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Estimation of the conformal factor under bounded Willmore energy

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Abstract We prove for closed, orientable surfaces in $\mathbb{R}^3$ with Willmore energy less that $8\pi - \delta$ and whose conformal structures are compactly contained in moduli space that after applying appropriate Möbius transformations the conformal factors between the induced metrics and conformal metrics of constant curvature are uniformly bounded by constants depending only on $\delta > 0$, the genus of the surfaces and the compact subset of the moduli space. Secondly, for a given sequence of closed, orientable surfaces as above, we prove that the conformal factor remains bounded without applying Möbius transformations if and only if no topology is lost. Similar estimates hold in higher codimension.

Keywords Willmore surfaces · Conformal parametrization · Geometric measure theory

Mathematics Subject Classification 53A05 · 53A30 · 53C21 · 49Q15

1 Introduction

For an immersion $f : \Sigma \to \mathbb{R}^n$ of a closed surface Σ, which we assume to be orientable, the Willmore functional is defined by

$$\mathcal{W}(f) = \frac{1}{4} \int_\Sigma |\vec{\mathbf{H}}|^2 \, \mathrm{d}\mu_g,$$

where $\vec{\mathbf{H}}$ denotes the mean curvature vector of f, $g = f^* g_{euc}$ the pull-back metric and μ_g the induced area measure on Σ. The main interest for the Willmore functional steams from its invariance under conformal transformations.

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We continue the work of [9] and denote by β_p^n the infimum of the Willmore energy of immersions $f : \Sigma \to \mathbb{R}^n$ of a closed, orientable surface Σ of genus p. We know $\mathcal{W}(f) \geq 4\pi$ with equality only for round spheres, in particular $\beta_p^n \geq 4\pi$. We put as in [9]

$$\tilde{\beta}_p^n := \min\left\{4\pi + \sum_{i=1}^k (\beta_{p_i}^n - 4\pi) : 1 \leq p_i < p, \sum_{i=1}^k p_i = p\right\}, \tag{1.1}$$

where $\tilde{\beta}_1^n = \infty$,

$$e_n := \begin{cases} 4\pi & \text{for} \quad n = 3, \\ 8\pi/3 & \text{for} \quad n = 4, \\ 2\pi & \text{for} \quad n \geq 5, \end{cases} \tag{1.2}$$

and define the constants

$$\mathcal{W}_{n,p} := \min(8\pi, \tilde{\beta}_p^n, \beta_p^n + e_n). \tag{1.3}$$

For $n = 3$, the last term could be ommitted as $\beta_p^3 + e_3 > 8\pi$.

By Poincaré's theorem any smooth metric g on $\Sigma \not\cong S^2$ is uniquely conformal to a unit volume constant curvature metric

$$g_{poin} = e^{-2u} g. \tag{1.4}$$

The compactness theorem [9] Theorem 4.1 or Theorem 5.3 in §5 for $n \geq 5$ estimates the conformal factor in (1.4) for the pull-back metric of a smooth immersion $f : \Sigma \to \mathbb{R}^n$ of a closed, orientable surface Σ of genus p with

$$\mathcal{W}(f) \leq \mathcal{W}_{n,p} - \delta \tag{1.5}$$

after applying an appropriate Möbius transformation by

$$\parallel u \parallel_{L^\infty(\Sigma)} \leq C(n, p, \delta). \tag{1.6}$$

In the definition (1.3), the bound 8π excludes by the Li-Yau inequality in [11] branch points, and the bound $\mathcal{W}_{n,p} \leq \tilde{\beta}_{n,p}$ prevents topological splitting in the sense that p handles of Σ do not group in p_1 and p_2 handles with $p_1 + p_2 = p, 1 \leq p_1, p_2 < p$. The last bound $\mathcal{W}_{n,p} \leq \beta_p^n + e_n$ is an algebraic condition in order to apply the Hardy space theory in the work of Müller and Sverak [12]. As $\beta_p^n \geq 4\pi$ and $e_3 = 4\pi$, this does not appear in (1.3) for $n = 3$. The constant $e_n = 2\pi$ for $n \geq 5$ is directly taken from the general situation in [12], see §4. In [9] Theorem 6.1, the constants were adapted from [12] to our situation to $e_4 = 8\pi/3$. Whether these constants are optimal for $n \geq 4$ is not clear.

The compactness theorem [9] Theorem 4.1 was used in [10] to obtain existence of conformally constrained Willmore minimizers, these are minimizers of the Willmore energy of immersions conformal to a smooth metric g_0 on Σ, when the infimum satisfies

$$\mathcal{W}(\Sigma, g_0, n) := \inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \text{ smooth immersion conformal to } g_0\} < \mathcal{W}_{n,p}. \tag{1.7}$$

The existence of conformally constrained Willmore minimizers was recently extended in [7] and [13] to $\mathcal{W}(\Sigma, g_0, n) < 8\pi$ in any codimension. Moreover in [10] smoothness of any conformally constrained Willmore minimizer was shown.

Actually even if we do not fix the metric g_0 or likewise the conformal class induced by f, the estimation of the conformal factor in (1.4) gives control on the pull-back metric after reparametrization, as it was shown in [9, Lemma 5.1] that a bound on the conformal factor and on the Willmore energy

$$\mathcal{W}(f), \max_{\Sigma} |u| \leq \Lambda$$

implies that the induced conformal structure lie in a compact subset of the moduli space depending on n, Λ and the genus of Σ.

The first aim of this article is to prove a partial converse of [9, Lemma 5.1].

Theorem 1.1 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a smooth immersion of a closed, orientable surface* Σ *of genus* $p \geq 1$ *with*

$$\mathcal{W}(f) \leq \begin{cases} 8\pi - \delta & \text{for} \quad n = 3, \\ \beta_p^n + e_n - \delta & \text{for} \quad n \geq 4, \end{cases}$$

for some $\delta > 0$ *and assume that the conformal structure induced by the pull-back metric of* f *lies in a compact subset K of the moduli space.*

Then after applying an Möbius transformation, the pull-back metric $g := f^* g_{euc}$ *is uniformly conformal to a unit volume constant curvature metric* $g_{poin} := e^{-2u} g$*, more precisely*

$$\| u \|_{L^\infty(\Sigma)} \leq C(n, p, K, \delta).$$

□

This converse is only partial as the energy bound is restricted by 8π and the algebraic energy condition in (1.3). Here the bound $\mathcal{W}_{n,p} \leq \tilde{\beta}_{n,p}$ is replaced by the compactness in moduli space, which is weaker by [7] Theorem 5.3 and Theorem 5.5 or [14] Theorem I.1.

The algebraic energy condition restricts the energy loss by comparing to the infimum β_p^n. This can be localized.

Theorem 1.2 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a smooth immersion of a closed, orientable surface* $\Sigma \ncong S^2$ *with*

$$\mathcal{W}(f) \leq \min(8\pi, \mathcal{W}(\Sigma, f^* g_{euc}, n) + e_n) - \delta$$

for some $\delta > 0$*. Moreover we assume that the conformal structure induced by the pull-back metric of f lies in a compact subset K of the moduli space.*

Then after applying an Möbius transformation, the pull-back metric $g := f^* g_{euc}$ *is uniformly conformal to a unit volume constant curvature metric* $g_{poin} := e^{-2u} g$*, more precisely*

$$\| u \|_{L^\infty(\Sigma)} \leq C(n, p, K, \delta).$$

□

Clearly Theorem 1.1 implies Theorem 1.2 as $\mathcal{W}(\Sigma, g_0, n) \geq 4\pi$. Theorem 1.1 is suited when working for example on a fixed Riemann surface. It extends the framework of [10] to $\mathcal{W}(\Sigma, g_0, n) < 8\pi$ in any codimension.

There is a second aim of this article. The compactness theorem [9] Theorem 4.1 and Theorems 1.1 and 1.2 in this article all estimate the conformal factor after applying appropriate Möbius transformations. The dividing out the invariance group of the Willmore functional is certainly necessary, and it is sufficient to obtain existence results of conformally constrained

Willmore minimizers. In applications, there is the stronger task to estimate the conformal factor for a given sequence without applying Möbius transformations. Therefore there is a need for a precise criterion, which can be checked on the given sequence, whether this sequence needs preparation by Möbius transformations or not. We recall that the bound $\mathcal{W}_{n,p} \leq \tilde{\beta}_p^n$ in (1.3) prevents topological splitting by an energy bound, and we actually see that preserving the topology is necessary and sufficient for the estimation of the conformal factor. To be more precise, we establish for a sequence of closed, orientable embedded surfaces $\Sigma_m \subseteq \mathbb{R}^n$ of fixed genus $p \geq 1$ with

$$\begin{gathered} \limsup_{m \to \infty} \mathcal{W}(\Sigma_m) < 8\pi, \\ \Sigma_m \subseteq B_1(0), \\ \mathcal{H}^2 \lfloor \Sigma_m \to \mu \neq 0 \quad \text{weakly as Radon measures,} \end{gathered} \tag{1.8}$$

that $spt\ \mu$ is a closed, orientable, embedded topological surface of $genus(spt\ \mu) \leq p$. Then without loss of topology in the sense that $genus(spt\ \mu) = p$ and with the algebraic energy condition, we prove an estimate of the conformal factor. In a second step, we replace no loss in topology by compactness of the conformal structures in moduli space and the non-triviality of the topology in the sense that $genus(spt\ \mu) \geq 1$. Actually these conditions are equivalent.

Theorem 1.3 *Let $\Sigma_m \subseteq \mathbb{R}^n$ be closed, orientable, embedded surfaces of fixed genus $p \geq 1$ with*

$$\begin{gathered} \limsup_{m \to \infty} \mathcal{W}(\Sigma_m) < 8\pi, \\ \Sigma_m \subseteq B_1(0), \\ \mathcal{H}^2 \lfloor \Sigma_m \to \mu \neq 0 \quad \textit{weakly as Radon measures.} \end{gathered}$$

Then $spt\ \mu$ is a closed, orientable, embedded topological surface of $genus(spt\ \mu) \leq p$. No topology is lost in the sense that

$$genus(spt\ \mu) = p$$

if and only if some topology is kept in the sense that

$$genus(spt\ \mu) \geq 1$$

and the conformal structues

$$[\Sigma_m] \textit{ lie in a compact subset of the moduli space.}$$

In this case if moreover

$$\limsup_{m \to \infty} \mathcal{W}(\Sigma_m) < \mathcal{W}(\mu) + e_n,$$

then the induced metrics $g_m := g_{euc}|\Sigma_m$ are uniformly conformal to unit volume constant curvature metrics $g_{poin,m} := e^{-2u_m} g_m$ for m large, more precisely

$$\limsup_{m \to \infty} \| u_m \|_{L^\infty(\Sigma)} < \infty.$$

□

We proceed from (1.8) and prove that some topology can always be kept in the sense of $genus(spt\ \mu) \geq 1$ after applying appropriate Möbius transformations. This yields Theorem 1.1 and in turn Theorem 1.2. Finally, we summarize the equivalence of no topological loss and the compactness in moduli space in the following theorem.

Theorem 1.4 *Let* $\Sigma_m \subseteq \mathbb{R}^n$ *be closed, orientable, embedded surfaces of genus* $p \geq 1$ *with*

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < 8\pi.$$

Then the conformal structures induced by Σ_m *lie in a compact subset of the moduli space if and only if no topology is lost after applying appropriate Möbius transformations, more precisely that any subsequence has a subsequence such that after applying appropriate Möbius transformations*

$$\mathcal{H}^2 \lfloor \Sigma_m \to \mu \quad \textit{weakly as Radon measures}$$

with $spt\ \mu$ *is a closed, orientable, embedded topological surface and*

$$genus(spt\ \mu) = p.$$

In this case after passing to a subsequence the conformal structures converge

$$[\Sigma_m] \to [spt\ \mu] \textit{ in moduli space}.$$

□

The inclusion that compactness in moduli space keeps together the topology after applying appropriate Möbius transformations was already observed in [7], where uniformly conformal weak limits in $W^{2,2}_{loc}(\Sigma - \mathcal{S})$ for some finite $\mathcal{S} \subseteq \Sigma$ were obtained.

2 Convergence without loss of topology

We start proving that measure theoretic limits under bounded Willmore energy have a topology and a genus.

Proposition 2.1 *Let* $\Sigma_m \subseteq \mathbb{R}^n$ *be closed, orientable, embedded surfaces of fixed genus* $p \geq 0$ *with*

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < 8\pi, \tag{2.1}$$

$$\begin{gathered} \Sigma_m \subseteq B_R(0), \\ \mathcal{H}^2 \lfloor \Sigma_m \to \mu \neq 0 \quad \textit{weakly as Radon measures}, \end{gathered} \tag{2.2}$$

for some $R < \infty$.

Then $spt\ \mu$ *is a closed, orientable, embedded topological surface of genus*

$$genus(spt\ \mu) \leq p. \tag{2.3}$$

Moreover for a closed, orientable surface of same genus, there exists a uniformly conformal $W^{2,2}-$*immersion, that is* $f^* g_{euc} = e^{2u} g_0$ *for some smooth metric* g_0 *on* Σ *and* $u \in L^\infty(\Sigma)$, *such that*

$$f : \Sigma \xrightarrow{\approx} spt\mu = f(\Sigma) \tag{2.4}$$

is a bi-lipschitz homeomorphism and for $\mu_f := f(\mu_g)$

$$\mu_f = \mu = \mathcal{H}^2 \lfloor spt\ \mu. \tag{2.5}$$

Proof By (2.1) and lower semicontinuity

$$\mathcal{W}(\mu) \le \limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < 8\pi \tag{2.6}$$

and by the Li-Yau inequality in [11] or [8] (A.17)

$$\theta^2(\mu, x) \le \mathcal{W}(\mu)/(4\pi) < 2 \quad \text{for all} \quad x \in \mathbb{R}^n.$$

We replace (2.1) by the weaker assumption

$$\begin{aligned} &\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < \infty, \\ &\theta^2(\mu, x) < 2 \quad \text{for all} \quad x \in \mathbb{R}^n, \end{aligned} \tag{2.7}$$

and proceed from here.

We may assume

$$\mathcal{W}(\Sigma_m) \le \mathcal{W} - \delta \tag{2.8}$$

for some $\mathcal{W} < \infty, \delta > 0$ and get by lower semicontinuity

$$\mathcal{W}(\mu) \le \limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < \infty. \tag{2.9}$$

By monotonicity formula, we get as in [16, Theorem 3.1]

$$\Sigma_m \to spt\ \mu \quad \text{in Hausdorff-distance.} \tag{2.10}$$

Next by the Gauß equations and the Gauß-Bonnet theorem

$$\int_{\Sigma_m} |A_{\Sigma_m}|^2 \, d\mathcal{H}^2 = 4\mathcal{W}(\Sigma_m) + 8\pi(p-1) \le 4\mathcal{W} + 8\pi(p-1) =: A. \tag{2.11}$$

Putting $\alpha_m := |A_{\Sigma_m}|^2 \mathcal{H}^2 \lfloor \Sigma_m$, we may assume after passing to a subsequence that $\alpha_m \to \alpha$ weakly as Radon measures. Clearly $\alpha(\mathbb{R}^n) \le A < \infty$, and there are at most finitely many bad point $z_1, \dots, z_N \in \mathbb{R}^n$ with

$$\alpha(\{x_k\}) \ge 8\pi - \delta \quad \text{for} \quad k = 1, \dots, N, \tag{2.12}$$

for any fixed $0 < \delta < \pi$ and $N \le A/(7\pi) =: N(p, \mathcal{W})$. As $\alpha(B_\varrho(x) - \{x\}) \to 0$ for $\varrho \to 0$ and any $x \in \mathbb{R}^n$, we can cover $spt\ \mu \subseteq \bigcup_{k=1}^K B_{\varrho_k/4}(x_k)$ with $x_k \in spt\ \mu$ and

$$\alpha(B_{2\varrho_k}(x_k) - \{x_k\}) < \varepsilon^2 \quad \text{for} \quad k = 1, \dots, K \tag{2.13}$$

and $\varepsilon \le \varepsilon_0(n, p, N, \delta)$ small enough chosen below. Further we may assume that $x_k = x_k$ for $k = 1, \dots, N$ and

$$\alpha(B_{2\varrho_k}(x_k)) < 8\pi - \delta \quad \text{for} \quad k = N+1, \dots, K. \tag{2.14}$$

By (2.12) and (2.13), we see that $x_l \notin B_{2\varrho_k}(x_k), k \ne l = 1, \dots, N$, hence

$$B_{\varrho_k}(x_k) \cap B_{\varrho_l}(x_l) = \emptyset \quad \text{for} \quad k = 1, \dots, N. \tag{2.15}$$

As $\mu \ne 0$, we may further assume that

$$spt\ \mu \not\subseteq B_{2\varrho_k}(x_k) \quad \text{for} \quad k = 1, \dots, K. \tag{2.16}$$

Next by the monotonicity formula, see [16, 1.2] or [8, A.3 and A.5], writing $\perp$ for the projection onto $(T_x\mu)^\perp$,

$$\int\limits_{B_\varrho(x)} frac|(\xi - x)^\perp|^2|\xi - x|^4 \, \mathrm{d}\mu(\xi) < \infty \quad \text{for all} \quad x \in \mathbb{R}^n, \varrho > 0,$$

and by (2.7), we can additionally assume for some $\theta^2(\mu, x_k) < \gamma_k < 2$ that

$$\begin{aligned} &mu(\overline{B_{7\varrho_k/8}(x_k)}) < \gamma_k \pi (7\varrho_k/8)^2, \\ &\int\limits_{B_{2\varrho_k}(x_k)} \frac{|(\xi - x_k)^\perp|^2}{|\xi - x_k|^4} \, \mathrm{d}\mu(\xi) < \varepsilon^2. \end{aligned}$$

We get from above (2.2) and (2.7) for $0 < r_m \ll \varrho_k/4, r_m \to 0$ and m large enough that

$$\begin{aligned} &\int\limits_{\Sigma_m \cap B_{\varrho_k}(x_k) - B_{r_m}(x_k)} |A_{\Sigma_m}|^2 \, \mathrm{d}\mathcal{H}^2 < \varepsilon^2, \\ &\mathcal{H}^2(\Sigma_m \cap B_{7\varrho_k/8}(x_k)) < \gamma_k \pi (7\varrho_k/8)^2, \\ &\int\limits_{\Sigma_m \cap B_{\varrho_k}(x_k) - B_{r_m}(x_k)} \frac{|(\xi - x_k)^\perp|^2}{|\xi - x_k|^4} \, \mathrm{d}\mathcal{H}^2(\xi) < \varepsilon^2, \end{aligned} \tag{2.17}$$

for $k = 1, \ldots, K$, and

$$\int\limits_{\Sigma_m \cap B_{\varrho_k}(x_k)} |A_{\Sigma_m}|^2 \, \mathrm{d}\mathcal{H}^2 < 8\pi - \delta/2 \quad \text{for} \quad k = N + 1, \ldots, K. \tag{2.18}$$

