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A new family of surfaces of general type with $K^2 = 7$ and $p_g = 0$

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Abstract We construct a new family of smooth minimal surfaces of general type with $K^2 = 7$ and $p_g = 0$. We show that a surface in this family has ample canonical divisor and birational bicanonical morphism. We also prove that these surfaces satisfy Bloch's conjecture.

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1 Introduction

Minimal surfaces of general type with $p_g = 0$ have been constructed and studied since the 1930's (cf. [7] and [12]). These surfaces have invariants $p_g = q = 0$ and $1 \leq K^2 \leq 9$. For each value of K^2, except the case $K^2 = 7$, there exists quite a list of examples. However, to the best of the author's knowledge, there is only one known family of minimal surfaces of general type with $K^2 = 7$ and $p_g = 0$ (cf. [4, Tables 1–3]). This family of surfaces is due to M. Inoue (cf. [13]). We will show that the surfaces with $K^2 = 7$ constructed in [18] are in fact Inoue surfaces (see Sect. 6).

Inoue surfaces with $K^2 = 7$ are quotients of complete intersections inside the product of four elliptic curves by a group isomorphic to $\mathbb{Z}_2^5$ acting freely (cf. [13]). Alternatively, Inoue surfaces can be realized as finite $\mathbb{Z}_2^2$-covers of the 4-nodal cubic surface (cf. [16, Example 4.1]). We refer to the recent article [5], where the authors use both constructions to study the deformations of Inoue surfaces.

The bicanonical morphism of an Inoue surface has degree 2 and is composed with exactly one involution of the Galois group $\mathbb{Z}_2^2$. The bicanonical morphism of a smooth minimal surface of general type with $K^2 = 7$ and $p_g = 0$ has degree either 1 or 2 (cf. [16] and [17]). Involutions on surfaces of general type with $K^2 = 7$ and $p_g = 0$ are studied in [15] and [18]. Lists of numerical possibilities are given in these articles. However, no new example is constructed (see Sect. 6). It is also claimed in a pre-version of [15] that three quotients of an

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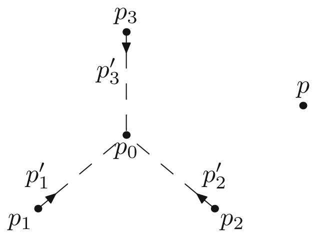


Fig. 1 Configurations of $p_0, p_1, \ldots, p_3'$ and p

Inoue surface by the involutions are rational surfaces. However, we point out that one of the quotients is birational to an Enriques surface (see Sect. 6).

In this article, we construct a family of surfaces with $K^2 = 7$ and $p_g = 0$, as finite $\mathbb{Z}_2^2$-covers of certain singular Del Pezzo surfaces of degree one. These surfaces have ample canonical divisor. For a surface S in our family, the bicanonical morphism of S is not composed with any involution of the Galois group $\mathbb{Z}_2^2$. Using the results of [17], we show that the bicanonical morphism of S is birational. So the family is indeed new.

We show that three quotients of S by the involutions have Kodaira dimensions $-\infty$, 0, 1, respectively, realizing some numerical possibilities on the lists of [15] and [18]. Applying the results of [3], we prove that S satisfies Bloch's conjecture.

Notation and conventions We make the convention that the indices $i \in \{1, 2, 3\}$ should be understood as residue classes modulo 3. We denote by g_1, g_2, g_3 the nonzero elements of the group $G := \mathbb{Z}_2^2$ and by $\chi_i \in G^*$ the nontrivial character orthogonal to g_i for $i = 1, 2, 3$. Linear equivalence is denoted by $\equiv$. The rest of the notation is standard in algebraic geometry.

2 Certain weak Del Pezzo surfaces of degree one

In this section, we construct a family of weak Del Pezzo surfaces of degree one as blowup of $\mathbb{P}^2$ at eight points. We use $(x_1 : x_2 : x_3)$ as the homogeneous coordinates for $\mathbb{P}^2$. Let $p_1 = (1 : 0 : 0)$, $p_2 = (0 : 1 : 0)$, $p_3 = (0 : 0 : 1)$, $p_0 = (1 : 1 : 1)$ and let p_j' be the infinitely near point over p_j corresponding to the line $\overline{p_j p_0}$ for $j = 1, 2, 3$. We state a lemma on conics passing through some of these points (Fig. 1).

Lemma 2.1 *For each $i = 1, 2, 3$, there is a unique conic c_i passing through the points $p_i, p_{i+1}, p_{i+1}', p_{i+2}$ and p_{i+2}'. Its defining polynomial is $x_i(x_{i+1} + x_{i+2}) - x_{i+1}x_{i+2} = 0$. Moreover, c_i does not pass through the point p_i'.*

Let $\sigma\colon W \to \mathbb{P}^2$ be the blowup at the eight points $p_0, p_1, p_1', p_2, p_2', p_3, p_3'$ and p, where the eighth point p satisfies the Zariski open conditions:

(I) $p \notin \cup_{i=1}^3 \{\overline{p_0 p_i} : x_{i+1} = x_{i+2}\} \cup_{i=1}^3 \{\overline{p_{i+1} p_{i+2}} : x_i = 0\}$;
(II) $p \notin c_1 \cup c_2 \cup c_3$.

Denote by E_j (respectively E_j', E) the **total transform** of the point p_j (respectively p_j', p) and by L the pullback of a general line by σ. Then

$$Pic(W) = \mathbb{Z}L \oplus \mathbb{Z}E_0 \oplus \oplus_{j=1}^3 (\mathbb{Z}E_j \oplus \mathbb{Z}E_j') \oplus \mathbb{Z}E$$

and $-K_W \equiv 3L - E_0 - \sum_{j=1}^3 (E_j + E_j') - E$.

