)\searrow & \det\mathcal{V}\otimes\mathcal{L}^2\otimes S^2\mathcal{U}(g)\searrow & & \\
 & & \mathcal{L}\otimes S^2\mathcal{U}\searrow & \det\mathcal{V}\otimes\mathcal{L}\otimes\mathcal{O}(g) & \\
 & & & \mathcal{O} &
\end{array}$$

The arrows here are induced by s. So, it is easy to see that the bottom chain is the Koszul complex of s, while the top chain is the same complex twisted by $\det\mathcal{V}\otimes\mathcal{L}(g)$. Hence the spectral sequence degenerates in the third term and shows that the cohomology of the above complex is supported in degree zero and is an extension of $\alpha_*\alpha^*(\det\mathcal{V}\otimes\mathcal{L}(g))$ by $\alpha_*\alpha^*\mathcal{O}$. Since we already know that it is supported on M, we conclude that it can be written as $\alpha_*\mathcal{S}_0$, where $\mathcal{S}_0$ is an extension of $\det\mathcal{V}\otimes\mathcal{L}(g)$ by $\mathcal{O}$ on M. This gives (4) and (6) for $\mathcal{S}_0$.

Analogously, consider the complex for $k=1$. The first term of the spectral sequence looks like

$$\begin{array}{ccccc}
\mathcal{O}(-2g)\otimes\Lambda^3\mathcal{V}\otimes\mathcal{L}^3 \longrightarrow \mathcal{U}(-g)\otimes\Lambda^4\mathcal{V}\otimes\mathcal{L}^3 & & & & \\
\mathcal{O}(-2g)\otimes\mathcal{V}\otimes\mathcal{L}^2 \longrightarrow \mathcal{U}(-g)\otimes\Lambda^2\mathcal{V}\otimes\mathcal{L}^2 \longrightarrow \mathcal{U}\otimes\Lambda^4\mathcal{V}\otimes\mathcal{L}^2 & & & & \\
& & \mathcal{U}(-g)\otimes\mathcal{L} \longrightarrow \mathcal{U}\otimes\Lambda^2\mathcal{V}\otimes\mathcal{L} \longrightarrow \Lambda^3\mathcal{V}\otimes\mathcal{L} & & \\
& & & & \mathcal{U} \longrightarrow \mathcal{V}
\end{array}$$

The maps are induced by the wedge multiplication, so one can check that the second term looks like

$$\mathcal{V}/\mathcal{U}\otimes\mathcal{L}^3(-3g) \to \mathcal{V}/\mathcal{U}\otimes\mathcal{L}^2\otimes S^2\mathcal{U}(-g) \to \mathcal{V}/\mathcal{U}\otimes\mathcal{L}\otimes S^2\mathcal{U} \to \mathcal{V}/\mathcal{U}$$

The maps are induced by s, so it is the Koszul complex of s tensored with $\mathcal{V}/\mathcal{U}$, hence $\mathcal{S}_1 \cong \mathcal{V}/\mathcal{U}$. This gives (3) and (6) for $\mathcal{S}_1$. For other $\mathcal{S}_k$ we deduce (3), (4) and (6) by a suitable twist. □

Applying the functor ρ_* to the resolutions (6) twisted by $\mathcal{O}(-g)$ and using the projection formula and the Borel–Bott–Weil Theorem to compute the pushforwards of $\mathcal{O}(-kg)$ and $\mathcal{U}(-kg)$ for $0 \leq k \leq 3$, we deduce the following

Corollary 3.3 *We have* $\rho_*(\mathcal{S}_k) \cong \mathcal{B}_k$, $\rho_*(\mathcal{S}_k(-g)) = 0$.

This gives an easy proof of the fact that

Lemma 3.4 *The extension in* (4) *is nontrivial.*

Proof Assume that $\mathcal{S}_0 \cong \mathcal{O}_M \oplus \det\mathcal{V}\otimes\mathcal{L}(g)$. Then

$$\rho_*(\mathcal{S}_0(-g)) \cong \rho_*((\mathcal{O}_M \oplus \det\mathcal{V}\otimes\mathcal{L}(g))(-g)) \cong \rho_*(\mathcal{O}_M(-g)\oplus\det\mathcal{V}\otimes\mathcal{L}).$$

By the projection formula we have $\rho_*(\det\mathcal{V}\otimes\mathcal{L}) = \det\mathcal{V}\otimes\mathcal{L}$, so we deduce that $\rho_*\mathcal{S}_0(-g) \neq 0$, which contradicts to Corollary 3.3. □

Now we can also compute the pushforwards of the duals of $\mathcal{S}_k$.

Corollary 3.5 *We have* $\rho_*(\mathcal{S}_k^\vee) = 0$.

Proof Indeed, since $\mathcal{S}_k$ is of rank 2 and $\det\mathcal{S}_k = \det\mathcal{V}\otimes\mathcal{L}^{1-k}(g)$ [this follows from (3) and (4)], we have $\mathcal{S}_k^\vee \cong \mathcal{S}_k(-g)\otimes\det\mathcal{V}^\vee\otimes\mathcal{L}^{k-1}$, hence its pushforward is a twist of $\rho_*(\mathcal{S}_k(-g))$ which is zero. □

Another consequence is the following

Corollary 3.6 *We have* $\rho_*(\mathcal{S}_l^\vee\otimes\mathcal{S}_k) \cong \mathcal{B}_{k-l}$.

Proof First of all consider the case $l = 0$. Then dualizing (4) we obtain an exact triple

$$0 \to \det\mathcal{V}^\vee\otimes\mathcal{L}^{-1}(-g) \to \mathcal{S}_0^\vee \to \mathcal{O}_M \to 0.$$

Tensoring it by $\mathcal{S}_k$, pushing forward and using Corollary 3.3, we obtain the claim. Now for arbitrary l the formula follows by tensoring with $\mathcal{B}_{-l}$ and using (5). □

Now we can describe the embedding $\mathcal{D}^b(Y, \mathcal{B}_0) \to \mathcal{D}^b(M)$.