By Hausdorff-convergence in (2.10) and $x_k \in spt\ \mu$, we get $\Sigma_m \cap B_{\varrho_k/4}(x_k) \neq \emptyset$ for m large. Moreover Σ_m is not contained in any $B_{\varrho_k}(x_k)$ for m large by (2.16), and, as Σ_m is connected, any component of $\Sigma_m \cap B_\sigma(x_k), \sigma < 9\varrho_k/16$, extends to $\partial B_{9\varrho_k/16}(x_k)$ in the sense of [9, Lemma 2.1 (a)]. Then we obtain as in [9, 4.16] for the multiplicity $M_k = m_k = 1$ as in [9, 2.5], that is there is exactly one component of $D_m^k := D_{m,\sigma}^k$ of $\Sigma_m \cap B_\sigma(x_k)$ for $\sigma \in [5\varrho_k/8, 7\varrho_k/8]$ appropriate as in Theorem 5.2 and that the multiplicity of its boundary entering in [9, 2.6] equals one in the sense

$$\left| \int\limits_{\partial D_m^k} k_{g_m} \, \mathrm{d}s - 2\pi \right| \leq C(n)\varepsilon^{\alpha(n)}$$

for appropriate $\alpha(n) > 0$. Denoting the genus of $D_m^k \oplus B_1(0)$ by $p_{m,k}$, we get by the Gauß-Bonnet theorem

$$\left| \int\limits_{D_m^k} K_{g_m} \, \mathrm{d}\mu_{g_m} + 4\pi p_{m,k} \right| \leq C(n)\varepsilon^{\alpha(n)}. \tag{2.19}$$

Passing to a subsequence, we may assume $p_k := p_{m,k}$ is independent of m and after renumbering x_k, we may assume that

$$\left.\begin{array}{l} D_m^k \text{ is not a disc, } p_k \geq 1, \\ \text{or} \\ \int_{\Sigma_m \cap B_{\varrho_k}(x_k)} |A_{\Sigma_m}|^2 \, d\mathcal{H}^2 > 4e_n - \delta, \end{array}\right\} \quad \text{for} \quad k = 1, \ldots, \tilde{N}, \tag{2.20}$$

and

$$\left.\begin{array}{l} D_m^k \text{ is a disc, } p_k = 0, \\ \int_{\Sigma_m \cap B_{\varrho_k}(x_k)} |A_{\Sigma_m}|^2 \, d\mathcal{H}^2 \leq 4e_n - \delta, \end{array}\right\} \quad \text{for} \quad k = \tilde{N}+1, \ldots, K, \tag{2.21}$$

when observing that (2.21) is certainly true for $k = N+1, \ldots, K$ by (2.18), as $e_n \geq 2\pi$, and

$$\int\limits_{\partial D_m^k} |K_{g_m}| \, d\mu_{g_m} \leq \frac{1}{2} \int\limits_{\partial D_m^k} |A_{\Sigma_m}|^2 \, d\mu_{g_m} \leq 4\pi - \delta/4,$$

hence $p_k = p_{m,k} = 0$ by (2.19) for $C(n)\varepsilon^{\alpha(n)} < \delta/4$.

Then as in [9, Lemma 2.1 (b)], we may replace Σ_m in $B_{Mr_m}(x_k)$ for $k=1, \ldots, \tilde{N}$ and M large observing (2.15) to obtain a closed, orientable, embedded surface $\tilde{\Sigma}_m$ for m large and $Mr_m \ll \varrho_k/4$ and

$$\Sigma_m = \tilde{\Sigma}_m \quad \text{in} \quad \mathbb{R}^n - \bigcup_{k=1}^{\tilde{N}} B_{Mr_m}(x_k),$$

$$\int\limits_{\tilde{\Sigma}_m \cap B_{\varrho_k}(x_k)} |A_{\tilde{\Sigma}_m}|^2 \, d\mathcal{H}^2 < C(n)\varepsilon^2 \quad \text{for} \quad k = 1, \ldots, \tilde{N}, \tag{2.22}$$

$$\tilde{D}_m^k := \tilde{\Sigma}_m \cap B_\sigma(x_k) \text{ is a disc for} \quad k = 1, \ldots, \tilde{N}.$$

Clearly

$$genus(\tilde{\Sigma}_m) = p - \sum_{k=1}^{\tilde{N}} p_k \leq p, \tag{2.23}$$

and we choose $\Sigma \cong \tilde{\Sigma}_m$ with $0 \leq genus(\Sigma) \leq p$.

Next

$$\mathcal{W}(\tilde{\Sigma}_m) \leq \mathcal{W}(\Sigma_m) + \sum_{k=1}^{\tilde{N}} \frac{1}{4} \int\limits_{\tilde{\Sigma}_m \cap B_{Mr_m}(x_k)} |\vec{\mathbf{H}}_{\tilde{\Sigma}_m}| \, d\mathcal{H}^2 \leq \mathcal{W}(\Sigma_m) + C(n)N\varepsilon^2 \leq \mathcal{W} - \delta/2 \tag{2.24}$$

by (2.8) for $\varepsilon = \varepsilon(n, N, \delta)$ small enough. Again by the Li-Yau inequality in [11] or [8] (A.16)

$$|\mathcal{H}^2(\Sigma_m \cap B_{Mr_m}(x_k)) - \mathcal{H}^2(\tilde{\Sigma}_m \cap B_{Mr_m}(x_k))| \leq C\mathcal{W}M^2 r_m^2 \quad \text{for} \quad k = 1, \ldots, \tilde{N},$$

hence by (2.2)

$$\mathcal{H}^2 \lfloor \tilde{\Sigma}_m \to \mu \tag{2.25}$$

and as in [16] Theorem 3.1

$$\tilde{\Sigma}_m \to spt\ \mu \quad \text{in Hausdorff-distance.} \tag{2.26}$$

Further we see from (2.11) and (2.22) that

$$\int_{\tilde{\Sigma}_m} |A_{\tilde{\Sigma}_m}| \, d\mathcal{H}^2 \leq \int_{\Sigma_m} |A_{\Sigma_m}| \, d\mathcal{H}^2 + \sum_{k=1}^{\tilde{N}} \int_{\tilde{\Sigma}_m \cap B_{Mr_m}(x_k)} |A_{\tilde{\Sigma}_m}| \, d\mathcal{H}^2$$
$$\leq A + C(n) N \varepsilon^2 \leq A + 1$$

for $\varepsilon = \varepsilon(n, N)$ small enough, and putting $\tilde{\alpha}_m := |A_{\tilde{\Sigma}_m}|^2 \mathcal{H}^2 \lfloor \tilde{\Sigma}_m$, we get after passing to a subsequence that $\tilde{\alpha}_m \to \tilde{\alpha}$ weakly as Radon measures and by (2.13), (2.21) and (2.22) that

$$\begin{aligned}
\tilde{\alpha}(B_{2\varrho_k}(x_k)) &\leq \alpha(B_{2\varrho_k}(x_k) - B_{\varrho_k}(x_k)) + C(n)\varepsilon^2 \leq C(n)\varepsilon^2 < \pi \quad \text{for} \quad k = 1, \ldots, \tilde{N},\\
\tilde{\alpha}(B_{2\varrho_k}(x_k)) &\leq \alpha(B_{2\varrho_k}(x_k) - B_{\varrho_k}(x_k)) + 4e_n - \delta < 4e_n - \delta/2 \quad \text{for } k = \tilde{N}+1, \ldots, K,\\
\tilde{\alpha}(B_{2\varrho_k}(x_k) - \{x_k\}) &= \alpha(B_{2\varrho_k}(x_k) - \{x_k\}) < \varepsilon^2 \quad \text{for } k = \tilde{N}+1, \ldots, K,
\end{aligned} \tag{2.27}$$

for $\varepsilon = \varepsilon(n, N, \delta)$ small enough. We get from (2.26) that $\tilde{\Sigma}_m \subseteq \bigcup_{k=1}^{K} B_{\varrho_k/2}(x_k)$ for m large and from (2.27)

$$\int_{\tilde{\Sigma}_m \cap B_{\varrho_k}(x_k)} |A_{\tilde{\Sigma}_m}|^2 \, d\mathcal{H}^2 < 4e_n - \delta/2,$$
$$\int_{\tilde{\Sigma}_m \cap B_{\varrho_k}(x_k) - B_{\varrho_k/2}(x_k)} |A_{\tilde{\Sigma}_m}|^2 \, d\mathcal{H}^2 < C(n)\varepsilon^2$$

for $k = 1, \ldots, K$. For $\varepsilon \leq \varepsilon(n, \mathcal{W}, \delta/2)/C(n)$ as in Theorem 5.2, this verifies (5.9) and (5.10) for $\Lambda = \mathcal{W}$, $\delta/2$ and m large. (5.11) is verified by (2.21) and (2.22), when observing (2.19). For $\tilde{\Sigma}_m \cong \Sigma \not\cong S^2$, we choose diffeomorphisms $\tilde{f}_m : \Sigma \xrightarrow{\approx} \tilde{\Sigma}_m$ and conclude by [9] Theorem 3.1 or Theorem 5.2 for the pull-back metrics $\tilde{g}_m := \tilde{f}_m^* g_{euc}$ and the conformal smooth unit volume constant curvature metrics $\tilde{g}_{poin,m} = e^{-2\tilde{u}_m} \tilde{g}_m$ that

$$\| \tilde{u}_m \|_{L^\infty(\Sigma)} \leq C(n, \mathcal{W}, \varrho_l/\varrho_k, K, p, \delta/2) \tag{2.28}$$

for m large. Then by Proposition 6.1 after appropriate reparametrization of $\tilde{f}_m$, we get for a subsequence that

$$\begin{aligned}
&\tilde{f}_m \to f \text{ weakly in } W^{2,2}(\Sigma), \text{ weakly}^* \text{ in } W^{1,\infty}(\Sigma),\\
&\tilde{u}_m \to u \text{ weakly in } W^{1,2}(\Sigma), \text{ weakly}^* \text{ in } L^\infty(\Sigma),\\
&\tilde{g}_{poin,m} \to g_{poin} \quad \text{smoothly},\\
&f^* g_{euc} = e^{2u} g_{poin},
\end{aligned} \tag{2.29}$$

in particular f is a $W^{2,2}$−immersion uniformly conformal to g_{poin}. By uniform convergence $\tilde{f}_m \to f$, bounded covergence $\sqrt{\tilde{g}_m} \to \sqrt{g}$ and (2.25), we see

$$\mu_f := f(\mu_g) \leftarrow \tilde{f}_m(\mu_{\tilde{g}_m}) = \mathcal{H}^2 \lfloor \tilde{\Sigma}_m \to \mu. \tag{2.30}$$

□

We continue with our original assumption (2.1) instead of (2.7) and get by Proposition 7.2 and (2.6) that

$$\mathcal{W}(f) = \mathcal{W}(\mu) < 8\pi$$

and

$$f : \Sigma \stackrel{\approx}{\longrightarrow} spt\ \mu = f(\Sigma)$$

is a bi-lipschitz homeomorphism and by (2.30)

$$\mu = \mu_f = \mathcal{H}^2 \lfloor spt\ \mu,$$

which is (2.4) and (2.5), hence $spt\ \mu \cong \Sigma$ is a closed, orientable, embedded topological surface of genus $\leq p$ by (2.23), which is as in (2.3), and the proposition is proved in the case that $\Sigma \not\cong S^2$.

In case $\tilde{\Sigma}_m \cong \Sigma \cong S^2$, we may assume after translation that $0 \in \tilde{\Sigma}_m$. We select $\varrho > 0$ with $\tilde{\alpha}(B_{2\varrho}(0) - \{0\}) < \varepsilon^2$ and choose some $x_k = 0, \varrho_k = \varrho$. Then as in [9, Lemma 2.1 (b)], we replace $\tilde{\Sigma}_m$ in $B_\varrho(0)$ for m large to obtain a closed, orientable, embedded surface S_m with

$$\begin{aligned} S_m = \tilde{\Sigma}_m \quad & \text{in } \mathbb{R}^n - B_\varrho(0), \\ \int\limits_{S_m \cap B_\varrho(0)} |A_{S_m}|^2 \, d\mathcal{H}^2 & < C(n)\varepsilon^2, \\ S_m = L_m \quad & \text{in } B_{\varrho/2}(0), \end{aligned} \tag{2.31}$$

for some $2-$plane $L_m \ni 0$, which we may assume to be fixed $L = L_m$ after suitable rotations. As the rotations and the above translations are compact, we still have (2.25) with a rotated and translated μ. By the estimate in (2.31), we see that $S_m \cap B_\sigma(0)$ is a disc for appropriate $\sigma = \sigma_{m,k} \in [5\varrho/8, 7\varrho/8]$, hence recalling $\tilde{\Sigma}_m \cong S^2$

$$S_m \cong S^2. \tag{2.32}$$

Assuming by (2.1) that

$$\mathcal{W}(\Sigma_m) \leq 8\pi - \delta$$

for some $\delta > 0$, we get as in (2.24)

$$\mathcal{W}(S_m) \leq \mathcal{W}(\Sigma_m) + C(n)(N+1)\varepsilon^2 \leq 8\pi - \delta/2$$

for $\varepsilon \leq \varepsilon(n, N, \delta)$. By the Li-Yau inequality in [11] or [8, A.16]

$$\varrho^{-2}\mathcal{H}^2(S_m \cap B_\varrho) \leq C,$$

hence after passing to a subsequence

$$\mathcal{H}^2 \lfloor S_m \to \beta \quad \text{weakly as Radon measures.} \tag{2.33}$$

Then by lower semicontinuity and the Li-Yau inequality in [11] or [8, A.17]

$$\theta^2(\beta) \leq \mathcal{W}(\beta)/(4\pi) < 2. \tag{2.34}$$

Now we take any orientable, connected surface $H \subseteq \mathbb{R}^2$ with $H - B_{\varrho/4}(0) = L - B_{\varrho/4}(0)$, and which is not a disc, say

$$q := genus(H \cup \{\infty\}) \geq 1. \tag{2.35}$$

We replace S_m in $B_{\varrho/4}(0)$ by $H \cap B_{\varrho/4}(0)$ to obtain a obtain a closed, orientable, embedded surface $\Gamma_m \not\cong S^2$ with

$$\begin{aligned} \Gamma_m &= \tilde{\Sigma}_m \quad \text{in } \mathbb{R}^n - B_\varrho(0), \\ \Gamma_m &= S_m \quad \text{in } \mathbb{R}^n - B_{\varrho/4}(0), \\ \Gamma_m &= H \quad \text{in } B_{\varrho/2}(0). \end{aligned} \tag{2.36}$$

Then clearly

$$\chi(\Gamma_m) = \chi(S_m - \overline{B_{\varrho/2}(0)}) + \chi(H \cap B_{\varrho/2}(0)),$$

and by (2.32) and (2.35)

$$genus(\Gamma_m) = genus(H \cup \{\infty\}) = q \geq 1. \tag{2.37}$$

As

$$\mathcal{W}(\Gamma_m) \leq \mathcal{W}(S_m) + \mathcal{W}(H) \leq 8\pi + \mathcal{W}(H) < \infty \tag{2.38}$$

is bounded, we get as above after passing to a subseqeuence

$$\mathcal{H}^2 \lfloor \Gamma_m \to \nu \quad \text{weakly as Radon measures.} \tag{2.39}$$

Clearly by (2.25), (2.33) and (2.36)

$$\begin{aligned} \nu\lfloor(\mathbb{R}^n - \overline{B_\varrho(0)}) &= \mu\lfloor(\mathbb{R}^n - \overline{B_\varrho(0)}), \\ \nu\lfloor(\mathbb{R}^n - \overline{B_{\varrho/4}(0)}) &= \beta\lfloor(\mathbb{R}^n - \overline{B_{\varrho/4}(0)}), \\ \nu\lfloor B_{\varrho/2}(0) &= \mathcal{H}^2\lfloor(H \cap B_{\varrho/2}(0)). \end{aligned} \tag{2.40}$$

We see $\theta^2(\nu) = \theta^2(\mathcal{H}^2\lfloor H) \leq 1$ in $B_{\varrho/2}(0)$ and by (2.34) that $\theta^2(\nu) = \theta^2(\beta) < 2$ in $\mathbb{R}^n - \overline{B_{\varrho/4}(0)}$, hence combining

$$\theta^2(\nu, x) < 2 \quad \text{for all} \quad x \in \mathbb{R}^n. \tag{2.41}$$

Therefore Γ_m satisfy with (2.38) and (2.41) the weaker assumption in (2.7), and we can proceed with Γ_m as in the beginning of the proof. As $\Gamma_m = H$ in $B_{\varrho/2}(0)$ by (2.36) is smooth, we see that there are no bad points for Γ_m in $B_{\varrho/2}(0)$. Then by (2.22) for Γ_m

$$\tilde{\Gamma}_m = \Gamma_m = H \quad \text{in } B_{\varrho/2 - Mr_m}(0) \tag{2.42}$$

with $r_m \to 0$, and we conclude for $Mr_m < \varrho/4$ that

$$\chi(\tilde{\Gamma}_m) = \chi(\tilde{\Gamma}_m - \overline{B_{\varrho/4}(0)}) + \chi(H \cap B_{\varrho/4}(0)) \leq 1 + \chi(H \cap B_{\varrho/4}(0)) = \chi(H \cap \{\infty\}),$$

hence $genus(\tilde{\Gamma}_m) \geq genus(H \cup \{\infty\}) = q$. As $genus(\tilde{\Gamma}_m) \leq genus(\Gamma_m) = q$ by (2.23) and (2.37), we get

$$genus(\tilde{\Gamma}_m) = q \geq 1, \tag{2.43}$$

in particular $\Gamma \cong \tilde{\Gamma}_m$ is not a sphere. Then by above there is a uniformly conformal $W^{2,2}$−immersion

$$h : \Gamma \xrightarrow{\approx} spt\ \nu \tag{2.44}$$

which is a bi-lipschitz homeomorphism. Moreover by (2.37) and (2.43)

$$genus(spt\ \nu) = genus(\Gamma) = genus(\tilde{\Gamma}_m) = q = genus(H \cup \{\infty\}),$$

and, as $spt\ \nu = H$ in $B_{\varrho/2}(0)$ by (2.42), we conclude as above

$$\chi(spt\ \nu - \overline{B_{\varrho/3}(0)}) = \chi(spt\ \nu) - \chi(spt\ \nu \cap B_{\varrho/3}(0)) = \chi(spt\ \nu) - (\chi(H \cup \{\infty\}) - 1) = 1,$$

hence

$$spt\ \nu - \overline{B_{\varrho/3}(0)} \text{ is a topological disc.} \tag{2.45}$$

Next as $\mu = \nu$ in $\mathbb{R}^n - \overline{B_\varrho(0)}$ by (2.40), we get that $spt\ \mu - \overline{B_\varrho(0)}$ is an open, orientable, topological 2−manifold. As ϱ can be arbitrarily small and $0 \in \tilde{\Sigma}_m$ was arbitrary, $spt\ \mu$ is an open, orientable, topological 2-manifold. Observing that $spt\ \mu$ is compact and connected by (2.2), (2.10) and conectedness of Σ_m, we get

$$spt\ \mu \text{ is a closed, orientable, topological surface.}$$

We select an open neighbourhood $U(0)$ in $spt\ \mu$ of 0 which is a disc and $\gamma := \partial U(p)$ is a closed Jordan arc. For ϱ small enough, we have $\gamma \cap B_{2\varrho}(0) = \emptyset$ and

$$\gamma \subseteq spt\ \mu - \overline{B_\varrho(0)} = spt\ \nu - \overline{B_\varrho(0)} \subseteq spt\ \nu - \overline{B_{\varrho/3}(0)}.$$

The last set is a disc by (2.45), hence the interior I_γ of γ in $spt\ \nu - \overline{B_{\varrho/3}(0)}$ is a disc as well. Now I_γ is connected and

$$\partial I_\gamma \cap \partial B_\varrho(0) = \gamma \cap \partial B_\varrho(0) = \emptyset$$

hence

$$I_\gamma \subseteq spt\ \nu - \overline{B_\varrho(0)} \quad \text{or} \quad I_\gamma \subseteq spt\ \nu \cap B_\varrho(0).$$

Observing that

$$\overline{I_\gamma} \cap spt\ \nu - \overline{B_\varrho(0)} \supseteq \gamma \neq \emptyset,$$

we get

$$I_\gamma \subseteq spt\ \nu - \overline{B_\varrho(0)} = spt\ \mu - \overline{B_\varrho(0)}.$$

We see that I_γ and $U(0)$ are open, closed and connected in $spt\ \mu - \gamma$, hence these are connetced components of $spt\ \mu - \gamma$. As $0 \in U(0), 0 \notin I_\gamma$ by above and $spt\ \mu - \gamma$ can have at most two components, we get

$$spt\ \mu = U(0) + \gamma + I_\gamma. \tag{2.46}$$

Sicne both $U(0)$ and I_γ are discs, we get

$$spt\ \mu \cong S^2,$$

which is (2.3) in the case $\Sigma \cong S^2$.