The surface W is a weak Del Pezzo surface of degree 1, i.e., $-K_W$ is nef and big and $K_W^2 = 1$. This follows from the fact that any four of the eight points $p_0, p_1, \dots, p'_3, p$ are not collinear (cf. [11, Proposition 8.1.7]). The following proposition describes the (-2)-curves on W.

Proposition 2.2 *The surface W has exactly six* (-2)*-curves. Their divisor classes are as follows:*

$$C_j \equiv L - E_0 - E_j - E'_j,\ C'_j \equiv E_j - E'_j \text{ for } j = 1, 2, 3. \tag{2.1}$$

Proof Assume that $C \subset W$ is a (-2)-curve and its divisor class is $xL - a_0E_0 - \sum_{j=1}^3 (a_jE_j + a'_jE'_j) - aE$. If $\sigma(C)$ is a point, then C is one of the curves C'_1, C'_2, C'_3. If $c := \sigma(C)$ is a curve, then c is an irreducible curve of degree x having multiplicity at least a_0 (respectively $a_1, \dots, a$) at the point p_0 (respectively $p_1, \dots, p$). Therefore $a_0, \dots, a$ are nonnegative integers. If $x = 1$, then C is one of the curves C_1, C_2, C_3. It suffices to exclude the case $x \geq 2$. Since $C^2 = -2$ and $K_WC = 0$,

$$x^2 + 2 = a_0^2 + \sum_{j=1}^{3} \left(a_j^2 + a_j'^2\right) + a^2, \qquad 3x = a_0 + \sum_{j=1}^{3} \left(a_j + a'_j\right) + a.$$

By Cauchy's inequality, $9x^2 \leq (x^2 + 2) \cdot 8$ and thus $x \leq 4$.

If $x = 4$, then equality holds and $a_0 = \cdots = a = 2$. It follows that $CC'_1 = x - a_0 - a_1 - a'_1 = -2$. Then $C = C'_1$, a contradiction. Hence $x \neq 4$.

If $x = 2$, then c is an irreducible smooth conic and thus $a_0, \cdots, a \in \{0, 1\}$. Moreover, $a_0 + \sum_{j=1}^3 (a_j + a'_j) + a = a_0^2 + \sum_{j=1}^3 (a_j^2 + a_j'^2) + a^2 = 6$. Therefore exactly six of $a_0, \cdots, a$ are 1. This contradicts Lemma 2.1 or condition (II). Hence $x \neq 2$.

If $x = 3$, then c is an irreducible cubic and thus $a_0, \dots, a \in \{0, 1, 2\}$. Moreover, $a_0 + \sum_{j=1}^3 (a_j + a'_j) + a = 9$ and $a(a-1) + \sum_{j=1}^3 (a_j(a_j - 1) + a'_j(a'_j - 1)) + a(a-1) = 2$. Therefore exactly one of $a_0, \dots, a$ is 2 and the others are 1. If $a_0 = 2$ or $a_j = 2$ or $a'_j = 2$, then $CC_j = -1$. This contradicts the fact that C is irreducible. So $a = 2$ and thus $C \equiv -K_W - E$. We exclude this case by the following lemma. □

Lemma 2.3 *The linear system* $|-K_W - E|$ *is empty.*

Proof Assume by contradiction that $|-K_W - E| \neq \emptyset$. Then an element of $|-K_W - E|$ corresponds to a cubic c on $\mathbb{P}^2$ passing through $p_0, p_1, p_2, p_3, p'_1, p'_2, p'_3$ and having p as a singularity. Let $F(x_1, x_2, x_3)$ be the defining polynomial of c. Since c passes through p_1, p_2, p_3, F does not contain the terms x_1^3, x_2^3, x_3^3. Since $\overline{p_0p_j} : x_{j+1} = x_{j+2}$ is the tangent line to c at the point p_j, the coefficient of the term $x_j^2x_{j+1}$ is the negative of the term $x_j^2x_{j+2}$. We may assume that

$$F(x_1, x_2, x_3) = Ax_1^2(x_2 - x_3) + Bx_2^2(x_3 - x_1) + Cx_3^2(x_2 - x_1) + Dx_1x_2x_3,$$

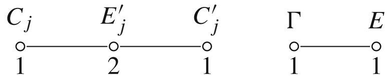
where $A, B, C, D \in \mathbb{C}$. Since c contains $p_0 = (1 : 1 : 1)$, $D = 0$. Assume that $p = (1 : \alpha : \beta)$, where $\alpha \neq 0, 1$, $\beta \neq 0, 1$ and $\alpha \neq \beta$ (see condition (I)). The singularity p of c imposes the following conditions on F:

$$\begin{aligned} (\alpha - \beta)A + \alpha^2(\beta - 1)B + \beta^2(\alpha - 1)C &= 0, \\ A + 2\alpha(\beta - 1)B + \beta^2C &= 0, \\ -A + \alpha^2B + 2\beta(\alpha - 1)C &= 0. \end{aligned}$$

The coefficient matrix has determinant $2\alpha\beta(\alpha - 1)(\beta - 1)(\alpha - \beta)$, which is nonzero, we have $A = B = C = 0$. Hence $|-K_W - E| = \emptyset$. □

The (-2)-curves $C_1, C_1', C_2, C_2', C_3, C_3'$ are pairwise disjoint. Let $\eta \colon W \to \Sigma$ be the morphism contracting these curves. By Proposition 2.2, Σ has six nodes and $-K_\Sigma$ is ample.