Theorem 3.7 *The functor $\Phi : \mathcal{D}^b(Y, \mathcal{B}_0) \to \mathcal{D}^b(M), \mathcal{F} \mapsto \mathcal{S}_0 \otimes_{\mathcal{B}_0} \rho^*\mathcal{F}$ is fully faithful. Moreover,*

$$\Phi(\mathcal{B}_k) \cong \mathcal{S}_k.$$

Proof First, note that

$$\begin{aligned}\mathsf{Hom}(\Phi(\mathcal{F}), \mathcal{G}) &= \mathsf{Hom}(\mathcal{S}_0 \otimes_{\mathcal{B}_0} \rho^*\mathcal{F}, \mathcal{G}) \cong \mathsf{Hom}_{\mathcal{B}_0}(\rho^*\mathcal{F}, \mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{G}) \\ &\cong \mathsf{Hom}_{\mathcal{B}_0}(\mathcal{F}, \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{G})).\end{aligned}$$

Thus the right adjoint functor $\Phi^! : \mathcal{D}^b(M) \to \mathcal{D}^b(Y, \mathcal{B}_0)$ is given by

$$\Phi^!(\mathcal{G}) = \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{G})$$

(the structure of $\mathcal{B}_0$-module is induced by that of $\mathcal{S}_0^\vee$). So, to check the full faithfulness of Φ it suffices to compute $\Phi^! \circ \Phi$. For this we note that by the projection formula abd Corollary 3.6 we have

$$\Phi^!(\Phi(\mathcal{F})) = \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{S}_0 \otimes_{\mathcal{B}_0} \rho^*\mathcal{F}) \cong \rho_*(\mathcal{S}_0^\vee \otimes_{\mathcal{O}_M} \mathcal{S}_0) \otimes_{\mathcal{B}_0} \mathcal{F} \cong \mathcal{B}_0 \otimes_{\mathcal{B}_0} \mathcal{F} \cong \mathcal{F}.$$

Thus $\Phi^! \circ \Phi \cong \mathsf{id}$, so Φ is fully faithful. Finally, $\Phi(\mathcal{B}_k) = \mathcal{S}_0 \otimes_{\mathcal{B}_0} \mathcal{B}_k \cong \mathcal{S}_k$ by (5). □

We conclude the section with the following simple calculation.

Lemma 3.8 *For each i and each k we have $(\mathcal{S}_k)_{|\Sigma_i^\pm} \cong \mathcal{O}_{\Sigma_i^\pm} \oplus \mathcal{O}_{\Sigma_i^\pm}(1)$.*

Proof Restrict (3) and (4) to $\Sigma = \Sigma_i^\pm$. Since $\mathcal{O}(g)$ restricts to Σ as $\mathcal{O}_\Sigma(1)$ we obtain the claim for even k. For odd k we have to describe the restriction of $\mathcal{V}_y/\mathcal{U}$ to Σ. Since $\Sigma = \mathsf{Gr}(2, W) \subset \mathsf{Gr}(2, \mathcal{V}_y)$, we have on Σ an exact sequence of vector bundles

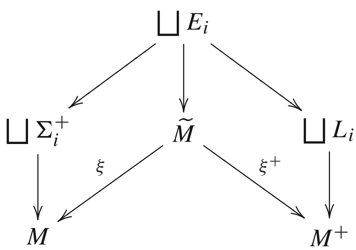
$$0 \to W/\mathcal{U} \to \mathcal{V}_y/\mathcal{U} \to \mathcal{V}_y/W \otimes \mathcal{O}_\Sigma \to 0.$$

The first term is $\mathcal{O}_\Sigma(1)$ and the third is $\mathcal{O}_\Sigma$. Hence $(\mathcal{V}_y/\mathcal{U})_{|\Sigma} \cong \mathcal{O}_\Sigma \oplus \mathcal{O}_\Sigma(1)$. □

4 The flip

A flip is a very important operation in the birational geometry. The definition and examples of flips can be found, e.g. in [5], or in [8]. In our situation we will meet the simplest example of a flip.

From now on we choose one of the planes $\mathbb{P}(W_i^\pm) \subset Q_{y_i}$ for each point y_i, say $\mathbb{P}(W_i^+)$, and the corresponding plane $\Sigma_i^+ = \mathsf{Gr}(2, W_i^+) \subset M$. Recall that by Proposition 2.6 the normal bundles of Σ_i^+ in M are $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$. Let us apply the composition of flips in all these planes and denote by M^+ the resulting Moishezon variety. More precisely, consider the blowup $\xi : \widetilde{M} \to M$ of M in the union of all Σ_i^+. Then each of the exceptional divisors $E_i = \xi^{-1}(\Sigma_i^+)$ is isomorphic to $\Sigma_i^+ \times \mathbb{P}^1$ and its normal bundle is $\mathcal{O}(-1, -1)$. Hence in the category of Moishezon varieties it can be blown down onto a line $L_i \cong \mathbb{P}^1 \subset M^+$. Thus we have a diagram