Finally by (2.44) and compactness of $spt\ \mu$, we get a finite atlas $\{\varphi_1^{-1}, \ldots, \varphi_L^{-1}\}$ of uniformal conformal $W^{2,2}$-immersions

$$\varphi_l : B_1(0) \xrightarrow{\approx} U_l \subseteq spt\ \mu \quad \text{for} \quad l = 1, \ldots, L,$$

which are further bi-lipschitz homeomorphisms. This means in particular that

$$g_l := \varphi_l^* g_{euc} = e^{2u_l} g_{0,l}$$

for smooth metrics g_l on $B_1(0)$ and $u_l \in L^\infty_{loc}(B_1(0))$. Introducing local conformal coordiantes for the smooth metrics $g_{0,l}$, we may assume after rearranging the atlas that φ_l are conformal with

$$g_l := \varphi_l^* g_{euc} = e^{2u_l} g_{euc}$$

and $u_l \in L^\infty(B_1(0))$. Then all transitions maps

$$\varphi_{k,l} := \varphi_k^{-1} \circ \varphi_l := \varphi_l^{-1}(U_l \cap U_k) \xrightarrow{\approx} \varphi_k^{-1}(U_l \cap U_k)$$

are conformal, hence holomorphic after possibly reversing the orientation, in particular smooth. As $u_l \in L^\infty(B_1(0))$ and $D\varphi_{k,l}$ is continuous, we see that $D\varphi_{k,l}$ have full rank everywhere. Then $\varphi_{k,l}$, being bijective, is a holomorphic diffeomorphism. With this atlas $spt\ \mu$ is a compact, simply sonnected Riemann surface. By [6, Lemma 2.3.3], there exists a smooth conformal metric g_0 on $spt\ \mu$, that is

$$\varphi_l : (B_1(0), g_{euc}) \xrightarrow{\approx} (U_l, g_0)$$

is conformal, and more precisely $\varphi_l^* g_0 = e^{2v_l} g_{euc}$ for some $v_l \in C^\infty(B_1(0))$, hence

$$\varphi_l^* g_0 = e^{2v_l} g_{euc} = e^{2(v_l - u_l)} \varphi_l^* g_{euc}$$

and $g_0 = e^{2(v_l - u_l)\circ\varphi_l^{-1}} g_{euc}$. As $u_l, v_l \in L^\infty_{loc}(B_1(0))$, we get

$$g_0 = e^{2u_0} g_{euc} \quad \text{on} spt\ \mu \text{ for some } u_0 \in L^\infty(spt\ \mu). \tag{2.47}$$

We remark, since only $u_l \in L^\infty(B_1(0))$, we cannot conclude that g_{euc} is smooth on $spt\ \mu$ with respect to the holomorphic atlas above.

By the uniformisation theorem for simply connected Riemann surfaces, see [4, Theorem IV.1.1], $(spt\ u, g_0)$ is conformally equivalent to the sphere S^2 with standard metric $g_{S^2} := g_{euc}|S^2$, hence there is a conformal diffeomorphism

$$f : (S^2, g_{S^2}) \xrightarrow{\approx} (spt\ \mu, g_0),$$

say

$$f^* g_0 = e^{2v} g_{S^2} \quad \text{for some } v \in C^\infty(S^2). \tag{2.48}$$

Now for any conformal chart $\psi : V \xrightarrow{\approx} B_1(0)$ of S^2 with $V \subseteq f_l^{-1}(U_l)$ for some l, we see that

$$\psi_l := \varphi_l^{-1} \circ (f|V) \circ \psi^{-1} : B_1(0) \to B_1(0)$$

is conformal, hence smooth, and we conclude that $f \in W^{2,2}(S^2, \mathbb{R}^n)$ and is lipschitz. Calculating the pull-back metrics by (2.47) and (2.48), we get

$$f^* g_{euc} = f^*(e^{-2u_0} g_0) = e^{2(v - (u_0 \circ f))} g_{S^2}$$

with $v - (u_0 \circ f) \in L^\infty(S^2)$, hence f is a $W^{2,2}$-immersion uniformly conformal to g_{S^2}. Then by Proposition 7.2 (7.7)

$$\mu_f = f(\mu_g) = \# f^{-1} \cdot \mathcal{H}^2 \lfloor f(S^2) = \mathcal{H}^2 \lfloor spt\ \mu,$$

as f is bijective.

Since H_{Σ_m} is bounded in $L^2(\mathcal{H}^2 \lfloor \Sigma_m)$ by (2.1), we get from (2.2) that μ has weak mean curvature in $L^2(\mu)$ and by Allard's integral compactness theorem, see [1, Theorem

6.4] or [15, Remark 42.8], that μ is an integral varifold. Moreover by (2.1), (2.2) and lower semicontinuity

$$\mathcal{W}(\mu) \leq \limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < 8\pi,$$

hence by [8] (A.10) and (A.17) that $\theta^2(\mu) \geq 1$ on $spt\ \mu$ and $\theta^2(\mu) \leq \mathcal{W}(\mu)/4\pi < 2$, hence μ has unit density $\mu-$almost everywhere and by above

$$\mu = \mathcal{H}^2 \lfloor spt\ \mu = \mu_f,$$

hence (2.5). Then by Proposition 7.2 and above

$$\mathcal{W}(f) = \mathcal{W}(\mu) < 8\pi$$

and

$$f : S^2 \xrightarrow{\approx} spt\ \mu = f(S^2)$$

is a bi-lipschitz homeomorphism, which is (2.4) in the case that $\Sigma \cong S^2$, and the proposition is fully proved. □

Remarks 1. The sphere case when $\Sigma \cong S^2$ in the above proposition is more elaborate, since we cannot estimate the conformal factor in (2.28) by [9] Theorem 3.1 or Theorem 5.2 as in the case when $\Sigma \ncong S^2$ due to the presence of non-trivial conformal transformations on the sphere. For $n = 3$, this could be done for $\mathcal{W}(\Sigma_m) \leq 6\pi$ by [2] and [3].

2. For a second homeomorphism $\hat{f} : \hat{\Sigma} \xrightarrow{\approx} spt\ \mu$ with $\hat{f}$ a uniformly conformal $W^{2,2}-$immersion, say $\hat{f}^* g_{euc} = e^{2\hat{u}} \hat{g}_{poin}$ for some smooth unit volume constant curvature metric $\hat{g}_{poin}$ and $\hat{u} \in L^\infty(\hat{\Sigma})$, we see that $\phi := f^{-1} \circ \hat{f} : \hat{\Sigma} \xrightarrow{\approx} \Sigma$ is a homeomorphism. Moreover as f is bi-lipschitz and $\hat{f}$ is lipschitz, we get that ϕ is lipschitz and calculate the pull-back metric with (2.29)

$$e^{2\hat{u}} \hat{g}_{poin} = \hat{f}^* g_{euc} = \phi^* f^* g_{euc} = \phi^*(e^{2u} g_{poin}) = e^{2u\circ\phi} \phi^* g_{poin}.$$

We see that

$$\phi : (\hat{\Sigma}, \hat{g}_{poin}) \to (\Sigma, g_{poin})$$

is conformal, hence holomorphic or anti-holomorphic, in particular smooth. As $u \in L^\infty(\Sigma), \hat{u} \in L^\infty(\hat{\Sigma})$ and $D\phi$ is continuous, we see that $D\phi$ hat full rank everywhere on $\hat{\Sigma}$, and, as ϕ is bijective, we get that ϕ is a diffeomorphism. Then the conformal structures induced by f respectively $\hat{f}$ coincide, and we can define the conformal structure induced by $spt\ \mu$ by putting

$$[spt\ \mu] := [(\Sigma, f^* g_{euc})] = [(\hat{\Sigma}, \hat{f}^* g_{euc})]. \tag{2.49}$$

□

Certainly by Möbius invariance of the Willmore funcitonal, the limit can be a sphere, and most of the information would be lost. Therefore it is important to have a device to ensure, after appropriately applying Möbius transformations, the non-triviality of the limit which means for us to keep some topology. This was for example done in the existence proof of Willmore minimizers under fixed genus in [16, Lemma 4.1] to avoid round spheres. Other examples are the arrangement lemma in [9, Lemma 4.1] or the 3-points normalization lemma in [13, Lemma III.1]. Here we successively apply [9, Lemma 4.1] to keep some topology in the general situation of the previous proposition.

Proposition 2.2 *Let* $\Sigma_m \subseteq \mathbb{R}^n$ *be closed, orientable, embedded surfaces of genus* $p \geq 1$ *with*

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < 8\pi. \tag{2.50}$$

Then after applying appropriate Möbius transformations and passing to a subsequence

$$\begin{gathered}\Sigma_m \subseteq B_1(0),\\ \mathcal{H}^2 \lfloor \Sigma_m \to \mu \neq 0 \quad \textit{weakly as Radon measures},\end{gathered} \tag{2.51}$$

with $spt\ \mu$ *is a closed, orientable, embedded topological surface and*

$$genus(spt\ \mu) \geq 1. \tag{2.52}$$

Proof We avoid the case $\Sigma \cong S^2$ in the previous proposition by applying appropriate Möbius transformations as in [9, Lemma 4.1] and may assume that

$$\begin{gathered}\Sigma_m \subseteq B_1(0),\\ \int\limits_{\Sigma_m \cap B_{\varrho_0}(x)} |A^0_{\Sigma_m}|^2 \,\mathrm{d}\mathcal{H}^2 \leq \frac{1}{2} \int\limits_{\Sigma_m} |A^0_{\Sigma_m}|^2 \,\mathrm{d}\mathcal{H}^2 \quad \text{for all} \quad x \in \mathbb{R}^n,\end{gathered} \tag{2.53}$$

$\varrho_0 = \varrho_0(n, E) > 0$ and where $E := \int_{\Sigma_m} |A^0_{\Sigma_m}|^2 \,\mathrm{d}\mathcal{H}^2$. By (2.50) we may assume

$$\mathcal{W}(\Sigma_m) < 8\pi - \delta \tag{2.54}$$

for some $\delta > 0$ and get by the Li-Yau inequality in [11] or [8, A.16]

$$\varrho^{-2}\mathcal{H}^2(\Sigma_m \cap B_\varrho) \leq C,$$

hence after passing to a subsequence

$$\mathcal{H}^2 \lfloor \Sigma_m \to \mu \quad \text{weakly as Radon measures.}$$

Now by (2.53), we cannot have $\Sigma_m \subseteq B_{\varrho_0}(x)$ for any $x \in \mathbb{R}^n$, as $E > 0$, since Σ_m are no round spheres. Therefore $diam\ \Sigma_m \geq \varrho_0$ and $\mathcal{H}^2(\Sigma_m) \geq c_0\varrho_0^2$ by [16, Lemma 1.1]. As $\Sigma_m \subseteq B_1(0)$ by (2.53), we have $\mu(\mathbb{R}^n) = \lim_{m\to\infty} \mathcal{H}^2(\Sigma_m) > 0$, in particular $\mu \neq 0$, hence together with (2.53), we get (2.51). Then we have (2.1) and (2.2) for $R = 1$, and we proceed as in Proposition 2.1 and use the notation there.

By the Gauß equations and the Gauß-Bonnet theorem, we have with (2.54)

$$\int\limits_{\Sigma_m} |A^0_{\Sigma_m}|^2 \,\mathrm{d}\mathcal{H}^2 = 2\mathcal{W}(\Sigma_m) + 8\pi(p-1) \in [8\pi p, 8\pi(p+1) - 2\delta[. \tag{2.55}$$

Extending $D^k_m = \Sigma_m \cap B_\sigma(x_k)$ outside $B_\sigma(x_k)$ to a flat plane near infinity as in [9, Lemma 2.1 (b)], we obtain

$$\left| \int\limits_{D^k_m} |A^0_{\Sigma_m}|^2 \,\mathrm{d}\mathcal{H}^2 - 2\mathcal{W}(D^k_m) - 8\pi p_k \right| \leq C(n)\varepsilon^{\alpha(n)}.$$

Observing by extension to the inside as in (2.22) that

$$\mathcal{W}(\Sigma_m - D^k_m) \geq 4\pi - C(n)\varepsilon^{\alpha(n)},$$

hence by (2.54)

$$\mathcal{W}(D_m^k) = \mathcal{W}(\Sigma_m) - \mathcal{W}(\Sigma_m - D_m^k) \leq 4\pi - \delta + C(n)\varepsilon^{\alpha(n)}, \tag{2.56}$$

we get for $C(n)\varepsilon^{\alpha(n)} < \delta$

$$\int\limits_{D_m^k} |A_{\Sigma_m}^0|^2 \, \mathrm{d}\mathcal{H}^2 \in]8\pi p_k - \delta, 8\pi(p_k + 1) - \delta[. \tag{2.57}$$

Choosing $\varrho_k \leq \varrho_0$, we get from (2.53), (2.55) and (2.57)

$$8\pi p_k - \delta < \int\limits_{\Sigma_m \cap B_{\varrho_k}(x)} |A_{\Sigma_m}^0|^2 \, \mathrm{d}\mathcal{H}^2 \leq 4\pi(p+1) - \delta,$$

hence

$$p_k < (p+1)/2 \leq p \quad \text{for} \quad k = 1, \ldots, \tilde{N}, \tag{2.58}$$

and we see that one disc cannot take all the topology. When

$$spt\ \mu \cong S^2, \tag{2.59}$$

we have $\Sigma \cong \tilde{\Sigma}_m \cong S^2$, and see from (2.23) that $\tilde{N} \geq 1$ and after renumbering in (2.20) that

$$p_1 \geq 1. \tag{2.60}$$

For $p = 1$, this is impossible by (2.58), and the proposition is proved for $p = 1$.

To proceed for $p \geq 2$, we do the replacement of Σ_m as in (2.22) but only in $B_{Mr_m}(x_k)$for $k = 2, \ldots, \tilde{N}$ and obtain intermediate closed, orientable, embedded surfaces Γ_m. We get as in (2.24) for $\mathcal{W} = 8\pi$ that

$$\limsup_{m \to \infty} \mathcal{W}(\Gamma_m) < 8\pi$$

and as in (2.25) and (2.26)

$$\begin{gathered} \mathcal{H}^2 \lfloor \Gamma_m \to \mu, \\ \Gamma_m \to spt\ \mu \quad \text{in Hausdorff-distance.} \end{gathered} \tag{2.61}$$

and calculate by our assumption $\tilde{\Sigma}_m \cong S^2$, (2.58) and (2.60) that

$$genus(\Gamma_m) = genus(\tilde{\Sigma}_m) + p_1 = p_1 \in \{1, \ldots, p-1\}.$$

Then by induction and compressing by a factor of $1/2$, there are Möbius transformations Φ_m such that after passing to a subseqeuence

$$\begin{aligned} &\Phi_m \Gamma_m \subseteq B_{1/2}(0), \\ &\mathcal{H}^2 \lfloor \Phi_m \Gamma_m \to \nu \neq 0 \quad \text{weakly as Radon measures,} \end{aligned} \tag{2.62}$$

with $spt\ \nu$ is a closed, orientable, embedded topological surface and

$$genus(spt\ \nu) \geq 1. \tag{2.63}$$

We claim

$$\begin{gathered} \Phi_m \Sigma_m \subseteq B_1(0), \\ \mathcal{H}^2 \lfloor \Phi_m \Sigma_m \to \nu \neq 0 \quad \text{weakly as Radon measures,} \end{gathered} \tag{2.64}$$

for m large, which yields (2.51) and (2.52) by (2.63).

To prove (2.64), we work with Hausdorff convergence and get from (2.10) for $\Phi_m \Gamma_m$ that

$$\Phi_m \Gamma_m \to spt\ \nu \quad \text{in Hausdorff-distance.} \tag{2.65}$$

We claim

$$\Phi_m \Sigma_m \to spt\ \nu \quad \text{in Hausdorff-distance.} \tag{2.66}$$

If this is true, we see $\Phi_m \Sigma_m \subseteq B_1(0)$ for m large, as $spt\ \nu \subseteq \overline{B_{1/2}(0)}$ by (2.62) and (2.65), which is the first part in (2.64). As $\mathcal{W}(\Phi_m \Sigma_m) = \mathcal{W}(\Sigma_m) < 8\pi - \delta$ by (2.54), we get by the Li-Yau inequality in [11] or [8] (A.16) after passing to a subsequence

$$\mathcal{H}^2 \lfloor \Phi_m \Sigma_m \to \beta \quad \text{weakly as Radon measures.}$$

As $diam(spt\ \nu) > 0$, we get $diam(\Phi_m \Sigma_m) \geq \varrho$ for some $\varrho > 0$, hence $\mathcal{H}^2(\Phi_m \Sigma_m) \geq c_0 \varrho^2$ by [16] Lemma 1.1. As $\Phi_m \Sigma_m \subseteq B_1(0)$ by above, we have $\beta(\mathbb{R}^n) = \lim_{m\to\infty} \mathcal{H}^2(\Phi_m \Sigma_m) > 0$, in particular $\beta \neq 0$. Therefore we have (2.1) and (2.2) for $\Phi_m \Sigma_m$ and proceed as in Proposition 2.1. Then $\Phi_m \Sigma_m \to spt\ \beta$ in Hausdorff-distance by (2.10), hence $spt\ \beta = spt\ \nu$ by (2.66). We get by (2.5)

$$\beta = \mathcal{H}^2 \lfloor spt\ \beta = \mathcal{H}^2 \lfloor spt\ \nu = \nu,$$

and (2.64) follows.

