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Self-shrinkers for the mean curvature flow in arbitrary codimension

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Abstract In this paper, we generalize Colding–Minicozzi's recent results about codimension-1 self-shrinkers for the mean curvature flow to higher codimension. In particular, we prove that the sphere $bf\,S^n(\sqrt{2n})$ is the only complete embedded connected F-stable self-shrinker in $\mathbf{R}^{n+k}$ with $\mathbf{H} \neq 0$, polynomial volume growth, flat normal bundle and bounded geometry. We also discuss some properties of symplectic self-shrinkers, proving that any complete symplectic self-shrinker in $\mathbf{R}^4$ with polynomial volume growth and bounded second fundamental form is a plane. As a corollary, we show that there is no finite time Type I singularity for symplectic mean curvature flow, which has been proved by Chen–Li using different method. We also study Lagrangian self-shrinkers and prove that for Lagrangian mean curvature flow, the blow-up limit of the singularity may be not F-stable.

Keywords Self-shrinkers · Mean curvature flow · F-stable · Symplectic

Mathematics Subject Classification (2000) Primary 53C44; Secondary 53C21

1 Introduction

Let Σ^n be an n-dimensional immersed submanifold of $\mathbf{R}^{n+k}$. We say Σ is a *self-shrinker for the mean curvature flow*, if it satisfies a quasi-linear elliptic system

$$\mathbf{H} = -\frac{1}{2}\mathbf{x}^{\perp}, \tag{1.1}$$

where $\mathbf{H}$ is the mean curvature vector of Σ in $\mathbf{R}^{n+k}$, $\mathbf{x}$ is the position vector, and $\mathbf{w}^{\perp}$ is the normal part of a vector $\mathbf{w}$ in $\mathbf{R}^{n+k}$.

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Self-shrinkers are special solutions for the mean curvature flow equation

$$\left(\frac{\partial}{\partial t}\mathbf{x}\right)^{\perp} = \mathbf{H}, \tag{1.2}$$

and what is more, they are very important singularities of the mean curvature flow.

In 1984, Huisken [15] proved that, if the initial hypersurface M_0 in $\mathbf{R}^{n+1}$ is strictly convex, then along the mean curvature flow, the surface will be strictly convex at each time, and the mean curvature flow will contract to a point at a finite time T. Moreover, the normalized mean curvature flow will converge exponentially to a round sphere, the simplest example of self-shrinker.

In 1990, Huisken [16] studied Type I singularities of the mean curvature flow, and using a crucial monotonicity formula, he proved that *any Type I singularity of the mean curvature flow must be a self-shrinker*. More precisely, if the mean curvature flow develops Type I singularity at finite time T, then the rescaled mean curvature flow will converge smoothly to a self-shrinker. In the same paper, Huisken gave a first classification theorem for the self-shrinkers in hypersurface case, proving that *the sphere is the only closed self-shrinker with nonnegative mean curvature*.

Later on, Huisken [17] classified complete self-shrinkers and proved that $\boldsymbol{S}^m \times \boldsymbol{R}^{n-m}$ *are the only smooth embedded hypersurfaces with nonnegative mean curvature H, polynomial volume growth, bounded second fundamental form, and satisfying the self-shrinker equation* (1.1).

In 1994, Ilmanen [18] studied singularities of mean curvature flow of surfaces. Suppose Σ_t is a solution of mean curvature flow, $t \in [0, T)$. Define

$$\Sigma_t^{\lambda} \equiv \Sigma_t^{(y,T),\lambda} = \lambda^{-1}(\Sigma_{\lambda^{-2}t+T} - y),\ t \in \left[-\frac{T}{\lambda^2}, 0\right). \tag{1.3}$$

Ilmanen proved that: *Any family of rescalings* $\{M_t^{\lambda}\}_{t\in[-\frac{T}{\lambda^2},0)}$ *converges subsequentially to a self-similar shrinking mean curvature flow* $\{\nu_t\}_{t<0}$ *in the sense of Radon measures for all t and ν_{-1} satisfies the self-shrinker equation* (1.1).

Recently, Colding and Minicozzi [8] studied generic singularities of generic mean curvature flow of hypersurfaces and proved that *shrinking spheres, cylinders and planes are the only stable self-shrinkers*. Let us first recall some notations and definitions. Given $\mathbf{x}_0 \in \mathbf{R}^{n+k}$, $t_0 > 0$, define the functional $F_{\mathbf{x}_0,t_0}$ by

$$F_{\mathbf{x}_0,t_0}(\Sigma) = (4\pi t_0)^{-\frac{n}{2}} \int\limits_{\Sigma} e^{-\frac{|\mathbf{x}-\mathbf{x}_0|^2}{4t_0}} d\mu. \tag{1.4}$$

We will observe (as in the hypersurface case) that Σ is a critical point of the functional $F_{\mathbf{x}_0,t_0}$ precisely when it is a self-shrinker (Proposition 2.6). The *entropy* $\lambda = \lambda(\Sigma)$ of Σ is then defined to be the supremum of the $F_{\mathbf{x}_0,t_0}$ functionals

$$\lambda = \sup_{\mathbf{x}_0,t_0} F_{\mathbf{x}_0,t_0}(\Sigma). \tag{1.5}$$

It is easy to see that the critical points of λ are self-shrinkers for the mean curvature flow. We will say that a self-shrinker is *entropy-stable* if it is a local minimum for the entropy functional. The main results of [8] is as follows:

Theorem 1.1 (Theorem 0.12 of [8]) *Suppose that $\Sigma^n \subset \boldsymbol{R}^{n+1}$ is a smooth complete embedded self-shrinker without boundary and with polynomial volume growth.*

(1) *If* Σ *is not equal to* $\boldsymbol{S}^m \times \boldsymbol{R}^{n-m}$*, then there is a graph* $\tilde{\Sigma}$ *over* Σ *of a function with arbitrary small* C^l *norm (for any fixed* l*) so that* $\lambda(\tilde{\Sigma}) < \lambda(\Sigma)$.
(2) *If* Σ *is not* $\boldsymbol{S}^n$ *and does not split off a line, then the function in (1) can be taken to have compact support.*

In particular, in either case, Σ *cannot arise as a tangent flow to the mean curvature flow starting from* $\tilde{\Sigma}$.

Roughly speaking, the proof of Theorem 1.1 can be divided into three parts:

(A) Suppose Σ does not split off a line isometrically and it is entropy-stable, then it is F-stable;
(B) F-stable implies mean convexity (i.e., $H \geq 0$);
(C) Classify the mean convex self-shrinkers.

For (C), they proved that

Theorem 1.2 (Theorem 0.17 of [8]) $\boldsymbol{S}^m \times \boldsymbol{R}^{n-m}$ *are the only smooth complete embedded self-shrinkers without boundary, with polynomial volume growth, and* $H \geq 0$ *in* $\boldsymbol{R}^{n+1}$.

Theorem 1.2 improves Huisken's classification theorem [17] by removing the assumption that "Σ has bounded second fundamental form". It is easy from the self-shrinker equation (1.1) to see that the factor $\mathbf{S}^m$ has radius $\sqrt{2m}$.

Then they classified all F-stable self-shrinkers in $\mathbf{R}^{n+1}$:

Theorem 1.3 (Theorem 4.30 of [8]) $\boldsymbol{S}^n(\sqrt{2n})$ *is the only smooth embedded closed* F*-stable hypersurface in* $\boldsymbol{R}^{n+1}$ *for any* $n \geq 2$.

Theorem 1.4 (Theorem 4.31 of [8]) $\boldsymbol{R}^n$ *is the only smooth complete embedded noncompact* F*-stable hypersurface in* $\boldsymbol{R}^{n+1}$ *without boundary and with polynomial volume growth.*

The F-functional recalled above plays then a key role in Colding–Minicozzi's classification and one might wonder which kind of geometric functional it really is. A beautiful aspect of this new approach is that F-stability is actually very closely related (in fact almost equivalent) to the classical Volume-stability of an associated minimal submanifold via a simple and elegant construction due to Smoczyk [24]. We will treat this relationship in a forthcoming note [4].

We have seen that self-shrinkers are important singularities for the mean curvature flow. Although there are many works in the hypersurface case, there are only few results in higher codimension. In 2005, Smoczyk [23] generalized Huisken's result [17] to higher codimensions and gave a classification theorem for self-shrinkers with parallel principle normal $\nu := \frac{\mathbf{H}}{|\mathbf{H}|}$:

Theorem 1.5 (Theorem 1.1 of [23]) *Let* $\Sigma^n \subset \boldsymbol{R}^{n+k}$, $n \geq 2$, *be a compact self-shrinker. Then* Σ *is spherical if and only if* $\boldsymbol{H} \neq 0$ *and* $\nabla^{\perp}\nu = 0$.

Smoczyk also classified complete self-shrinkers with $\mathbf{H} \neq 0$, parallel principle normal and bounded geometry (Theorem 1.3 of [23]).

Based on Smoczyk's work, we prove the following classification of F-stable self-shrinkers:

Main Theorem 1 $\boldsymbol{S}^n(\sqrt{2n})$ *is the only* n*-dimensional* F*-stable embedded closed self-shrinker with* $\boldsymbol{H} \neq 0$ *and flat normal bundle in* $\boldsymbol{R}^{n+k}$.

More generally, we have:

Main Theorem 2 *Let $\Sigma^n \subset \boldsymbol{R}^{n+k}$ be a complete embedded connected self-shrinker with $\boldsymbol{H} \neq 0$, polynomial volume growth and flat normal bundle. Suppose further that Σ has uniform bounded geometry and is F-stable. Then Σ must be $\boldsymbol{S}^n(\sqrt{2n})$.*

We can see that most of the results on high codimensional case are obtained under the assumption "with flat normal bundle". Indeed, it is an interesting question that "whether or under which conditions the blow up flow of the mean curvature flow is normal flat". In fact, as we know [1], the type II blow-up flow of a curve shrinking flow for space curves is a planar curve (thus normal flat).

Two important mean curvature flows with high codimension are the symplectic and the Lagrangian mean curvature flows. In these cases we can prove:

Main Theorem 3 *There does not exist any two-dimensional complete embedded symplectic self-shrinker in $\boldsymbol{R}^4$ with flat normal bundle, bounded geometry and $|\boldsymbol{H}| \neq 0$.*

Recently, Le–Sesum [19] gave a gap theorem for the self-shrinker which states that a complete self-shrinker with polynomial volume growth and $|\mathbf{A}| < \frac{1}{2}$ must be a plane. Later, Cao and Li [5] generalized their result to high codimensional case. (Note that, our definition of self-shrinker is slightly different from their's. In [19] and [5], the assumption is $|\mathbf{A}| \leq 1$, which in our notation becomes $|\mathbf{A}| \leq \frac{1}{2}$.) Note also that all the following theorems do not need the assumption "with flat normal bundle".

For symplectic self-shrinkers, using the elliptic equation satisfied by the Kähler angle and the self-adjoint property of the stability operator, we can obtain a stronger gap theorem:

Main Theorem 4 *Suppose Σ^2 is a complete symplectic self-shrinker with polynomial volume growth in $\boldsymbol{R}^4$. If $|\boldsymbol{A}|^2 \leq 1$, then Σ must be a plane.*

It is proved in [14] that a symplectic translating soliton for the symplectic mean curvature flow with polynomial volume growth, flat normal bundle and bounded second fundamental form must be minimal (thus a plane). We have similar result for symplectic self-shrinker. Indeed, we can remove the "normal flat" assumption in this case.

Main Theorem 5 *Suppose Σ^2 is a complete symplectic self-shrinker in $\boldsymbol{R}^4$ with polynomial volume growth and Kähler angle α. If $|\boldsymbol{A}|^2$ is bounded and $\cos\alpha \geq \delta > 0$, then Σ must be a plane.*

We can then give a direct proof of the following fact first proved by Chen and Li [6] by a different approach.

Corollary 1.6 *There is no finite time Type I singularity for the symplectic mean curvature flow.*

For the Lagrangian mean curvature flow, Chen and Li [7] and Wang [26] proved that there is no finite time Type I singularity in the almost calibrated case. On the contrary, in 2007, Groh et al. [11] constructed examples of monotone, equivariant Lagrangian mean curvature flow which can develop Type-I singularity. The blow up flow converges to some equivariant Lagrangian self-shrinkers classified by Anciaux [2]. Neves [21] also constructed Lagrangian mean curvature flow with trivial Maslov class which develops singularity at finite time.

One can show that any equivariant Lagrangian self-shrinkers can never be F-stable. In fact, motivated by Hamiltonian stability of minimal Lagrangian minimal surfaces [22], we introduce Hamiltonian F-stability (see Appendix A) and prove that

Main Theorem 6 *Complete equivariant Lagrangian self-shrinkers are never Hamiltonian F-stable. In particular, they are never F-stable in the usual sense.*

As a corollary, we have

Corollary 1.7 *The blow-up limit of the singularity of a Lagrangian mean curvature flow may be not F-stable.*

On the other hand, Wang [25] considered the graphic self-shrinker and gave a Bernstein type theorem. Indeed, Ecker–Huisken [10] showed that a self-shrinker is a plane if it is an entire graph with polynomial volume growth. Wang [25] can remove the assumption "with polynomial volume growth".

Recently, Zhang [27] also considered F-stability of self-shrinker solutions to the harmonic map heat flow.

Added in proof After the submission of our paper, two preprints by Andrews et al. [3] and Lee and Lue [20] appeared on arxiv with significant overlap with our results.

The following sections are organized as follows: In Sects. 2 and 3, we compute the first variation and second variation formulas for F-functional, respectively; in Sect. 4, we consider minimal submanifolds of spheres, proving that they are all self-shrinkers, which is known to expert (for example, [23]); in Sect. 5, we prove Main Theorem 1 and Main Theorem 2; and in Sects. 6 and 7, we prove the properties of symplectic self-shrinkers and Lagrangian self-shrinkers, respectively.

2 The first variation formula

In this section, we will compute the first variation of the F-functional and recall some of the results of [8].