By a result of Bondal and Orlov we have the following

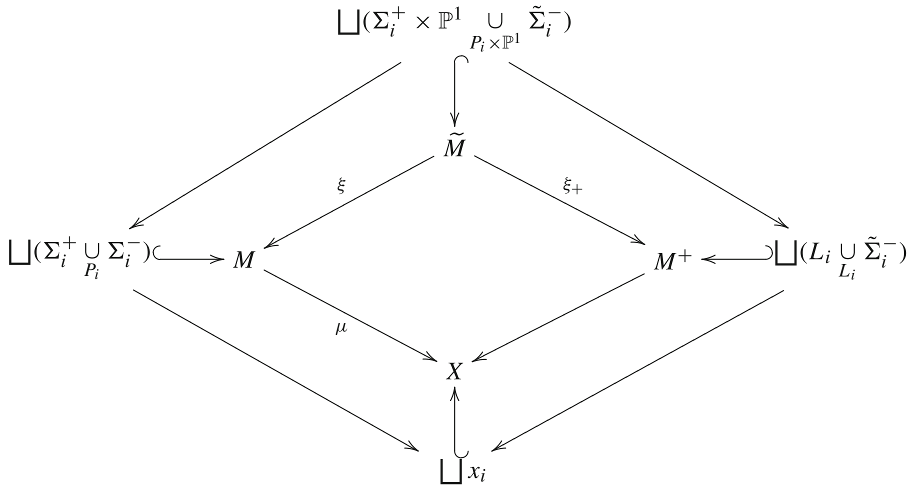
Proposition 4.1 ([4]) *The functor $\xi_*(\xi^+)^* : \mathcal{D}^b(M^+) \to \mathcal{D}^b(M)$ is fully faithful. Moreover, there is a semiorthogonal decomposition*

$$\mathcal{D}^b(M) = \langle \xi_*(\xi^+)^*(\mathcal{D}^b(M^+)), \{\mathcal{O}_{\Sigma_i^+}\}_{i=1}^N \rangle.$$

Further we will need a detailed description of the fibers of M^+ over X. Let $x_i = f^{-1}(y_i)$ be the nodal points of X and $X_{sm} = X \setminus \{x_i\}_{i=1}^N$ be the smooth locus of X.

Since the plane Σ_i^- intersects Σ_i^+ transversally at point P_i and does not intersect other planes Σ_j^+, the proper preimage of Σ_i^- with respect to the map ξ is the blowup $\tilde{\Sigma}_i^-$ of Σ_i^- at P_i, i.e. a Hirzebruch surface F_1. Moreover, $\tilde{\Sigma}_i^-$ intersects the exceptional divisor $E_i = \Sigma_i^+ \times \mathbb{P}^1$ of the blowup ξ along the line $P_i \times \mathbb{P}^1$, which is the exceptional line of the Hirzebruch surface. It follows that the contraction ξ_+ maps $\tilde{\Sigma}_i^-$ into M^+ isomorphically, and its exceptional line maps onto $L_i := \xi_+(E_i)$.

Lemma 4.2 *There is a regular morphism $M^+ \to X$ such that the diagram*



commutes. Moreover, over the smooth locus X_{sm} the maps $M \to X$ and $M^+ \to X$ coincide. Finally, the fiber $M^+_{x_i}$ of M^+ over x_i is the blowup $\tilde{\Sigma}_i^-$ of Σ_i^- in the point P_i and the line $L_i = \xi_+(E_i)$ is the (-1)-curve on $\tilde{\Sigma}_i^-$.

Proof The first claim is evident—since the map $\xi_+ : \widetilde{M} \to M^+$ is a contraction of divisors E_i, and the map $\mu \circ \xi : \widetilde{M} \to X$ contracts each of these divisors to a point, we conclude that

$\mu \circ \xi$ factors through ξ_+. Moreover, since ξ and ξ_+ are identities over X_{sm}, it follows that the morphisms $M \to X$ and $M^+ \to X$ coincide over X_{sm}. Finally, note that the fiber of $\widetilde{M}$ over x_i is the union of E_i and the proper preimage of Σ_i^-. Since the map ξ_+ contracts Σ_i^+ onto the line L_i which is contained in the image of the embedding $\tilde{\Sigma}_i^- \subset M^+$, we conclude that $M_{x_i}^+ = \tilde{\Sigma}_i^-$. The fact that L_i is the exceptional line of the Hirzebruch surface $\tilde{\Sigma}_i^- \cong F_1$ was already explained above. □

As we already have seen, $\tilde{\Sigma}_i^-$ is isomorphic to a Hirzebruch surface F_1. In particular, it has a canonical contraction $\tilde{\Sigma}_i^- \to \mathbb{P}^1$ which induces an isomorphism of the exceptional line $L_i \subset \tilde{\Sigma}_i^-$ onto $\mathbb{P}^1$. Denote (the pullback to $\tilde{\Sigma}_i^-$ of) the generator of the Picard group of Σ_i by h and the class of the exceptional line $L_i \subset \tilde{\Sigma}_i^-$ by l. Then the class of the fiber of the projection $\tilde{\Sigma}_i^- \to \mathbb{P}^1$ is $h - l$.

Lemma 4.3 *We have* $\omega_{M^+|\tilde{\Sigma}_i^-} \cong \mathcal{O}_{\tilde{\Sigma}_i^-}(-h - l)$.

Proof Note that $\omega_{\widetilde{M}} = \xi^*\omega_M(\sum E_i) = \xi_+^*\omega_{M^+}(2\sum E_i)$. Hence $\xi_+^*\omega_{M^+} = \xi^*\omega_M(-\sum E_i)$. Using Corollary 2.7 we obtain

$$\xi_+^*\omega_{M^+|\tilde{\Sigma}_i^-} \cong \xi^*\omega_{M|\tilde{\Sigma}_i^-} \otimes \mathcal{O}(-E_i)_{|\tilde{\Sigma}_i^-} \cong \mathcal{O}_{\tilde{\Sigma}_i^-}(-h) \otimes \mathcal{O}_{\tilde{\Sigma}_i^-}(-l) \cong \mathcal{O}_{\tilde{\Sigma}_i^-}(-h - l).$$

But ξ_+ is an isomorphism on $\tilde{\Sigma}_i^-$, hence the claim. □

It turns out that M^+ has a very simple structure—it is a $\mathbb{P}^1$-fibration over a small resolution of X.