It remains to establish (2.66). We first prove

$$\Phi_m(\infty) \not\to \infty. \tag{2.67}$$

Otherwise we rotate under sterographic projection $\mathbb{R}^n \cup \{\infty\} \cong S^n$ slightly from $\Phi_m(\infty)$ to ∞ and get Möbius transformations Ψ_m with

$$\Psi_m(\Phi_m(\infty)) = \infty,$$
$$\Psi_m \to id_{\mathbb{R}^n} \quad \text{smoothly on compact subsets of } \mathbb{R}^n.$$

Then by (2.62) and (2.65)

$$\Psi_m \Phi_m \Gamma_m \subseteq B_{3/4}(0) \quad \text{for } m \text{ large},$$
$$\mathcal{H}^2 \lfloor \Psi_m \Phi_m \Gamma_m \to \nu \neq 0 \quad \text{weakly as Radon measures,}$$
$$\Psi_m \Phi_m \Gamma_m \to spt\ \nu \quad \text{in Hausdorff-distance.}$$

As $\Psi_m \Phi_m(\infty) = \infty$, we see that $\Psi_m \Phi_m$ is an isometry multiplied by some factor $\lambda_m > 0$. Then by (2.61) and above

$$\lambda_m = \mathcal{H}^2(\Psi_m \Phi_m \Gamma_m)/\mathcal{H}^2(\Gamma_m) \to \nu(\mathbb{R}^n)/\mu(\mathbb{R}^n) =: \lambda \in]0, \infty[,$$

and we see for the isometries $U_m := \lambda_m^{-1} \Psi_m \Phi_m$ that

$$U_m \Gamma_m \to \lambda^{-1} spt\ \nu \quad \text{in Hausdorff-distance.}$$

We select any $x_m \in \Gamma_m$ and get by (2.61) after passing to a subsequence that $x_m \to x \in spt\ \mu$ and $U_m x_m \to y \in \lambda^{-1} spt\ \nu$. We concldue that

$$|U_m(0)| \leq |U_m(x_m)| + |U_m(x_m) - U_m(0)| = |U_m(x_m)| + |x_m| \to |y| + |x|,$$

hence after passing to a subsequence $U_m \to U$. Then by above and (2.61)

$$U(spt\ \mu) \leftarrow U_m \Gamma_m \to \lambda^{-1} spt\ \nu,$$

hence $spt\ \mu \cong spt\ \nu$. This contradicts (2.59) and (2.63), hence we get (2.67).

From (2.67), we get after passing to a subsequence that $\Phi_m(\infty) \to b \in \mathbb{R}^n$, and (2.62) is true for Φ_m replaced by $\Phi_m - \Phi_m(\infty)$ and ν replaced by its translation $(x \mapsto x - b)_\# \nu$, in particular we can assume $\Phi_m(\infty) = 0$. Let I be the inversion at the unit sphere, that is $I(x) := x/|x|^2$, then $I \circ \Phi_m$ are Möbius transformations with $I\Phi_m(\infty) = \infty$, hence are isometries multiplied by a factor $\lambda_m > 0$ and

$$I\Phi_m(x) = \lambda_m O_m(x - a_m) \quad \text{for all } x \in \mathbb{R}^n$$

and some $a_m \in \mathbb{R}^n$ and some orthogonal O_m. After passing to a subsequence we get $O_m \to O$, and (2.62) remains true for Φ_m replaced by $O_m^T \Phi_m$ and ν replaced by its rotation $O_\#^T \nu$. Since $O_m^T I = I O_m^T$ and $I^{-1} = I$, we calculate

$$O_m^T \Phi_m(x) = I(\lambda_m(x - a_m)) \quad \text{for all} \quad x \in \mathbb{R}^n,$$

hence we may assume

$$\Phi_m(x) = I(\lambda_m(x - a_m)) \quad \text{for all } x \in \mathbb{R}^n. \tag{2.68}$$

If $0 \notin spt\ \nu$, then $spt\ \nu \subseteq \overline{B_{1/2}(0)} - B_\varepsilon(0)$ for some $\varepsilon > 0$ by (2.62), and we get that (2.62) remains true for Φ_m replaced by $I\Phi_m$ and ν replaced by its inversion $I_\# \nu$. Then we have $I\Phi_m(\infty) = \infty$, which was already excluded in (2.67).

Therefore $0 \in spt\ \nu$, and we select by (2.65) and $spt\ \nu \neq \{0\}$ by (2.63) sequences $x_m, y_m \in \Gamma_m$ with $x_m \to x \neq 0$, $y_m \to 0$. Then clearly by (2.68)

$$I(x_m), I(y_m) \in I\Phi_m \Gamma_m = \lambda_m(\Gamma_m - a_m),$$

hence

$$\infty \leftarrow |I(x_m) - I(y_m)| \leq diam(\lambda_m(\Gamma_m - a_m)) = \lambda_m diam(\Gamma_m).$$

By (2.61), we see $diam(\Gamma_m) \to diam(spt\ \mu) < \infty$ and conclude $\lambda_m \to \infty$. From (2.68), we calculate the modulus of the derivative

$$|D\Phi_m(x)| = \lambda_m |DI(\lambda_m(x - a_m))| = \frac{1}{\lambda_m |x - a_m|^2} \quad \text{for} \quad x \in \mathbb{R}^n. \tag{2.69}$$

For a subsequence, we may assume that $a_m \to a \in \mathbb{R}^n \cup \{\infty\}$ and get

$$|D\Phi_m| \to 0 \quad \text{uniformly on compact subsets of } \mathbb{R}^n - \{a\}.$$

Now we are coming back to our construction from (2.22) and see

$$\begin{aligned} \Sigma_m &= \tilde{\Sigma}_m \quad \text{in } \mathbb{R}^n - \bigcup_{k=1}^{\tilde{N}} B_{Mr_m}(x_k), \\ \Sigma_m &= \Gamma_m \quad \text{in } \mathbb{R}^n - \bigcup_{k=2}^{\tilde{N}} B_{Mr_m}(x_k), \\ \Gamma_m &= \tilde{\Sigma}_m \quad \text{in } \mathbb{R}^n - B_{Mr_m}(x_1). \end{aligned} \tag{2.70}$$

If $a \neq x_1$, then

$$diam(\Phi_m(B_{Mr_m}(x_1))) \to 0$$

and $\Phi_m \Gamma_m$ and $\Phi_m \tilde{\Sigma}_m$ have the same limit in Hausdorff-distance, hence by (2.65)

$$\Phi_m \tilde{\Sigma}_m \to spt\ \nu \quad \text{in Hausdorff-distance.}$$

Then as above by the Li-Yau inequality and (2.5)

$$\mathcal{H}^2 \lfloor \Phi_m \tilde{\Sigma}_m \to \nu \quad \text{weakly as Radon measures.}$$

Next $\limsup_{m\to\infty} \mathcal{W}(\tilde{\Sigma}_m) < 8\pi$ by (2.24) for $\mathcal{W} = 8\pi$, and we conclude by Proposition 2.1

$$genus(spt\ \nu) \leq genus(\tilde{\Sigma}_m) = genus(spt\ \mu),$$

which contradicts (2.59) and (2.63), hence we get $a = x_1$.

Then as above $\Phi_m \Gamma_m$ and $\Phi_m \Sigma_m$ have the same limit in Hausdorff-distance, hence by (2.65)

$$\Phi_m \Sigma_m \to spt\ \nu \quad \text{in Hausdorff-distance,}$$

which is (2.66), and the proposition is proved. □

Remark We strengthen (2.56) to

$$\mathcal{W}(D_m^k) \geq \beta_{p_k}^n - 4\pi - C(n)\varepsilon^{\alpha(n)},$$
$$\mathcal{W}(\Sigma_m - \cup_{k=1}^{\tilde{N}} D_m^k) \geq \beta_{genus(\Sigma)}^n - C(n)\varepsilon^{\alpha(n)},$$

hence

$$\begin{aligned} \mathcal{W}(\Sigma_m) &= \mathcal{W}\left(\Sigma_m - \cup_{k=1}^{\tilde{N}} D_m^k\right) + \sum_{k=1}^{\tilde{N}} \mathcal{W}(D_m^k) \\ &\geq \beta_{genus(\Sigma)}^n - C(n)\varepsilon^{\alpha(n)} + \sum_{k=1}^{\tilde{N}} (\beta_{p_k}^n - 4\pi - C(n)\varepsilon^{\alpha(n)}) \\ &\geq \beta_{genus(\Sigma)}^n + \sum_{k=1}^{\tilde{N}} (\beta_{p_k}^n - 4\pi) - C(n,N)\varepsilon^{\alpha(n)}). \end{aligned}$$

Now

$$p = genus(\Sigma) + \sum_{k=1}^{\tilde{N}} p_k$$

by (2.23), and if the limit keeps some topology in the sense that $genus(\Sigma) = genus(spt\ \mu) \geq 1$ then $p_k < p$, and we conclude

$$1 \leq genus(spt\ \ \mu) < p \Longrightarrow \mathcal{W}(\Sigma_m) \geq \tilde{\beta}_p^n - C(n,N)\varepsilon^{\alpha(n)}$$

for the constant defined in (1.1).

We conclude, if we strengthen (2.1) to

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < \min(8\pi, \tilde{\beta}_p^n), \tag{2.71}$$

then we have after applying appropriate Möbius transformations that no topology is lost in the sense that

$$genus(spt\ \mu) = p.$$

□

Estimation of the conformal factor is only possible, if no topology is lost. In the next proposition, we prove that if no topology is lost then the induced conformal structures lie in a compact subset of moduli space. Actually we do not need this proposition for the estimation of the conformal factor, but it will give together with the following section the equivalence of no topological loss after applying appropriate Möbius transformations and compactness in moduli space.

Proposition 2.3 *If in Proposition 2.1 no topology is lost in the sense that*

$$genus(spt\ \mu) = p \geq 1, \tag{2.72}$$

then the conformal structures induced by Σ_m *lie in a compact subset of the moduli space.*

Proof We proceed with the notation of Proposition 2.1. If no topology is lost as in (2.72), we get from (2.23) that

$$p_k = 0 \quad \text{for} \quad k = 1, \ldots, K, \tag{2.73}$$

in particular $D_m^k \cong \tilde{D}_m^k$ are discs, and by construction in (2.22) there are diffeomorphisms

$$\begin{gathered} \psi_m : \tilde{\Sigma}_m \xrightarrow{\approx} \Sigma_m, \\ \psi_m = id \quad \text{on} \ \Sigma_m - \cup_{k=1}^{\tilde{N}} B_{Mr_m}(x_k), \\ \psi_m(\tilde{\Sigma}_m \cap B_{Mr_m}(x_k)) \subseteq B_{Mr_m}(x_k). \end{gathered} \tag{2.74}$$

We define $f_m := \psi_m \circ \tilde{f}_m : \Sigma \xrightarrow{\approx} \Sigma_m, g_m := f_m^* g_{euc}$ and consider the unit volume constant curvature metrics $g_{poin,m} = e^{-2u_m} g_m$. By (2.4) and $x_k \in spt\ \mu = f(\Sigma)$, we define $q_k := f^{-1}(x_k)$ and see for any $\Omega' \subset\subset \Omega := \Sigma - \{q_1, \ldots, q_{\tilde{N}}\}$ by uniform convergence in (2.29) and $r_m \to 0$ that $d(\tilde{f}_m(\Omega'), x_k) > Mr_m$, hence $\tilde{f}_m(\Omega') \cap B_{Mr_m}(x_k) = \emptyset$, and

$$f_m = \tilde{f}_m, g_m = \tilde{g}_m \quad \text{on} \ \Omega' \tag{2.75}$$

for m large depending on Ω', in particular by (2.29)

$$g_m \to f^* g_{euc} = e^{2u} g_{poin} \quad \text{pointwise in} \ \Omega.$$

By elementary differential geometry, we know

$$-\Delta_{g_m} u_m + 2\pi \chi(\Sigma) e^{-2u_m} = K_{g_m}, \quad -\Delta_{\tilde{g}_m} \tilde{u}_m + 2\pi \chi(\Sigma) e^{-2\tilde{u}_m} = K_{\tilde{g}_m}$$

on Σ, hence

$$-\Delta_{\tilde{g}_{poin,m}}(u_m - \tilde{u}_m) = -e^{2\tilde{u}_m} \Delta_{\tilde{g}_m}(u_m - \tilde{u}_m) = -2\pi \chi(\Sigma)(e^{-2(u_m - \tilde{u}_m)} - 1) \quad \text{in}\ \Omega', \tag{2.76}$$

and, as $\chi(\Sigma) = 2(1-p) \leq 0$,

$$\Delta_{\tilde{g}_{poin,m}}(\tilde{u}_m - u_m)_+ \geq 0 \quad \text{in} \ \Omega'$$

for m large. For any $q \in \Omega$ there exists an open neighbourhood $U(q) \subset\subset \Omega$ of q which is a disc by $\varphi : U(q) \cong B_1(0)$. Then by local maximum estimates, see [5, Theorem 8.17], (2.29) and writing $\varphi : U_\varrho(q) \cong B_\varrho(0)$ that

$$\sup_{U_{1/2}(q)} (\tilde{u}_m - u_m) \leq C(g_{poin}, q)\ \| (\tilde{u}_m - u_m)_+ \|_{L^2(U_{3/4}(q))}$$

for m large. To estimate the norm on the right-side, we observe

$$1 = \mu_{g_{poin,m}}(\Sigma) = \int\limits_{\Sigma} e^{-2u_m} \, d\mu_{g_m} \geq \int\limits_{U(q)} e^{-2u_m} \, d\mu_{\tilde{g}_m}$$
$$= \int\limits_{U(q)} e^{2(\tilde{u}_m - u_m)} \, d\mu_{\tilde{g}_{poin,m}} \geq \int\limits_{U(q)} (1 + \tilde{u}_m - u_m)_+^2 \, d\mu_{\tilde{g}_{poin,m}}$$

and get using (2.29)

$$u_m - \tilde{u}_m \geq -C(g_{poin}, q) \quad \text{on } U_{1/2}(q) \tag{2.77}$$

for m large.

Then by (2.76)

$$|\Delta_{\tilde{g}_{poin,m}}(u_m - \tilde{u}_m)| \leq C(g_{poin}, p, q) \quad \text{in } U_{1/2}(q),$$

and $u_m - \tilde{u}_m - \Gamma \geq 0$ on $U_{1/2}(q)$ for $\Gamma = C(g_{poin}, q)$, and we get by the Harnack inequality, see [5, Theorem 8.17 and 8.18], and (2.29)

$$\sup_{U_{1/4}(q)} (u_m - \tilde{u}_m - \Gamma) \leq C(g_{poin}, q) \inf_{U_{1/4}(q)} (u_m - \tilde{u}_m - \Gamma) + C(g_{poin}, p, q)$$

and

$$\sup_{U_{1/4}(q)} (u_m - \tilde{u}_m) \leq C(g_{poin}, q) \inf_{U_{1/4}(q)} (u_m - \tilde{u}_m) + C(g_{poin}, p, q) \tag{2.78}$$

for m large.

Now if $\limsup_{m\to\infty} \sup_{U_{1/4}(q)} u_m < \infty$, we see from (2.28) and (2.77) that u_m is bounded from above and below on $U_{1/4}(q)$. Otherwise for a subsequence $\sup_{U_{1/4}(q)} u_m \to \infty$. Then by (2.28) and (2.78)

$$\inf_{U_{1/4}(q)} u_m \to \infty,$$

hence $u_m \to \infty$ uniformly on $U_{1/4}(q)$.

Covering appropriately, we get after passing to a subsequence either

$$\begin{aligned} &u_m \to \infty \text{ uniformly on compact subsets of } \Omega, \\ \text{or} \\ &u_m \text{ is uniformly bounded on compact subsets of } \Omega. \end{aligned} \tag{2.79}$$

We define

$$\ell_m := \inf\{l_{g_{poin,m}}(\gamma) \mid \gamma \text{ is a closed geodesic in } (\Sigma, g_{poin,m})\} > 0,$$

where $l_{g_{poin,m}}$ denotes the length with respect to $g_{poin,m}$. By the Mumford compactness theorem, see e.g. [17] Theorem C.1, the conformal structures induced by $g_{poin,m}$ respectively by g_m lie in a compact subset of the moduli space, if

$$\liminf_{m\to\infty} \ell_m > 0. \tag{2.80}$$

Therefore we prove (2.80) and assume $\ell_m \leq 1$ in the following.

If (2.80) is not true, we get after passing to a subsequence that $\ell_m \to 0$. We first prove this implies the first case in (2.79), that is

$$u_m \to \infty \text{ uniformly on compact subsets of } \Omega. \tag{2.81}$$

By definition of ℓ_m, there exists a closed geodesic γ_m in $(\Sigma, g_{poin,m})$ of length $\ell_m \leq \ell_{m,0} = l_{g_{poin,m}}(\gamma_m) \leq 2\ell_m$. Further γ_m is non-nullhomotopic, see [17, pp. 184–185].

We consider any non-nullhomotopic curve γ_m in Σ with

$$\ell_{m,0} = l_{g_{poin,m}}(\gamma_m) \to 0. \tag{2.82}$$

As $D_m^k = B_\sigma(x_k) \cap \Sigma_m, \sigma \in [5\varrho_k/8, 7\varrho_k/8]$ appropriate, are discs by (2.73), we see that $\tilde{\gamma}_m := f_m(\gamma_m)$ cannot stay in any $B_{5\varrho_k/8}(x_k) \cap \Sigma_m$. As $\Sigma_m = f_m(\Sigma) \subseteq \cup_{k=1}^K B_{\varrho_k/2}(x_k)$ for m large, we get after passing to a subsequence

$$\tilde{\gamma}_m \cap B_{\varrho_l/2}(x_l) \neq \emptyset \quad \text{for some } l \in \{1, \ldots, K\} \tag{2.83}$$

and

$$l_{g_m}(\gamma_m) \geq \varrho_l/8 \quad \text{for some } l \in \{1, \ldots, K\} \tag{2.84}$$

and m large. If

$$\tilde{\gamma}_m \cap B_{\varrho_k/2}(x_k) = \emptyset \quad \text{for all} \quad k \in \{1, \ldots, \tilde{N}\} \tag{2.85}$$

and m large, in particular $l \notin \{1, \ldots, \tilde{N}\}$, we see by (2.74) and $Mr_m \leq \varrho_k$ that $f_m = \tilde{f}_m$ on γ_m and by uniform convergence in (2.29)

$$\liminf_{m \to \infty} d(f(\gamma_m), x_k) \geq \varrho_k/2 \quad \text{for all} \quad k \in \{1, \ldots, \tilde{N}\}.$$

Putting

$$\Omega_\tau := f^{-1}\left(\mathbb{R}^n - \cup_{k=1}^{\tilde{N}} \overline{B_{\tau\varrho_k}(x_k)}\right) \subset\subset \Omega \quad \text{for} \quad 0 < \tau < 1,$$

we see

$$\gamma_m \subseteq \Omega_{1/4} \subset\subset \Omega \quad \text{for } m \text{ large.} \tag{2.86}$$

We get from (2.82) and (2.84) that the first case in (2.79) is true, that is (2.81).