Suppose the variation vector field is $\Sigma_0' = \mathbf{V}$, where $\mathbf{V}$ is a normal vector field. Set $\mathbf{x}_0' = \mathbf{y}$, $t_0' = h$. Furthermore, suppose $\mathbf{x}_0 = 0$, $t_0 = 1$, such that

$$F_{0,1}(\Sigma_0) = (4\pi)^{-\frac{n}{2}} \int\limits_{\Sigma_0} e^{-\frac{|\mathbf{x}|^2}{4}} d\mu_0. \tag{2.1}$$

From the first variation formula for area [9], we know that

$$d\mu' = -\langle \Sigma_0', \mathbf{H}\rangle d\mu = -\langle \mathbf{V}, \mathbf{H}\rangle d\mu. \tag{2.2}$$

Here, $\mathbf{H}$ is the mean curvature vector of Σ_0. By direct computation, we have

$$\begin{aligned} \frac{\partial}{\partial s}\log\left\{(4\pi t_s)^{-\frac{n}{2}} e^{-\frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s}}\right\} &= \frac{\partial}{\partial s}\left\{-\frac{n}{2}\log 4\pi - \frac{n}{2}\log t_s - \frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s}\right\} \\ &= -\frac{n}{2t_s}t_s' - \frac{\langle \mathbf{x}-\mathbf{x}_s, \mathbf{x}'-\mathbf{x}_s'\rangle}{2t_s} + \frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s^2}t_s' \\ &= \left(\frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s^2} - \frac{n}{2t_s}\right)t_s' - \frac{\langle \mathbf{x}-\mathbf{x}_s, \mathbf{V}\rangle}{2t_s} + \frac{\langle \mathbf{x}-\mathbf{x}_s, \mathbf{x}_s'\rangle}{2t_s}. \end{aligned} \tag{2.3}$$

Thus we have

$$\begin{aligned} &\frac{\partial}{\partial s}\left\{(4\pi t_s)^{-\frac{n}{2}} e^{-\frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s}}\right\} \\ &= (4\pi t_s)^{-\frac{n}{2}} e^{-\frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s}} \left\{\left(\frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s^2} - \frac{n}{2t_s}\right)t_s' - \frac{\langle \mathbf{x}-\mathbf{x}_s, \mathbf{V}\rangle}{2t_s} + \frac{\langle \mathbf{x}-\mathbf{x}_s, \mathbf{x}_s'\rangle}{2t_s}\right\}. \end{aligned} \tag{2.4}$$

Therefore, we obtain **the first variation formula:**

$$F'(\Sigma)=\frac{\partial}{\partial s}|_{s=0}F_{x_s,t_s}(\Sigma_s)=(4\pi)^{-\frac{n}{2}}\int_\Sigma\left\{\left(\frac{|\mathbf{x}|^2}{4}-\frac{n}{2}\right)h-\left\langle\frac{\mathbf{x}}{2},\mathbf{V}\right\rangle+\left\langle\frac{\mathbf{x}}{2},\mathbf{y}\right\rangle-\left\langle\mathbf{V},\mathbf{H}\right\rangle\right\}e^{-\frac{|\mathbf{x}|^2}{4}}d\mu$$
$$=(4\pi)^{-\frac{n}{2}}\int_\Sigma\left\{-\left\langle\mathbf{V},\mathbf{H}+\frac{\mathbf{x}^\perp}{2}\right\rangle+\left(\frac{|\mathbf{x}|^2}{4}-\frac{n}{2}\right)h+\frac{1}{2}\left\langle\mathbf{x},\mathbf{y}\right\rangle\right\}e^{-\frac{|\mathbf{x}|^2}{4}}d\mu. \tag{2.5}$$

Σ_0 is said to be a **self-shrinker** if

$$\mathbf{H}+\frac{\mathbf{x}^\perp}{2}=0. \tag{2.6}$$

Next, we will prove that a critical point of the functional is indeed a self-shrinker. To prove this, we first prove some identities on the self-shrinker.

Following Colding–Minicozzi [8], we introduce the operator

$$\mathcal{L}v\equiv\Delta v-\frac{1}{2}\langle x,\nabla v\rangle=e^{\frac{|\mathbf{x}|^2}{4}}div\left(e^{-\frac{|\mathbf{x}|^2}{4}}\nabla v\right). \tag{2.7}$$

Now let us recall some properties of the operator $\mathcal{L}$ proved in [8]. It is easy to see that Colding–Minicozzi's proofs for these results can be generalized to high codimensional case easily. So we will omit most of the proofs here.

Lemma 2.1 (Lemma 3.8 of [8]) *If $\Sigma^n\subset\boldsymbol{R}^{n+k}$ is a submanifold of $\boldsymbol{R}^{n+k}$, u is a C^1 function with compact support, and v is a C^2 function, then*

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|\mathbf{x}|^2}{4}}=-\int_\Sigma\langle\nabla u,\nabla v\rangle e^{-\frac{|\mathbf{x}|^2}{4}}. \tag{2.8}$$

Corollary 2.2 (Corollary 3.10 of [8]) *Suppose $\Sigma^n\subset\boldsymbol{R}^{n+k}$ is a complete submanifold without boundary. If u and v are C^2 functions with*

$$\int_\Sigma\left(|u\nabla v|+|\nabla u||\nabla v|+|u\mathcal{L}v|\right)e^{-\frac{|\mathbf{x}|^2}{4}}<\infty, \tag{2.9}$$

then we get

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|\mathbf{x}|^2}{4}}=-\int_\Sigma\langle\nabla u,\nabla v\rangle e^{-\frac{|\mathbf{x}|^2}{4}}. \tag{2.10}$$

To keep short, we will say that a function u is "in the weighted $W^{2,2}$ space" if

$$\int_\Sigma\left(|u|^2+|\nabla u|^2+|\mathcal{L}u|^2\right)e^{-\frac{|\mathbf{x}|^2}{4}}<\infty. \tag{2.11}$$

By Corollary 2.2, if u and v are both in the weighted $W^{2,2}$ space, then we have

$$\int_\Sigma u(\mathcal{L}v)e^{-\frac{|\mathbf{x}|^2}{4}}=\int_\Sigma v(\mathcal{L}u)e^{-\frac{|\mathbf{x}|^2}{4}}. \tag{2.12}$$

Lemma 2.3 (Lemma 3.20 of [8]) *If* $\Sigma^n \subset \boldsymbol{R}^{n+k}$ *is a submanifold of* $\boldsymbol{R}^{n+k}$ *with* $\boldsymbol{H} + \frac{\boldsymbol{x}^\perp}{2} = 0$, *then*

$$\mathcal{L}x_i = -\frac{1}{2}x_i, \tag{2.13}$$

$$\mathcal{L}|\boldsymbol{x}|^2 = 2n - |\boldsymbol{x}|^2. \tag{2.14}$$

Here, x_i *is the* i*-th component of the position vector* $\boldsymbol{x}$, *i.e.,* $x_i = \langle \boldsymbol{x}, \partial_i \rangle$.

Lemma 2.4 (Lemma 3.25 of [8]) *If* $\Sigma^n \subset \boldsymbol{R}^{n+k}$ *is a complete submanifold of* $\boldsymbol{R}^{n+k}$ *without boundary, with polynomial volume growth, and* $\boldsymbol{H} + \frac{\boldsymbol{x}^\perp}{2} = 0$, *then*

$$\int_\Sigma (|\boldsymbol{x}|^2 - 2n)e^{-\frac{|\boldsymbol{x}|^2}{4}} = 0 \tag{2.15}$$

$$\int_\Sigma \boldsymbol{x}e^{-\frac{|\boldsymbol{x}|^2}{4}} = 0 = \int_\Sigma \boldsymbol{x}|\boldsymbol{x}|^2 e^{-\frac{|\boldsymbol{x}|^2}{4}} \tag{2.16}$$

$$\int_\Sigma (|\boldsymbol{x}|^4 - 2n(2n+4) - 16|\boldsymbol{H}|^2)e^{-\frac{|\boldsymbol{x}|^2}{4}} = 0 \tag{2.17}$$

Furthermore, if $\boldsymbol{w}$ *is a constant vector in* $\boldsymbol{R}^{n+k}$, *then*

$$\int_\Sigma \langle \boldsymbol{x}, \boldsymbol{w}\rangle^2 e^{-\frac{|\boldsymbol{x}|^2}{4}} = 2\int_\Sigma |\boldsymbol{w}^T|^2 e^{-\frac{|\boldsymbol{x}|^2}{4}}. \tag{2.18}$$

Corollary 2.5 (Corollary 3.34 of [8]) *If* Σ *is as in Lemma* 2.4, *then*

$$\int_\Sigma \left\{ \left(\frac{|\boldsymbol{x}|^2}{4} - \frac{n}{2} \right)^2 - \frac{n}{2} \right\} e^{-\frac{|\boldsymbol{x}|^2}{4}} = -\int_\Sigma |\boldsymbol{H}|^2 e^{-\frac{|\boldsymbol{x}|^2}{4}}. \tag{2.19}$$

Now we can conclude the main result in this section.

Proposition 2.6 Σ *is a critical point of* $F_{0,1}$ *if and only if* $\boldsymbol{H} + \frac{\boldsymbol{x}^\perp}{2} = 0$.

Proof By the first variation formula (2.5), we see that Σ is a critical point of the functional $F_{0,1}$ if and only if

$$\begin{cases} \mathbf{H} + \frac{\mathbf{x}^\perp}{2} = 0, \ \textit{on}\ \Sigma, \\ \int_\Sigma \left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2} \right) e^{-\frac{|\mathbf{x}|^2}{4}} d\mu = 0, \\ \int_\Sigma \mathbf{x} e^{-\frac{|\mathbf{x}|^2}{4}} d\mu = 0, \end{cases}$$

which is equivalent to $\mathbf{H} + \frac{\mathbf{x}^\perp}{2} = 0$ by (2.15) and (2.16). □

3 The second variation formula

In this section, we will compute the second variation of the functional $F_{0,1}$. Following Colding–Minicozzi [8], we will use square brackets $[\cdot]$ to denote weighted integrals

$$[f] = \int_\Sigma f e^{-\frac{|\mathbf{x}|^2}{4}}.$$

We suppose

$$\partial_s|_{s=0}\Sigma_s = \mathbf{V}, \quad \partial_s|_{s=0}\mathbf{x}_s = \mathbf{y}, \quad \partial_s|_{s=0}t_s = h,$$
$$\partial_{ss}|_{s=0}\Sigma_s = \mathbf{V}', \quad \partial_{ss}|_{s=0}\mathbf{x}_s = \mathbf{y}', \quad \partial_{ss}|_{s=0}t_s = h'.$$

Then by (2.4) and (2.5), we have

$$\begin{aligned} F'' = & \left[\left(-\left\langle \mathbf{V}, \mathbf{H} + \frac{\mathbf{x}^\perp}{2}\right\rangle + \left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right) h + \frac{1}{2}\langle \mathbf{x}, \mathbf{y}\rangle\right)^2\right] \\ & + \left[-\left\langle \mathbf{V}', \mathbf{H} + \frac{\mathbf{x}^\perp}{2}\right\rangle + \left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right) h' + \frac{1}{2}\langle \mathbf{x}, \mathbf{y}'\rangle\right] \\ & + \left[-\left\langle \mathbf{V}, \left(\mathbf{H} + \frac{(\mathbf{x}-\mathbf{x}_s)^\perp}{2t_s}\right)'\right\rangle + \left(\frac{|\mathbf{x}-\mathbf{x}_s|^2}{4t_s^2} - \frac{n}{2t_s}\right)' h + \left\langle \left(\frac{\mathbf{x}-\mathbf{x}_s}{2t_s}\right)', \mathbf{y}\right\rangle\right]. \end{aligned} \tag{3.1}$$

We compute the second variation at a critical point of $F_{0,1}$, i.e., $\mathbf{H} + \frac{\mathbf{x}^\perp}{2} = 0$. Then using Lemma 2.4, we have

$$\begin{aligned} F'' = & \left[\left(\left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right) h + \frac{1}{2}\langle \mathbf{x}, \mathbf{y}\rangle\right)^2\right] - [\langle \mathbf{V}, \mathbf{H}'\rangle] - \left[\left\langle \mathbf{V}, \frac{((\mathbf{x}-\mathbf{x}_s)^\perp)'}{2t_s} - \frac{(\mathbf{x}-\mathbf{x}_s)^\perp}{2t_s^2} t_s'\right\rangle\right] \\ & + \left[\left(\frac{\langle \mathbf{x}-\mathbf{x}_s, \mathbf{x}'-\mathbf{x}_s'\rangle}{2t_s^2} - \frac{|\mathbf{x}-\mathbf{x}_s|^2}{2t_s^3} t_s' + \frac{n}{2t_s^2} t_s'\right) h\right] + \left[\left\langle \frac{\mathbf{x}'-\mathbf{x}_s'}{2t_s} - \frac{\mathbf{x}-\mathbf{x}_s}{2t_s^2} t_s', \mathbf{y}\right\rangle\right] \\ = & \left[\left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right)^2 h^2 + h\left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right)\langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{4}\langle \mathbf{x}, \mathbf{y}\rangle^2\right] \\ & - [\langle \mathbf{V}, \mathbf{H}'\rangle] - \frac{1}{2}\left[\langle \mathbf{V}, ((\mathbf{x}-\mathbf{x}_s)^\perp)'\rangle\right] + \frac{1}{2}\left[\langle \mathbf{V}, \mathbf{x}^\perp\rangle h\right] \\ & + \left[\left(\frac{1}{2}\langle \mathbf{x}, \mathbf{V}\rangle - \frac{1}{2}\langle \mathbf{x}, \mathbf{y}\rangle - \frac{|\mathbf{x}|^2}{2} h + \frac{n}{2} h\right) h\right] + \left[\frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle - \frac{1}{2}\langle \mathbf{y}, \mathbf{y}\rangle - \frac{1}{2}\langle \mathbf{x}, \mathbf{y}\rangle h\right]. \end{aligned} \tag{3.2}$$

Note that by Lemma 2.4,

$$\left[h\left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right)\langle \mathbf{x}, \mathbf{y}\rangle\right] = h\left\langle \left[\frac{|\mathbf{x}|^2}{4}\mathbf{x}\right], \mathbf{y}\right\rangle - h\frac{n}{2}\langle [\mathbf{x}], \mathbf{y}\rangle = 0.$$

Thus we have

$$\begin{aligned} F'' = & \left[\left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right)^2 h^2 + \frac{1}{4}\langle \mathbf{x}, \mathbf{y}\rangle^2 - \langle \mathbf{V}, \mathbf{H}'\rangle - \frac{1}{2}\langle \mathbf{V}, ((\mathbf{x}-\mathbf{x}_s)^\perp)'\rangle \right. \\ & \left. + h\langle \mathbf{x}, \mathbf{V}\rangle - h\langle \mathbf{x}, \mathbf{y}\rangle + h^2\left(\frac{n}{2} - \frac{|\mathbf{x}|^2}{2}\right) + \frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle - \frac{1}{2}|\mathbf{y}|^2\right] \\ = & \left[\left(\left(\frac{|\mathbf{x}|^2}{4} - \frac{n}{2}\right)^2 - \frac{n}{2}\right) h^2 + \frac{1}{4}\langle \mathbf{x}, \mathbf{y}\rangle^2 - \langle \mathbf{V}, \mathbf{H}'\rangle - \frac{1}{2}\langle \mathbf{V}, ((\mathbf{x}-\mathbf{x}_s)^\perp)'\rangle \right. \\ & \left. + h\langle \mathbf{x}, \mathbf{V}\rangle - h\langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle - \frac{1}{2}|\mathbf{y}|^2\right] \end{aligned}$$

$$= \Bigg[-h^2|\mathbf{H}|^2 + \frac{1}{2}|\mathbf{y}^T|^2 - \langle \mathbf{V}, \mathbf{H}' \rangle - \frac{1}{2}\langle \mathbf{V}, ((\mathbf{x}-\mathbf{x}_s)^{\perp})' \rangle + h\langle \mathbf{x}, \mathbf{V}\rangle$$
$$- h\langle \mathbf{x}, \mathbf{y}\rangle + \frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle - \frac{1}{2}|\mathbf{y}|^2\Bigg]$$
$$= \Bigg[-\langle \mathbf{V}, \mathbf{H}' \rangle - \frac{1}{2}\langle \mathbf{V}, ((\mathbf{x}-\mathbf{x}_s)^{\perp})' \rangle + h\langle \mathbf{x}, \mathbf{V}\rangle - h^2|\mathbf{H}|^2 + \frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle - \frac{1}{2}|\mathbf{y}^{\perp}|^2\Bigg]. \quad (3.3)$$

Here, the second equality used (2.15), the third equality used (2.18) and Corollary 2.5, and the fourth equality used (2.16).