Proposition 4.4 *The map $M^+ \to X$ factors as a composition $M^+ \xrightarrow{\mu_+} X^+ \xrightarrow{\sigma_+} X$, where the map $\mu_+ : M^+ \to X^+$ is a $\mathbb{P}^1$-fibration and $\sigma^+ : X^+ \to X$ is a small resolution of singularities. The restriction of the map μ_+ to the fiber $M_{x_i}^+ = \tilde{\Sigma}_i^-$ coincides with the projection $\tilde{\Sigma}_i^- \to \mathbb{P}^1$. The curve $C_i = \mu_+(\tilde{\Sigma}_i^-) \cong \mathbb{P}^1$ is the exceptional locus of X^+ over $x_i \in X$.*

Proof We apply to M^+ the relative Minimal Model Program over X, see [12]. Since the relative MMP commutes with the base change, let us first look at $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^-$ which is the preimage of X_{sm}. The map $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^- \to X_{sm}$ is a $\mathbb{P}^1$-fibration, so its relative Picard group is $\mathbb{Z}$, and the relative canonical class is ample, hence the first (and the last) step of the MMP for $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^-$ is the $\mathbb{P}^1$-fibration $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^- \to X_{sm}$.

Now consider what happens over analytic neighborhoods of singular points. Let $x = x_i$ be one of singular points. Consider an analytic neighborhood U of x in X and its preimage $M_U^+ \subset M^+$. Then the relative (over U) effective cone of M_U^+ is generated by curves in the special fiber $M_x^+ = \tilde{\Sigma}^-$, that is by the (-1)-curve L and by the fiber of the projection $\tilde{\Sigma}^- \to \mathbb{P}^1$. By Lemma 4.3 the canonical class $K_{M^+/X}$ restricts to $\tilde{\Sigma}^-$ as $-h - l$, hence L is K-positive, while the fiber is K-negative. Hence the first step in MMP is the contraction of the ray generated by the fiber of the projection $\tilde{\Sigma}^- \to \mathbb{P}^1$. By MMP this contraction should be either

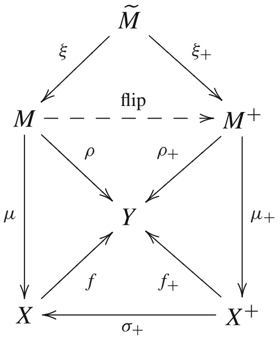
1. a flip, or
2. a divisorial contraction, or
3. a conic bundle.

By a result of Kawamata [7] the case of a flip is impossible, since the center of a flip in dimension 4 is always a $\mathbb{P}^2$, while in our case the only compact surface in M_U^+ is $M_x^+ = \tilde{\Sigma}^-$ which is a Hirzebruch surface F_1. Similarly, a divisorial contraction is impossible, since then one of the steps of MMP on $M_U^+ \setminus \tilde{\Sigma}_i^-$ would also be a divisorial contraction, while as we have shown above the only nontrivial step of this MMP is a $\mathbb{P}^1$-fibration.

Thus the first step of MMP for M_U^+ is a conic bundle $M_U^+ \to X_U^+$. Once again, over $U \setminus \{x\}$ this conic bundle should coincide with the $\mathbb{P}^1$-fibration $M^+ \setminus \bigsqcup \tilde{\Sigma}_i^- \to X_{sm}$, hence gluing all these conic bundles for all singular points x_i, we obtain a global conic bundle structure on M^+, that is a global map $\mu_+ : M^+ \to X^+$ for some Moishezon variety X^+. Now we apply [13] and conclude that X^+ is necessarily smooth, hence X^+ is a resolution of singularities of X. Further, the restriction of the map μ_+ to $\tilde{\Sigma}_i^-$ is a conic bundle which contracts all the fibers of the projection $\tilde{\Sigma}_i^- \to \mathbb{P}^1$, hence the fiber of X_+ over x_i is the image C_i of $\tilde{\Sigma}_i^-$. Since X^+ is smooth and the fiber of X^+ over x_i is $C_i \cong \mathbb{P}^1$, the map $\sigma_+ : X^+ \to X$ is a small resolution of singularities. So, it remains to check that $\mu_+ : M^+ \to X^+$ is a $\mathbb{P}^1$-fibration.

Since we already know that μ_+ is a conic bundle, we should check that its degeneration locus is empty. But the degeneration locus of a conic bundle is a divisor, while $M^+ \to X^+$ is nondegenerate over the complement $X_{sm} = X^+ \setminus (\sqcup C_i)$ of a finite number of curves, hence the degeneration locus is empty. □

Denoting $f_+ = f \circ \sigma_+$, $\rho_+ = f_+ \circ \mu_+$, we obtain a commutative diagram



(7)

Since the map $\mu_+ : \mathcal{D}^b(M^+) \to \mathcal{D}^b(X^+)$ is a $\mathbb{P}^1$-fibration, the pullback functor $\mu_+^* : \mathcal{D}^b(X^+) \to \mathcal{D}^b(M^+)$ is fully faithful. Composing with the functor given by the flip we obtain

Corollary 4.5 *The functor $\xi_* \xi_+^* \mu_+^* : \mathcal{D}^b(X^+) \to \mathcal{D}^b(M)$ is fully faithful.*

Thus we have constructed all the required components in $\mathcal{D}^b(M)$. It remains to check that they generate the whole category. This is done in the next section.