Now we turn to study the (-1)-curves on W. Denote by Γ the strict transform of the line $\overline{p_0 p}$. Then Γ is a (-1)-curve and $\Gamma \equiv L - E_0 - E$. Note that the pencil of lines on $\mathbb{P}^2$ passing through p_0 induces a the fibration $g \colon W \to \mathbb{P}^1$. Denote by F a general fiber of g. Then $F \equiv L - E_0$. Moreover, g has exactly four singular fibers:



for $j = 1, 2, 3$. Here numbers on nodes represent the multiplicities.

Starting from the two (-1)-curves Γ and E, we will find two more (-1)-curves. We first need some properties of the linear system $|-2K_W|$.

Proposition 2.4 ([11, Theorem 8.3.2]) *The linear system $|-2K_W|$ defines a morphism $\phi \colon W \to \mathbb{P}^3$. It factors as a birational morphism $\eta \colon W \to \Sigma$ contracting exactly the six (-2)-curves and a finite morphism $q \colon \Sigma \to Q$ of degree 2, where Q is a quadric cone.*

See [11, Theorem 8.3.2] for a general statement on weak Del Pezzo surfaces of any degree and for a proof.

Proposition 2.5 (1) *The linear system $|-2K_W - \Gamma|$ consists of a single (-1)-curve. Denote this (-1)-curve by $\tilde{B}_2$. Then $\tilde{B}_2\Gamma = 3$ and $\tilde{B}_2 E = 1$.*
(2) *The linear system $|-2K_W - E|$ consists of a single (-1)-curve. Denote this (-1)-curve by $\tilde{B}_3$. Then $\tilde{B}_3\Gamma = 1$ and $\tilde{B}_3 E = 3$.*
(3) *The curve $\Gamma + E + \tilde{B}_2 + \tilde{B}_3$ has only nodes.*

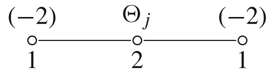
Proof (1) There is an exact sequence

$$0 \to \mathcal{O}_W(-2K_W - \Gamma) \to \mathcal{O}_W(-2K_W) \to \mathcal{O}_\Gamma(-2K_W) \to 0.$$

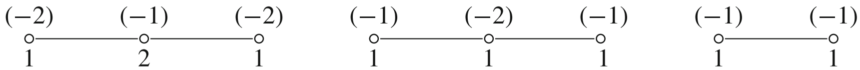
Since $-2K_W\Gamma = 2$, by Proposition 2.4, $h^0(W, \mathcal{O}_W(-2K_W - \Gamma)) \geq 1$.
Now assume that $A \in |-2K_W - \Gamma|$. Then $A^2 = -1$ and $K_W A = -1$. Since $-K_W$ is nef and big, we may assume that $A = A_1 + A_2$, where A_1 is an irreducible curve with $-K_W A_1 = 1$ and $Supp(A_2)$ is contained in the union of the (-2)-curves. Since Γ is disjoint from the (-2)-curves, we have $A_1A_2 + A_2^2 = (-2K_W - \Gamma)A_2 = 0$. Since $(A_1 + A_2)^2 = A^2 = -1$, it follows that $A_1^2 + A_1A_2 = -1$. In particular, $A_1^2 \leq -1$. By the adjunction formula, A_1 is a (-1)-curve and thus $A_1A_2 = A_2^2 = 0$. Hence $A_2 = 0$ and $A = A_1$ is a (-1)-curve.
Hence $|-2K_W - \Gamma|$ consists of a (-1)-curve $\tilde{B}_2$. Moreover, $\tilde{B}_2\Gamma = (-2K_W - \Gamma)\Gamma = 3$ and $\tilde{B}_2 E = (-2K_W - \Gamma)E = 1$.
(2) The proof is similar to (1).
(3) Recall that $\Gamma + E$ is disjoint from the (-2)-curves, since they are contained in different fibers of g. It follows that $\tilde{B}_2 \equiv -2K_W - \Gamma$ and $\tilde{B}_3 \equiv -2K_W - E$ are also disjoint from the (-2)-curves. Note that $\tilde{B}_2\tilde{B}_3 = (-2K_W - \Gamma)(-2K_W - E) = 1$ and $\tilde{B}_2 E = \tilde{B}_3\Gamma = \Gamma E = 1$. It suffices to prove that

(a) Γ (respectively E) intersects $\tilde{B}_2 + \tilde{B}_3$ transversely.
(b) $\tilde{B}_2$ (respectively $\tilde{B}_3$) intersects $\Gamma + E$ transversely.

For (a), let $M := \tilde{B}_2 + \tilde{B}_3$. Then $|M|$ induces a genus 0 fibration $h \colon W \to \mathbb{P}^1$. Since $MC_j = MC_j' = 0$ for $j = 1, 2, 3$, the six (-2)-curves are contained in the singular fibers of h. We claim that h has exactly four singular fibers: $\tilde{B}_2 + \tilde{B}_3$ and $M_j (j = 1, 2, 3)$:



where the (-2)-curves are $C_1, \ldots, C_3'$, and Θ_j is a (-1)-curve for $j = 1, 2, 3$.
Actually, since $-K_W$ is nef, any irreducible component A in a singular fiber is either a (-2)-curve or a (-1)-curve. Since $-K_W M = -K_W(\tilde{B}_2 + \tilde{B}_3) = 2$, any singular fiber contains either one (-1)-curve with multiplicity 2, or two (-1)-curves with each multiplicity 1. Since the (-2)-curves of W are pairwise disjoint, any singular fiber is of one of the following types:



Each fiber of the first two types contributes 2 to the Picard number $\rho(W)$. Note that W has six (-2)-curves and $\rho(W) = 9$. By considering how the (-2)-curves are distributed along the singular fibers, we see that there are four singluar fibers, one of which is $\tilde{B}_2 + \tilde{B}_3$, the other three are of the first type. Our claim is proved.
Since $M\Gamma = (-4K_W - \Gamma - E)\Gamma = 4$, $h|_\Gamma : \Gamma \to \mathbb{P}^1$ is of degree 4. Since Γ is disjoint from C_j and C_j', $\Gamma\Theta_j = \frac{1}{2}\Gamma M = 2$. Thus M_j induces ramification points of $h|_\Gamma$. The Riemann-Hurwitz formula implies that $h|_\Gamma$ does not have any other ramification points than those on M_1, M_2, M_3. In particular, Γ intersects the fiber $\tilde{B}_2 + \tilde{B}_3$ transversely. A similar argument shows that E intersects $\tilde{B}_2 + \tilde{B}_3$ transversely.
For (b), we return to the fibration $g : W \to \mathbb{P}^1$. Note that the singular fibers of g are of the same types of those of h. Also note that $\tilde{B}_2 F = \tilde{B}_2(\Gamma + E) = 4$ and $\tilde{B}_3 F = \tilde{B}_3(\Gamma + E) = 4$. A similar argument as the proof of (a) shows that $\tilde{B}_2$ (respectively $\tilde{B}_3$) intersects $\Gamma + E$ transversely.

For later use, we prove the following lemma.

Lemma 2.6 *The linear system* $|-2K_W + \Gamma|$ *defines a birational morphism* $\phi : W \to \mathbb{P}^5$ *and* ϕ *contracts exactly the six* (-2)*-curves* $C_1, \ldots, C_3'$.

Proof Since $H^1(W, \mathcal{O}_W(-2K_W)) = 0$ and $(-2K_W + \Gamma)\Gamma = 1$, the trace of $|-2K_W + \Gamma|$ on Γ is complete and base point free. By Proposition 2.4, $|-2K_W + \Gamma|$ is base point free. By Ramanujam's vanishing theorem and Riemann–Roch theorem, $h^0(W, \mathcal{O}_W(-2K_W + \Gamma)) = 6$. Since $(-2K_W + \Gamma)^2 = 7$ and $Im(\phi)$ is nondegenerate, ϕ is birational.

Now assume that C is an irreducible curve and $(-2K_W + \Gamma)C = 0$. Then $C^2 < 0$. Since $-K_W$ is nef, C is either a (-2)-curve or a (-1)-curve. If C is a (-1)-curve, then $\Gamma C = 2K_W C = -2$. This contradicts the fact that Γ is a (-1)-curve. Therefore C is one of the (-2)-curves $C_1, \ldots, C_3'$. Hence ϕ contracts exactly the (-2)-curves $C_1, \ldots, C_3'$. □

3 Construction of surfaces of general type

In this section, we construct a family of surfaces of general type as finite $\mathbb{Z}_2^2$-covers of the weak Del Pezzo surface W. First, we define three effective divisors on W

$$\begin{aligned}
\Delta_1 &:= F_b + \Gamma + (C_1 + C_1' + C_2 + C_2') \equiv 4L - 4E_0 - 2E_1' - 2E_2' - E, \\
\Delta_2 &:= \tilde{B}_2 + (C_3 + C_3') \equiv -2K_W - 2E_3' + E, \qquad (3.1)\\
\Delta_3 &:= \tilde{B}_3 \equiv -2K_W - E.
\end{aligned}$$

Here we require that

(A) The curve F_b is a smooth fiber of the fibration $g\colon W \to \mathbb{P}^1$ (see Sect. 2);
(B) The divisor $\Delta := \Delta_1 + \Delta_2 + \Delta_3$ has only nodes.
By Proposition 2.5, $\tilde{B}_2 + \tilde{B}_3 + \Gamma$ has only nodes. Since F_b, Γ, $\tilde{B}_2$, $\tilde{B}_3$ are disjoint from the (-2)-curves $C_1, \ldots, C_3'$, (B) is equivalent to the requirement that F_b intersects the curve $\tilde{B}_2 + \tilde{B}_3$ transversely. By Bertini's theorem, this is the case for a general fiber of g.

We also define three divisors

$$\begin{aligned}\mathcal{L}_1 &= -2K_W - E_3', \\ \mathcal{L}_2 &= -K_W + (2L - 2E_0 - E_1' - E_2' - E), \\ \mathcal{L}_3 &= -K_W + (2L - 2E_0 - E_1' - E_2' - E_3').\end{aligned} \tag{3.2}$$

It follows that $2\mathcal{L}_i \equiv \Delta_{i+1} + \Delta_{i+2}$, $\mathcal{L}_i + \Delta_i \equiv \mathcal{L}_{i+1} + \mathcal{L}_{i+2}$ for $i = 1, 2, 3$.