From (3.3), we see that in order to compute the second variation of $F_{0,1}$, it suffices to compute $\mathbf{H}'$ and $((\mathbf{x}-\mathbf{x}_s)^{\perp})'$. The computation will use the variations of normal vector fields and mean curvature vectors. The proofs are standard. For the purpose of completeness, we give the proofs of them in the Appendix A. Assuming them, we will continue our computation.

By Lemma 8.4, we have

$$\begin{aligned} -\langle \mathbf{V}, \mathbf{H}' \rangle &= -\langle V^{\beta} e_{\beta}, (\Delta V^{\alpha} + V^{\beta} h^{\beta}_{ij} h^{\alpha}_{ij}) e_{\alpha} \rangle \\ &= -V^{\alpha}(\Delta V^{\alpha} + V^{\beta} h^{\beta}_{ij} h^{\alpha}_{ij}). \end{aligned} \quad (3.4)$$

By Lemma 8.2 and the fact that $x_0 = 0$, we compute

$$\begin{aligned} -\frac{1}{2}\langle \mathbf{V}, ((\mathbf{x}-\mathbf{x}_s)^{\perp})' \rangle &= -\frac{1}{2}\langle \mathbf{V}, (\langle \mathbf{x}-\mathbf{x}_s, e_{\alpha}\rangle e_{\alpha})' \rangle \\ &= -\frac{1}{2}\langle \mathbf{V}, \langle \mathbf{x}'-\mathbf{x}_s', e_{\alpha}\rangle e_{\alpha} \rangle - \frac{1}{2}\langle \mathbf{V}, \langle \mathbf{x}-\mathbf{x}_s, e_{\alpha}'\rangle e_{\alpha} \rangle \\ &\quad -\frac{1}{2}\langle \mathbf{V}, \langle \mathbf{x}-\mathbf{x}_s, e_{\alpha}\rangle e_{\alpha}' \rangle \\ &= -\frac{1}{2}\langle \mathbf{V}, \langle \mathbf{V}-\mathbf{y}, e_{\alpha}\rangle e_{\alpha} \rangle \\ &\quad -\frac{1}{2}\langle \mathbf{V}, e_{\alpha}\rangle \langle \mathbf{x}, -\nabla V^{\alpha} - V^{\beta}\langle \bar{\nabla}_{e_i} e_{\beta}, e_{\alpha}\rangle e_i + b^{\beta}_{\alpha} e_{\beta} \rangle \\ &\quad -\frac{1}{2}\langle \mathbf{x}, e_{\alpha}\rangle \langle \mathbf{V}, b^{\beta}_{\alpha} e_{\beta} \rangle \\ &= -\frac{1}{2}\langle \mathbf{V}, \mathbf{V}-\mathbf{y}^{\perp} \rangle + \frac{1}{2}V^{\alpha}\langle \mathbf{x}, \nabla V^{\alpha} \rangle + \frac{1}{2}V^{\alpha}V^{\beta}\langle \bar{\nabla}_{e_i} e_{\beta}, e_{\alpha}\rangle \langle \mathbf{x}, e_i \rangle \\ &\quad -\frac{1}{2}V^{\alpha} b^{\beta}_{\alpha}\langle \mathbf{x}, e_{\beta}\rangle - \frac{1}{2}V^{\beta} b^{\beta}_{\alpha}\langle \mathbf{x}, e_{\alpha}\rangle. \end{aligned} \quad (3.5)$$

Note that

$$V^{\alpha}V^{\beta}\langle \bar{\nabla}_{e_i} e_{\beta}, e_{\alpha}\rangle = V^{\beta}V^{\alpha}\langle \bar{\nabla}_{e_i} e_{\alpha}, e_{\beta}\rangle = -V^{\alpha}V^{\beta}\langle \bar{\nabla}_{e_i} e_{\beta}, e_{\alpha}\rangle,$$

thus

$$V^{\alpha}V^{\beta}\langle \bar{\nabla}_{e_i} e_{\beta}, e_{\alpha}\rangle = 0.$$

On the other hand,

$$-\frac{1}{2}V^{\alpha} b^{\beta}_{\alpha}\langle \mathbf{x}, e_{\beta}\rangle = -\frac{1}{2}V^{\beta} b^{\alpha}_{\beta}\langle \mathbf{x}, e_{\alpha}\rangle = \frac{1}{2}V^{\beta} b^{\beta}_{\alpha}\langle \mathbf{x}, e_{\alpha}\rangle.$$

Hence, we obtain that

$$\begin{aligned}-\frac{1}{2}\langle \mathbf{V}, ((\mathbf{x}-\mathbf{x}_s)^{\perp})'\rangle &= -\frac{1}{2}\langle \mathbf{V}, \mathbf{V}-\mathbf{y}^{\perp}\rangle + \frac{1}{2}V^{\alpha}\langle \mathbf{x}, \nabla V^{\alpha}\rangle \\ &= -\frac{1}{2}|\mathbf{V}|^2 + \frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle + \frac{1}{2}V^{\alpha}\langle \mathbf{x}, \nabla V^{\alpha}\rangle. \end{aligned} \tag{3.6}$$

Putting (3.4) and (3.6) into (3.3), we obtain

$$\begin{aligned} F'' &= \Big[-V^{\alpha}(\Delta V^{\alpha} + V^{\beta}h^{\beta}_{ij}h^{\alpha}_{ij}) - \frac{1}{2}|\mathbf{V}|^2 + \frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle + \frac{1}{2}V^{\alpha}\langle \mathbf{x}, \nabla V^{\alpha}\rangle \\ &\quad + h\langle \mathbf{x}, \mathbf{V}\rangle - h^2|\mathbf{H}|^2 + \frac{1}{2}\langle \mathbf{V}, \mathbf{y}\rangle - \frac{1}{2}|\mathbf{y}^{\perp}|^2\Big] \\ &= \Big[-V^{\alpha}\Big(\Delta V^{\alpha} + V^{\beta}h^{\beta}_{ij}h^{\alpha}_{ij} - \frac{1}{2}\langle \mathbf{x}, \nabla V^{\alpha}\rangle + \frac{1}{2}V^{\alpha}\Big) + \langle \mathbf{V}, \mathbf{y}\rangle - h^2|\mathbf{H}|^2 + h\langle \mathbf{x}, \mathbf{V}\rangle - \frac{1}{2}|\mathbf{y}^{\perp}|^2\Big]. \end{aligned} \tag{3.7}$$

Recall that the self-shrinker equation is given by

$$\mathbf{H} = -\frac{1}{2}\mathbf{x}^{\perp},$$

thus we have **the second variation formula:**

$$F'' = \Big[-\langle \mathbf{V}, L\mathbf{V}\rangle + \langle \mathbf{V}, \mathbf{y}\rangle - h^2|\mathbf{H}|^2 - 2h\langle \mathbf{H}, \mathbf{V}\rangle - \frac{1}{2}|\mathbf{y}^{\perp}|^2\Big]. \tag{3.8}$$

Here L is an operator from $N\Sigma$ to $N\Sigma$ defined by

$$L(V^{\alpha}e_{\alpha}) \equiv \Big(\Delta V^{\alpha} + V^{\beta}h^{\beta}_{ij}h^{\alpha}_{ij} - \frac{1}{2}\langle \mathbf{x}, \nabla V^{\alpha}\rangle + \frac{1}{2}V^{\alpha}\Big)e_{\alpha}. \tag{3.9}$$

When $k = 1$, i.e., Σ is a hypersurface in $\mathbf{R}^{n+1}$, the above operator reduces to be

$$L(v) \equiv \Delta v + |\mathbf{A}|^2 v - \frac{1}{2}\langle \mathbf{x}, \nabla v\rangle + \frac{1}{2}v. \tag{3.10}$$

This is just the operator defined by Colding–Minicozz [8].

Definition 3.1 We say that a self-shrinker Σ is **F-s***table* if for every normal variation $\boldsymbol{V}$, there exist variations of x_0 and t_0 that make $F'' \geq 0$.

4 Minimal submanifolds in spheres

In this section, we will show that any minimal submanifold of the sphere is a self-shrinker for the mean curvature flow.

In the following, we will first give a generalized definition of self-shrinker.

Definition 4.1 We say a manifold $\Sigma^n \subset \boldsymbol{R}^{n+k}$ is a **self-shrinker** if it is a time $t = -\frac{1}{2\lambda}$ slice of a self-shrinking mean curvature flow that disappears at (0,0), i.e., of a mean curvature flow satisfying

$$\Sigma_t = \sqrt{-2\lambda t}\,\Sigma_{-\frac{1}{2\lambda}}. \tag{4.1}$$

Here, λ is a positive constant.

It is easy to show that a self-shrinker defined in Definition 4.1 is equivalent to

$$\mathbf{H} = -\lambda \mathbf{x}^{\perp}, \tag{4.2}$$

where $\mathbf{x}$ is the position vector of Σ in $\mathbf{R}^{n+k}$.

Theorem 4.1 *A complete submanifold $\Sigma^n \subset \boldsymbol{S}^{n+k-1}(r) \subset \boldsymbol{R}^{n+k}$ is a minimal submanifold of $\boldsymbol{S}^{n+k-1}(r)$ if and only if it is a self-shrinker in $\boldsymbol{R}^{n+k}$ for $\lambda = \frac{n}{r^2}$, where λ is the constant in Definition 4.1.*

Proof We choose a local orthonormal frame $\{e_A\}_{A=1}^{n+k}$ in $\mathbf{R}^{n+k}$ such that $\{e_i\}_{i=1}^{n}$ are tangent to Σ, $\{e_\alpha\}_{\alpha=n+1}^{n+k-1}$ are in the normal bundle of Σ in $\mathbf{S}^{n+k-1}(r)$, and e_{n+k} is normal to $\mathbf{S}^{n+k-1}(r)$. We denote by $\tilde{\nabla}$ and $\bar{\nabla}$ the Levi-Civita connections on $\mathbf{S}^{n+k-1}(r)$ and $\mathbf{R}^{n+k}$, respectively, and $\tilde{\mathbf{H}}$ and $\bar{\mathbf{H}}$ the mean curvature vector of Σ in $\mathbf{S}^{n+k-1}(r)$ and $\mathbf{R}^{n+k}$, respectively. Then we have

$$\begin{aligned} \bar{\mathbf{H}} &= \sum_{i=1}^{n} \sum_{\alpha=n+1}^{n+k-1} \langle \bar{\nabla}_{e_i} e_i, e_\alpha \rangle e_\alpha + \sum_{i=1}^{n} \langle \bar{\nabla}_{e_i} e_i, e_{n+k} \rangle e_{n+k} \\ &= \sum_{i=1}^{n} \sum_{\alpha=n+1}^{n+k-1} \langle \tilde{\nabla}_{e_i} e_i, e_\alpha \rangle e_\alpha + \sum_{i=1}^{n} \langle \bar{\nabla}_{e_i} e_i, e_{n+k} \rangle e_{n+k} \\ &= \tilde{\mathbf{H}} + \left\langle \sum_{i=1}^{n} \bar{\nabla}_{e_i} e_i, e_{n+k} \right\rangle e_{n+k}. \end{aligned} \tag{4.3}$$

Thus, Σ is minimal in $\mathbf{S}^{n+k-1}$, i.e., $\tilde{\mathbf{H}} = 0$, if and only if

$$\bar{\mathbf{H}} = \left\langle \sum_{i=1}^{n} \bar{\nabla}_{e_i} e_i, e_{n+k} \right\rangle e_{n+k}. \tag{4.4}$$

As $\{e_1, \ldots, e_{n+k-1}\}$ is an orthonormal frame of $\mathbf{S}^{n+k-1}(r)$, and that e_{n+k} is normal to $\mathbf{S}^{n+k-1}(r)$, we know that for each i

$$-\langle \bar{\nabla}_{e_i} e_i, e_{n+k} \rangle = \left\langle e_i, \bar{\nabla}_{e_i} \left(\frac{1}{r} \mathbf{x} \right) \right\rangle = \left\langle e_i, \frac{1}{r} e_i \right\rangle = \frac{1}{r}. \tag{4.5}$$

Here, we have used the fact that $\bar{\nabla}_{\mathbf{V}} \mathbf{x} = \mathbf{V}$ for each vector $\mathbf{V}$. Putting (4.5) into (4.4) and using the fact that $e_{n+k} = \frac{\mathbf{x}}{r}$, we know that (4.4) is equivalent to

$$\bar{\mathbf{H}} = -\frac{n}{r} e_{n+k} = -\frac{n}{r^2} \mathbf{x} = -\frac{n}{r^2} \mathbf{x}^{\perp}. \tag{4.6}$$

Comparing (4.2) and (4.6) yields the conclusion with $\lambda = \frac{n}{r^2}$. □

5 Classification results

In this section, we will prove the classification results.

Proof of Main Theorem 1 First note that, by Colding–Minicozzi's result (Theorem 4.23 and (11.10) of [8]), we know that the sphere is F-stable.

It is known that (for example, Remark 1.2 of [23]) normal flat implies parallel principle normal. By our assumptions and Theorem 1.5, we know that Σ^n is a minimal submanifold of

$\mathbf{S}^{n+k-1} \subset \mathbf{R}^{n+k}$. By Theorem 4.1 and the fact that $\lambda = \frac{1}{2}$ in our case, we know that $\mathbf{S}^{n+k-1}$ is a sphere of radius $\sqrt{2n}$.

We choose a local orthonormal frame $\{e_A\}_{A=1}^{n+k}$ such that $e_1, \ldots, e_n$ are tangential to Σ, $e_{n+1}, \ldots, e_{n+k}$ are in the normal bundle, and e_{n+k} is normal to the sphere. We denote $\tilde{\mathbf{A}}$ and $\mathbf{A}$ the second fundamental form of Σ in $\mathbf{S}^{n+k-1}$ and $\mathbf{R}^{n+k}$, respectively. Similarly, we denote $\tilde{\nabla}$ and $\bar{\nabla}$ the Levi-Civita connection of $\mathbf{S}^{n+k-1}$ and $\mathbf{R}^{n+k}$, respectively. Then we have for each $n+1 \leq \alpha \leq n+k-1$

$$h_{ij}^{\alpha} = -\langle \bar{\nabla}_{e_i} e_j, e_\alpha \rangle = -\langle \tilde{\nabla}_{e_i} e_j, e_\alpha \rangle = \tilde{h}_{ij}^{\alpha},$$

and

$$h_{ij}^{n+k} = -\langle \bar{\nabla}_{e_i} e_j, e_{n+k} \rangle = \langle e_j, \bar{\nabla}_{e_i} e_{n+k} \rangle = \left\langle e_j, \bar{\nabla}_{e_i} \left(\frac{\mathbf{x}}{\sqrt{2n}} \right) \right\rangle = \frac{1}{\sqrt{2n}} \langle e_i, e_j \rangle = \frac{1}{\sqrt{2n}} \delta_{ij}.$$

Thus we know that

$$H^{\alpha} = \sum_{i=1}^{n} h_{ii}^{\alpha} = \sum_{i=1}^{n} \tilde{h}_{ii}^{\alpha} = \tilde{H}^{\alpha} = 0,$$

and

$$H^{n+k} = \sum_{i=1}^{n} h_{ii}^{n+k} = \sum_{i=1}^{n} \frac{1}{\sqrt{2n}} = \sqrt{\frac{n}{2}}.$$

Therefore, the mean curvature vector is given by

$$\mathbf{H} = -\sum_{\alpha=n+1}^{n+k-1} H^{\alpha} e_{n+1} - H^{n+k} e_{n+k} = -\sqrt{\frac{n}{2}} e_{n+k} = -\frac{1}{2}\mathbf{x}. \tag{5.1}$$

This is just the equation for self-shrinker.