In case (2.85) is not true, we can choose $l \in \{1, \ldots, \tilde{N}\}$ in (2.83). As $\tilde{\gamma}_m$ cannot stay in $B_{5\varrho_l/8}(x_l)$, we get as in (2.84) a subarc $\gamma_{m,0}$ of γ_m whose image $\tilde{\gamma}_{m,0} := f_m(\gamma_{m,0})$ has its endpoints in $\partial B_{\varrho_l/2}(x_l)$ and $\partial B_{5\varrho_l/8}(x_l)$ and is contained in $\overline{B_{5\varrho_l/8}(x_l)} - B_{\varrho_l/2}(x_l)$. Then we see by (2.15) that

$$\tilde{\gamma}_{m,0} \cap B_{\varrho_k}(x_k) = \emptyset \quad \text{for all} \quad k \neq l, k \in \{1, \ldots, \tilde{N}\},$$

hence as above

$$\gamma_{m,0} \subseteq \Omega_{1/4} \subset\subset \Omega \quad \text{for} \quad m \text{ large.} \tag{2.87}$$

Clearly $l_{g_m}(\gamma_{m,0}) \geq \varrho_l/8$ and again by (2.82), we get that the first case in (2.79) is true, that is (2.81).

We conclude from (2.81) together with (2.28), (2.29) and (2.75) for any $\Omega' \subset\subset \Omega$ that

$$\begin{aligned} diam(\Omega', g_{poin,m}) &\leq e^{-\inf_{\Omega'} u_m} diam(\Omega', g_m) = e^{-\inf_{\Omega'} u_m} diam(\Omega', \tilde{g}_m) \\ &\leq e^{-\inf_{\Omega'} u_m} e^{\sup_\Sigma \tilde{u}_m} diam(\Sigma, \tilde{g}_{poin,m}) \leq Ce^{-\inf_{\Omega'} u_m} \to 0. \end{aligned} \tag{2.88}$$

For $p = 1$, that is Σ_m are tori, there exist lattices $\Gamma_m = \mathbb{Z} + (a_m + ib_m)\mathbb{Z} \subseteq \mathbb{C}$ with $0 \leq a_m \leq 1/2, b_m > 0, a_m^2 + b_m^2 \geq 1$ and such that $(\Sigma_m, g_{poin,m})$ is isometric to $(\mathbb{C}/\Gamma_m, b_m^{-1} g_{euc})$, as $g_{poin,m}$ have unit volume, see [6, §2.7]. Here we see $\ell_m = 1/\sqrt{b_m}$, and at each point of $(\Sigma_m, g_{poin,m})$ there exists a non-nullhomotopic curve γ_m of length $\ell_m \to 0$. Then by (2.86) and (2.87), we get $\gamma_m \cap \Omega_{1/4} \neq \emptyset$ for m large, and we conclude by (2.88) that

$$diam(\Sigma_m, g_{poin,m}) \leq \ell_m + diam(\Omega_{1/4}, g_{poin,m}) \to 0.$$

But

$$diam(\Sigma_m, g_{poin,m}) = diam(\mathbb{C}/\Gamma_m, b_m^{-1} g_{euc}) \geq \sqrt{b_m}/2 = 1/(2\ell_m) \to \infty,$$

which is a contradiction, and the proposition is proved for $p = 1$.

For $p \geq 2$, we proceed similarly and obtain from the collar lemma, see [17, Lemma D.1], for any closed geodesic γ_m in $(\Sigma, g_{poin,m})$ of length $\ell_m \leq \ell_{m,0} = l_{g_{poin,m}}(\gamma_m) \leq 2\ell_m$, which exists by definition of ℓ_m, a neighbourhood U_m of γ_m in $(\Sigma, g_{poin,m})$ isometric to $T/\sim$, where

$$T := \{(re^{i\theta} \mid 1 \leq r \leq e^{\ell_{m,0}(4\pi(p-1))^{1/2}}, |\theta - \pi/2| < \theta_0\}$$

is considered in the hyperbolic plane with hyperbolic metric divided by $4\pi(p-1) > 0$ in order to adjust to our convention of curvature $K_{g_{poin,m}} = -4\pi(1-p)$, $\sim$ identifies $e^{i\theta}$ and $e^{\ell_{m,0}+i\theta}$, and θ_0 is a fixed positive constant, as we assume $\ell_m \leq 1$. Here γ_m corresponds to $\theta = \pi/2$, say $\gamma_m(t) :\cong e^{t+i\pi/2}$. We select a second closed geodesic $\hat{\gamma}_m(t) :\cong e^{t+i((\pi+\theta_0)/2)}, 0 \leq t \leq \ell_{m,0}(4\pi(p-1))^{1/2}$, with length $l_{g_{poin,m}}(\hat{\gamma}_m) = \ell_{m,0}$ and which is homotopic to γ_m, hence is non-nullhomotopic, see [17, pp. 184–185].

Since $\theta \mapsto e^{i\theta}$ is a geodesic in the hyperbolic plane and geodesics in the hyperbolic plane are globally miniminzing, we see with hyperbolic distance that

$$d_{g_{poin,m}}(\gamma_m(t), \hat{\gamma}_m(t)) = d_H\left(e^{t+i\pi/2}, e^{t+i(\pi+\theta_0)/2}\right)/\sqrt{4\pi(p-1)} \geq \delta$$

for some fixed $\delta = \delta(\theta_0, p) > 0$, as $\ell_m \leq 1$, in particular

$$d_{g_{poin,m}}(\gamma_m, \hat{\gamma}_m) \geq \delta - 2\ell_{m,0} \geq \delta/2 \quad \text{for } m \text{ large,} \tag{2.89}$$

as $\ell_{m,0} \leq 2\ell_m \to 0$. As both γ_m and $\hat{\gamma}_m$ are non-nullhomotopic, we see by (2.86) and (2.87) that $\gamma_m \cap \Omega_{1/4}, \hat{\gamma}_m \cap \Omega_{1/4} \neq \emptyset$ for m large and conclude by (2.88) that

$$d_{g_{poin,m}}(\gamma_m, \hat{\gamma}_m) \leq diam(\Omega_{1/4}, g_{poin,m}) \to 0,$$

which contradicts (2.89), and the proposition is fully proved.

Remark Combining Proposition 2.3 with the previous remark after Proposition 2.2, we see that the conformal structures induced by smooth immersions $f : \Sigma \to \mathbb{R}^n$ of a closed, orientable surface Σ of genus $p \geq 1$ with

$$\mathcal{W}(f) \leq \min(8\pi, \tilde{\beta}_p^n) - \delta$$

lie in a compact subset of moduli space depending on $n, p, \delta > 0$. This was already proved in [7, Theorems 5.3 and 5.5] and [14, Theorem 1.1]. □

Now we estimate the conformal factor if no topology is lost under the algebraic energy restriction in order to apply the Hardy space theory. This gives a precise criterion, when the conformal factor can be estimated for a given sequence without applying Möbius transformations.

Proposition 2.4 *If in Proposition 2.1 no topology is lost, in the sense that*

$$genus(spt\ \mu) = p \geq 1, \tag{2.90}$$

and

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < \mathcal{W}(\mu) + e_n, \tag{2.91}$$

then the induced metrics $g_m := g_{euc}|\Sigma_m$ *are uniformly conformal to unit volume constant curvature metrics* $g_{poin,m} := e^{-2u_m} g_m$ *for* m *large, more precisely*

$$\limsup_{m\to\infty} \parallel u_m \parallel_{L^\infty(\Sigma)} < \infty. \tag{2.92}$$

Proof We continue with the notation of the Proposition 2.1 and want to exclude the real bad points in (2.20). Firstly by (2.90), we get $genus(\Sigma) = genus(\tilde{\Sigma}_m) = p = genus(\Sigma_m)$ and from (2.23) that

$$p_k = 0 \quad \text{for} \quad k = 1, \ldots, K. \tag{2.93}$$

Next as f is a uniformly conformal $W^{2,2}$-immersion, we get from [10] Proposition 5.2, Theorem 7.2 and (2.5)

$$\frac{1}{4} \int_\Sigma |A_f|^2 \, d\mu_f + 2\pi(1 - genus(\Sigma)) = \mathcal{W}(f) = \mathcal{W}(\mu_f) = \mathcal{W}(\mu).$$

Next by (2.1) and (2.91), we may assume

$$\mathcal{W}(\Sigma_m) \leq \min(8\pi, \mathcal{W}(\mu) + e_n) - \delta$$

for some $\delta > 0$ and m large, and get by the Gauß equations and the Gauß–Bonnet Theorem as in (2.11) and $genus(\Sigma) = genus(\Sigma_m) = p$ that

$$\int_{\Sigma_m} |A_{\Sigma_m}|^2 \, d\mathcal{H}^2 \leq \int_{\mathbb{R}^2} |A_f|^2 \, d\mu_f + 4e_n - 4\delta$$

for m large. Putting $\alpha_f := |A_f|^2 \mu_f$ and recalling $\alpha_m := |A_{f_m}|^2 \mathcal{H}^2 \lfloor \Sigma_m \to \alpha$ and $spt\ \alpha_m \subseteq B_R(0)$ by (2.2), we get

$$\alpha(\mathbb{R}^n) = \lim_{m\to\infty} \alpha_m(\mathbb{R}^n) \leq \alpha_f(\mathbb{R}^n) + 4e_n - 4\delta. \tag{2.94}$$

Recalling $\tilde{\alpha}_m := |A_{\tilde{f}_m}|^2 \mathcal{H}^2 \lfloor \tilde{\Sigma}_m \to \tilde{\alpha}$, we get by (2.29) for any $\eta \in C_0^0(\mathbb{R}^n), \eta \geq 0$, that

$$\begin{aligned} \int \eta \, d\alpha_f &= \int_\Sigma (\eta \circ f)|A_f|^2 \, d\mu_g \leq \liminf_{m\to\infty} \int_\Sigma (\eta \circ \tilde{f}_m)|A_{\tilde{f}_m}|^2 \, d\mu_{\tilde{g}_m} \\ &= \lim_{m\to\infty} \int \eta \, d\tilde{\alpha}_m = \int \eta \, d\tilde{\alpha}, \end{aligned}$$

hence

$$\alpha_f \le \tilde{\alpha}. \tag{2.95}$$

By (2.22), we see that

$$\alpha_m = \tilde{\alpha}_m \quad \text{in } \mathbb{R}^n - \cup_{k=1}^N \overline{B_{Mr_m}(z_k)}$$

and as $r_m \to 0$ and with (2.95)

$$\alpha = \tilde{\alpha} \ge \alpha_f \quad \text{in } \mathbb{R}^n - \{z_1, \dots, z_N\}.$$

Since $\alpha_f(\{x\}) = 0$ for any $x \in \mathbb{R}^n$, we get from (2.94)

$$\begin{aligned} \alpha(\mathbb{R}^n) &\le \alpha_f(\mathbb{R}^n) + 4e_n - 4\delta = \alpha_f(\mathbb{R}^n - \{z_1, \dots, z_N\}) + 4e_n - 4\delta \\ &= \alpha(\mathbb{R}^n - \{z_1, \dots, z_N\}) + 4e_n - 4\delta \end{aligned}$$

and

$$\alpha(\{z_1, \dots, z_N\}) \le 4e_n - 4\delta. \tag{2.96}$$

Combining with (2.93), we see that (2.20) is vacant for δ as above. This yields by (2.22) that $\Sigma_m = \tilde{\Sigma}_m \cong \Sigma$, hence $\tilde{g}_m = \tilde{f}_m^* g_m$, $g_{poin,m} = \tilde{f}_m^* \tilde{g}_{poin,m}$, $u_m = \tilde{u}_m \circ f_m$, and (2.92) follows from (2.28), and the proposition is proved. □

3 Compactness in moduli space

In [7], it was proved that compactness in moduli space gives uniformly conformal weak limits in $W^{2,2}_{loc}(\Sigma - \mathcal{S})$ for some finite $\mathcal{S} \subseteq \Sigma$. In our set up, we clarify that if some topology is kept under compactness in moduli space then no additional choice of Möbius transformations is necessary to keep all topology.

Proposition 3.1 *If in Proposition 2.1 some topology is kept in the sense that*

$$genus(spt\ \mu) \ge 1, \tag{3.1}$$

and the conformal structures induced by Σ_m *lie in a compact subset of the moduli space, then*

$$genus(spt\ \mu) = p, \tag{3.2}$$

that is no topology is lost, and

$$[\Sigma_m] \to [spt\ \mu] \text{ in moduli space} \tag{3.3}$$

in the sense of (2.49).

Proof We use the notation of Proposition 2.1 and consider a closed, orientable surface Σ_p of genus $p \ge 1$ and diffeomorphisms $f_m : \Sigma_p \xrightarrow{\approx} \Sigma_m$. As the conformal structures induced by the pull-back metrics $g_m := f_m^* g_{euc}$ lie in a compact subset of the moduli space choosing the parametrizations f_m appropriately and passing to a subsequnce, we may assume that the unit volume constant curvature metrics

$$g_{poin,m} := e^{-2u_m} g_m \to g_{poin,0} \quad \text{smoothly on } \Sigma_p. \tag{3.4}$$

By (2.1), we may assume

$$\mathcal{W}(f_m) \leq 8\pi - \delta \tag{3.5}$$

for some $\delta > 0$. As $f_m(\Sigma_p) = \Sigma_m \subseteq B_R(0)$ by (2.2), we get by the Li-Yau inequality in [11] or [8, A.16]

$$\mu_{g_m}(\Sigma_p) \leq C. \tag{3.6}$$

We define $\nu_m := |A_{f_m}|^2 \mu_{g_m}$ and see by the Gauß equations and the Gauß–Bonnet theorem as in (2.11) that

$$\nu_m(\Sigma_p) = \int_{\Sigma_p} |A_{f_m}|^2 \, d\mu_{g_m} = 4\mathcal{W}(f_m) + 8\pi(p-1) \leq 8\pi(p+3) =: A_0 < \infty,$$

hence after passing to a subsequence $\nu_m \to \nu$ weakly as Radon measures on Σ_p. Clearly $\nu(\Sigma_p) \leq A_0 < \infty$, and there are at most finitely many bad points $q_1, \ldots, q_L \in \Sigma_p$ with

$$\nu(\{q_l\}) \geq \varepsilon_0(n) \quad \text{for} \quad l = 1, \ldots, L, \tag{3.7}$$

for $\varepsilon_0(n)$ small enough choosen below, and we consider the open set $\Omega_0 := \Sigma_p - \{q_1, \ldots, q_L\}$.

For any $q \in \Omega_0$ there exists an open neighbourhood $\varphi : U(q) \cong B_1(0)$ of q with

$$\nu(\overline{U(q)}) < \varepsilon_0(n) \tag{3.8}$$

and for m large that

$$\int_{U(q)} |A_{f_m}|^2 \, d\mu_{g_m} \leq \varepsilon_0(n) - \delta \tag{3.9}$$

for some $\delta = \delta(q) > 0$. By elementary differential geometry and the Gauß–Bonnet theorem, we know

$$-\Delta_{g_m} u_m + 2\pi\chi(\Sigma_p) e^{-2u_m} = K_{g_m} \quad \text{on } \Sigma_p. \tag{3.10}$$

By the uniformisation theorem for simply connected Riemann surfaces, see [4, Theorem IV.1.1], we can parametrize $f_m \circ \varphi_m^{-1} : B_1(0) \cong U(q) \to \mathbb{R}^n$ conformally with respect to the euclidean metric on $B_1(0)$, possibly after replacing $U(q)$ by a slightly smaller ball. Then by [12] Theorem 4.2.1 and (3.9) for $\varepsilon_0(n)$ small enough, there exists $v_m \in C^\infty(U(q))$ with

$$-\Delta_{g_m} v_m = K_{g_m} \quad \text{on } U(q) \tag{3.11}$$

satisfying

$$\|v_m\|_{L^\infty(U(q))} \leq C(n,\delta) \int_{B_1(0)} |A_{f_m}|^2 \, d\mu_{g_m} \leq C(n,q). \tag{3.12}$$

Actually one can choose $\varepsilon_0(n) = 4\pi$, see Proposition 5.1. By (3.10), we see

$$-\Delta_{g_{poin,m}}(u_m - v_m) = -2\pi\chi(\Sigma_p) \quad \text{in } U(q), \tag{3.13}$$

hence by local maximum estimates, see [5, Theorem 8.17], and (3.4), writing $\varphi : U_\varrho(q) \cong B_\varrho(0)$ that

$$\sup_{U_{1/2}(q)} (u_m - v_m) \leq C(g_{poin,0}, q) \Big(\| 2\pi\chi(\Sigma_p) \|_{L^2(U_{3/4}(q))} + \| (u_m - v_m)_+ \|_{L^2(U_{3/4}(q))} \Big)$$

and by (3.12)

$$\sup_{U_{1/2}(q)} u_m \leq C(n, p, g_{poin,0}, q) \left(1+ \parallel u_{m,+} \parallel_{L^2(U_{3/4}(q))}\right)$$

for m large. To estimate the norm on the right-side, we observe by (3.6)

$$C \geq \mu_{g_m}(\Sigma_p) \geq \int\limits_{U_{3/4}(q)} e^{2u_m} \,d\mu_{g_{poin,m}} \geq \int\limits_{U_{3/4}(q)} (1+u_m)_+^2 \,d\mu_{g_{poin,m}}$$

and get for m large

$$\sup_{U_{1/2}(q)} u_m, |Df_m| \leq C(n, p, g_{poin,0}, q). \tag{3.14}$$

Then $\Gamma - u_m + v_m \geq 0$ in $U_{1/2}(q)$ for $\Gamma = C(n, p, g_{poin,0}, q)$, hence by (3.13) and the Harnack inequality, see [5] Theorem 8.17 and 8.18, and (3.4) that

$$\sup_{U_{1/4}(q)} (\Gamma - u_m - v_m) \leq C(g_{poin,0}, q) \left(\inf_{U_{1/4}(q)} (\Gamma - u_m - v_m) + \parallel 2\pi\chi(\Sigma_p) \parallel_{L^2(U_{1/2}(q))}\right)$$

and by (3.12)

$$\sup_{U_{1/4}(q)} (\Gamma - u_m) \leq C(n, p, g_{poin,0}, q) \inf_{U_{1/4}(q)} (\Gamma - u_m) + C(n, p, g_{poin,0}, q), \tag{3.15}$$

Now if $\liminf_{m\to\infty} \inf_{U_{1/4}(q)} u_m > -\infty$, we see from (3.14) that u_m is bounded from below and above on $U_{1/4}(q)$. Otherwise for a subsequence $\inf_{U_{1/4}(q)} u_m \to -\infty$. Then

$$\sup_{U_{1/4}(q)} (\Gamma - u_m) = \Gamma - \inf_{U_{1/4}(q)} u_m \to \infty$$

and by (3.15)

$$\infty \leftarrow \inf_{U_{1/4}(q)} (\Gamma - u_m) = \Gamma - \sup_{U_{1/4}(q)} u_m,$$

hence $u_m \to -\infty$ uniformly on $U_{1/4}(q)$.