By [8, Section 1] or [9, Theorem 2], the data (3.1) and (3.2) define a finite G-cover $\tilde{\pi}\colon V \to W$. Conditions (A) and (B) imply that V is smooth. By the formulae in [9, Section 2],

$$2K_V \equiv \tilde{\pi}^*(2K_W + \Delta) \equiv \tilde{\pi}^*\left(-2K_W + \Gamma + \sum_{j=1}^{3}(C_j + C_j')\right), \tag{3.3}$$

$$p_g(V) = p_g(W) + \sum_{i=1}^{3} h^0(W, \mathcal{O}_W(K_W + \mathcal{L}_i)). \tag{3.4}$$

Since C_j or C_j' (for $j = 1, 2, 3$) is a connected component of Δ, the (set-theoretic) inverse image $\tilde{\pi}^{-1}C_j$ or $\tilde{\pi}^{-1}C_j'$ is a disjoint union of two (-1)-curves. Let $\varepsilon\colon V \to S$ be the blowdown of these twelve (-1)-curves. By construction, there is a finite G-cover $\pi\colon S \to \Sigma$ such that the following diagram commutes:

$$\begin{array}{ccc} V & \xrightarrow{\varepsilon} & S \\ \downarrow{\scriptstyle\tilde{\pi}} & & \downarrow{\scriptstyle\pi} \\ W & \xrightarrow{\eta} & \Sigma \end{array} \tag{3.5}$$

The discussion above shows that

$$2K_S \equiv \pi^*(-2K_\Sigma + \gamma), \tag{3.6}$$

where $\gamma = \eta(\Gamma)$ is a (-1)-curve contained in the smooth locus of Σ.

Theorem 3.1 *S is a smooth minimal surface of general type with $K_S^2 = 7$ and $p_g(S) = 0$. Moreover, K_S is ample.*

Proof By (3.6), $K_S^2 = \frac{1}{4}4(-2K_\Sigma + \gamma)^2 = 7$. To show that $p_g(S) = p_g(V) = 0$, we use (3.4). Since $p_g(W) = 0$, it suffices to show that $h^0(W, \mathcal{O}_W(K_W + \mathcal{L}_i)) = 0$ for $i = 1, 2, 3$.

By (3.2) and the configurations of the eight points on $\mathbb{P}^2$, it is clear that $|K_W + \mathcal{L}_2|$ and $|K_W + \mathcal{L}_3|$ are empty. Now assume by contradiction that $|K_W + \mathcal{L}_1| \neq \emptyset$. Let $D \in |K_W + \mathcal{L}_1|$. By (3.2) and (2.1), $DC_3 = DC_3' = -1$ and thus $D \geq C_3 + C_3'$. Note that $D - C_3 - C_3' \equiv 2L - E_1 - E_1' - E_2 - E_2' - E_3 - E$. This contradicts condition (II). Therefore $|K_W + \mathcal{L}_1| = \emptyset$ and thus $p_g(S) = 0$.

Lemma 2.6 implies that $-2K_\Sigma + \gamma$ is ample. Since π is a finite morphism, K_S is ample by (3.6). In particular, S is minimal and of general type. ◻

We have constructed a family of surfaces with a $\mathbb{Z}_2^2$-action, parameterized by a 3-dimensional open subset $\{(p, F_b)|\ p \in \mathbb{P}^2$ satisfying conditions (I) and (II), $F_b \in |F|$satisfying conditions (A) and (B)$\}$ of $\mathbb{P}^2 \times \mathbb{P}^1$. A natural question arises: is this a new family? To the best of the author's knowledge, the Inoue surfaces are the only known smooth minimal surfaces of general type with $K^2 = 7$ and $p_g = 0$. So we intend to show that the surfaces constructed here satisfy different properties from the Inoue surfaces.

4 The bicanonical map

It is known that the bicanonical morphism of an Inoue surface has degree 2 (cf. [16, Example 4.1]). In this section, we prove that our surfaces have birational bicanonical morphism. So they are indeed new surfaces.

Proposition 4.1 *For a surface S as in Theorem 3.1, the dimensions of the eigenspaces of $H^0(S, \mathcal{O}_S(2K_S))$ for the G-action are as follows:* $\dim H^0(S, \mathcal{O}_S(2K_S))^{inv} = 6$, $\dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_1} = 1$, $\dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_2} = 1$ *and* $\dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_3} = 0$.

Proof By the formulae in [9, Section 2],

$$\dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_i} = h^0(W, \mathcal{O}_W(2K_W + \mathcal{L}_{i+1} + \mathcal{L}_{i+2}))$$

for $i = 1, 2, 3$ and $\dim H^0(S, \mathcal{O}_S(2K_S))^{inv} = h^0(W, \mathcal{O}_W(2K_W + \Delta))$. By (3.2) and (2.1),

$$2K_W + \mathcal{L}_1 + \mathcal{L}_3 \equiv (2L - E_1 - E_2 - E_3 - E_3' - E) + \sum_{j=1}^{3}(C_j + C_j'),$$

$$2K_W + \mathcal{L}_1 + \mathcal{L}_2 \equiv (2L - E_1 - E_2 - E_3 - 2E) + \sum_{j=1}^{3}(C_j + C_j').$$

It is immediate that $\dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_2} = 1$ and $\dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_3} = 0$.

By (3.2), (3.3) and Lemma 2.6, we have $\dim H^0(S, \mathcal{O}_S(2K_S))^{inv} = 6$. Since $h^0(S, \mathcal{O}_S(2K_S)) = K_S^2 + 1 = 8$, it follows that $\dim H^0(S, \mathcal{O}_S(2K_S))^{\chi_1} = 1$. □

Corollary 4.2 *The bicanonical morphism $\varphi \colon S \to \mathbb{P}^7$ is not composed with any involution of the group G.*

Theorem 4.3 *For a surface S as in Theorem 3.1, the bicanonical morphism $\varphi \colon S \to \mathbb{P}^7$ is birational.*

Proof By [16] and [17], φ has degree either 1 or 2. By Lemma 2.6, $|-2K_\Sigma + \gamma|$ defines a birational morphism of Σ. Since $2K_S = \pi^*(-2K_\Sigma + \gamma)$ and S is a $\mathbb{Z}_2^2$-Galois cover of Σ, φ is birational if and only if φ is not composed with any involution of the Galois group. Thus φ is a birational by Corollary 4.2. □

5 The intermediate double covers and Bloch's conjecture

By construction, we see that the automorphism group of the surface S in Theorem 3.1 contains at least three involutions. Involutions on surfaces of general type with $K^2 = 7$ and $p_g = 0$ are studied in [18] and [15]. Lists of numerical possibilities are given in these articles. The surfaces constructed here realize some possibilities of their lists.