Fix $n+1 \leq \alpha \leq n+k-1$. We choose a normal vector field $\mathbf{V}$ such that locally it is given by e_α, i.e., $V^{\beta} = \delta_{\alpha\beta}$ for $n+1 \leq \beta \leq n+k-1$ and $V^{n+k} \equiv 0$. Then by the definition of the stability operator (3.9), we have

$$\begin{aligned}(L\mathbf{V})^{\alpha} &= \Delta V^{\alpha} + V^{\beta} h_{ij}^{\beta} h_{ij}^{\alpha} - \frac{1}{2}\langle \mathbf{x}, \nabla V^{\alpha} \rangle + \frac{1}{2} V^{\alpha} \\ &= \sum_{i,j} (h_{ij}^{\alpha})^2 + \frac{1}{2} = \sum_{i,j} (\tilde{h}_{ij}^{\alpha})^2 + \frac{1}{2} = |\tilde{\mathbf{A}}^{\alpha}|^2 + \frac{1}{2}.\end{aligned}$$

Here, $\tilde{\mathbf{A}}^{\alpha} = h_{ij}^{\alpha} \omega^i \otimes \omega^j$ is the component of the second fundamental form $\tilde{\mathbf{A}}$. For $n+1 \leq \beta \leq n+k-1$ and $\beta \neq \alpha$,

$$\begin{aligned}(L\mathbf{V})^{\beta} &= \Delta V^{\beta} + V^{\gamma} h_{ij}^{\gamma} h_{ij}^{\beta} - \frac{1}{2}\langle \mathbf{x}, \nabla V^{\beta} \rangle + \frac{1}{2} V^{\beta} \\ &= \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{\beta},\end{aligned}$$

and

$$\begin{aligned}(L\mathbf{V})^{n+k} &= \Delta V^{n+k} + V^{\gamma} h_{ij}^{\gamma} h_{ij}^{n+k} - \frac{1}{2}\langle \mathbf{x}, \nabla V^{n+k} \rangle + \frac{1}{2} V^{n+k} \\ &= \sum_{i,j} h_{ij}^{\alpha} h_{ij}^{n+k} = \frac{1}{\sqrt{2n}} \sum_{i} h_{ii}^{\alpha} = \frac{1}{\sqrt{2n}} H^{\alpha} = 0.\end{aligned}$$

From this, we see that $L\mathbf{V} = (|\tilde{\mathbf{A}}^{\alpha}|^2 + \frac{1}{2})\mathbf{V} + \sum_{n+1\leq\beta\leq n+k-1,\beta\neq\alpha}(\sum_{i,j} h_{ij}^{\alpha}h_{ij}^{\beta})e_{\beta}$. Putting this into the second variation formula (3.8) and noting that $\langle \mathbf{V}, e_{\beta}\rangle = 0$ for $n+1 \leq \beta \leq n+k-1$, $\beta \neq \alpha$ and $\langle \mathbf{V}, \mathbf{H}\rangle = 0$, we obtain

$$\begin{aligned} F''(\Sigma) &= \left[-\langle \mathbf{V}, L\mathbf{V}\rangle + \langle \mathbf{V}, \mathbf{y}\rangle - h^2|\mathbf{H}|^2 - 2h\langle \mathbf{H}, \mathbf{V}\rangle - \frac{1}{2}|\mathbf{y}^{\perp}|^2 \right] \\ &= \left[-\langle \mathbf{V}, (|\tilde{\mathbf{A}}^{\alpha}|^2 + \frac{1}{2})\mathbf{V}\rangle + \langle \mathbf{V}, \mathbf{y}\rangle - h^2|\mathbf{H}|^2 - \frac{1}{2}|\mathbf{y}^{\perp}|^2 \right] \\ &= \left[-|\tilde{\mathbf{A}}^{\alpha}|^2|\mathbf{V}|^2 - \frac{1}{2}|\mathbf{V} - \mathbf{y}^{\perp}|^2 - h^2|\mathbf{H}|^2 \right]. \end{aligned}$$

We claim that $\tilde{\mathbf{A}}^{\alpha} \equiv 0$. Indeed, suppose it does not hold, then for this $\mathbf{V}$ and any $h \in \mathbf{R}$, $\mathbf{y} \in \mathbf{R}^{n+2}$

$$F''(\Sigma) \leq \left[-|\tilde{\mathbf{A}}^{\alpha}|^2|\mathbf{V}|^2 \right] < 0.$$

This contradicts the fact that Σ is F-stable. Thus the claim holds.

As the above argument holds for any $n+1 \leq \alpha \leq n+k-1$, we know that Σ^n must be a totaly geodesic submanifold of $\mathbf{S}^{n+k-1}$, thus it must be $\mathbf{S}^n$. By the self-shrinker equation, we know that it must be of radius $\sqrt{2n}$. This finishes the proof of the theorem. □

Proof of Main Theorem 2 By Theorem 1.3 of [23], Σ must belong to one of the following classes:

$$\Sigma^n = \Gamma \times \mathbf{R}^{n-1}, \quad \Sigma^n = \tilde{\Sigma}^r \times \mathbf{R}^{n-r}.$$

Here, Γ is one of the homothetically shrinking curves in $\mathbf{R}^2$ found by Abresh and Langer and $\tilde{\Sigma}^r$ is a complete minimal submanifold of the sphere $\mathbf{S}^{k+r-1}(\sqrt{2r}) \subset \mathbf{R}^{k+r}$, where $0 < r = rank(\mathbf{A}^{\nu}) \leq n$ denotes the rank of the principle second fundamental form $\mathbf{A}^{\nu} = \langle \nu, \mathbf{A}\rangle$.

In the first case, we know that the only embedded one found by Abresh-Langer is the circle. Thus in this case Σ is $\mathbf{S}^1 \times \mathbf{R}^{n-1}$ in $\mathbf{R}^{n+1}$. But Colding–Minicozzi has shown in Section 11 of [8] that the cylinder is not F-stable. Thus this case is impossible in our case.

In the second case, for each $0 < r \leq n$, $\tilde{\Sigma}$ is a minimal submanifold of $S^{r+k-1}(\sqrt{2r})$. It is easy to see that (for example, the proof of Theorem 0.12 of [8]), $\tilde{\Sigma}$ is F-stable if Σ is so. But in the proof of Main Theorem 1, we have proved that the only F-stable one is just $\mathbf{S}^r(\sqrt{2r})$. Thus Σ must be $\mathbf{S}^r(\sqrt{2r}) \times \mathbf{R}^{n-r}$. As above, Colding–Minicozzi has shown in Section 11 of [8] that $\mathbf{S}^r(\sqrt{2r}) \times \mathbf{R}^{n-r}$ is not F-stable for all $1 < r < n$.

Combining the above two cases, we see that Σ must be $\mathbf{S}^n(\sqrt{2n})$. This proves the theorem. □

6 Symplectic self-shrinkers

In this section, we prove the properties of symplectic self-shrinkers and apply them to show that there is no finite time Type I singularity for the symplectic mean curvature flow. Before that, we first fix some notations and recall some basic facts on symplectic mean curvature flow.

Suppose M is a Kähler-Einstein surface. Let Σ be a smooth surface in M, and ω, $\langle\cdot,\cdot\rangle$ be the Kähler form and the Kähler metric on M respectively. The Kähler angle α of Σ in M is defined by

$$\omega|_{\Sigma} = \cos\alpha d\mu_{\Sigma}$$

where $d\mu_\Sigma$ is the area element of the induced metric from $\langle\cdot,\cdot\rangle$. We call Σ a *symplectic* surface if $\cos\alpha > 0$, a *Lagrangian* surface if $\cos\alpha = 0$, a *holomorphic* curve if $\cos\alpha = 1$.

Assume that Σ is a symplectic surface and we consider the immersions

$$F_0 : \Sigma \longrightarrow M$$

of smooth surface Σ in M. Suppose that Σ evolves along the mean curvature in M, then there is a one-parameter family $F_t = F(\cdot, t)$ of immersions which satisfy the mean curvature flow equation:

$$\begin{cases} \frac{d}{dt}F(x,t) = \mathbf{H}(x,t) \\ F(x,0) = F_0(x). \end{cases}$$

Here $\mathbf{H}(x,t)$ is the mean curvature vector of $\Sigma_t = F_t(\Sigma)$ at $F(x,t)$ in M.

Recall that [5] the Kähler angle α of Σ in M satisfies the parabolic equation:

$$\left(\frac{\partial}{\partial t} - \Delta\right)\cos\alpha = |\overline{\nabla} J|^2\cos\alpha + R\sin^2\alpha\cos\alpha,$$

where J is the complex structure of $\mathbf{R}^4$ and in local orthonormal frame $|\overline{\nabla} J|^2 = |h^2_{1i} + h^1_{2i}|^2 + |h^2_{2i} - h^1_{1i}|^2$ which depends only on the orientation of Σ and does not depend on the choice of the frame. If the initial surface is symplectic, i.e. $\cos\alpha(\cdot, 0)$ has a positive lower bound, then by applying the parabolic maximum principle to this evolution equation, one concludes that $\cos\alpha$ remains positive as long as the mean curvature flow has a smooth solution. In this case, the mean curvature flow is called *symplectic mean curvature flow*.

A two dimensional self-shrinker in $\mathbf{R}^4$ is called *symplectic self-shrinker* if it is a symplectic surface in $\mathbf{R}^4$.

Next we derive the elliptic equation satisfied by the Kähler angle on a symplectic self-shrinker.

Lemma 6.1 *On a symplectic self-shrinker Σ^2 in $\boldsymbol{R}^4$, we have*

$$\Delta\cos\alpha - \frac{1}{2}\langle \boldsymbol{x}, \nabla\cos\alpha\rangle = -|\overline{\nabla} J|^2\cos\alpha. \tag{6.1}$$

Proof We may choose a local orthonormal frame $\{e_1, e_2, e_3, e_4\}$ on $\mathbf{R}^4$ along Σ such that e_1, e_2 are tangent to Σ, e_3, e_4 are in the normal bundle of Σ., and the Kähler form ω takes the form

$$\omega = \cos\alpha u_1 \wedge u_2 + \cos\alpha u_3 \wedge u_4 + \sin\alpha u_1 \wedge u_3 - \sin\alpha u_2 \wedge u_4, \tag{6.2}$$

where $\{u_1, u_2, u_3, u_4\}$ is the dual of $\{e_1, e_2, e_3, e_4\}$. Then along Σ, the complex structure on $\mathbf{R}^4$ takes the form

$$J = \begin{pmatrix} 0 & \cos\alpha & \sin\alpha & 0 \\ -\cos\alpha & 0 & 0 & -\sin\alpha \\ -\sin\alpha & 0 & 0 & \cos\alpha \\ 0 & \sin\alpha & -\cos\alpha & 0 \end{pmatrix}. \tag{6.3}$$

The Kähler angle is given by

$$\cos\alpha = \omega(e_1, e_2) = \langle e_1 \wedge e_2, \omega\rangle. \tag{6.4}$$

For the sake of simplicity, we can assume that the covariant derivative of the orthonormal frame satisfy (at a fixed point $p \in \Sigma$)

$$\nabla_{e_i} e_j = 0. \tag{6.5}$$

In the following we will compute at the point p. Recall that on a symplectic surface Σ in $\mathbf{R}^4$, the Kähler angle satisfies (Proposition 3.3 of [12])

$$\Delta\cos\alpha = -|\overline{\nabla}J|^2\cos\alpha - \sin\alpha(H^4_{,1} + H^3_{,2}), \tag{6.6}$$

where

$$H^\beta_{,i} = -\langle\bar{\nabla}^N_{e_i}\mathbf{H}, e_\beta\rangle = \langle\bar{\nabla}^N_{e_i}(H^\gamma e_\gamma), e_\beta\rangle. \tag{6.7}$$

Note that there is a different sign in the above formula from [12] because in our notation $\mathbf{H} = -H^\alpha e_\alpha$. Now, we suppose further that Σ is a self-shrinker, i.e.,

$$\mathbf{H} = -\frac{1}{2}\mathbf{x}^\perp, \tag{6.8}$$

which is equivalent to

$$H^\beta = \frac{1}{2}\langle\mathbf{x}, e_\beta\rangle, \quad \beta = 3, 4. \tag{6.9}$$

Then by (6.7), we have

$$\begin{aligned}
H^\beta_{,i} &= \langle\bar{\nabla}^N_{e_i}(H^\gamma e_\gamma), e_\beta\rangle = \langle(\bar{\nabla}_{e_i}H^\gamma)e_\gamma + H^\gamma\bar{\nabla}_{e_i}e_\gamma, e_\beta\rangle\\
&= \frac{1}{2}\bar{\nabla}_{e_i}\langle\mathbf{x}, e_\beta\rangle + H^\gamma\langle\bar{\nabla}_{e_i}e_\gamma, e_\beta\rangle\\
&= \frac{1}{2}\langle\bar{\nabla}_{e_i}\mathbf{x}, e_\beta\rangle + \frac{1}{2}\langle\mathbf{x}, \bar{\nabla}_{e_i}e_\beta\rangle + H^\gamma\langle\bar{\nabla}_{e_i}e_\gamma, e_\beta\rangle\\
&= \frac{1}{2}\langle e_i, e_\beta\rangle + \frac{1}{2}\langle\mathbf{x}, \langle\bar{\nabla}_{e_i}e_\beta, e_j\rangle e_j + \langle\bar{\nabla}_{e_i}e_\beta, e_\gamma\rangle e_\gamma\rangle + H^\gamma\langle\bar{\nabla}_{e_i}e_\gamma, e_\beta\rangle\\
&= \frac{1}{2}\langle\mathbf{x}, e_j\rangle\langle\bar{\nabla}_{e_i}e_\beta, e_j\rangle + \frac{1}{2}\langle\mathbf{x}, e_\gamma\rangle\langle\bar{\nabla}_{e_i}e_\beta, e_\gamma\rangle + H^\gamma\langle\bar{\nabla}_{e_i}e_\gamma, v_\beta\rangle\\
&= \frac{1}{2}h^\beta_{ij}\langle\mathbf{x}, e_j\rangle + H^\gamma\langle\bar{\nabla}_{e_i}e_\beta, e_\gamma\rangle + H^\gamma\langle\bar{\nabla}_{e_i}e_\gamma, e_\beta\rangle\\
&= \frac{1}{2}h^\beta_{ij}\langle\mathbf{x}, e_j\rangle + H^\gamma\bar{\nabla}_{e_i}\langle e_\beta, e_\gamma\rangle\\
&= \frac{1}{2}h^\beta_{ij}\langle\mathbf{x}, e_j\rangle.
\end{aligned} \tag{6.10}$$

Putting (6.10) into (6.6), we obtain

$$\begin{aligned}
\Delta\cos\alpha &= -|\overline{\nabla}J|^2\cos\alpha - \frac{1}{2}\sin\alpha\left\{h^4_{1j}\langle\mathbf{x}, e_j\rangle + h^3_{2j}\langle\mathbf{x}, e_j\rangle\right\}\\
&= -|\overline{\nabla}J|^2\cos\alpha - \frac{1}{2}\sin\alpha\left\{(h^4_{11} + h^3_{21})\langle\mathbf{x}, e_1\rangle + (h^4_{12} + h^3_{22})\langle\mathbf{x}, e_2\rangle\right\}.
\end{aligned} \tag{6.11}$$