Covering appropriately, we get after passing to a subsequence either

$$\begin{aligned} &u_m \to -\infty \text{ uniformly on compact subsets of } \Omega_0, \\ \text{or}& \\ &u_m \text{ is uniformly bounded on compact subsets of } \Omega_0. \end{aligned} \tag{3.16}$$

We choose open neighbourhoods $U(q_l)$ of q_l which are pairwise disjoint discs and put

$$\Omega' := \Sigma_p - \cup_{l=1}^L U(q_l) \subset\subset \Omega_0. \tag{3.17}$$

In the first case in (3.16), we see by (3.4) that $diam_{g_m}(\Omega') \to 0$ for the intrinsic diameter, hence

$$diam(f_m(\Omega')) \to 0.$$

In the notation of Proposition 2.1, as $\Sigma_m = f_m(\Sigma_p) \subseteq \cup_{k=1}^K B_{\varrho_k/2}(x_k)$ for m large, we get after passing to a subsequence

$$f_m(\Omega') \subseteq B_{5\varrho_k/8}(x_k) \cap \Sigma_m \subseteq D_m^k = \Sigma_m \cap B_\sigma(x_k) \quad \text{for some } k \in \{1, \ldots, K\} \tag{3.18}$$

for some appropriate $\sigma \in [5\varrho_k/8, 7\varrho_k/8]$ and m large. We know that D_m^k is connected, and its boundary ∂D_m^k consists of a single Jordan curve. Therefore the complement of $\Sigma_m - \partial D_m^k = \Sigma_m - \partial B_\sigma(x_k)$ consists of exactly two components which are

$$D_m^k \quad \text{and} \quad \Sigma_m - \overline{B_\sigma(x_k)}.$$

As f_m is a diffeomorphism, we know that $\gamma_m := f_m^{-1}(\partial D_m^k)$ is a single Jordan curve in Σ_p, in particular connected, and, as $\gamma_m \cap \Omega' = \emptyset$ by (3.18), we conclude

$$\gamma_m \subseteq U(q_l) \quad \text{for some } l \in \{1, \ldots, L\}$$

after passing to a subsequence. As $U(q_l)$ is a disc, the complement of γ_m in Σ_p consists of exactly two components, one of which is the interior I_m of γ_m in $U(q_l)$ which moreover is a disc as well. We call the other component the exterior E_m, and see that E_m is not a disc, as $I_m \oplus E_m = \Sigma_p \not\cong S^2$. These components correspond under f_m to the components of $\Sigma_m - \partial D_m^k$.

By (3.18), we see $f_m^{-1}(D_m^k) \not\subseteq U(q_l)$, hence $D_m^k \cong f_m^{-1}(D_m^k) = E_m$ is not a disc and

$$\Sigma_m - \overline{B_\sigma(x_k)} = f_m^{-1}(I_m) \text{ is a disc.}$$

Then D_m^k appears in (2.20) and is replaced in the construction of Proposition 2.1 by a disc, and we get

$$genus(\Sigma) = genus(\tilde{\Sigma}_m) \le genus\left((\Sigma_m - \overline{B_\sigma(x_k)}) \oplus B_1(0)\right) = 0.$$

But $genus(\Sigma) = genus(spt\ \mu) \ge 1$ by (3.1), hence can exclude the first case in (3.16).

Therefore we have the second case in (3.16) and prove that no topology is lost in Proposition 2.1 in the sense of (3.2). If (3.2) is not satisfied, we get from (2.23) that $\tilde{N} \ge 1$ and after renumbering in (2.20) that

$$p_1 \ge 1. \tag{3.19}$$

This means that D_m^1 is not a disc and by construction in (2.22), there exists a closed curve γ_m in $B_{Mr_m}(x_1) \cap \Sigma_m$ which is not null-homotopic in Σ_m. As $\gamma_m^{\Sigma_p} := f_m^{-1}(\gamma_m)$ is connected and $U(q_l)$ are discs, it has to meet Ω' in (3.17), hence

$$q_m \in \gamma_m^{\Sigma_p} \cap \Omega' \neq \emptyset.$$

Passing to a subsequence, we get $q_m \to q \in \overline{\Omega'} \subseteq \Omega_0$, and we select an open neighbourhood $U(q) \subset\subset \Omega_0$ of q which is a disc, say $\varphi : U(q) \xrightarrow{\approx} B_1(0), \varphi : U_\varrho(q) \cong B_\varrho(0)$. Clearly for m large, we have $q_m \in U(q) \cap \gamma_m^{\Sigma_p}$. As $U(q)$ is a disc and $\gamma_m^{\Sigma_p}$ is not null-homotopic, $\gamma_m^{\Sigma_p}$ cannot stay in $U(q)$, hence there is

$$q_{m,\varrho} \in \gamma_m^{\Sigma_p} \cap \partial U_\varrho(q) \quad \text{for} \quad 0 < \varrho < 1. \tag{3.20}$$

As $f_m(\gamma_m^{\Sigma_p}) = \gamma_m \subseteq B_{Mr_m}(x_1)$, we get

$$|f_m(q_{m,\varrho}) - x_1| \to 0 \quad \text{for} \quad 0 < \varrho < 1. \tag{3.21}$$

Next choosing

$$\Omega' \cup U(q) \subset\subset \Omega'' \subset\subset \Omega''' \subset\subset \Omega_0$$

and writing

$$\Delta_{g_{poin,m}} f_m = e^{2u_m} \Delta_{g_m} f_m = e^{2u_m} \vec{\mathbf{H}}_{f_m} \quad \text{on } \Sigma_p,$$

we estimate by (3.16)

$$\int_{\Omega'''} |e^{2u_m} \overrightarrow{\mathbf{H}}_{f_m}|^2 \, d\mu_{g_{poin,m}} \le C \int_{\Sigma_p} |\overrightarrow{\mathbf{H}}_{f_m}|^2 \, d\mu_{g_m} \le C$$

and get by standard elliptic theory, see [5, Theorem 8.8], (2.2) and (3.4) for m large

$$\| f_m \|_{W^{2,2}(\Omega'')} \le C,$$

hence after passing to a subsequence using (3.4)

$$\begin{gathered} f_m \to f_0 \text{ weakly in } W^{2,2}_{loc}(\Omega_0), \text{ weakly}^* \text{ in } W^{1,\infty}_{loc}(\Omega_0), \\ g_m \to g_0 := f_0^* g_{euc} \text{ weakly in } W^{1,2}_{loc}(\Omega_0), \\ u_m \to u_0 \text{ weakly in } W^{1,2}_{loc}(\Omega_0), \text{ weakly}^* \text{ in } L^\infty_{loc}(\Omega_0), \\ g_0 = f_0^* g_{euc} = e^{2u_0} g_{poin,0}, \end{gathered} \tag{3.22}$$

with $u_0 \in L^\infty_{loc}(\Omega_0)$ and in particular, we have $f_m \to f_0$ uniformly on compact subsets of Ω_0. Passing to a subsequence in (3.20) and (3.21) for fixed $0 < \varrho < 1$, we get $q_{m,\varrho} \to q_\varrho$ for this subsequence and

$$q_\varrho \in \partial U_\varrho(q), \quad f_0(q_\varrho) = x_1, \tag{3.23}$$

in particular $q_\varrho \neq q_{\tilde{\varrho}}$ for $\varrho \neq \tilde{\varrho}$.

Now for any $\eta \in C^0(\mathbb{R}^n), \eta \ge 0,$ and Fatou's lemma by (3.22)

$$\int_{\Omega_0} (\eta \circ f_0) \, d\mu_{g_0} \le \liminf_{m \to \infty} \int_{\Omega_0} (\eta \circ f_m) \, d\mu_{g_m} = \liminf_{m \to \infty} \int \eta \, d\mu_{f_m}$$

hence for $\mu_{f_0} := f_0(\mu_{g_0} | \Omega_0)$ and observing $\mu_{f_m} = \mathcal{H}^2 \lfloor \Sigma_m \to \mu$ by (2.2)

$$\int_{\Omega_0} \eta \, d\mu_{f_0} \le \int \eta \, d\mu,$$

and

$$\mu_{f_0} \le \mu. \tag{3.24}$$

As $u_0 \in L^\infty_{loc}(\Omega_0)$, we get by Proposition 7.1 and (2.7) that

$$\# f_0^{-1}(x) \le \theta^2(\mu_{f_0}, x) \le \theta^2(\mu, x) < 2 \quad \text{for all} \quad x \in \mathbb{R}^n,$$

hence

$$f_0 : \Omega_0 \to \mathbb{R}^n \text{ is injective.} \tag{3.25}$$

As $\{q_\varrho | 0 < \varrho < 1\} \subseteq f_0^{-1}(x_1) \cap \Omega''$, this is a contradiction, and we get (3.2) and may choose $\Sigma = \Sigma_p$.

It remains to prove (3.3). We know by (3.4) that

$$[\Sigma_m] = [g_{poin,m}] \to [g_{poin,0}],$$

and, as $[spt \ \mu] = [f^* g_{euc}] = [g_{poin}]$ by (2.29) and (2.49), we have to prove

$$[g_{poin,0}] = [g_{poin}]. \tag{3.26}$$

From Proposition 7.1, we know that $\theta^2(\mu_{f_0}) > 0$ on $f_0(\Omega_0)$, hence $f_0(\Omega_0) \subseteq spt\ \mu$ by (3.24). Since $f : \Sigma \xrightarrow{\approx} spt\ \mu$ is bi-lipschitz by Proposition 2.1 and f_0 is locally lipschitz, as $u_0 \in L^\infty_{loc}(\Omega_0)$ in (3.22), we see that

$$\phi : f^{-1} \circ f_0 : \Omega_0 \to \Sigma$$

is locally lipschitz and is injective by (3.25). Clearly $f \circ \phi = f_0$ and by (2.29) and (3.22)

$$e^{2u_0} g_{poin,0} = f_0^* g_{euc} = \phi^* f^* g_{euc} = \phi^*(e^{2u} g_{poin}) = e^{2u\circ\phi} \phi^* g_{poin}, \tag{3.27}$$

hence

$$\phi : (\Omega_0, g_{poin,0}) \to (\Sigma, g_{poin})$$

is conformal, hence holomorphic after possibly reversing the orientation, in particular smooth. As $u \in L^\infty(\Sigma), u_0 \in L^\infty_{loc}(\Omega_0)$ and $D\phi$ is continuous, we see that $D\phi$ has full rank everywhere on Ω_0. Then ϕ, being injective, is a diffeomorphism onto the open set $\Omega := \phi(\Omega_0) \subseteq \Sigma$.

Recalling $\Omega_0 = \Sigma - \{q_1, \ldots, q_l\}$, we choose open neighbourhoods $U(q_l)$ of q_l, which are pariwise disjoint discs, and smooth closed Jordan curves γ_l^0 in $U(q_l)$ such that q_l lie in the interior I_l^0 of γ_l^0 in $U(q_l)$. Then $U_0 := \Omega_0 - \cup_{l=1}^L \overline{I_l^0} \subset\subset \Omega_0$ is open with l boundary components and $\chi(U_0) = \chi(\Sigma) - l$. As ϕ is a diffeomorphism, we see that $U := \phi(U_0) \subseteq \Sigma$ is open and

$$\chi(U) = \chi(\Sigma) - l. \tag{3.28}$$

Clearly $\phi(\partial U_0) \subseteq \partial U$. On the other hand for $w \in \partial U$, there exists $y_k \in U_0$ with $\phi(y_k) \to w$. For a subsequence we see $y_k \to y \in \overline{U_0} \subseteq \Omega_0$, hence $w = \phi(y) \in \Omega$. If $y \in U_0$, then $w \in U$, which is a contradiction. Therefore $y \in \partial U_0$ and $w \in \phi(\partial U_0)$. Together

$$\partial U = \phi(\partial U_0) = \cup_{l=1}^L \phi(\gamma_l^0),$$

and the boundary of U in Σ consists of l Jordan curves $\gamma_l := \phi(\gamma_l^0) \subseteq \Omega$. The boundary of the exterior $V := \Sigma - \overline{U}$ of U lies in $\partial U = \cup_{l=1}^L \gamma_l$, hence ∂V consists of at most l Jordan curves. As $\chi(V) = \chi(\Sigma) - \chi(U) = l$ by (3.28), we see that each component of $V = \Sigma - \overline{U}$ is a disc.

Since $\phi(I_l^0 - \{q_l\})$ is connected and lies in $\Omega - \overline{U} \subseteq \Sigma - \overline{U} = V$, it is contained in one of these discs, which we call D_l. By the uniformisation theorem for simply connected Riemann surfaces, see [4, Theorem IV.1.1], these are conformally equivalent to the disc or the plane in $\mathbb{C}$. Extending beyond the Jordan curves γ_l^0 and γ_l, we get conformal diffeomorphisms $\varphi_l : B_1(0) \xrightarrow{\approx} (I_l^0, g_{poin,0}), \varphi_l(0) = q_l, \psi_l : B_1(0) \xrightarrow{\approx} (D_l, g_{poin})$. Then

$$h_l := \psi_l^{-1} \circ \phi \circ \varphi_l : B_1(0) - \{0\} \to B_1(0)$$

is holomorphic and injective. Therefore h_l extends to a holomorphic function on $B_1(0)$ with $h_l'(0) \neq 0$, and ϕ extends to a holomorphic function $\phi : (\Sigma, g_{poin,0}) \to (\Sigma, g_{poin})$ and $D\phi$ has full rank everywhere on Σ. Then $\phi(\Sigma)$ is open and compact, hence $\phi(\Sigma) = \Sigma$ and ϕ is surjective. Also ϕ is a local diffeomorphism, hence a covering projection, as Σ is compact. Then $\#\phi^{-1}(q)$ is finite and constant for $q \in \Sigma$. As ϕ is injective on $\Omega_0 = \Sigma - \{q_1, \ldots, q_l\}$, we see that ϕ is injective, hence a diffeomorphism. From (3.27), we see $\phi^* g_{poin} = e^{2u_0 - 2(u\circ\phi)} g_{poin,0}$, hence $\phi^* g_{poin} = g_{poin,0}$, which yields (3.26), and the proposition is proved. □

Combining the previous proposition with §2, we estimate the conformal factor and get the equivalence of no topologiccal loss after applying appropriate Möbius transformations and compactness in moduli space.

Theorem 3.1 *Let* $\Sigma_m \subseteq \mathbb{R}^n$ *be closed, orientable, embedded surfaces of fixed genus* $p \geq 1$ *with*

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < 8\pi, \tag{3.29}$$

$$\Sigma_m \subseteq B_1(0),$$
$$\mathcal{H}^2 \lfloor \Sigma_m \to \mu \neq 0 \quad \textit{weakly as Radon measures}. \tag{3.30}$$

Then $spt\ \mu$ *is a closed, orientable, embedded topological surface of* $genus(spt\ \mu) \leq p$. *No topology is lost in the sense that*

$$genus(spt\ \mu) = p \tag{3.31}$$

if and only if some topology is kept in the sense that

$$genus(spt\ \mu) \geq 1 \tag{3.32}$$

and the conformal structues

$$[\Sigma_m] \textit{ lie in a compact subset of the moduli space}. \tag{3.33}$$

In this case if moreover

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < \mathcal{W}(\mu) + e_n, \tag{3.34}$$

then the induced metrics $g_m := g_{euc}|\Sigma_m$ *are uniformly conformal to unit volume constant curvature metrics* $g_{poin,m} := e^{-2u_m} g_m$ *for* m *large, more precisely*

$$\limsup_{m\to\infty} \parallel u_m \parallel_{L^\infty(\Sigma)} < \infty. \tag{3.35}$$

Proof Proposition 2.1 implies that $spt\ \mu$ is a closed, orientable, embedded topological surface of $genus(spt\ \mu) \leq p$. By Proposition 2.4 the assumptions (3.31) and (3.34) imply (3.35). It remains to prove the equivalence of (3.31) on the ond side and (3.32) and the relative compactness of the conformal structures $[\Sigma_m]$ on the other side.

When no topology is lost in the sense of (3.31), the conformal structures lie in a compact subset of moduli space by Proposition 2.3 and obviously we have (3.32).

Conversely if some topology is kept in the sense of (3.32) and if the conformal structures lie in a compact subset of moduli, we get (3.31) by Propsition 3.1. □

Theorem 3.2 *Let* $\Sigma_m \subseteq \mathbb{R}^n$ *be closed, orientable, embedded surfaces of genus* $p \geq 1$ *with*

$$\limsup_{m\to\infty} \mathcal{W}(\Sigma_m) < 8\pi.$$

Then the conformal structures induced by Σ_m *lie in a compact subset of the moduli space if and only if no topology is lost after applying appropriate Möbius transformations, more precisely that any subsequence has a subsequence such that after applying appropriate Möbius transformations*

$$\mathcal{H}^2 \lfloor \Sigma_m \to \mu \quad \textit{weakly as Radon measures}$$

with $spt\ \mu$ *is a closed, orientable, embedded topological surface and*

$$genus(spt\ \mu) = p.$$

In this case after passing to a subsequence the conformal structures converge

$$[\Sigma_m] \to [spt\ \mu] \text{ in moduli space}.$$

Proof When no topology is lost for all subsequences, the conformal structures lie in a compact subset of moduli space by Proposition 2.3.