Proposition 5.1 *Let S be a surface as in Theorem 3.1.*

(1) *The involution g_1 has 9 isolated fixed points on S and S/g_1 is a rational surface.*
(2) *The involution g_2 has 9 isolated fixed points on S and S/g_2 is birational to an Enriques surface.*
(3) *The involution g_3 has 7 isolated fixed points on S. Moreover, S/g_3 has Kodaira dimension 1 and K_{S/g_3} is nef.*

Proof (1) The $\mathbb{Z}_2^2$-cover $\tilde{\pi}: V \to W$ of (3.5) factors through the intermediate double cover $\tilde{\pi}_1: V_1 \to W$. The covering data associated to $\tilde{\pi}_1$ is $\Delta_2 + \Delta_3 \equiv 2\mathcal{L}_1$. Note that V_1 has exactly one node, lying over the node of $\tilde{B}_2 + \tilde{B}_3$. The inverse image $\tilde{\pi}_1^{-1} C_3$ or $\tilde{\pi}_1^{-1} C_3'$ is a (-1)-curve, while the inverse image $\tilde{\pi}_1^{-1} C_k$ or $\tilde{\pi}_2^{-1} C_k'$ $(k = 1, 2)$ is a disjoint union of two (-2)-curves. Contracting all these curves, we obtain the quotient surface S/g_1. By construction, S/g_1 has exactly 9 nodes (the images of the node of V_1 and the 8 (-2)-curves $\tilde{\pi}_1^{-1} C_k, \tilde{\pi}_1^{-1} C_k'$ $(k = 1, 2)$). Hence g_1 has 9 isolated fixed points on S.
In the proof of Proposition 2.4 (3), we show that $|M| = |\tilde{B}_2 + \tilde{B}_3|$ gives a genus 0 fibration $h: W \to \mathbb{P}^1$. For a general M, we have $M(\Delta_2 + \Delta_3) = 0$. So the pullback of M by $\tilde{\pi}_1$ is a disjoint union of two smooth rational curves. Applying Stein factorization to the morphism $h \circ \tilde{\pi}: V_1 \to \mathbb{P}^1$, we conclude that V_1 has a genus 0 fibration. Since V_1 is a quotient of V, $q(V_1) = 0$. Hence V_1 is a rational surface and so is S/g_1.

(2) Consider the intermediate double cover $\tilde{\pi}_2: V_2 \to W$ associated to the data $\Delta_1 + \Delta_3 \equiv 2\mathcal{L}_2$. Then V_2 has exactly 5 nodes, lying over the 5 nodes of $F_b + \Gamma + \tilde{B}_3$. Contracting the inverse image $\tilde{\pi}_2^{-1}(C_j)$ and $\tilde{\pi}_2^{-1}(C_j')$ $(j = 1, 2, 3)$, we obtain S/g_2. It has 9 nodes (the images of the 5 nodes and the 4 (-2)-curves $\tilde{\pi}_2^{-1} C_3, \tilde{\pi}_2^{-1} C_3'$). Hence g_2 has 9 isolated fixed points on S.
Since V_2 is a quotient of V, $p_g(V_2) = q(V_2) = 0$. To show that V_2 is birational to an Enriques surface, it suffices to show that $P_{2m+1}(V_2) = 0$ and $P_{2m}(V_2) = 1$ for $m \geq 1$. Note that $K_{V_2} = \tilde{\pi}_2^*(K_W + \mathcal{L}_2)$. Therefore,

$$P_{2m}(V_2) = h^0(W, \mathcal{O}_W(2mK_W + (2m-1)\mathcal{L}_2) + h^0(W, \mathcal{O}_W(2mK_W + 2m\mathcal{L}_2))$$

By (3.2),

$$2mK_W + 2m\mathcal{L}_2 \equiv 2m\Gamma + m(C_1 + C_1' + C_2 + C_2'). \tag{5.1}$$

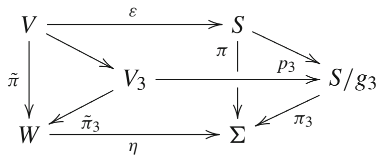
It is immediate that $h^0(W, \mathcal{O}_W(2mK_W + 2m\mathcal{L}_2)) = 1$.
By (3.2), it is clear that $|2K_W + \mathcal{L}_2| = \emptyset$. For $m \geq 2$, by (5.1),

$$2mK_W + (2m-1)\mathcal{L}_2 \equiv 2(m-1)\Gamma + (m-1)(C_1 + C_1' + C_2 + C_2') + (2K_W + \mathcal{L}_2).$$

Note that $\Gamma(2K_W + \mathcal{L}_2) = -2$, $C_k(2K_W + \mathcal{L}_2) = C_k'(2K_W + \mathcal{L}_2) = -1 (k = 1, 2)$. Similarly to the proof of Theorem 3.1, it is easy to show $|2mK_W + (2m-1)\mathcal{L}_2| = \emptyset$. Hence $P_{2m}(V_2) = 1$.
A similar argument using (5.1) shows that $P_{2m+1}(V_2) = 0$ for $m \geq 1$. Hence V_2 is birational to an Enriques surface.