On the other hand, by (6.3), (6.4) and (6.5), we have

$$\begin{aligned}
\nabla_{e_1}\cos\alpha &= \bar{\nabla}_{e_1}\cos\alpha = \omega(\bar{\nabla}_{e_1}e_1, e_2) + \omega(e_1, \bar{\nabla}_{e_1}e_2)\\
&= \omega(-h^\beta_{11}e_\beta, e_2) + \omega(e_1, -h^\beta_{12}e_\beta)\\
&= -(h^4_{11} + h^3_{12})\sin\alpha
\end{aligned} \tag{6.12}$$

and

$$\begin{aligned}\nabla_{e_2}\cos\alpha &= \bar{\nabla}_{e_2}\cos\alpha = \omega(\bar{\nabla}_{e_2}e_1, e_2) + \omega(e_1, \bar{\nabla}_{e_2}e_2)\\ &= \omega(-h^{\beta}_{21}e_\beta, e_2) + \omega(e_1, -h^{\beta}_{22}e_\beta)\\ &= -(h^4_{21} + h^3_{22})\sin\alpha.\end{aligned} \tag{6.13}$$

Plugging (6.12) and (6.13) into (6.11), we obtain

$$\begin{aligned}\Delta\cos\alpha &= -|\overline{\nabla}J|^2\cos\alpha + \frac{1}{2}\nabla_{e_1}\cos\alpha\langle\mathbf{x}, e_1\rangle + \frac{1}{2}\nabla_{e_2}\cos\alpha\langle\mathbf{x}, e_2\rangle\\ &= -|\overline{\nabla}J|^2\cos\alpha + \frac{1}{2}\langle\mathbf{x}, \nabla_{e_1}\cos\alpha e_1 + \nabla_{e_2}\cos\alpha e_2\rangle\\ &= -|\overline{\nabla}J|^2\cos\alpha + \frac{1}{2}\langle\mathbf{x}, \nabla\cos\alpha\rangle,\end{aligned}$$

i.e,

$$\Delta\cos\alpha - \frac{1}{2}\langle\mathbf{x}, \nabla\cos\alpha\rangle = -|\overline{\nabla}J|^2\cos\alpha. \tag{6.14}$$

□

As a corollary, we have

Corollary 6.2 *Every complete symplectic self-shrinker must be noncompact.*

Proof Suppose Σ is a closed symplectic self-shrinker in $\mathbf{R}^4$. As $\cos\alpha > 0$, we know from (6.1) that

$$\Delta\cos\alpha - \frac{1}{2}\langle\mathbf{x}, \nabla\cos\alpha\rangle \le 0. \tag{6.15}$$

By the strong maximum principle, $\cos\alpha$ must be a positive constant. Then (6.1) implies that $|\overline{\nabla}J| \equiv 0$ on Σ. Thus Σ is a holomorphic curve in $\mathbf{R}^4$. In particular, it must be a minimal surface in $\mathbf{R}^4$. But every minimal surface in $\mathbf{R}^4$ is noncompact. This is a contradiction. □

Next, we proceed to prove the Main Theorem 4. As before, we define the operator $\mathcal{L}$ by (2.7). Then (6.1) becomes

$$\mathcal{L}\cos\alpha = -|\overline{\nabla}J|^2\cos\alpha. \tag{6.16}$$

Before proving the Main Theorems, we first give one identity satisfied by the mean curvature vector on a self-shrinkers in any dimension and codimension. This generalizes Theorem 5.2 of [8]. (In Appendix B, we give another two geometric identities satisfied on a self-shrinker in arbitrary codimension which are not needed in this paper.)

Suppose $\Sigma^n \subset \mathbf{R}^{n+k}$ is a self-shrinker. We choose a frame $\{e_A\}_{A=1}^{n+k}$ on $\mathbf{R}^{n+k}$ along Σ such that $\{e_i\}_{i=1}^n$ are tangent to Σ and $\{e_\alpha\}_{\alpha=n+1}^{n+p}$ are in the normal bundle. We will compute pointwise. So we will always choose the frame $\{e_i\}_{i=1}^n$ such that $\bar{\nabla}^T_{e_i}e_j(p) = 0$, i.e., at p, $\bar{\nabla}_{e_i}e_j = -h^{\alpha}_{ij}e_\alpha$.

Lemma 6.3 *Let L be defined by* (3.9)*, then we have*

$$LH = H. \tag{6.17}$$

Proof By definition, we only need to prove that

$$(L\mathbf{H})^\alpha = \mathbf{H}^\alpha. \tag{6.18}$$

As $\mathbf{H} = -H^\alpha e_\alpha$, it suffices to show that

$$\Delta H^\alpha + H^\beta h^\beta_{ij} h^\alpha_{ij} - \frac{1}{2}\langle \mathbf{x}, \nabla H^\alpha \rangle + \frac{1}{2} H^\alpha = H^\alpha. \tag{6.19}$$

Recall the self-shrinker equation:

$$-H^\alpha e_\alpha = \mathbf{H} = -\frac{1}{2}\mathbf{x}^\perp = -\frac{1}{2}\langle \mathbf{x}, e_\alpha \rangle e_\alpha.$$

Thus we have

$$2H^\alpha = \langle \mathbf{x}, e_\alpha \rangle. \tag{6.20}$$

Applying (6.20) gives (at the fixed point p)

$$\begin{aligned} H^\alpha_{,i} &= \langle \bar\nabla^N_{e_i}(H^\gamma e_\gamma), e_\alpha \rangle = e_i(H^\gamma)\langle e_\gamma, e_\alpha \rangle + H^\gamma \langle \bar\nabla^N_{e_i} e_\gamma, e_\alpha \rangle \\ &= \bar\nabla_{e_i} H^\alpha + \frac{1}{2}\langle \mathbf{x}, e_\gamma \rangle \langle \bar\nabla^N_{e_i} e_\gamma, e_\alpha \rangle \\ &= \frac{1}{2}\bar\nabla_{e_i}\langle \mathbf{x}, e_\alpha \rangle - \frac{1}{2}\langle \mathbf{x}, e_\gamma \rangle \langle e_\gamma, \bar\nabla^N_{e_i} e_\alpha \rangle \\ &= \frac{1}{2}\langle \bar\nabla_{e_i}\mathbf{x}, e_\alpha \rangle + \frac{1}{2}\langle \mathbf{x}, \bar\nabla_{e_i} e_\alpha \rangle - \frac{1}{2}\langle \mathbf{x}, \bar\nabla^N_{e_i} e_\alpha \rangle \\ &= \frac{1}{2}\langle e_i, e_\alpha \rangle + \frac{1}{2}\langle \mathbf{x}, h^\alpha_{ij} e_j \rangle = \frac{1}{2} h^\alpha_{ij}\langle \mathbf{x}, e_j \rangle. \end{aligned} \tag{6.21}$$

Here, we have used the fact that $\bar\nabla_{e_i}\mathbf{x} = e_i$. Next, we compute the second covariant derivative at p. By (6.20) and (6.21) and our choice of the frame:

$$\begin{aligned} H^\alpha_{,ik} &= \langle \bar\nabla^N_{e_k}\bar\nabla^N_{e_i}(H^\gamma e_\gamma), e_\alpha \rangle = \langle \bar\nabla^N_{e_k}(H^\gamma_{,i} e_\gamma), e_\alpha \rangle \\ &= e_k(H^\gamma_{,i})\langle e_\gamma, e_\alpha \rangle + H^\gamma_{,i}\langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha \rangle \\ &= e_k(H^\alpha_{,i}) + \frac{1}{2} h^\gamma_{ij}\langle \mathbf{x}, e_j \rangle \langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha \rangle \\ &= \frac{1}{2} e_k(h^\alpha_{ij})\langle \mathbf{x}, e_j \rangle + \frac{1}{2} h^\alpha_{ij}\bar\nabla_{e_k}\langle \mathbf{x}, e_j \rangle + \frac{1}{2} h^\gamma_{ij}\langle \mathbf{x}, e_j \rangle \langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha \rangle \\ &= \frac{1}{2}\left(e_k(h^\alpha_{ij}) + h^\gamma_{ij}\langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha \rangle \right)\langle \mathbf{x}, e_j \rangle + \frac{1}{2} h^\alpha_{ik} - \frac{1}{2} h^\alpha_{ij} h^\beta_{kj}\langle \mathbf{x}, e_\beta \rangle. \end{aligned} \tag{6.22}$$

By the definition of covariant derivative of the second fundamental form (Section 7 of [26]) and the choice of frame, we have at p

$$\begin{aligned} h^\alpha_{ij,k} &= \langle (\bar\nabla_{e_k}\mathbf{A})(e_i, e_j), e_\alpha \rangle \\ &= \langle \bar\nabla^N_{e_k}(\mathbf{A}(e_i, e_j)) - \mathbf{A}((\bar\nabla_{e_k} e_i)^T, e_j) - \mathbf{A}(e_i, (\bar\nabla_{e_k} e_j)^T), e_\alpha \rangle \\ &= \langle \bar\nabla^N_{e_k}(h^\gamma_{ij} e_\gamma), e_\alpha \rangle = e_k(h^\alpha_{ij}) + h^\gamma_{ij}\langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha \rangle. \end{aligned} \tag{6.23}$$

Therefore, we have by Codazzi equation

$$H^\alpha_{,ik} = \frac{1}{2}h^\alpha_{ik,j}\langle \mathbf{x}, e_j\rangle + \frac{1}{2}h^\alpha_{ik} - \frac{1}{2}h^\alpha_{ij}h^\beta_{kj}\langle \mathbf{x}, e_\beta\rangle. \tag{6.24}$$

Taking trace of (6.22) and using (6.20), we obtain

$$\Delta H^\alpha = \frac{1}{2}H^\alpha_{,j}\langle \mathbf{x}, e_j\rangle + \frac{1}{2}H^\alpha - \frac{1}{2}h^\alpha_{ij}h^\beta_{ij}\langle \mathbf{x}, e_\beta\rangle = \frac{1}{2}\langle \mathbf{x}, \nabla H^\alpha\rangle + \frac{1}{2}H^\alpha - H^\beta h^\alpha_{ij}h^\beta_{ij}, \tag{6.25}$$

which is just (6.19). This proves the lemma. □

Next, we assume that the self-shrinker Σ^n has flat normal bundle in $\mathbf{R}^{n+k}$. From Lemma 6.3, we can easily obtain that

$$\mathcal{L}|\mathbf{H}|^2 = -2H^\beta H^\gamma h^\beta_{ij}h^\gamma_{ij} + |\mathbf{H}|^2 + 2|\nabla \mathbf{H}|^2. \tag{6.26}$$

Now, we can prove the Main Theorems.

Proof of Main Theorem 4 By (6.26), we have

$$\mathcal{L}|\mathbf{H}|^2 \geq -2|\mathbf{A}|^2|\mathbf{H}|^2 + |\mathbf{H}|^2 + 2|\nabla \mathbf{H}|^2. \tag{6.27}$$

As Σ is normal flat, we know that

$$|\overline{\nabla} J|^2 = |\mathbf{A}|^2. \tag{6.28}$$

Choosing $u = |\mathbf{H}|^2$ and $v = \cos\alpha$ in (2.12) and using (6.16) and (6.27), we have

$$[-\cos\alpha|\mathbf{A}|^2|\mathbf{H}|^2] = [|\mathbf{H}|^2\mathcal{L}\cos\alpha] = [\cos\alpha\mathcal{L}|\mathbf{H}|^2] \geq [\cos\alpha(1-2|\mathbf{A}|^2)|\mathbf{H}|^2 + 2\cos\alpha|\nabla\mathbf{H}|^2], \tag{6.29}$$

from which we can obtain that

$$[\cos\alpha(1 - |\mathbf{A}|^2)|\mathbf{H}|^2 + 2\cos\alpha|\nabla\mathbf{H}|^2] \leq 0. \tag{6.30}$$

If $|\mathbf{A}|^2 < 1$, then $\mathbf{H} \equiv 0$. From the self-shrinker equation (6.8), we know that Σ is a plane.

If $|\mathbf{A}|^2 \equiv 1$, then we have $|\nabla\mathbf{H}| \equiv 0$. Similar to the argument of Theorem 1.1 in [5], we know that Σ is a sphere $\mathbf{S}^2(2)$ or a cylinder $\mathbf{S}^1(\sqrt{2}) \times \mathbf{R}^1$. By Corollary 6.2, Σ cannot be a sphere.

Claim *The cylinder $\boldsymbol{S}^1(\sqrt{2}) \times \boldsymbol{R}^1$ is not symplectic with respect to any complex structure of $\boldsymbol{R}^4$.*

Proof of the Claim First, we suppose the cylinder is embedded in $\mathbf{R}^4$ in the standard way. Namely, it is given by

$$\begin{aligned} F_0 : \mathbf{S}^1 \times \mathbf{R}^1 &\to \mathbf{R}^4 \\ F_0(u, v) &= (\sqrt{2}\cos u, \sqrt{2}\sin u, v, 0). \end{aligned} \tag{6.31}$$

Then

$$(F_0)_u = \sqrt{2}(-\sin u, \cos u, 0, 0), \quad (F_0)_v = (0, 0, 1, 0).$$

We can choose an orthonormal frame by

$$e_1 = (-\sin u, \cos u, 0, 0), \quad e_2 = (0, 0, 1, 0).$$

We will prove that the embedding is not symplectic with respect to any complex structure on $\mathbf{R}^4$. It is well know that the space of orientation preserving complex structure in $\mathbf{R}^4$ is given by

$$\mathcal{J}=\left\{J\in SO(4): J=\begin{pmatrix}0 & x & y & z\\ -x & 0 & -z & y\\ -y & z & 0 & -x\\ -z & -y & x & 0\end{pmatrix} \text{ or } J=\begin{pmatrix}0 & x & y & z\\ -x & 0 & z & -y\\ -y & -z & 0 & x\\ -z & y & -x & 0\end{pmatrix},\quad x^2+y^2+z^2=1.\right\}$$

Now, we take any $J\in\mathcal{J}$, if J takes the first form then it is easy to check that

$$Je_2=(y,-z,0,x).$$

Therefore,

$$\cos\alpha_{J,F_0}=\omega(e_1,e_2)=\langle Je_1,e_2\rangle=-\langle e_1,Je_2\rangle=y\sin u+z\cos u,$$

which obvious has zeroes. Therefore, F_0 cannot be symplectic with respect to such complex structures.

It J takes the second form then it is easy to check that

$$Je_2=(y,z,0,-x).$$

Therefore,

$$\cos\alpha_{J,F_0}=\omega(e_1,e_2)=\langle Je_1,e_2\rangle=-\langle e_1,Je_2\rangle=y\sin u-z\cos u,$$

which also obvious has zeroes. Therefore, F_0 can also not be symplectic with respect to such complex structures. Thus, we have proved that F_0 cannot be symplectic with respect to any complex structure in $\mathbf{R}^4$.

Now, suppose we embed the cylinder in any way $F:\mathbf{S}^1\times\mathbf{R}^1\to\mathbf{R}^4$ and we choose any complex structure $J\in\mathcal{J}$. Then we know that the differences between F and F_0 are just one translation and one rotation in $\mathbf{R}^4$. By an elementary computation, we can see that

$$\cos\alpha_{J,F}=\cos\alpha_{J',F_0}, \tag{6.32}$$

where J' is another complex structure in $\mathcal{J}$. By the above argument for F_0, we know that $\cos\alpha_{J,F}$ must have zeroes. By definition, F is not symplectic with respect to the complex structure J in $\mathbf{R}^4$.