Conversely by Proposition 2.2 any subsequence has a subsequence such that after applying appropriate Möbius transformations some topology is kept in the sense that $genus(spt\ \mu) \geq 1$, and when the conformal structures lie in a compact subset of moduli, we get by Propsition 3.1 that $genus(spt\ \mu) = p$ and the convergence of the conformal structues as above. □

4 Main results

For a closed, orientable surface Σ with smooth metric g_0 or at least uniformly conformal to a smooth metric, we recall the definition in (1.7)

$$\mathcal{W}(\Sigma, g_0, n) := \inf\{\mathcal{W}(f) \mid f : \Sigma \to \mathbb{R}^n \text{ smooth immersion conformal to } g_0\}.$$

Theorem 4.1 *Let* $f : \Sigma \to \mathbb{R}^n$ *be a smooth immersion of a closed, orientable surface* $\Sigma \not\cong S^2$ *with*

$$\mathcal{W}(f) \leq \min(8\pi, \mathcal{W}(\Sigma, f^* g_{euc}, n) + e_n) - \delta \tag{4.1}$$

for some $\delta > 0$. *Moreover we assume that the conformal structure induced by the pull-back metric of* f *lies in a compact subset* K *of the moduli space.*

Then after applying an Möbius transformation, the pull-back metric $g := f^* g_{euc}$ *is uniformly conformal to a unit volume constant curvature metric* $g_{poin} := e^{-2u} g$, *more precisely*

$$\parallel u \parallel_{L^\infty(\Sigma)} \leq C(n, p, K, \delta).$$

Proof We consider a sequence of smooth immersions $f_m : \Sigma \to \mathbb{R}^n$ with pull-back metrics $g_m := f_m^* g_{euc}$,

$$\mathcal{W}(f_m) \leq \min(8\pi, \mathcal{W}(\Sigma, g_m, n) + e_n) - \delta \tag{4.2}$$

and with conformal structures induced by g_m converging in moduli space. As $\mathcal{W}(f_m) \leq 8\pi - \delta$ by (4.2), we apply Proposition 2.2 and can proceed after applying appropriate Möbius transformations and passing to a subsequence as in Proposition 2.1 with some topology kept, that is $genus(spt\ \mu) \geq 1$. By Proposition 3.1 no topology is lost in the sense that $genus(spt\ \mu) = genus(\Sigma)$, and we get a uniformly conformal $W^{2,2}-$immersion $f : \Sigma \to \mathbb{R}^n$ with

$$[g_m] = [f_m(\Sigma)] \to [spt\ \mu] = [f^* g_{euc}] \text{ in moduli space}$$

by (3.3). Then by [10, Proposition 5.1 and Theorem 5.1]

$$\limsup_{m \to \infty} \mathcal{W}(\Sigma, g_m, n) \leq \mathcal{W}(\Sigma, f^* g_{euc}, n) \leq \mathcal{W}(f),$$

hence by (4.2)

$$\limsup_{m\to\infty} \mathcal{W}(f_m) < \mathcal{W}(f) + e_n,$$

which verifies (2.91). Then by Proposition 2.4

$$\limsup_{m\to\infty} \| u_m \|_{L^\infty(\Sigma)} < \infty,$$

and the theorem follows, as f_m was an arbitrary sequence. □

As $\mathcal{W}(\Sigma, g, n) \geq \beta_p^n$ and $e_3 = 4\pi$, we obtain the following corollary, which may be considered as a pratial converse of [9, Lemma 5.1].

Theorem 4.2 *Let $f : \Sigma \to \mathbb{R}^n$ be a smooth immersion of a closed, orientable surface Σ of genus $p \geq 1$ with*

$$\mathcal{W}(f) \leq \begin{cases} 8\pi - \delta & \text{for } n = 3, \\ \beta_p^n + e_n - \delta & \text{for } n \geq 4, \end{cases} \tag{4.3}$$

for some $\delta > 0$ and assume that the conformal structure induced by the pull-back metric of f lies in a compact subset K of the moduli space.

Then after applying an Möbius transformation, the pull-back metric $g := f^* g_{euc}$ is uniformly conformal to a unit volume constant curvature metric $g_{poin} := e^{-2u} g$, more precisely

$$\| u \|_{L^\infty(\Sigma)} \leq C(n, p, K, \delta). \tag{4.4}$$

□

Also we generalize the lower semicontinuity of $\mathcal{W}(\Sigma, g, n)$with respect to$g$ of [10] Proposition 5.1 below the energy level $\mathcal{W}_{n,p}$ to the energy level 8π. We write $\mathcal{W}(\Sigma, c, n) := \mathcal{W}(\Sigma, g, n)$ for the conformal structure c induced by g.

Proposition 4.1 *Let $\Sigma \not\cong S^2$ be a closed, orientable surface and consider conformal structures $c_m \to c$ converging in moduli space. Then*

$$\liminf_{m\to\infty} \mathcal{W}(\Sigma, c_m, n) < 8\pi \Longrightarrow \mathcal{W}(\Sigma, c, n) \leq \liminf_{m\to\infty} \mathcal{W}(\Sigma, c_m, n).$$

In particular $c \mapsto \mathcal{W}(\Sigma, c, n)$is continuous at c with $\mathcal{W}(\Sigma, c, n) \leq 8\pi$.

Proof We select unit volume constant curvature metrics $g_{poin,m}, g_{poin}$ inducing c_m, c with

$$g_{poin,m} \to g_{poin} \quad \text{smoothly on } \Sigma$$

and smooth immersion $f_m : \Sigma \to \mathbb{R}^n$ conformal to $g_{poin,m}$ and with

$$\mathcal{W}(f_m) \leq \mathcal{W}(\Sigma, c_m, n) + 1/m.$$

Clearly by assumption and $e_n \geq 2\pi$, we see

$$\mathcal{W}(f_m) \leq \min(8\pi, \mathcal{W}(\Sigma, c_m, n) + e_n) - \delta$$

for $\delta > 0$ small enough, m large, and get by Theorem 4.1 after applying appropriate Möbius transformations for the pull-back metrics $g_m := f_m^* g_{euc} = e^{2u_m} g_{poin,m}$ that

$$\limsup_{m\to\infty} \| u_m \|_{L^\infty(\Sigma)} < \infty.$$

Then by Proposition 6.1 and the remark following after passing to an appropriate subsequence

$$f_m \to f \text{ weakly in } W^{2,2}(\Sigma),$$
$$f^* g_{euc} = e^{2u} g_{poin},$$

for some $u \in L^\infty(\Sigma)$. Therefore f is a $W^{2,2}-$immersion uniformly conformal to g_{poin}, and we get from [10, Theorem 5.1]

$$\mathcal{W}(\Sigma, c, n) \leq \mathcal{W}(f) \leq \liminf_{m\to\infty} \mathcal{W}(f_m) = \liminf_{m\to\infty} \mathcal{W}(\Sigma, c_m, n).$$

Finally, if $c \mapsto \mathcal{W}(\Sigma, c, n)$ were not continuous at c with $\mathcal{W}(\Sigma, c, n) \leq 8\pi$, by upper semicontinuity of $c \mapsto \mathcal{W}(\Sigma, c, n)$ in [10, Proposition 5.1], there exists a sequence $c_m \to c$ with

$$\liminf_{m\to\infty} \mathcal{W}(\Sigma, c_m, n) < \mathcal{W}(\Sigma, c, n) \leq 8\pi.$$

Then by above

$$\mathcal{W}(\Sigma, c, n) \leq \liminf_{m\to\infty} \mathcal{W}(\Sigma, c_m, n),$$

which is a contradiction, and the proposition is proved. □

Appendix A: Estimates in higher dimension

In this section, we prove a higher dimensional version of [9, Theorem 6.1]. We recall the definition of the constants in (1.2)

$$e_n := \begin{cases} 4\pi & \text{for } n = 3, \\ 8\pi/3 & \text{for } n = 4, \\ 2\pi & \text{for } n \geq 5. \end{cases}$$

Theorem 5.1 *Let $f : \mathbb{R}^2 \to \mathbb{R}^n$ be a complete conformal immersion with induced metric $g = e^{2u} g_{euc}$ and square integrable second fundamental form satisfying*

$$\int_{\mathbb{R}^2} K_g \, d\mu_g = 0, \tag{5.1}$$

$$\int_{\mathbb{R}^2} |A|^2 \, d\mu_g \leq 4e_n - \delta := \begin{cases} 16\pi - \delta & \textit{for } n = 3, \\ 32\pi/3 - \delta & \textit{for } n = 4, \\ 8\pi - \delta & \textit{for } n \geq 5, \end{cases} \tag{5.2}$$

for some $\delta > 0$. Then the limit $\lambda = \lim_{z\to\infty} u(z) \in \mathbb{R}$ exists, and

$$\|u - \lambda\|_{L^\infty(\mathbb{R}^2)}, \|Du\|_{L^2(\mathbb{R}^2)}, \|D^2 u\|_{L^1(\mathbb{R}^2)} \leq C(n, \delta) \int_{\mathbb{R}^2} |A|^2 \, d\mu_g. \tag{5.3}$$

Proof For $n = 3$, we estimate with $|K| \leq |A|^2/2$ that

$$\int_{\mathbb{R}^2} |K| \, d\mu_g \leq \frac{1}{2} \int_{\mathbb{R}^2} |A|^2 \, d\mu_g \leq 8\pi - \delta/2,$$

and the result follows from [9, Theorem 6.1]. For $n = 4$, we estimate with $|A^0|^2 = |A|^2/2 - K$ and (5.1) that

$$\int_{\mathbb{R}^2} |K| \, d\mu_g + \frac{1}{2} \int_{\mathbb{R}^2} |A^0|^2 \, d\mu_g \leq \frac{3}{4} \int_{\mathbb{R}^2} |A|^2 \, d\mu_g \leq 8\pi - 3\delta/4,$$

and again the result follows from [9, Theorem 6.1].

For $n \geq 5$, we get by elementary differential geometry from $g_{euc} = e^{-2u} g$ and $K_{g_{euc}} = 0$ that

$$-\Delta_g u = K_g \quad \text{on } \mathbb{R}^2. \tag{5.4}$$

Let $\varphi : \mathbb{R}^2 \to G_{n,2} \subseteq \mathbb{P}^{n-1}(\mathbb{C})$ be the Gauß map of f on $\mathbb{R}^2$ into the Grassmanian of oriented two planes as subset of the complex projective space, see [12, §2.2]. We know from [12, §2.3]

$$|D\varphi|^2 = \frac{1}{2}|A|^2, \tag{5.5}$$

in particular $D\varphi \in L^2(\mathbb{R}^2)$ by (5.2), hence $\varphi \in W_0^{1,2}(\mathbb{R}^2, \mathbb{P}^{n-1}(\mathbb{C}))$ in the sense of [12, §2.1]. Further from [12, §2.2], we know for the standard Kähler two-form ω on $\mathbb{P}^{n-1}(\mathbb{C})$ that

$$\varphi^* \omega = K_g vol_g. \tag{5.6}$$

We get by (5.1)

$$\int_{\mathbb{R}^2} \varphi^* \omega = 0$$

and by (5.2)

$$\int_{\mathbb{R}^2} J\varphi \, d\mu_g \leq \frac{1}{2} \int_{\mathbb{R}^2} |D\varphi|^2 \, d\mu_g = \frac{1}{4} \int_{\mathbb{R}^2} |A|^2 \, d\mu_g \leq 2\pi - \delta/4.$$

Then by [12, Corollary 3.5.7] there exists a smooth $v : \mathbb{R}^2 \to \mathbb{R}$ with

$$-\Delta v = *\varphi^* \omega \quad \text{in } \mathbb{R}^2, \ \lim_{z \to \infty} v(z) = 0, \tag{5.7}$$

and satisfying the estimates

$$\|v\|_{L^\infty(\mathbb{R}^2)}, \|Dv\|_{L^2(\mathbb{R}^2)}, \|D^2 v\|_{L^1(\mathbb{R}^2)} \leq C(n, \delta) \int_{\mathbb{R}^2} |D\varphi|^2 \, d\mu_g \leq C(n, \delta) \int_{\mathbb{R}^2} |A|^2 \, d\mu_g. \tag{5.8}$$

We rewrite (5.7) by (5.6) into

$$-\Delta_g v = -e^{-2u} \Delta v = e^{-2u} * \varphi^* \omega = K_g$$

and see from (5.4) that $u - v$ is an entire harmonic function. But [12, Theorem 4.2.1, Corollary 4.2.5] combined with (5.1), imply that u is bounded. Therefore $u - v$ is also bounded and reduces to a constant λ. Then (5.3) follows from (5.8), which proves the theorem. □

With this theorem, we obtain extensions of [9, Theorems 3.1 and 4.1] along the proofs given there.

Theorem 5.2 *Let* $f : \Sigma \to \mathbb{R}^n$ *be an immersion of a closed, orientable surface* Σ *of genus* $p \geq 1$ *with* $\mathcal{W}(f) \leq \Lambda$. *Assume that* $f(\Sigma) \subseteq \bigcup_{k=1}^K B_{\varrho_k/2}(x_k)$ *with* $\varrho_l/\varrho_k \leq R$, *such that for all* $k = 1, \ldots, K$ *and some* $\delta > 0$ *the following conditions hold:*

$$\int_{B_{\varrho_k}(x_k)} |A|^2 \, d\mu < 4e_n - \delta = \begin{cases} 16\pi - \delta & \text{for } n = 3, \\ 32\pi/3 - \delta & \text{for } n = 4, \\ 8\pi - \delta & \text{for } n \geq 5, \end{cases} \tag{5.9}$$

$$\int_{B_{\varrho_k}(x_k) - B_{\varrho_k/2}(x_k)} |A|^2 \, d\mu < \varepsilon^2. \tag{5.10}$$

Denoting by $D_\sigma^{k,\alpha}$, $1 \leq \alpha \leq m_k$, *the components of* $f^{-1}(B_\sigma(x_k))$ *which meet* $\partial B_{9\varrho_k/16}(x_k)$, *we further assume for all* $\sigma \in [5\varrho_k/8, 7\varrho_k/8]$ *up to a set of measure at most* $\varrho_k/16$ *that*

$$\int_{D_\sigma^{k,\alpha}} K_g \, d\mu_g > -2\pi + \delta \quad \text{for all } \alpha = 1, \ldots, m_k. \tag{5.11}$$

Then for $\varepsilon \leq \varepsilon(n, \Lambda, \delta)$ *and* $C_0 \geq C_0(\Lambda)$, *there is a constant curvature metric* $g_0 = e^{-2u} g$ *such that*

$$\sup_\Sigma |u| \leq C(n, \Lambda, R, K, p, \delta).$$

□

Remark We see that (5.9) satisfies for $n = 3, 4$ the weaker conditions [9] Theorem 3.1 (3.1) and (3.2) with different δ and for ε small enough. This follows from $|K| \leq |A|^2/2$ and $|A^0|^2 = |A|^2/2 - K$ and the observation by (5.11) and [9, 2.6] that

$$\left| \int_{D_\sigma^{k,\alpha}} K_g \, d\mu_g \right| \leq C\varepsilon^\alpha.$$

□

We recall the definitions of the constants in (1.1), see also [9],

$$\tilde{\beta}_p^n := \min \left\{ 4\pi + \sum_{i=1}^k (\beta_{p_i}^n - 4\pi) : 1 \leq p_i < p, \sum_{i=1}^k p_i = p \right\},$$

where $\tilde{\beta}_1^n = \infty$, and in (1.3)

$$\mathcal{W}_{n,p} := \min(8\pi, \tilde{\beta}_p^n, \beta_p^n + e_n).$$

For $n = 3$, the last term could be ommitted as $\beta_p^3 + e_3 > 8\pi$.

Theorem 5.3 *For* $p \geq 1$, *let* $\mathcal{C}(n, p, \delta)$ *be the class of closed, orientable, genus* p *surfaces* $f : \Sigma \to \mathbb{R}^n$ *satisfying* $\mathcal{W}(f) \leq \mathcal{W}_{n,p} - \delta$ *for some* $\delta > 0$. *Then for any* $f \in \mathcal{C}(n, p, \delta)$ *there is a Möbius transformation* Φ *and a constant curvature metric* g_0, *such that the metric* g *induced by* $\Phi \circ f$ *satisfies*

$$g = e^{2u} g_0 \quad \text{where} \quad \sup_\Sigma |u| \leq C(n, p, \delta) < \infty.$$

□

As already pointed out in [9], the bound $\mathcal{W}_{3,p}$ is sharp for $n = 3$, but for $n \geq 4$ there is no indication that the terms $\beta_p^4 + (8\pi/3)$ or $\beta_p^n + 2\pi$ are necessary.

There is also a version to solve (5.7) when f is an immersion of a disc, but we do not use it in the text.

Proposition 5.1 *Let $f : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^n$, be a conformal immersion with induced metric $g = e^{2u} g_{euc}$ and square integrable second fundamental form satisfying*

$$\int_{B_1(0)} |A|^2 \, d\mu_g \leq \begin{cases} 8\pi - \delta & \text{for} \quad n = 3, \\ 4\pi - \delta & \text{for} \quad n \geq 4, \end{cases} \tag{5.12}$$

for some $\delta > 0$. Then there exists a smooth solution $v : B_1(0) \to \mathbb{R}$ of

$$-\Delta_g v = K_g \quad \text{on } B_1(0) \tag{5.13}$$

satisfying

$$\|v\|_{L^\infty(B_1(0))}, \|Dv\|_{L^2(B_1(0))}, \|D^2 v\|_{L^1(B_1(0))} \leq C(n, \delta) \int_{B_1(0)} |A|^2 \, d\mu_g. \tag{5.14}$$

Proof For $n \geq 4$, we let as in the proof of the previous theorem $\varphi : B_1(0) \to G_{n,2} \subseteq \mathbb{P}^{n-1}(\mathbb{C})$ be the Gauß map of f on $B_1(0)$ into the Grassmanian. Extending by $\varphi(z) := \varphi(1/\bar{z})$ for $|z| > 1$, we see $\varphi \in W_0^{1,2}(\mathbb{R}^2, \mathbb{P}^{n-1}(\mathbb{C}))$ in the sense of [12, §2.1] by (5.5) and (5.12), and for the standard Kähler two-form ω on $\mathbb{P}^{n-1}(\mathbb{C})$, see [12, §2.2], that $\int_{\mathbb{R}^2} \varphi^* \omega = 0$. Next by (5.5) and (5.12)

$$\int_{\mathbb{R}^2} J\varphi \, d\mu_g = 2 \int_{B_1(0)} J\varphi \, d\mu_g \leq \int_{B_1(0)} |D\varphi|^2 \, d\mu_g = \frac{1}{2} \int_{B_1(0)} |A|^2 \, d\mu_g \leq 2\pi - \delta/2.$$

Then by [12, Corollary 3.5.7] there exists a smooth $v : \mathbb{R}^2 \to \mathbb{R}$ with

$$-\Delta v = *\varphi^* \omega \quad \text{in } \mathbb{R}^2, \quad \lim_{z \to \infty} v(z) = 0, \tag{5.15}$$

and satisfying the estimates

$$\|v\|_{L^\infty(\mathbb{R}^2)}, \|Dv\|_{L^2(\mathbb{R}^2)}, \|D^2 v\|_{L^1(\mathbb{R}^2)} \leq C(n, \delta) \int_{\mathbb{R}^2} |D\varphi|^2 \, d\mu_g \leq C(n, \delta) \int_{B_1(0)} |A|^2 \, d\mu_g. \tag{5.16}$$

As above (5.15) implies by (5.6) that $-\Delta_g v = K_g$ on $B_1(0)$, hence (5.13), and (5.16) gives (5.14), which proves the propsition for $n \geq 4$.