(3) Consider the intermediate double cover $\tilde{\pi}_3: V_3 \to W$ associated to the data $\Delta_1 + \Delta_2 \equiv 2\mathcal{L}_3$. Then V_3 has exactly 7 nodes, lying over the 7 nodes of the curve $F_b + \Gamma + \tilde{B}_2$. The inverse image $\tilde{\pi}_3^{-1} C_j$ or $\tilde{\pi}_3^{-1} C_j'$ $(j = 1, 2, 3)$ is a (-1)-curve. Contracting these (-1)-curves, we obtain S/g_3, which has 7 nodes. Thus g_3 has 7 isolated fixed points on S. By construction, there are double covers $\pi_3: S/g_3 \to \Sigma$ and $p_3: S \to S/g_3$ such

that the following diagram commutes.



By (3.2), $2K_{V_3} \equiv \tilde{\pi}_3^*(2K_W + 2\mathcal{L}_3) \equiv \tilde{\pi}_3^*(L - E_0 + C_1 + C_2 + C_3 + C_1' + C_2' + C_3')$. In Sect. 2, we show that $|L - E_0|$ gives a genus 0 fibration $g\colon W \to \mathbb{P}^1$, and the (-2)-curves $C_1, \ldots, C_3'$ are contained in the fibers of g. This induces a fibration on $g'\colon \Sigma \to \mathbb{P}^1$. Denote the general fiber of g' by F'. From the diagram, $2K_{S/g_3} \equiv \pi_3^*(F')$. Thus K_{S/g_3} is nef, $K_{S/g_3}^2 = 0$ and S/g_3 has Kodaira dimension 1. □

Remark 5.1 We remark that (2) (respectively (3)) realize some numerical possibilities of case a) (respectively case b)) of [18, Theorem 4.1]. (1), (2) and (3) realize respectively the following possible cases in the list of [15]:

(1) $k = 9$, $K_W^2 = -2$, W is a rational surface and $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$.

(2) $k = 9$, $K_W^2 = -2$, W is birational to an Enriques surface and $B_0 = \underset{(3,-2)}{\Gamma_0}$.

(3) $k = 7$, $K_W^2 = 0$, W is minimal proper elliptic and $B_0 = \underset{(2,-2)}{\Gamma_0}$.

These cases are different from those of the Inoue surfaces. See [15, Section 5] and Sect. 6.

Recently, it is shown in [3] that the Bloch's conjecture ([6]) holds for Inoue surfaces with $K^2 = 7$ and $p_g = 0$, by using the method of "enough automorphisms" ([14] and [1]). We observe that the results of [3] apply to our surfaces.

Theorem 5.2 *Let S be a surface as in Theorem 3.1. Then S satisfies the Bloch's conjecture, i.e., the kernel $T(S)$ of the natural morphism $A_0^0(S) \to Alb(S)$ is trivial. In particular, $A_0^0(S) = 0$.*

Proof The first statement follows directly from [3, Proposition 1.3,Corollary 1.5] and Proposition 5.1. Since $Alb(S)$ is trivial in our case, Bloch's conjecture says that $A_0^0(S) = 0$.

6 Remarks on related topics

In a pre-version of [15], it is claimed that the three quotients of an Inoue surface by the involutions are rational. The claim turns out to be wrong. We will point out that one of the quotients is birational to an Enriques surface. In [18], a family of surfaces of general type with $K^2 = 7$ is constructed as bidouble planes. However, here we show that this family consists of Inoue surfaces.

We first stick to the same notation of [18, Section 4.2] (Fig. 2).

Let p, p_1, p_2, p_3 be four points in general position of $\mathbb{P}^2$, and let p_k' $(k = 1, 2)$ be the infinitely near point of p_k corresponding to the line $\overline{p_k p}$. Denote by T_j $(j = 1, 2, 3)$ the line $\overline{p_j p}$ and by T_4 a general line passing through p. Denote by C_1, C_2 two distinct smooth conics passing through p_1, p_2, p_1', p_2'. Denote by L a quintic passing through p, having p_k

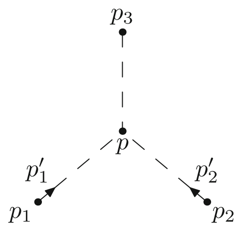


Fig. 2 Configurations of p, p_1, p_2, p_3, p_1' and p_2'

($k = 1, 2$) as a (2, 2)-singularity and having p_3 as an ordinary triple point (See the last paragraph in [18, Subsection 4.2.1]).

We claim that L is a union of a conic C and a cubic Γ, where C is the conic passing through $p_1, p_2, p_1', p_2', p_3$, and Γ is a cubic passing through p, p_1, p_2, p_1', p_2' and having p_3 as an ordinary double point. Note that $LC = 11$. The claim follows from Bézout's Theorem.

In [18], it is claimed that the smooth minimal model of the bidouble plane associated to the following branch divisors is a surface of general type with $K^2 = 7$ and $p_g = 0$:

$$D_1 = L = C + \Gamma,\ D_2 = T_1 + C_1 + C_2,\ D_3 = T_2 + T_3 + T_4. \tag{6.1}$$

We explain how to find the smooth minimal model of the bidouble plane and show that this is indeed an Inoue surface with $K^2 = 7$.

Let $\sigma : \widetilde{Y} \to \mathbb{P}^2$ be the blowup at the six points p, p_1, p_2, p_3, p_1' and p_2'. Denote by $\hat{L}$ the pullback of a general line of $\mathbb{P}^2$ and by E (respectively $E_j,\ E_k'$) the total transform of p (respectively $p_j (\ j = 1, 2, 3),\ p_k' (\ k = 1, 2)$). We also denote by $\widetilde{T}_1$ the strict transform of T_1, and similarly for other curves.