As F and J are arbitrary, this proves the claim.

By the previous argument and the claim, we must have that Σ is a hyperplane in $\mathbf{R}^4$. □

Next, we prove the Main Theorem 3.

Proof of Main Theorem 3 By Corollary 6.2, we know that any symplectic self-shrinker in $\mathbf{R}^4$ must be complete noncompact. Suppose Σ^2 is a complete self-shrinker in $\mathbf{R}^4$ with flat normal bundle, bounded geometry and $|\mathbf{H}|\neq 0$, then by Theorem 1.3 of [23], we know that Σ must be one of the following:

(i) $\Gamma\times\mathbf{R}^1$, where Γ is one of the self-shrinking curves classified by Abresch and Langer;
(ii) $\tilde{\Sigma}^1\times\mathbf{R}^1$, where $\tilde{\Sigma}^1$ is a minimal submanifold of $\mathbf{S}^2$;
(iii) $\tilde{\Sigma}^2$, where $\tilde{\Sigma}^2$ is a minimal submanifold of $\mathbf{S}^3$.

It is known that the only embedded one obtained by Abresch and Langer is only $\mathbf{S}^1$. Thus in the first case, $\Sigma=\mathbf{S}^1\times\mathbf{R}^1$. In the second case, as it is easy to see that any (1-dimensional) minimal submanifold of $\mathbf{S}^2$ is totally geodesic in $\mathbf{S}^2$, which is also $\mathbf{S}^1$. Thus in the second case, Σ is also $\mathbf{S}^1\times\mathbf{R}^1$. In the proof of the Main Theorem 4, we have proved that $\mathbf{S}^1\times\mathbf{R}^1$ is not symplectic. On the other hand, as we said as above, by Corollary 6.2, the closed surface $\tilde{\Sigma}^2$ can also not be symplectic. Therefore, all of the three cases above cannot be symplectic. This proves the theorem. □

Now we can prove the Main Theorem 5.

Proof of Main Theorem 5 By (6.16), we have

$$\mathcal{L}\frac{1}{\cos\alpha} = -\frac{\mathcal{L}\cos\alpha}{\cos^2\alpha} + 2\cos\alpha\left|\nabla\frac{1}{\cos\alpha}\right|^2 = \frac{|\overline{\nabla}J|^2}{\cos\alpha} + 2\cos\alpha\left|\nabla\frac{1}{\cos\alpha}\right|^2. \tag{6.33}$$

As $|\mathbf{A}|^2 \le C$ for some C by our assumption, we see that

$$|\nabla\alpha|^2 \le |\overline{\nabla}J|^2 \le 2|\mathbf{A}|^2 \le 2C.$$

Here, we used that fact that on a symplectic surface, $|\nabla\alpha|^2 \le |\overline{\nabla}J|^2$ (see [12]). Therefore,

$$\begin{aligned}
&1 \le \frac{1}{\cos\alpha} \le \frac{1}{\delta},\\
&\left|\nabla\frac{1}{\cos\alpha}\right|^2 = \frac{\sin^2\alpha}{\cos^2\alpha}|\nabla\alpha|^2 \le \frac{2C}{\delta^2},\\
&\left|\mathcal{L}\frac{1}{\cos\alpha}\right|^2 = \left|\frac{|\overline{\nabla}J|^2}{\cos\alpha} + 2\cos\alpha\left|\nabla\frac{1}{\cos\alpha}\right|^2\right|^2 \le \left(\frac{2C}{\delta} + \frac{4C}{\delta^2}\right)^2.
\end{aligned}$$

By our assumption on the volume growth and Corollary 2.2, we know from (6.33) that

$$-\left[\left|\nabla\frac{1}{\cos\alpha}\right|^2\right] = \left[\frac{1}{\cos\alpha}\mathcal{L}\frac{1}{\cos\alpha}\right] = \left[\frac{|\overline{\nabla}J|^2}{\cos^2\alpha} + 2\left|\nabla\frac{1}{\cos\alpha}\right|^2\right],$$

which is equivalent to

$$\left[\frac{|\overline{\nabla}J|^2}{\cos^2\alpha} + 3\left|\nabla\frac{1}{\cos\alpha}\right|^2\right] = 0.$$

Therefore, we have that

$$\overline{\nabla}J \equiv 0 \;\; and \;\; \cos\alpha = const.$$

In particular, Σ is minimal. By the self-shrinker equation (6.8), we know that Σ must be a plane. □

Finally, we prove that there is no finite time Type I singularity for the symplectic mean curvature flow. This follows from the standard monotonicity formula and the Main Theorem 5. For the purpose of completeness, we give the proof here. First we recall the following modified monotonicity formula obtained firstly by Huisken [16].

Let $H(X, X_0, t, t_0)$ be the backward heat kernel on $\mathbf{R}^4$. Define

$$\rho(X,t) = 4\pi(t_0 - t)H(X, X_0, t, t_0) = \frac{1}{4\pi(t_0 - t)}\exp -\frac{|X - X_0|^2}{4(t_0 - t)},$$

for $t < t_0$. Choose one cut-off function $\phi \in C_0^\infty(B_{2r}(X_0))$ with $\phi \equiv 1$ in $B_r(X_0)$, where $X_0 \in M$, $0 < 2r < i_M$. Choose a normal coordinates in $B_{2r}(X_0)$ and express F using the coordinates $(F^1, \dots, F^4)$ as a submanifold in $\mathbf{R}^4$. We define

$$\Phi(X_0, t_0, t) = \int_{\Sigma_t} \phi(F)\rho(F,t)d\mu_t.$$

Then we have

Proposition 6.4 (Proposition 4.2 of [6]) *Let M^4 be a Riemannian manifold. Then there are positive constants c_1 and c_2 depending only on M, F_0 and r which is the constant in the definition of ϕ such that along the mean curvature flow, we have*

$$\frac{\partial}{\partial t}\left(e^{c_1\sqrt{t_0-t}}\Phi(X_0,t_0,t)\right) \leq -e^{c_1\sqrt{t_0-t}}\int\limits_{M_t}\phi\rho(F,t)\left|\boldsymbol{H}+\frac{(F-X_0)^{\perp}}{2(t_0-t)}\right|^2 d\mu_t$$
$$+c_2e^{c_1\sqrt{t_0-t}}. \tag{6.34}$$

Definition 6.1 ([16]) We say that the mean curvature flow develops **T**ype I singularity at $T>0$, if

$$\limsup_{t\to T}(T-t)\max_{M_t}|A|^2\leq C,$$

for some positive constant C. Otherwise, we say the mean curvature flow develops Type II singularity.

Using the evolution equation for the second fundamental form, we can easily see that

Lemma 6.5 (Lemma 4.6 of [6]) *Let $U(t)\equiv\max_{\Sigma_t}|\boldsymbol{A}|^2$. If the mean curvature flow blows up at $T>0$, there is a positive constant c depending only on M^4, such that if $0<T-t<\frac{\pi}{16\sqrt{c}}$, the function $U(t)$ satisfies*

$$U(t)\geq\frac{1}{4\sqrt{2}(T-t)}. \tag{6.35}$$

Proof of Corollary 1.6 Our proof is based on the blow up argument of the mean curvature flow, which is similar to that of the main theorem of [6]. Suppose that the mean curvature flow develops Type I singularity at a finite time T. Assume that

$$\lambda_k^2=|A|^2(x_k,t_k)=\max_{t\leq t_k}|A|^2\to\infty,\quad as\ k\to\infty.$$

Since Σ is closed, we may assume that $x_k\to p\in\Sigma$ and $t_k\to T$ as $k\to\infty$. As the singularity is of Type I, we know that for $0<t_k<t_l<T$,

$$|F(p,t_k)-F(p,t_l)|\leq\int\limits_{t_k}^{t_l}|\frac{\partial F}{\partial t}|=\int\limits_{t_k}^{t_l}|\mathbf{H}|dt\leq C\sqrt{t_l-t_k}\to 0,$$

as $k,l\to\infty$. Therefore, we know that $F(p,T)$ exists.

Now, we choose a local coordinate on M^4 such that $F(p,T)=0$. Then we rescale the mean curvature flow as follows:

$$F_k(x,s)=\lambda_k(F(x,t_k+\lambda_k^{-2}s)-F(p,t_k)),\quad s\in[-\lambda_k^2t_k,0].$$

Denote by Σ_s^k the scaled surface $F_k(\cdot,s)$ and take

$$g_{ij}^k=\lambda_k^2g_{ij},\quad (g^k)^{ij}=\lambda_k^{-2}g^{ij}.$$

We have

$$\frac{\partial F_k}{\partial s}=\mathbf{H}_k,$$
$$|\mathbf{A}_k|^2=\lambda_k^{-2}|\mathbf{A}|^2.$$

Thus, we get that

$$|\mathbf{A}_k| \leq 1, \qquad |\mathbf{A}_k|(x_k, 0) = 1.$$

By Arzela-Ascoli theorem there is a subsequence of F_k which we also denote by F_k, such that $F_k \to F_\infty$ as $k \to \infty$ in any ball $B_R(0) \subset \mathbf{R}^4$, and F_∞ satisfies

$$\frac{\partial F_\infty}{\partial s} = \mathbf{H}_\infty,$$

with

$$|\mathbf{A}_\infty|(p, 0) = 1.$$

In other words, $\Sigma_s^k \to \Sigma_s^\infty$ in $C^2(B_R(0) \times [-R, 0])$ for any $R > 0$ and any ball $B_R(0) \subset \mathbf{R}^4$. We call Σ_s^∞ *the blow up flow* at 0.

For any $R > 0$, we choose a cut-off function $\phi_R \in C_0^\infty(B_{2R}(0))$ with $\phi_R \equiv 1$ in $B_R(0)$, where $B_\rho(0)$ is the metric ball centered at 0 with radius ρ in $\mathbf{R}^4$. It is easy to see that

$$\begin{aligned}
&\int_{\Sigma_s^k} \phi_R(F_k) \frac{1}{0 - s} \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - s)}\right) d\mu_s^k \\
&= \int_{\Sigma_{t_k + \lambda_k^{-2} s}} \phi(F) \frac{1}{t_k - (t_k + \lambda_k^{-2} s)} \exp\left(-\frac{|F(x, t_k + \lambda_k^{-2} s)|^2}{4(t_k - (t_k + \lambda_k^{-2} s))}\right) d\mu_s,
\end{aligned}$$

where ϕ is the function defined ni the definition of Φ. Notice that $t_k + \lambda_k^{-2} s \to T$ for any fixed s. By Proposition 6.4,

$$\frac{\partial}{\partial t}\left(e^{c_1\sqrt{T-t}}\Phi(X_0, T, t)\right) \leq c_2 e^{c_1\sqrt{T-t}},$$

and it then follows that $\lim_{t\to T} e^{c_1\sqrt{T-t}}\Phi$ exists. This implies that, for any fixed s_1 and s_2 with $-\infty < s_1 < s_2 < 0$, we have

$$\begin{aligned}
&e^{c_1\sqrt{t_k - (t_k + \lambda_k^{-2} s_2)}} \int_{\Sigma_{s_2}^k} \phi_R \frac{1}{(0 - s_2)^{\frac{n}{2}}} \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - s_2)}\right) d\mu_{s_2}^k \\
&\quad - e^{c_1\sqrt{t_k - (t_k + \lambda_k^{-2} s_1)}} \int_{\Sigma_{s_1}^k} \phi_R \frac{1}{(0 - s_1)^{\frac{n}{2}}} \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - s_1)}\right) d\mu_{s_1}^k \\
&\quad \to 0 \ \ as\ k \to \infty.
\end{aligned}$$

Integrating (6.34) from s_1 to s_2, we obtain

$$\begin{aligned}
&-e^{c_1\sqrt{-\lambda_k^{-2} s_2}} \int_{\Sigma_{s_2}^k} \phi_R \frac{1}{(0 - s_2)^{\frac{n}{2}}} \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - s_2)}\right) d\mu_{s_2}^k \\
&\quad + e^{c_1\sqrt{-\lambda_k^{-2} s_1}} \int_{\Sigma_{s_1}^k} \phi_R \frac{1}{(0 - s_1)^{\frac{n}{2}}} \exp\left(-\frac{|F_k + \lambda_k F(p, t_k)|^2}{4(0 - s_1)}\right) d\mu_{s_1}^k
\end{aligned}$$

$$\geq \int_{s_1}^{s_2} e^{c_1\sqrt{-\lambda_k^{-2}s}} \int_{\Sigma_s^k} \phi_R \rho(F_k, t) \left|\mathbf{H}_k + \frac{(F_k + \lambda_k F(p, t_k))^\perp}{2(0-s)}\right|^2 d\mu_s^k ds$$
$$-c_2\lambda_k^{-2}(s_2 - s_1)e^{c_1\lambda_k^{-1}\sqrt{-s_1}}. \quad (6.36)$$

Since the singularity is of Type I, we know that

$$|F(p, t_k)| \leq \int_{t_k}^{T} |\frac{\partial F}{\partial t}| = \int_{t_k}^{T} |\mathbf{H}|dt \leq C\sqrt{T - t_k} \leq \frac{C}{\lambda_k}.$$

Without loss of generality, we can assume that $\lambda_k F(p, t_k) \to Q$ as $k \to \infty$. Letting $k \to \infty$ in (6.36) we know that Σ_s^∞ satisfies

$$\mathbf{H}_\infty(s) + \frac{(F_\infty + Q)^\perp}{2(0-s)} = 0, \quad s \in (-\infty, 0].$$

This means that Σ_∞^s is generated by a self-shrinker, which we denote by $\hat{\Sigma}$. As the Kähler angle is scaling invariant, we know from the evolution equation of the Kähler angle and the maximum principle that $\cos\alpha \geq \delta$ for some $\delta > 0$ on $\hat{\Sigma}$. On the other hand, by the monotonicity formula, it is easy to see that the blow up limit must have polynomial volume growth (see Lemma 2.9 and Corollary 2.13 of [8]). Therefore, $\hat{\Sigma}$ is a complete symplectic self-shrinker in $\mathbf{R}^4$ with polynomial volume growth, bounded second fundamental form and $\cos\alpha \geq \delta > 0$. By the Main Theorem 5, it must be a plane. Thus each Σ_∞^s must be a plane. But this contradicts the fact that $|\mathbf{A}_\infty|(p, 0) = 1$. □

Remark 6.6 Chen and Li [6] proved Corollary 1.6 by finding a new monotonicity formula for symplectic mean curvature flow. Applying the new monotonicity formula, they can show that the blow up limit must be a plane. Here, we first use the classical monotonicity formula to show that the blow up limit must be a symplectic self-shrinker. Then we prove that it must be a plane.

7 Equivariant Lagrangian self-shrinkers

In this section, we study the properties of equivariant Lagrangian self-shrinkers. Some of our notations and backgrounds follow from [11].