5 Derived category of M^+

As it was shown in the previous section, M^+ is a $\mathbb{P}^1$-bundle over X^+, that is a Severi–Brauer variety, see [1,11]. It is well known that Severi-Brauer varieties over a scheme S are in bijection with Morita-equivalence classes of Azumaya algebras over S (recall that an Azumaya algebra on S is a sheaf of $\mathcal{O}_S$-algebras which étale locally is isomorphic to the endomorphism algebra of a vector bundle). One of the ways to construct an Azumaya algebra from a $\mathbb{P}^1$-bundle $T \to S$ is to find a vector bundle $\mathcal{E}$ on T which restricts to any fiber of $T \to S$ as $\mathcal{O}(1)^{\oplus n}$ for some n. Then the sheaf of algebras $\mathcal{A}_T = \mathcal{E}nd(\mathcal{E})$ restricts trivially to any fiber of $T \to S$, hence it is the pullback of a sheaf of algebras $\mathcal{A}$ on S. This is the corresponding Azumaya algebra.

To construct such a vector bundle for the $\mathbb{P}^1$-bundle $M^+ \to X^+$ we modify the vector bundle $\mathcal{S}_k$ defined in Proposition 3.2. Recall that the exceptional divisors of the blowups $\xi : \widetilde{M} \to M$ and $\xi_+ : \widetilde{M} \to M^+$ are $E_i \cong \Sigma_i^+ \times L_i \cong \mathbb{P}^2 \times \mathbb{P}^1$.

Lemma 5.1 *There are vector bundles $\mathcal{R}_k$ of rank 2 on M^+ such that there is a short exact sequence*

$$0 \to \xi^*\mathcal{S}_k \to \xi_+^*\mathcal{R}_k \to \bigoplus_{i=1}^{N} \mathcal{O}_{E_i}(0,-1) \to 0. \tag{8}$$

Proof By Lemma 3.8 we have an isomorphism $(\xi^*\mathcal{S}_k^\vee)_{|E_i} \cong \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(-1,0)$. Consider the composition $\xi^*\mathcal{S}_k^\vee \to (\xi^*\mathcal{S}_k^\vee)_{|E_i} \to \mathcal{O}_{E_i}(-1,0)$, where the second map is the unique projection. This map is clearly surjective. Denote the kernel of the sum of these maps over i by F, so that we have an exact triple

$$0 \to F \to \xi^*\mathcal{S}_k^\vee \to \oplus\mathcal{O}_{E_i}(-1,0) \to 0.$$

Let us check that F is a pullback of a vector bundle from M^+. Since $\xi_+ : \widetilde{M} \to M^+$ is a smooth blowup, it suffices to check that $F_{|E_i}$ is a pullback of a vector bundle from L_i. Let us restrict the above exact sequence to E_i. Since $\mathcal{N}_{E_i/\widetilde{M}} \cong \mathcal{O}_{E_i}(-1,-1)$ we obtain an exact sequence

$$0 \to \mathcal{O}_{E_i}(0,1) \to F_{|E_i} \to \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(-1,0) \to \mathcal{O}_{E_i}(-1,0) \to 0.$$

The last map is the projection to the second summand, hence we have an exact triple

$$0 \to \mathcal{O}_{E_i}(0,1) \to F_{|E_i} \to \mathcal{O}_{E_i} \to 0.$$

Since $\mathsf{Ext}^1_{\mathcal{D}^b(E_i)}(\mathcal{O}_{E_i}, \mathcal{O}_{E_i}(0,1)) \cong H^1(E_i, \mathcal{O}_{E_i}(0,1)) = 0$, we see that there is a decomposition $F_{|E_i} \cong \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(0,1)$. So, the bundle $F_{|E_i}$ is a pullback of $\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)$, hence the bundle F is a pullback of a vector bundle on M^+ which restricts to L_i as $\mathcal{O}_{L_i} \oplus \mathcal{O}_{L_i}(1)$. Now we define $\mathcal{R}_k$ as the dual of this vector bundle. So, by definition we have the following exact sequence

$$0 \to \xi_+^*\mathcal{R}_k^\vee \to \xi^*\mathcal{S}_k^\vee \to \oplus\mathcal{O}_{E_i}(-1,0) \to 0. \tag{9}$$

Dualizing this sequence and taking into account that

$$\mathsf{R}\mathcal{H}om(\mathcal{O}_{E_i}(-1,0), \mathcal{O}_{\widetilde{M}}) \cong \mathcal{O}_{E_i}(1,0) \otimes \mathcal{N}_{E_i/\widetilde{M}}[-1] \cong \mathcal{O}_{E_i}(0,-1)[-1]$$

we obtain (8). □

Remark 5.2 Note that we also proved an isomorphism $\mathcal{R}_{k|E_i} \cong \mathcal{O}_{E_i} \oplus \mathcal{O}_{E_i}(0,-1)$.

The bundles $\mathcal{R}_k$ enjoy a lot of interesting properties.

Lemma 5.3 *We have $(\rho_+)_*\mathcal{R}_k^\vee = 0$.*

Proof Indeed, using commutativity of (7) we deduce

$$(\rho_+)_*\mathcal{R}_k^\vee = (\rho_+)_*(\xi_+)_*\xi_+^*\mathcal{R}_k^\vee = \rho_*\xi_*\xi_+^*\mathcal{R}_k^\vee.$$

Applying the functor $\rho_*\xi_*$ to (9) we obtain a triangle

$$\rho_*\xi_*\xi_+^*\mathcal{R}_k^\vee \to \rho_*\xi_*\xi^*\mathcal{S}_k^\vee \to \oplus\rho_*\xi_*\mathcal{O}_{E_i}(-1,0).$$

The second term equals to $\rho_*\mathcal{S}_k^\vee$ which is zero by Corollary 3.5. Since $\rho\circ\xi$ contracts E_i to the point y_i, the third term is $\oplus H^\bullet(E_i,\mathcal{O}_{E_i}(-1,0))\otimes\mathcal{O}_{y_i}$, so it is also zero. Hence $(\rho_+)_*\mathcal{R}_k^\vee=0$. □