We improve for $n = 3$ by considering the Gauß map of f on $B_1(0)$ into the sphere as $\nu : B_1(0) \to S^2$. Extending as above by $\nu(z) := \nu(1/\bar{z})$ for $|z| > 1$, we see $\nu \in W_0^{1,2}(\mathbb{R}^2, S^2)$ and $\int_{\mathbb{R}^2} \nu^* vol_{S^2} = 0$. Next $J\nu = |K_g|$ and by (5.12)

$$\int_{\mathbb{R}^2} J\nu \, d\mu_g = 2 \int_{B_1(0)} |K_g| \, d\mu_g \leq \int_{B_1(0)} |A|^2 \, d\mu_g < 8\pi - \delta.$$

Then proceeding as in [9, Theorem 6.1] there exists a smooth $v : \mathbb{R}^2 \to \mathbb{R}$ with

$$-\Delta v = *\nu^* vol_{S^2} \quad \text{in } \mathbb{R}^2, \quad \lim_{z \to \infty} v(z) = 0,$$

and satisfying the estimates

$$\|v\|_{L^\infty(\mathbb{R}^2)}, \|Dv\|_{L^2(\mathbb{R}^2)}, \|D^2v\|_{L^1(\mathbb{R}^2)} \le C(\delta) \int\limits_{\mathbb{R}^2} |Dv|^2 \, d\mu_g \le C(\delta) \int\limits_{B_1(0)} |A|^2 \, d\mu_g.$$

Observing that $*\nu^* vol_{S^2} = K_g$ on $B_1(0)$, this proves the proposition for $n = 3$ as above. □

Remark We mention that (5.12) can be replaced along [9] Theorem 6.1 by

$$\int\limits_{B_1(0)} |K_g| \, d\mu_g \le 4\pi - \delta \quad \text{for} \quad n = 3,$$

$$\left.\begin{array}{l} \int_{B_1(0)} |K_g| \, d\mu_g + \frac{1}{2} \int_{B_1(0)} |A^0|^2 \, d\mu_g \le 4\pi - \delta, \\ \int_{B_1(0)} |A^0|^2 \, d\mu_g < 4\pi, \end{array}\right\} \quad \text{for} \quad n = 4.$$

□

Appendix B: Convergence in $W^{2,2}$

In this appendix, we prove a useful convergence proposition.

Proposition 6.1 *Let* $f_m : \Sigma \to \mathbb{R}^n$ *be smooth immersions of a closed, orientable surface* Σ *uniformly conformal to some smooth unit volume constant curvature metrics* $g_{poin,m}$ *and*

$$f_m^* g_{euc} = e^{2u_m} g_{poin,m},$$
$$\mathcal{W}(f_m), \| u_m \|_{L^\infty(\Sigma)} \le \Lambda$$

for some $\Lambda < \infty$.

Then for a subsequence there exist diffeomorphisms $\phi_m : \Sigma \xrightarrow{\approx} \Sigma$ *such that replacing* f_m *by* $f_m \circ \phi_m$

$$\begin{aligned} f_m &\to f \text{ weakly in } W^{2,2}(\Sigma), \text{weakly}^* \text{ in } W^{1,\infty}(\Sigma), \\ u_m &\to u \text{ weakly in } W^{1,2}(\Sigma), \text{weakly}^* \text{ in } L^\infty(\Sigma), \\ g_{poin,m} &\to g_{poin} \quad \text{smoothly}, \end{aligned} \tag{6.1}$$

$$f^* g_{euc} = e^{2u} g_{poin}. \tag{6.2}$$

If

$$\pi(f_m^* g_{euc}) \to \tau \text{ in } \mathcal{T}, \tag{6.3}$$

then ϕ_m *can be chosen with* $\phi_m \simeq id_\Sigma$ *and*

$$\pi(f^* g_{euc}) = \pi(g_{poin}) = \tau.$$

Proof By [9, Lemma 5.1] the conformal structures induced by $g_m := f_m^* g_{euc}$ respectively by $g_{poin,m}$ are compactly contained in moduli space, hence there exist diffeomorphisms $\phi_m : \Sigma \xrightarrow{\approx} \Sigma$ such that for a subsequence

$$\phi_m^* g_{poin,m} \to g_{poin} \quad \text{smoothly}$$

to a smooth unit volume constant curvature metric g_{poin}. After reparametrizing, we may assume $\phi_m = id_\Sigma$. We get by elementary differential geometry and the Gauß–Bonnet theorem

$$- \Delta_{g_m} u_m + 2\pi \chi(\Sigma) e^{-2u_m} = K_{g_m}. \tag{6.4}$$

Therefore

$$\begin{aligned}
&\int_\Sigma |Du_m|^2_{g_{poin,m}} \, \mathrm{d}\mu_{g_{poin,m}} = \int_\Sigma |Du_m|^2_{g_m} \, \mathrm{d}\mu_{g_m} \\
&= - \int_\Sigma \Delta_{g_m} u_m \cdot u_m \, \mathrm{d}\mu_{g_m} = \int_\Sigma \left(K_{g_m} - 2\pi \chi(\Sigma) e^{-2u_m} \right) u_m \, \mathrm{d}\mu_{g_m} \\
&\le C(\Sigma, \Lambda) \left(\int_\Sigma 1 \, \mathrm{d}\mu_{g_{poin,m}} + \int_\Sigma |K_{g_m}| \, \mathrm{d}\mu_{g_m} \right) \le C(\Sigma, \Lambda) \left(1 + \int_\Sigma |A_{f_m}|^2 \, \mathrm{d}\mu_{f_m} \right) \\
&\le C(\Sigma, \Lambda)(1 + \mathcal{W}(f_m)) \le C(\Sigma, \Lambda),
\end{aligned}$$

as $\mathcal{W}(f_m), |u_m| \le \Lambda, |K| \le |A|^2/2$ and the Gauß–Bonnet theorem. Therefore

$$\| Du_m \|_{L^2(\Sigma, g_{poin,m})} \le C(\Sigma, \Lambda),$$

and, as $g_{poin,m} \to g_{poin}$, we get for a subsequence

$$u_m \to u \begin{cases} \text{weakly in } W^{1,2}(\Sigma), \\ \text{weakly}^* \text{ in } L^\infty(\Sigma), \\ \text{and pointwise almost everywhere on } \Sigma. \end{cases}$$

Next

$$\Delta_{g_{poin,m}} f_m = e^{2u_m} \vec{\mathbf{H}}_{f_m} \quad \text{on } \Sigma$$

and, as $|u_m| \le \Lambda$,

$$\int_\Sigma |e^{2u_m} \vec{\mathbf{H}}_{f_m}|^2 \, \mathrm{d}\mu_{g_{poin,m}} \le C(\Lambda) \int_\Sigma |\vec{\mathbf{H}}_{f_m}|^2 \, \mathrm{d}\mu_{f_m} = C(\Lambda) \mathcal{W}(f_m) \le C(\Lambda),$$

and get by standard elliptic theory, see [5] Theorem 8.8, and $g_{poin,m} \to g_{poin}$ smoothly that f_m is bounded in $W^{2,2}(\Sigma)$, hence after passing to a subsequence

$$f_m \to f \text{ weakly in } W^{2,2}(\Sigma).$$

As $f_m^* g_{euc} = e^{2u_m} g_{poin,m}$ and $|u_m| \le \Lambda, g_{poin,m} \to g_{poin}$, we see that ∇f_m is bounded in $L^\infty(\Sigma)$, hence $\nabla f_m \to \nabla f$ is weakly* in $L^\infty(\Sigma)$. By the above convergences

$$f^* g_{euc} = e^{2u} g_{poin}.$$

If $\pi(f_m^* g_{euc}) \to \tau$ in Teichmüller space, we may further assume that $\phi_m \simeq id_\Sigma$ and

$$\tau \leftarrow \pi(f_m^* g_{euc}) = \pi(g_{poin,m}) \to \pi(g_{poin}),$$

hence $\pi(f^* g_{euc}) = \pi(g_{poin}) = \tau$, and the proposition is proved. □

Remarks 1. Clearly (6.1) is equivalent to [10, (2.3) – (2.5)], (6.2) is [10, 2.6], and (6.3) is [10, 2.2] with τ_0 replaced by τ.

2. Convergence of the Willmore energy

$$\mathcal{W}(f_m) \to \mathcal{W}(f)$$

gives even strong convergence $f_m \to f$ in $W^{2,2}$, see [10, Proposition 5.3].

3. If we add the assumption that $g_{poin,m} \to g_{poin}$ smoothly, the statement of the proposition is true without diffeomorphisms ϕ_m, as it is immediately seen from the beginning of the proof.

□

Appendix C: $W^{2,2}$-immersions

Proposition 7.1 *Let* $f : B_1(0) \subseteq \mathbb{R}^2 \to \mathbb{R}^n$ *with uniformly positive definite pull-back metric in the sense that*

$$c_0 g_{euc} \leq g := f^* g_{euc} \leq C g_{euc} \tag{7.1}$$

for some $0 < c_0 \leq C < \infty$.

Then for $\mu_f := f(\mu_g)$

$$\# f^{-1}(x) \leq (2C/c_0)\, \theta^2_*(\mu_f, x) \quad \textit{for any } x \in \mathbb{R}^n, \tag{7.2}$$

and if further $f \in W^{2,2}(B_1(0))$, *then*

$$\# f^{-1}(x) \leq \theta^2(\mu_f, x) \quad \textit{for all } x \in \mathbb{R}^n. \tag{7.3}$$

Proof We consider finitely many distinct $p_1, \ldots, p_N \in f^{-1}(x)$ and see for ϱ small that $B_\varrho(p_k) \subseteq B_1(0)$ are pairwise disjoint. By (7.1), we get $lip\ f \leq \sqrt{2C} < \infty$ and

$$\mu_f(B_{L\varrho}(x)) = \mu_g\left(f^{-1}(B_{L\varrho}(x))\right) \geq \mu_g(\cup_{i=1}^N B_\varrho(p_i)) \geq c_0 N \pi \varrho^2,$$

hence

$$\theta^2_*(\mu_f, x) = \liminf_{\varrho \to 0} \mu_f(B_\varrho(x))/(\pi \varrho^2) \geq c_0 N / L^2,$$

which is (7.2).

To proceed for $f \in W^{2,2}$, it suffices to consider $\theta^2_*(\mu_f, x) < \infty$, in particular $f^{-1}(x)$ is finite. We see for ϱ small enough that $x \notin f(\partial B_\varrho(p_i))$ and claim for $\mu_i := f(\mu_g \lfloor B_\varrho(p_i))$ that

$$\theta^2(\mu_i, x) \geq 1. \tag{7.4}$$

This implies

$$N \leq \sum_{i=1}^N \theta^2(\mu_i, x) \leq \theta^2(\mu_f, x),$$

and (7.3) follows.

To prove (7.4), we consider $f(0) = 0$ with $\liminf_{|p| \to 1} |f(p)| \geq r > 0$. As f is lipschitz, the image $\mu_f = f(\mu_g) = f(J_{g_0} f \cdot \mu_{g_0})$ is an integral varifold, see [15, 15.7]. We prove that μ_f has square integrable weak mean curvature in $B_r(0)$. For $\eta \in C_0^1(B_r(0), \mathbb{R}^n)$, we see that $\eta \circ f$ has compact support in $B_1(0)$ and calculate the first variation

$$\delta\mu_f(\eta) = \int div_{\mu_f}\eta \,\mathrm{d}\mu_f = \int_{B_1(0)} (div_{\mu_f}\eta) \circ f \,\mathrm{d}\mu_g,$$

where the divergence is given by

$$(div_{\mu_f}\eta) \circ f = g^{ij}\partial_i f^T((D\eta) \circ f)\partial_j f = g^{ij}\langle \partial_i f, \partial_j(\eta \circ f)\rangle.$$

We continue

$$\begin{aligned}\delta\mu_f(\eta) &= \int_{B_1(0)} g^{ij}\langle \partial_i f, \partial_j(\eta \circ f)\rangle \sqrt{g}\,\mathrm{d}\mathcal{L}^2 = -\int_{B_1(0)} \partial_j(g^{ij}\sqrt{g}\partial_i f)(\eta \circ f)\,\mathrm{d}\mathcal{L}^2 \\ &= -\int_{B_1(0)} (\Delta_g f)(\eta \circ f)\,\mathrm{d}\mu_g = -\int_{B_1(0)} \vec{\mathbf{H}}_f(\eta \circ f)\,\mathrm{d}\mu_g.\end{aligned}$$

Therefore μ_f has weak mean curvature in $B_r(0)$ given by

$$\vec{\mathbf{H}}_{\mu_f} = \sum_{p \in f^{-1}} \vec{\mathbf{H}}_f(p) \in L^2(\mu_f). \tag{7.5}$$

Then from [8, A.10], we get $\theta^2(\mu_f, x) \geq 1$ for any $x \in spt\ \mu_f \cap B_r(0)$. As $f(0) = 0$, we get from (7.2) that $\theta^2_*(\mu_f, 0) > 0$, in particular $0 \in spt\ \ \mu_f$, and (7.4) follows. □

Proposition 7.2 *Let $f \in W^{2,2}(\Sigma, \mathbb{R}^n)$ for some closed surface Σ with uniformly positive definite pull-back metric in the sense that*

$$c_0 g_0 \leq g := f^* g_{euc} \leq C g_0 \tag{7.6}$$

for some smooth metric g_0 on Σ and $0 < c_0 \leq C < \infty$.

Then

$$\begin{aligned}\mu_f &:= f(\mu_g) = \#f^{-1} \cdot \mathcal{H}^2 \lfloor f(\Sigma) = \#f^{-1} \cdot \mathcal{H}^2 \lfloor spt\ \mu_f, \\ spt\ \mu_f &= f(\Sigma).\end{aligned} \tag{7.7}$$

Moreover μ_f is an integral varifold with square integrable weak mean curvature and

$$\begin{aligned}&\mathcal{W}(\mu_f) = \mathcal{W}(f), \\ &\#f^{-1}(x) \leq \theta^2(\mu_f, x) \leq \mathcal{W}(f)/(4\pi) \quad \text{for all} \quad x \in \mathbb{R}^n.\end{aligned} \tag{7.8}$$

If $\mathcal{W}(f) < 8\pi$, then f is injective, in particular

$$f : \Sigma \xrightarrow{\approx} spt\ \mu_f = f(\Sigma)$$

is a homeomorphism and

$$\mu_f = \mathcal{H}^2 \lfloor spt\ \mu_f = \mathcal{H}^2 \lfloor f(\Sigma).$$

If moreover f is uniformly conformal to a smooth metric g_0 on Σ, that is $f^* g_{euc} = e^{2u} g_0$ with $u \in L^\infty(\Sigma)$, then f is a bi-lipschitz homeomorphism.

Proof Again as f is lipschitz from (7.6), we see that the image $\mu_f = f(\mu_g) = f(J_{g_0} f \cdot \mu_{g_0})$ is an integral varifold and

$$\mu_f = \#f^{-1} \cdot \mathcal{H}^2 \lfloor f(\Sigma), \tag{7.9}$$

see [15, 15.7]. Clearly $spt\ \mu_f \subseteq f(\Sigma)$, as $f(\Sigma)$ is compact, hence closed. On the other hand by (7.2), we get $\theta^2_*(\mu_f) > 0$ on $f(\Sigma)$, hence $spt\ \mu_f = f(\Sigma)$, and (7.7) follows.

As $\eta \circ f$ has compact support in Σ for any $\eta \in C^1_0(\mathbb{R}^n, \mathbb{R}^n)$, we see as in (7.5) that μ_f has weak mean curvature given by

$$\vec{\mathbf{H}}_{\mu_f} = \sum_{p \in f^{-1}} \vec{\mathbf{H}}_f(p) \in L^2(\mu_f)$$

and

$$\mathcal{W}(\mu_f) = \frac{1}{4} \int |\vec{\mathbf{H}}_{\mu_f}|^2 \, \mathrm{d}\mu_f = \frac{1}{4} \int_\Sigma |\vec{\mathbf{H}}_f|^2 \, \mathrm{d}\mu_g = \mathcal{W}(f) < \infty,$$

which gives the first part in (7.8). Then from [8, A.17], we get $\theta^2(\mu_f) \leq \mathcal{W}(\mu_f)/(4\pi)$, and the second part follows from (7.3).

If $\mathcal{W}(f) < 8\pi$, we see that $\#f^{-1} \leq 1$, hence f is injective and by (7.7)

$$\mu_f = \mathcal{H}^2 \lfloor spt\ \mu_f = \mathcal{H}^2 \lfloor f(\Sigma).$$

As obviusly $f : \Sigma \to spt\ \mu_f = f(\Sigma)$ is surjective, we get that f is bijective, and, as f is continuous and Σ is compact, we see that f is a homeomorphism.

Finally we assume f to be uniformly conformal. We already know that f is lipschitz, and it remains to prove that its inverse is lipschitz. We will use [12, Lemma 4.2.8]. If f^{-1} is not lipschitz, we have $p_k, q_k \in \Sigma$ with

$$|f(p_k) - f(q_k)| < d_{\Sigma, g_0}(p_k, q_k)/k. \tag{7.10}$$

We get for a subsequence $p_k \to p, q_k \to q$ in Σ, and, as $diam_{g_0} \Sigma$ is finite, $f(p) = f(q)$, hence $p = q$, as f is injective. Introducing local conformal coordinates for g_0 around p, we get an open neighbourhood $U(p)$ of p and $\varphi : B_1(0) \cong U(p)$ with $\varphi(0) = p, \varphi^* g_0 = e^{2v} g_{euc}, v \in L^\infty(B_1(0))$. Then

$$(f \circ \varphi)^* g_{euc} = \varphi^*(e^{2u} g_0) = e^{2(v + u \circ \varphi)} g_{euc}.$$

and $f_0 := f \circ \varphi : B_1(0) \to \mathbb{R}^n$ is conformal and $f_0 \in W^{2,2}_{loc}(B_1(0))$. Then for $M := \| v \|_{L^\infty(B_1(0))} + \| u \|_{L^\infty(\Sigma)} < \infty$, we can choose $0 < \varrho < 1$ with

$$\int_{B_\varrho(0)} |D^2 f_0|^2 \, \mathrm{d}\mathcal{L}^2 < c_0 e^{-2M},$$

where $c_0 = (\pi \tanh \pi)/2$ is given in [12, Lemma 4.2.8]. For k large, a square with vertices $\varphi^{-1}(p_k), \varphi^{-1}(q_k)$ is contained in $B_\varrho(0)$, and we get from [12, Lemma 4.2.8]

$$d_{\Sigma, g_0}(p_k, q_k) \leq e^M d_{\Sigma, f^* g_{euc}}(p_k, q_k) \leq \sqrt{2} e^M |f(p_k) - f(q_k)|,$$

which contradicts (7.10), and the proposition is proved. □

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