Assume

$$z : I \to \mathbf{C}^*$$
$$z(\phi) := u(\phi) + iv(\phi)$$

is some smooth regular curve in $\mathbf{C}^* = \mathbf{C}\backslash\{0\}$. The equivariant Lagrangian immersion $L = F(I \times \mathbf{S}^{n-1})$ is of the following form:

$$F : I \times \mathbf{S}^{n-1} \to \mathbf{C}^n$$
$$F(\phi, \tilde{\mathbf{x}}) := (u(\phi)G(\tilde{\mathbf{x}}), v(\phi)G(\tilde{\mathbf{x}})),$$

where we assume that the complex structure J is acting on $\mathbf{C}^n$ by

$$J(x^1, \ldots, x^n, y^1, \ldots, y^n) := (-y^1, \ldots, -y^n, x^1, \ldots, x^n)$$

and that $G : \mathbf{S}^{n-1} \to \mathbf{R}^n$ is the standard embedding of the sphere of radius 1 in $\mathbf{R}^n$. We will denote the coordinate on I by ϕ and local coordinates on $\mathbf{S}^{n-1}$ by $x^1, \cdots, x^{n-1}$. Latin indices $i, j, k, \cdots$ will be in the range between 1 and $n-1$, whereas greek indices $\alpha, \beta, \cdots$ are taken between 0 and $n-1$. In particular, we define coordinates y^α on $I \times \mathbf{S}^{n-1}$ by $y^0 := \phi$, $y^i := x^i$ for all $i \in \{1, \cdots, n-1\}$. Doubled indices will be summed according to the Einstein convention.

We want to compute the induced metric and the second fundamental form. To this end, let us denote any partial derivative of u, v with respect to ϕ by prime, and in addition we set $G_i := \frac{\partial G}{\partial x^i}$ and $G_{ij} = \frac{\partial G}{\partial x^i \partial x^j}$. With this notation, we have

$$\begin{aligned} F_0 &= (u'G, v'G), \quad F_i = (uG_i, vG_i), \\ \nu_0 &= JF_0 = (-v'G, u'G), \quad \nu_i = JF_i = (-vG_i, uG_i), \\ F_{00} &= (u''G, v''G), \quad F_{0i} = (u'G_i, v'G_i), \quad F_{ij} = (uG_{ij}, vG_{ij}). \end{aligned}$$

The induced metric $g_{\alpha\beta}$ and the second fundamental form $h_{\alpha\beta\gamma}$ on L are given by $g_{\alpha\beta} = \langle F_\alpha, F_\beta \rangle$ and $h_{\alpha\beta\gamma} = \langle \nu_\alpha, F_{\beta\gamma} \rangle$. (Note that for Lagrangian submanifold, we have the full symmetries for the three indices α, β and γ, i.e., $h_{\alpha\beta\gamma} = h_{\beta\gamma\alpha} = h_{\gamma\alpha\beta}$.) The standard metric on $\mathbf{S}^{n-1}$will be denoted by σ_{ij}. Thus

$$g_{00} = (u')^2 + (v')^2, \ g_{0i} = 0, \ g_{ij} = (u^2 + v^2)\sigma_{ij} \tag{7.1}$$

and

$$h_{000} = u'v'' - v'u'', \ h_{00i} = 0, \ h_{0ij} = (uv' - vu')\sigma_{ij}, \ h_{ijk} = 0. \tag{7.2}$$

For the mean curvature $H_\alpha = g^{\beta\gamma} h_{\alpha\beta\gamma}$, we have

$$H_0 = (n-1)\frac{uv' - vu'}{u^2 + v^2} + \frac{u'v'' - v'u''}{(u')^2 + (v')^2}, \tag{7.3}$$

$$H_i = 0. \tag{7.4}$$

In particular, the mean curvature vector is given by

$$\mathbf{H} = g^{\alpha\beta} H_\alpha \nu_\beta = g^{00} H_0 \nu_0. \tag{7.5}$$

It is known that [2,11] L is a self-shrinker, i.e. $\mathbf{H} = -\frac{1}{2}F^\perp$ if and only if

$$k + (n-1)\frac{\langle z, \nu \rangle}{|z|^2} = \frac{1}{2}\langle z, \nu \rangle. \tag{7.6}$$

Here, ν denotes the outward pointing unit normal along the curve $\gamma := z(I)$, and k is the curvature of γ. Using the above notation, we have

$$\nu = -J\left(\frac{z'}{|z'|}\right).$$

This implies that

$$\nu = \frac{1}{\sqrt{(u')^2 + (v')^2}} \begin{pmatrix} v' \\ -u' \end{pmatrix}$$

and

$$\langle z, \nu \rangle = \frac{uv' - vu'}{\sqrt{(u')^2 + (v')^2}}. \tag{7.7}$$

The curvature k of the curve γ is determined by

$$k = -\frac{1}{|z'|^2}\langle z'', \nu\rangle = \frac{u'v'' - v'u''}{((u')^2 + (v')^2)^{\frac{3}{2}}}. \tag{7.8}$$

Recall that F-stability is defined in Definition 3.1. It is not hard to see that every equivariant Lagrangian self-shrinker is not F-stable in the sense of Definition 3.1 (see the proof below). Motivated by Oh's study of minimal Lagrangian submanifold in Käler manifold [22], we introduce the concept of Hamiltonian F-stability. Let us first recall the definition of Hamiltonian variation.

Definition 7.1 ([22]) Let L be a Lagrangian submanifold and $\boldsymbol{V}$ be a vector field along L. $\boldsymbol{V}$ is call a Lagrangian (resp. Hamiltonian) variation if it satisfies that the one form on L

$$i^*(\boldsymbol{V}\rfloor\omega)$$

is closed (resp. exact).

Lemma 7.1 (Lemma 2.3 of [22]) *A normal variation* $\boldsymbol{V}$ *on* L *is Hamiltonian if and only if*

$$\boldsymbol{V} = J\cdot\nabla f$$

where f *is a function on* L *and* ∇ *is the gradient on* L *with respect to the induced metric.*

We can define Hamiltonian F-stability as follows:

Definition 7.2 We say that a Lagrangian self-shrinker L is $Hamiltonian$***F-stable*** if for every Hamiltonian variation $\boldsymbol{V}$, there exist variations of x_0 and t_0 that make $F'' \geq 0$.

Now we can prove the Main Theorem 6.

Proof of Main Theorem 6 Recall that the second variation of the F-functional is given by (3.8), where the stability operator L is defined by (3.9). Note that (3.9) is given using orthonormal basis for the tangent bundle and the normal bundle. Using the frame we are chosen as above, we can easily see that for $\mathbf{V} = g^{\alpha\beta}V_\alpha\nu_\beta$, we have

$$L(g^{\alpha\beta}V_\alpha\nu_\beta) = g^{\alpha\beta}\left(\Delta V_\alpha + g^{\gamma\delta}g^{rs}g^{tu}V_\gamma h_{\alpha rt}h_{\delta su} - \frac{1}{2}\langle\mathbf{x}, \nabla V_\alpha\rangle + \frac{1}{2}V_\alpha\right)\nu_\beta. \tag{7.9}$$

As before, we denote by

$$\mathcal{L} = \Delta - \frac{1}{2}\langle\mathbf{x}, \nabla(\cdot)\rangle.$$

As $\mathbf{V} = g^{\alpha\beta}V_\alpha\nu_\beta$, we have

$$\begin{aligned}\langle L\mathbf{V}, \mathbf{V}\rangle &= g^{\alpha\beta}\left(\Delta V_\alpha + g^{\gamma\delta}g^{rs}g^{tu}V_\gamma h_{\alpha rt}h_{\delta su} - \frac{1}{2}\langle\mathbf{x}, \nabla V_\alpha\rangle + \frac{1}{2}V_\alpha\right)V_\beta\\ &= g^{\alpha\beta}(\mathcal{L}V_\alpha)V_\beta + g^{\alpha\beta}g^{\gamma\delta}g^{rs}g^{tu}V_\beta V_\gamma h_{\alpha rt}h_{\delta su} + \frac{1}{2}|\mathbf{V}|^2.\end{aligned}$$

Therefore, by (3.8),

$$F'' = \Big[-g^{\alpha\beta}(\mathcal{L}V_\alpha)V_\beta - \frac{1}{2}|\mathbf{V}|^2 - g^{\alpha\beta}g^{\gamma\delta}g^{rs}g^{tu}V_\beta V_\gamma h_{\alpha rt}h_{\delta su}$$
$$+\langle \mathbf{V}, \mathbf{y}\rangle - h^2|\mathbf{H}|^2 - 2h\langle \mathbf{H}, \mathbf{V}\rangle - \frac{1}{2}|\mathbf{y}^\perp|^2\Big]$$
$$= \Big[g^{\alpha\beta}\langle \nabla V_\alpha, \nabla V_\beta\rangle_L - \frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 - g^{\alpha\beta}g^{\gamma\delta}g^{rs}g^{tu}V_\beta V_\gamma h_{\alpha rt}h_{\delta su}$$
$$-h^2|\mathbf{H}|^2 - 2h\langle \mathbf{H}, \mathbf{V}\rangle\Big].$$

$$= \int_L \Big\{g^{\alpha\beta}\langle \nabla V_\alpha, \nabla V_\beta\rangle_L - \frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 - g^{\alpha\beta}g^{\gamma\delta}g^{rs}g^{tu}V_\beta V_\gamma h_{\alpha rt}h_{\delta su}$$
$$-h^2|\mathbf{H}|^2 - 2h\langle \mathbf{H}, \mathbf{V}\rangle\Big\} e^{-\frac{|\mathbf{x}|^2}{4}}d\mu. \tag{7.10}$$

Note that we have (7.5), thus

$$\langle \mathbf{H}, \mathbf{V}\rangle = g^{\alpha\beta}H_\alpha V_\beta = g^{00}H_0V_0 = \frac{H_0V_0}{(u')^2+(v')^2},$$

and

$$|\mathbf{H}|^2 = \langle \mathbf{H}, \mathbf{H}\rangle = \frac{(H_0)^2}{(u')^2+(v')^2},$$

Therefore,

$$-h^2|\mathbf{H}|^2 - 2h\langle \mathbf{H}, \mathbf{V}\rangle = -\frac{h^2(H_0)^2+2hH_0V_0}{(u')^2+(v')^2} = -\frac{(hH_0+V_0)^2}{(u')^2+(v')^2} + \frac{(V_0)^2}{(u')^2+(v')^2}. \tag{7.11}$$

Using (7.1) and (7.2), we can compute directly that

$$g^{\alpha\beta}g^{\gamma\delta}g^{rs}g^{tu}V_\beta V_\gamma h_{\alpha rt}h_{\delta su} = \left(\frac{(u'v''-v'u'')^2}{((u')^2+(v')^2)^4} + \frac{(uv'-vu')^2}{((u')^2+(v')^2)^2(u^2+v^2)^2}\right)(V_0)^2$$
$$+\frac{2(uv'-vu')^2}{((u')^2+(v')^2)(u^2+v^2)^3}\sigma^{ij}V_iV_j. \tag{7.12}$$

By definition, we also have

$$|\mathbf{x}|^2 = u^2+v^2 \tag{7.13}$$

Putting (7.11), (7.12) and (7.13) into (7.10) gives

$$F'' = \int_L \Big\{g^{\alpha\beta}g^{\xi\eta}\frac{\partial V_\alpha}{\partial y^\xi}\frac{\partial V_\beta}{\partial y^\eta} - \frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 - \frac{(hH_0+V_0)^2}{(u')^2+(v'2} - \frac{2(uv'-vu')^2}{((u')^2+(v')^2)(u^2+v^2)^3}\sigma^{ij}V_iV_j$$
$$+\left(\frac{1}{(u')^2+(v')^2} - \frac{(u'v''-v'u')^2}{((u')^2+(v')^2)^4} - \frac{(uv'-vu')^2}{((u')^2+(v')^2)^2(u^2+v^2)^2}\right)(V_0)^2\Big\} e^{-\frac{u^2+v^2}{4}}d\mu. \tag{7.14}$$

It is easy from (7.14) to see that L is not F-stable in the usual sense. In fact, if we choose locally

$$V_\alpha = \begin{cases} 1, & \alpha = 1; \\ 0, & \alpha \neq 1, \end{cases}$$

then $F'' < 0$ for this choice of $\mathbf{V}$ no matter what h and $\mathbf{y}$ are.

In order to show that L is not Hamiltonian F-stable, we need to find a Hamiltonian variation $\mathbf{V}$ such that $F'' < 0$ for any choice of h and $\mathbf{y}$. Now, we take any Hamiltonian variation $\mathbf{V}$. By Lemma 7.1, we know that there exists a smooth function f on L such that

$$\mathbf{V} = J\nabla f = J\left(g^{\alpha\beta}\frac{\partial f}{\partial y^\alpha}F_\beta\right) = g^{\alpha\beta}\frac{\partial f}{\partial y^\alpha}\nu_\beta.$$

Therefore,

$$V_\alpha = \frac{\partial f}{\partial y^\alpha},$$

i.e.,

$$V_0 = \frac{\partial f}{\partial \phi}, \quad V_i = \frac{\partial f}{\partial x^i}.$$

Now we choose f to be a function in L such that it is independent of ϕ. (In other word, it is a function on $\mathbf{S}^{n-1}$.) Then $V_0 = 0$. With this choice of $\mathbf{V}$, we see from (7.14) that

$$\begin{aligned}
F'' &= \int_L \left\{ g^{ij}g^{kl}\frac{\partial V_i}{\partial x^k}\frac{\partial V_j}{\partial x^l} - \frac{2(uv'-vu')^2}{((u')^2+(v')^2)(u^2+v^2)^3}\sigma^{ij}V_iV_j \right. \\
&\quad \left. -\frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 - \frac{h^2(H_0)^2}{(u')^2+(v')^2} \right\} e^{-\frac{u^2+v^2}{4}}d\mu \\
&= \int_L \left\{ (u^2+v^2)^{-2}\sigma^{ij}\sigma^{kl}\frac{\partial V_i}{\partial x^k}\frac{\partial V_j}{\partial x^l} - \frac{2(uv'-vu')^2}{((u')^2+(v')^2)(u^2+v^2)^3}\sigma^{ij}V_iV_j \right. \\
&\quad \left. -\frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 - \frac{h^2(H_0)^2}{(u')^2+(v')^2} \right\} e^{-\frac{u^2+v^2}{4}}d\mu \\
&= \int_L \left\{ (u^2+v^2)^{-2}\sigma^{ij}\langle \nabla_{\mathbf{S}^{n-1}}V_i, \nabla_{\mathbf{S}^{n-1}}V_j\rangle - \frac{2(uv'-vu')^2}{((u')^2+(v')^2)(u^2+v^2)^3}|\mathbf{V}|^2_{\mathbf{S}^{n-1}} \right. \\
&\quad \left. -\frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 - \frac{h^2(H_0)^2}{(u')^2+(v')^2} \right\} e^{-\frac{u^2+v^2}{4}}d\mu \\
&= \int_L \left\{ (u^2+v^2)^{-2}|\nabla^2_{\mathbf{S}^{n-1}}f|^2_{\mathbf{S}^{n-1}} - \frac{2(uv'-vu')^2}{((u')^2+(v')^2)(u^2+v^2)^3}|\nabla_{\mathbf{S}^{n-1}}f|^2_{\mathbf{S}^{n-1}} \right. \\
&\quad \left. -\frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 - \frac{h^2(H_0)^2}{(u')^2+(v')^2} \right\} e^{-\frac{u^2+v^2}{4}}d\mu. \qquad (7.15)
\end{aligned}$$