Proposition 5.4 *The bundle $\mathcal{R}_k$ restricts to any fiber of $\mu_+ : M^+\to X^+$ as $\mathcal{O}(1)\oplus\mathcal{O}(1)$. Moreover*

$$\mathcal{R}_{k|\tilde{\Sigma}_i^-}\cong\mathcal{O}_{\tilde{\Sigma}_i^-}(h)\oplus\mathcal{O}_{\tilde{\Sigma}_i^-}(l). \tag{10}$$

Proof We are going to prove instead that $\mathcal{R}_k^\vee$ restricts to all fibers as $\mathcal{O}(-1)\oplus\mathcal{O}(-1)$. For fibers of $\mu^+ : M^+\setminus\bigsqcup\tilde{\Sigma}_i^-\to X^+\setminus\bigsqcup C_i$ this follows from Lemma 5.3. So it remains to investigate the restriction of $\mathcal{R}_k^\vee$ to $\tilde{\Sigma}_i^-$. For this we restrict (9) to $\tilde{\Sigma}_i^-$ and taking into account that by Lemma 3.8 we have $\mathcal{S}^\vee_{k\,|\tilde{\Sigma}_i^-}\cong\mathcal{O}_{\tilde{\Sigma}_i^-}\oplus\mathcal{O}_{\tilde{\Sigma}_i^-}(-h)$, we obtain

$$0\to(\mathcal{R}_k^\vee)_{|\tilde{\Sigma}_i^-}\to\mathcal{O}_{\tilde{\Sigma}_i^-}\oplus\mathcal{O}_{\tilde{\Sigma}_i^-}(-h)\to\mathcal{O}_{L_i}\to 0.$$

It follows that either (10) holds, or $(\mathcal{R}_k^\vee)_{|\tilde{\Sigma}_i^-}\cong\mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l)\oplus\mathcal{O}_{\tilde{\Sigma}_i^-}$. In the former case we are done since both $\mathcal{O}_{\tilde{\Sigma}_i^-}(h)$ and $\mathcal{O}_{\tilde{\Sigma}_i^-}(l)$ restrict as $\mathcal{O}(1)$ to any fiber of $\tilde{\Sigma}_i^-$ over C_i. Let us check that the case $(\mathcal{R}_k^\vee)_{|\tilde{\Sigma}_i^-}\cong\mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l)\oplus\mathcal{O}_{\tilde{\Sigma}_i^-}$ is impossible.

For this we note that by Lemma 5.3

$$0=\mathsf{Ext}^\bullet(\mathcal{O}_{y_i},(\rho_+)_*\mathcal{R}_k^\vee)=\mathsf{Ext}^\bullet(\rho_+^*\mathcal{O}_{y_i},\mathcal{R}_k^\vee).$$

On the other hand, the cohomology sheaves $\mathcal{H}^l=\mathcal{H}^l(\rho_+^*\mathcal{O}_{y_i})$ are supported on $\tilde{\Sigma}_i^-$. Moreover, $\mathcal{H}^l=0$ for $l>0$ (since ρ_+^* is right exact) and $\mathcal{H}^0\cong\mathcal{O}_{\tilde{\Sigma}_i^-}$. Consider the spectral sequence

$$\mathsf{Ext}^q(\mathcal{H}^p,\mathcal{R}_k^\vee)\Rightarrow\mathsf{Ext}^{q-p}(\rho_+^*\mathcal{O}_{y_i},\mathcal{R}_k^\vee)=0.$$

Note that by Serre duality on M^+ we have

$$\mathsf{Ext}^q(\mathcal{H}^i,\mathcal{R}_k^\vee)\cong\mathsf{Ext}^{4-q}(\mathcal{R}_k^\vee,\mathcal{H}^i\otimes\omega_{M^+})^\vee\cong H^{4-q}(M^+,\mathcal{H}^i\otimes\mathcal{R}_k\otimes\omega_{M^+})^\vee.$$

The right-hand side vanishes for $q\notin\{2,3,4\}$ since the sheaf $\mathcal{H}^i$ is supported on $\tilde{\Sigma}_i^-$ and $\dim\tilde{\Sigma}_i^-=2$. Hence the line $q=3$ does not change in the spectral sequence. But if $(\mathcal{R}_k^\vee)_{|\tilde{\Sigma}_i^-}\cong\mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l)\oplus\mathcal{O}_{\tilde{\Sigma}_i^-}$ then

$$\begin{aligned}\mathsf{Ext}^3(\mathcal{H}^0,\mathcal{R}_k^\vee)&\cong H^1(M^+,\mathcal{O}_{\tilde{\Sigma}_i^-}\otimes\mathcal{R}_k\otimes\omega_{M^+})^\vee\\&\cong H^1(\tilde{\Sigma}_i^-,(\mathcal{O}_{\tilde{\Sigma}_i^-}(h+l)\oplus\mathcal{O}_{\tilde{\Sigma}_i^-})\otimes\mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l))\\&\cong H^1(\tilde{\Sigma}_i^-,\mathcal{O}_{\tilde{\Sigma}_i^-}\oplus\mathcal{O}_{\tilde{\Sigma}_i^-}(-h-l))\cong\mathsf{k}\end{aligned}$$

gives a nontrivial element in $\mathsf{Ext}^3(\rho_+^*\mathcal{O}_{y_i},\mathcal{R}_k^\vee)$, which is impossible. □

As we mentioned at the beginning of the section, the bundle $\mathcal{R}_0$ allows to construct the Azumaya algebra on X^+ corresponding to the $\mathbb{P}^1$-bundle $M^+\to X^+$.

Proposition 5.5 ([11]) *The Brauer class of the $\mathbb{P}^1$-fibration $M^+\to X^+$ is given by an Azumaya algebra $\mathcal{B}^+$ on X^+ such that $\mu_+^*\mathcal{B}^+\cong\mathcal{E}nd(\mathcal{R}_0)$.*

The Azumaya algebra $\mathcal{B}^+$ in its turn provides a description of the derived category of M^+ due to the following result of Bernardara.