The surface $\widetilde{Y}$ is the minimal resolution of the 4-nodal cubic surface Y. We explain some geometry of $\widetilde{Y}$.

(1) $\widetilde{Y}$ has exactly four (-2)-curves: $\widetilde{T}_1 = \hat{L} - E_1 - E_1' - E$, $\widetilde{T}_2 = \hat{L} - E_2 - E_2' - E$, $N_1 = E_1 - E_1'$ and $N_2 = E_2 - E_2'$. These curves correspond to the four nodes of Y.
(2) $\widetilde{Y}$ contains nine (-1)-curves, which correspond to nine lines on the 4-nodal cubic surface Y. Among these curves, there are exactly three, which are disjoint from the (-2)-curves: $\widetilde{T}_3 = \hat{L} - E_3 - E$, $\widetilde{C} = 2\hat{L} - E_1 - E_1' - E_2 - E_2' - E_3$ and E_3. They correspond to three lines on Y which do not pass through any nodes. In particular, they are determined by the 4-nodal cubic surface Y.
(3) Note that

$$\begin{aligned} &\widetilde{T}_4 \in |\hat{L} - E|, && \widetilde{T}_4 + \widetilde{C} \equiv -K_{\widetilde{Y}}, \\ &\widetilde{C}_1, \widetilde{C}_2 \in |2\hat{L} - E_1 - E_1' - E_2 - E_2'|, && \widetilde{C}_1 + \widetilde{T}_3 \equiv \widetilde{C}_2 + \widetilde{T}_3 \equiv -K_{\widetilde{Y}}, \\ &\tilde{\Gamma} \in |3\hat{L} - E_1 - E_1' - E_2 - E_2' - 2E_3 - E|, && \tilde{\Gamma} + E_3 \equiv -K_{\widetilde{Y}}. \end{aligned}$$

So the divisor classes of $\widetilde{T}_4, \widetilde{C}_1, \widetilde{C}_2$ and $\tilde{\Gamma}$ are also determined by the 4-nodal cubic surface.

The total transforms of D_1, D_2, D_3 on $\widetilde{Y}$ are

$$\begin{aligned} \sigma^*(D_1) &= \widetilde{C} + \tilde{\Gamma} + 2E_1 + 2E_1' + 2E_2 + 2E_2' + 3E_3 + E, \\ \sigma^*(D_2) &= \widetilde{T}_1 + \widetilde{C}_1 + \widetilde{C}_2 + N_1 + 2E_1' + E + 2\left(E_1 + E_1' + E_2 + E_2'\right), \\ \sigma^*(D_3) &= \widetilde{T}_2 + \widetilde{T}_3 + \widetilde{T}_4 + N_2 + 2E_2' + E_3 + 3E. \end{aligned}$$

Applying the normalization procedure in the theory of bidouble covers (cf. [9, Section 2, Remark 3]), we obtain three new divisors:

$$\begin{aligned}\tilde{D}_1 &= \widetilde{C} + \tilde{\Gamma},\\ \tilde{D}_2 &= \widetilde{T}_1 + \widetilde{C}_1 + \widetilde{C}_2 + N_1 + E_3,\\ \tilde{D}_3 &= \widetilde{T}_2 + \widetilde{T}_3 + \widetilde{T}_4 + N_2.\end{aligned} \tag{6.2}$$

The bidouble cover $\tilde{\pi} : \widetilde{S} \to \widetilde{Y}$ associated to (6.2) is birational to the bidouble plane associated to (6.1). Using the above explanation of the geometry of $\widetilde{Y}$ and comparing (6.2) with [16, Example 4.1 (I)], we conclude that the smooth minimal model of $\widetilde{S}$ (and thus of the bidouble plane) is an Inoue surface.

Now we point out a mistake in the pre-version of [15]. This observation is due to Carlos Rito. Here we use the notation of [16, Example 4.1], as [15] uses almost the same notation (except denoting the minimal resolution Σ of the 4-nodal cubic surface by P). In [15, Section 5, paragraph 4], the authors claim "Also, $H^0(T_2, \mathcal{O}_{T_2}(2K_{T_2})) = 0$ by a similar argument as the case $i = 1$". Here T_2 is a double cover of Σ associated to the data $D_1 + D_3 \equiv 2L_2$. However, we will show that $H^0(T_2, \mathcal{O}_{T_2}(2K_{T_2})) = 1$.

It suffices to show $h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + L_2)) = 0$ and $h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + 2L_2)) = 1$, where $L_2 = 6l - 2e_1 - 2e_2 - 2e_3 - 2e_4 - 3e_5 - 3e_6$ ([16, Example 4.1] (II)). Since $2K_\Sigma + L_2 = -e_5 - e_6$, clearly $h^0(\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + L_2)) = 0$. Note that

$$\begin{aligned}2K_\Sigma + 2L_2 \equiv{}& (l - e_1 - e_2 - e_5) + (l - e_3 - e_4 - e_5) + (l - e_1 - e_4 - e_6)\\ &+ (l - e_2 - e_3 - e_6) + 2(l - e_5 - e_6).\end{aligned}$$

From the configuration of the six points on $\mathbb{P}^2$ ([16, Figure 1]), it is immediate that $h^0(\mathcal{O}_\Sigma, \mathcal{O}_\Sigma(2K_\Sigma + 2L_2)) = 1$.

Finally, we remark that T_2 is birational to an Enriques surface as described in [18] and it realizes the case $k = 9$, $K_W^2 = -2$ and $B_0 = \underset{(3,0)}{\Gamma_0} + \underset{(1,-2)}{\Gamma_1}$ on the list of [15].

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