Proposition 5.6 ([3]) *There is a semiorthogonal decomposition*

$$\mathcal{D}^b(M^+) = \langle \mathcal{D}^b(X^+), \mathcal{D}^b(X^+, \mathcal{B}^+) \rangle$$

with the embedding functors of the components given by $F \mapsto \mu_+^* F$ *and* $F \mapsto \mu_+^* F \otimes_{\mathcal{B}^+} \mathcal{R}_0$ *respectively.*

Combining this with Proposition 4.1 we see, that to prove Theorem 1.1 it remains to check that the functor $\xi_* \xi_+^*$ identifies $\mathcal{D}^b(X^+, \mathcal{B}^+)$ with the subcategory $\Phi(\mathcal{D}^b(Y, \mathcal{B}_0)) \subset \mathcal{D}^b(M)$. For this we find a relation between the algebras $\mathcal{B}^+$ on X^+ and $\mathcal{B}_0$ on Y.

Lemma 5.7 *We have* $(f_+)_* \mathcal{B}^+ \cong \mathcal{B}_0$*, an isomorphism of sheaves of algebras.*

Proof First of all

$$(f_+)_* \mathcal{B}^+ \cong (f_+)_* (\mu_+)_* (\xi_+)_* \xi_+^* \mu_+^* \mathcal{B}^+ \cong f_* \mu_* \xi_* \xi_+^* \, \mathcal{E}nd(\mathcal{R}_0) \cong \rho_* \xi_* \xi_+^* \, \mathcal{E}nd(\mathcal{R}_0).$$

On the other hand, tensoring (8) with $\xi^* \mathcal{S}_0^\vee$ and (9) with $\xi_+^* \mathcal{R}_0$ and using Lemma 3.8 and Remark 5.2 we obtain exact sequences

$$0 \to \xi^* \mathcal{E}nd(\mathcal{S}_0) \to \xi^* \mathcal{S}_0^\vee \otimes \xi_+^* \mathcal{R}_0 \to \oplus(\mathcal{O}_{E_i}(0,-1) \oplus \mathcal{O}_{E_i}(-1,-1)) \to 0,$$
$$0 \to \xi_+^* \mathcal{E}nd(\mathcal{R}_0) \to \xi^* \mathcal{S}_0^\vee \otimes \xi_+^* \mathcal{R}_0 \to \oplus(\mathcal{O}_{E_i}(-1,0) \oplus \mathcal{O}_{E_i}(-1,-1)) \to 0.$$

Pushing these sequences along $\rho \circ \xi$ and noting that E_i is contracted to a point we conclude that

$$\rho_* \xi_* \xi_+^* \mathcal{E}nd(\mathcal{R}_0) \cong \rho_* \xi_* (\xi^* \mathcal{S}_0^\vee \otimes \xi_+^* \mathcal{R}_0) \cong \rho_* \xi_* \xi^* \mathcal{E}nd(\mathcal{S}_0).$$

On the other hand $\rho_* \xi_* \xi^* \mathcal{E}nd(\mathcal{S}_0) \cong \rho_* \mathcal{E}nd(\mathcal{S}_0) \cong \mathcal{B}_0$ (see Corollary 3.6 for the last isomorphism). □

Now we are ready to prove our main result.

Proof of Theorem 1.1 As we already mentioned, by Proposition 4.1 and Proposition 5.6 we only have to identify the image of $\mathcal{D}^b(X^+, \mathcal{B}^+)$ under the functor $\xi_* \xi_+^*$ with the subcategory $\Phi(\mathcal{D}^b(Y, \mathcal{B}_0))$.

The isomorphism $(f_+)_* \mathcal{B}^+ \cong \mathcal{B}_0$ of Lemma 5.7 gives by adjunction a morphism $f_+^* \mathcal{B}_0 \to \mathcal{B}^+$ which equips $\mathcal{B}^+$ with a structure of a $\mathcal{B}_0$-module. Now we define the functor $\mathcal{D}^b(Y, \mathcal{B}_0) \to \mathcal{D}^b(X^+, \mathcal{B}^+)$ by $F \mapsto f_+^* F \otimes_{\mathcal{B}_0} \mathcal{B}^+$. The right adjoint functor then is $(f_+)_* : \mathcal{D}^b(X^+, \mathcal{B}^+) \to \mathcal{D}^b(Y, \mathcal{B}_0)$. Their composition takes F to

$$(f_+)_* (f_+^* F \otimes_{\mathcal{B}_0} \mathcal{B}^+) \cong F \otimes_{\mathcal{B}_0} (f_+)_* \mathcal{B}^+ = F \otimes_{\mathcal{B}_0} \mathcal{B}_0 = F.$$

This implies that this functor is fully faithful, and the orthogonal to its image consists of all objects G such that $(f_+)_* G = 0$. Since f_+ is the contraction of $(-1,-1)$ curves C_i, any object G in $\mathcal{D}^b(X^+)$ such that $(f_+)_* G = 0$ is a complex with cohomology supported on the union of curves C_i and being the direct sums of sheaves $\mathcal{O}_{C_i}(-1)$. Since $\mathcal{B}^+$ is an Azumaya algebra, the forgetful functor $\mathcal{D}^b(X^+, \mathcal{B}^+) \to \mathcal{D}^b(X^+)$ commutes with sheaf cohomology, hence each of these direct sums of $\mathcal{O}_{C_i}(-1)$ should be a $\mathcal{B}^+$-module. So, it remains to check that there is no such $\mathcal{B}^+$-modules.

For this we note that (10) implies $\mathcal{E}nd(\mathcal{R}_0)_{|\tilde{\Sigma}_i^-} \cong \mathcal{O}_{\tilde{\Sigma}_i^-} \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}(h-l) \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}(l-h) \oplus \mathcal{O}_{\tilde{\Sigma}_i^-}$. Therefore we have $\mathcal{B}^+_{|C_i} \cong \mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(1) \oplus \mathcal{O}_{C_i}(-1) \oplus \mathcal{O}_{C_i} \cong \mathcal{E}nd(\mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(-1))$, hence the category of $\mathcal{B}^+$-modules supported on C_i is equivalent to the category of sheaves on C_i, the equivalence taking a sheaf F to $F \otimes (\mathcal{O}_{C_i} \oplus \mathcal{O}_{C_i}(-1)) \cong F \oplus F(-1)$. It is clear that if $F \oplus F(-1)$ is a direct sum of $\mathcal{O}(-1)$ then $F = 0$. Thus we have checked that $\mathcal{D}^b(Y, \mathcal{B}_0) \cong \mathcal{D}^b(X^+, \mathcal{B}^+)$.

Finally, we compute the composition of the equivalence $\mathcal{D}^b(Y, \mathcal{B}_0) \cong \mathcal{D}^b(X^+, \mathcal{B}^+)$, and embeddings $\mathcal{D}^b(X^+, \mathcal{B}^+) \to \mathcal{D}^b(M^+) \to \mathcal{D}^b(M)$. It acts on $F \in \mathcal{D}^b(Y, \mathcal{B}_0)$ as

$$F \mapsto \xi_* \xi_+^* (\mu_+^* (f_+^* F \otimes_{\mathcal{B}_0} \mathcal{B}^+) \otimes_{\mathcal{B}^+} \mathcal{R}_0) \cong \xi_* \xi_+^* (\rho_+^* F \otimes_{\mathcal{B}_0} \mathcal{R}_0) \cong \xi_* (\xi_+^* \rho_+^* F \otimes_{\mathcal{B}_0} \xi_+^* \mathcal{R}_0).$$

Note that $\xi_+^* \rho_+^* F \cong \xi^* \rho^* F$, so tensoring (8) by it we obtain

$$\xi^* \rho^* F \otimes_{\mathcal{B}_0} \xi^* \mathcal{S}_0 \to \xi^* \rho^* F \otimes_{\mathcal{B}_0} \xi_+^* \mathcal{R}_0 \to \oplus (\xi^* \rho^* F \otimes_{\mathcal{B}_0} \mathcal{O}_{E_i}(0, -1)).$$

Since $\xi_* \mathcal{O}_{E_i}(0, -1) = 0$, applying ξ_* we see that

$$\xi_* (\xi^* \rho^* F \otimes_{\mathcal{B}_0} \xi^* \mathcal{S}_0) \cong \rho^* F \otimes_{\mathcal{B}_0} \mathcal{S}_0,$$

which gives the required isomorphism of functors and finishes the proof of Theorem 1.1. □

6 Concluding remarks and further questions

Remark 6.1 There is another way of proving Theorem 1.1, avoiding use of Moishezon varieties. For this one has to perform another birational modification of M. First, consider the blowup $M' \to M$ in the N points $P_i = \Sigma_i^+ \cap \Sigma_i^-$. Let $E_i' \cong \mathbb{P}^3$ be the exceptional divisors of this blowup. Then the proper preimages of the planes Σ_i^+ and Σ_i^- are Hirzebruch surfaces $\tilde{\Sigma}_i^-, \tilde{\Sigma}_i^+ \subset M'$ which do not intersect. Moreover, the (-1)-curves $L_i^\pm$ on $\tilde{\Sigma}_i^\pm$ are skew-lines in E_i'. One can check that the normal bundle to $\tilde{\Sigma}_i^\pm$ in M' restricts as $\mathcal{O}(-1) \oplus \mathcal{O}(-1)$ to any fiber of $\tilde{\Sigma}_i^\pm$ over $\mathbb{P}^1$. Hence one can make a (relative over $\mathbb{P}^1$) flop in all $2N$ surfaces $\tilde{\Sigma}_i^\pm$ simultaneously. We will obtain an algebraic variety M'' over X. The special fibers M''_{x_i} will coincide with blowups E_i'' of E_i' in the lines $L_i^\pm$ (and each of the surfaces $\tilde{\Sigma}_i^\pm$ will be replaced by $\mathbb{P}^1 \times \mathbb{P}^1$, coinciding with the exceptional divisor of E_i'' over $L_i^\pm$). Then by the same arguments as in Proposition 4.4 one can show that the map $M'' \to X$ factors as a $\mathbb{P}^1$-fibration $M'' \to X'$, where X' is the blowup of X in all points x_i. Then a careful analysis of the relation of the categories $\mathcal{D}^b(M'')$ and $\mathcal{D}^b(M)$ allows to prove Theorem 1.1.

An interesting question for the further investigation is to describe the category $\mathcal{D}^b(M)$ without restrictions on the dimension of Y. A natural approach would be to consider first the universal family of quadrics and then to apply a base change argument (see [9]) to obtain a decomposition in general case. Unfortunately, the approach of this paper does not work in this general setup because of the following effect — assume for simplicity that $D_3 = \emptyset$, but $\dim D_2 > 0$. Since the fibers of M over D_2 are the unions of two planes, we have an unramified double covering $\tilde{D}_2 \to D_2$. This covering in general is connected. Therefore, we cannot pick up one of the planes Σ^+ in all the fibers over D_2 and make a flip in them. However, the approach suggested in Remark 6.1 may work, and I guess that in case $D_3 = \emptyset$ should work without big changes. As for the case of nonempty D_3, a deeper analysis of the behavior of M over D_3 (and possibly more birational transformations) is required.

Another question which may prove interesting is investigation of the derived category of the relative scheme of lines (or other isotropic Grassmannians) of a family of quadrics of dimension bigger than 2.

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