By (7.1), we have that

$$\begin{aligned}
d\mu &= \sqrt{det(g_{\alpha\beta})}dy = ((u')^2+(v')^2)^{\frac{1}{2}}(u^2+v^2)^{\frac{n-1}{2}}\sqrt{det(\sigma_{ij})}d\phi d\tilde{\mathbf{x}} \\
&= ((u')^2+(v')^2)^{\frac{1}{2}}(u^2+v^2)^{\frac{n-1}{2}}d\phi d\sigma, \qquad (7.16)
\end{aligned}$$

where $d\sigma$ is the standard volume form on $\mathbf{S}^{n-1}$. Therefore, we have

$$F'' = \int_I ((u')^2+(v')^2)^{\frac{1}{2}}(u^2+v^2)^{\frac{n-5}{2}} e^{-\frac{u^2+v^2}{4}} d\phi \int_{\mathbf{S}^{n-1}} |\nabla^2_{\mathbf{S}^{n-1}} f|^2_{\mathbf{S}^{n-1}} d\sigma$$
$$-2\int_I \frac{(uv'-vu')^2(u^2+v^2)^{\frac{n-7}{2}}}{((u')^2+(v')^2)^{\frac{1}{2}}} e^{-\frac{u^2+v^2}{4}} d\phi \int_{\mathbf{S}^{n-1}} |\nabla_{\mathbf{S}^{n-1}} f|^2_{\mathbf{S}^{n-1}} d\sigma$$
$$-\int_{I\times\mathbf{S}^{n-1}} \left\{\frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 + \frac{h^2(H_0)^2}{(u')^2+(v')^2}\right\} ((u')^2+(v')^2)^{\frac{1}{2}}(u^2+v^2)^{\frac{n-1}{2}} e^{-\frac{u^2+v^2}{4}} d\phi d\sigma. \tag{7.17}$$

Claim

$$\int_I \frac{(uv'-vu')^2(u^2+v^2)^{\frac{n-7}{2}}}{((u')^2+(v')^2)^{\frac{1}{2}}} e^{-\frac{u^2+v^2}{4}} d\phi > 0.$$

Proof of the Claim Suppose the integration is 0, then we must have that

$$uv' - vu' \equiv 0.$$

From (7.7), we have

$$\langle z, \nu\rangle \equiv 0.$$

Then by the self-shrinker equation (7.6), we have

$$k \equiv 0,$$

which combining (7.8) yields

$$u'v'' - v'u'' \equiv 0.$$

Therefore, by (7.3), we have

$$H_0 \equiv 0,$$

which in turn implies by (7.5) that

$$\mathbf{H} \equiv 0.$$

Therefore, L is a minimal self-shrinker, which must be a plane passing through the origin. But from our construction of L, we see that L does not contain the origin. This contradiction proves the claim.

Now, we continue our proof. We will further choose a special f. Recall the Bochner formula on

$$\Delta|\nabla f|^2 = 2|\nabla^2 f|^2 + 2\langle\nabla f, \nabla\Delta f\rangle + 2Ric(\nabla f, \nabla f).$$

For $\mathbf{S}^{n-1}$, the Ricci curvature is $n-1$. Therefore, the above formula becomes

$$\Delta_{\mathbf{S}^{n-1}}|\nabla_{\mathbf{S}^{n-1}} f|^2 = 2|\nabla^2_{\mathbf{S}^{n-1}} f|^2 + 2\langle\nabla_{\mathbf{S}^{n-1}} f, \nabla_{\mathbf{S}^{n-1}}\Delta_{\mathbf{S}^{n-1}} f\rangle + 2(n-1)|\nabla_{\mathbf{S}^{n-1}} f|^2. \tag{7.18}$$

It is well known that the first positive eigenvalue for $\mathbf{S}^{n-1}$ is $n-1$. We take f_1 to be the eigenfunction with eigenvalue $n-1$, i.e.

$$\Delta_{\mathbf{S}^{n-1}} f_1 = -(n-1) f_1.$$

By (7.18), we have

$$\Delta_{\mathbf{S}^{n-1}}|\nabla_{\mathbf{S}^{n-1}}f_1|^2 = 2|\nabla^2_{\mathbf{S}^{n-1}}f_1|^2.$$

Therefore,

$$\int_{\mathbf{S}^{n-1}} |\nabla^2_{\mathbf{S}^{n-1}}f_1|^2_{\mathbf{S}^{n-1}}d\sigma = 0. \tag{7.19}$$

Of course, f_1 is not a constant. Therefore, for this choice of f_1 and $\mathbf{V} = J\nabla f_1$, the second variation formula (7.17) becomes

$$\begin{aligned} F'' &= -2\int_I \frac{(uv'-vu')^2(u^2+v^2)^{\frac{n-7}{2}}}{((u')^2+(v')^2)^{\frac{1}{2}}}e^{-\frac{u^2+v^2}{4}}d\phi \int_{\mathbf{S}^{n-1}} |\nabla_{\mathbf{S}^{n-1}}f_1|^2_{\mathbf{S}^{n-1}}d\sigma \\ &\quad - \int_{I\times\mathbf{S}^{n-1}} \left\{\frac{1}{2}|\mathbf{V}-\mathbf{y}^\perp|^2 + \frac{h^2(H_0)^2}{(u')^2+(v')^2}\right\}((u')^2+(v')^2)^{\frac{1}{2}}(u^2+v^2)^{\frac{n-1}{2}}e^{-\frac{u^2+v^2}{4}}d\phi d\sigma \\ &\le -2\int_I \frac{(uv'-vu')^2(u^2+v^2)^{\frac{n-7}{2}}}{((u'2+(v')^2)^{\frac{1}{2}}}e^{-\frac{u^2+v^2}{4}}d\phi \int_{\mathbf{S}^{n-1}} |\nabla_{\mathbf{S}^{n-1}}f_1|^2_{\mathbf{S}^{n-1}}d\sigma \\ &< 0. \end{aligned}$$

In other word, we proved that there is a Hamiltonian variation $\mathbf{V}$, such that for this variation, $F'' < 0$ for any choice of h and $\mathbf{y}$. From Definition 7.2, we see that L is not Hamiltonian F-stable. □

Proof of Corollary 1.7 By Theorem 1.15 (ii) of [11], starting from some equivariant Lagrangian immersion in $\mathbf{C}^n$, the mean curvature flow will develop Type-I singularity and the the rescaled flow will converge to some closed equivariant Lagrangian self-shrinker $\tilde{L}_\infty$. By the Main Theorem 6, $\tilde{L}_\infty$ is not F-stable. □

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Appendix A

In this appendix, we will prove the variations of normal vector field and mean curvature we need in Sect. 3. The proof is standard. When the variation vector $\mathbf{V}$ is the mean curvature vector $\mathbf{H}$, they are proved in [12]. We will follow the computations in [12].

We begin with fixing our notation. In a normal coordinate around some point in Σ, the induced metric on Σ is given by

$$g_{ij} = \langle \partial_i F, \partial_j F\rangle, \tag{8.1}$$

where $\partial_i (i = 1, \ldots, n)$ are the partial derivatives with respect to the local coordinate. Here, $\langle\cdot,\cdot\rangle$ is the inner product of $\mathbf{R}^{n+k}$.

We choose a local field of orthonormal frames $e_1, \cdots, e_n, e_{n+1}, \cdots, e_{n+k}$ of $\mathbf{R}^{n+k}$ along Σ_s such that $e_1, \ldots, e_n$ are tangent vectors of Σ_s and $e_{n+1}, \ldots, e_{n+k}$ are in the normal bundle over Σ_s. From now on, we will agree on the following index ranges:

$$1 \le i, j, k, l \le n, \quad n+1 \le \alpha, \beta, \gamma \le n+k, \quad 1 \le A, B, C \le n+k.$$

We can write

$$\mathbf{A} = \mathbf{A}^\alpha e_\alpha, \quad \mathbf{H} = -H^\alpha e_\alpha.$$

Let $\mathbf{A}^\alpha = (h^\alpha_{ij})$, where (h^α_{ij}) is a matrix. By the Weingarten equation, we have

$$h^\alpha_{ij} = \langle \bar{\nabla}_{e_i} e_\alpha, e_j \rangle = -\langle e_\alpha, \bar{\nabla}_{e_i} e_j \rangle = h^\alpha_{ji},$$

where $\bar{\nabla}$ is the Levi-Civita connection on $\mathbf{R}^{n+k}$. Furthermore,

$$H^\alpha = g^{ij} h^\alpha_{ij} = h^\alpha_{ii}.$$

Suppose the variation vector filed is $\mathbf{V} = V^\alpha e_\alpha$, i.e.,

$$F(\cdot, s) : \Sigma \to \mathbf{R}^{n+k}$$

satisfies

$$\frac{\partial F}{\partial s} = \mathbf{V}.$$

Then we have

Lemma 8.1 *The induced metric satisfies*

$$\frac{\partial}{\partial s} g_{ij} = 2V^\alpha h^\alpha_{ij}, \tag{8.2}$$

and

$$\frac{\partial}{\partial s} g^{ij} = -2V^\alpha h^\alpha_{ij}. \tag{8.3}$$

Proof We prove it at a fixed point. We have

$$\begin{aligned}
\frac{\partial}{\partial s} g_{ij} &= \bar{\nabla}_{\mathbf{V}} \langle \partial_i F, \partial_j F \rangle = \langle \bar{\nabla}_{\mathbf{V}} \partial_i F, e_j \rangle + \langle e_i, \bar{\nabla}_{\mathbf{V}} \partial_j F \rangle \\
&= \langle \bar{\nabla}_{e_i} \mathbf{V}, e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} \mathbf{V} \rangle \\
&= \langle \bar{\nabla}_{e_i} (V^\alpha e_\alpha), e_j \rangle + \langle e_i, \bar{\nabla}_{e_j} (V^\alpha e_\alpha) \rangle \\
&= V^\alpha \langle \bar{\nabla}_{e_i} e_\alpha, e_j \rangle + V^\alpha \langle e_i, \bar{\nabla}_{e_j} e_\alpha \rangle \\
&= 2V^\alpha h^\alpha_{ij}.
\end{aligned}$$

Here we have used the fact that

$$\bar{\nabla}_{\mathbf{V}} \partial_i F = \frac{\partial^2}{\partial s \partial x_i} F = \bar{\nabla}_{\partial_i F} \mathbf{V}.$$

As at the fixed point p, $g_{ij}(p) = \delta_{ij}$, we know that

$$\frac{\partial}{\partial s} g^{ij} = -2V^\alpha h^\alpha_{ij}.$$

□

Lemma 8.2 *Denote* $\langle \frac{\partial}{\partial s} e_\alpha, e_\beta \rangle = \langle \bar{\nabla}_V e_\alpha, e_\beta \rangle = b^\beta_\alpha$, *then* $b^\beta_\alpha = -b^\alpha_\beta$, *and we have*

$$\frac{\partial}{\partial s} e_\alpha = -\nabla V^\alpha - V^\beta \langle \bar{\nabla}_{e_i} e_\beta, e_\alpha \rangle e_i + b^\beta_\alpha e_\beta. \tag{8.4}$$

Here, ∇V^α *is the covariant differential for the induced metric on* Σ_s.

Proof We have

$$\begin{aligned}\frac{\partial}{\partial s}e_\alpha &= \langle\frac{\partial}{\partial s}e_\alpha, e_i\rangle e_i + \langle\frac{\partial}{\partial s}e_\alpha, e_\beta\rangle e_\beta\\&= \langle\bar{\nabla}_{\mathbf{V}}e_\alpha, e_i\rangle e_i + b_\alpha^\beta e_\beta\\&= -\langle e_\alpha, \bar{\nabla}_{\mathbf{V}}\partial_i F\rangle e_i + b_\alpha^\beta e_\beta\\&= -\langle e_\alpha, \bar{\nabla}_{e_i}\mathbf{V}\rangle e_i + b_\alpha^\beta e_\beta\\&= -\langle e_\alpha, \bar{\nabla}_{e_i}(V^\beta e_\beta)\rangle e_i + b_\alpha^\beta e_\beta\\&= -\langle e_\alpha, (\bar{\nabla}_{e_i}V^\beta)e_\beta + V^\beta\bar{\nabla}_{e_i}e_\beta\rangle e_i + b_\alpha^\beta e_\beta\\&= -(\bar{\nabla}_{e_i}V^\alpha)e_i - V^\beta\langle\bar{\nabla}_{e_i}e_\beta, e_\alpha\rangle e_i + b_\alpha^\beta e_\beta\\&= -\nabla V^\alpha - V^\beta\langle\bar{\nabla}_{e_i}e_\beta, e_\alpha\rangle e_i + b_\alpha^\beta e_\beta.\end{aligned}$$

□

Lemma 8.3 *The second fundamental form satisfies*

$$\frac{\partial}{\partial s}h_{ij}^\alpha = -V_{,ji}^\alpha + V^\beta h_{ik}^\alpha h_{jk}^\beta + h_{ij}^\beta\langle e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle. \tag{8.5}$$

Here, $V_{,ji}^\alpha$ *denotes the second covariant derivative for the connection on the normal bundle.*

Proof We compute at a fixed point $p\in\Sigma$. We can choose a frame e_i so that $\bar{\nabla}_{e_i}^T e_j(p) = 0$, i.e., at p, $\bar{\nabla}_{e_i}e_j = -h_{ij}^\beta e_\beta$. From

$$h_{ij}^\alpha = -\langle\bar{\nabla}_{e_i}e_j, e_\alpha\rangle,$$

and the fact that $\mathbf{R}^{n+k}$ is flat, we have

$$\begin{aligned}\frac{\partial}{\partial s}h_{ij}^\alpha &= -\langle\bar{\nabla}_{\mathbf{V}}\bar{\nabla}_{e_i}e_j, e_\alpha\rangle - \langle\bar{\nabla}_{e_i}e_j, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= -\langle\bar{\nabla}_{e_i}\bar{\nabla}_{\mathbf{V}}e_j, e_\alpha\rangle - \langle\bar{\nabla}_{[\mathbf{V},e_i]}e_j, e_\alpha\rangle - \langle\bar{\nabla}_{e_i}e_j, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= -\langle\bar{\nabla}_{e_i}\bar{\nabla}_{e_j}\mathbf{V}, e_\alpha\rangle - \langle -h_{ij}^\beta e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= -\langle\bar{\nabla}_{e_i}(\bar{\nabla}_{e_j}^T\mathbf{V} + \bar{\nabla}_{e_j}^\perp\mathbf{V}), e_\alpha\rangle + \langle h_{ij}^\beta e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= -\langle\bar{\nabla}_{e_i}(\bar{\nabla}_{e_j}^T\mathbf{V}), e_\alpha\rangle - \langle\bar{\nabla}_{e_i}^\perp\bar{\nabla}_{e_j}^\perp\mathbf{V}, e_\alpha\rangle + \langle h_{ij}^\beta e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= \langle\bar{\nabla}_{e_j}^T\mathbf{V}, \bar{\nabla}_{e_i}e_\alpha\rangle - \langle V_{,ji}^\beta e_\beta, e_\alpha\rangle + \langle h_{ij}^\beta e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= \langle\bar{\nabla}_{e_j}^T(V^\beta e_\beta), h_{ik}^\alpha e_k\rangle - V_{,ji}^\beta\langle e_\beta, e_\alpha\rangle + \langle h_{ij}^\beta e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= -V_{,ji}^\alpha + V^\beta\langle h_{jl}^\beta e_l, h_{ik}^\alpha e_k\rangle + \langle h_{ij}^\beta e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle\\&= -V_{,ji}^\alpha + V^\beta h_{ik}^\alpha h_{jk}^\beta + h_{ij}^\beta\langle e_\beta, \bar{\nabla}_{\mathbf{V}}e_\alpha\rangle.\end{aligned}$$

□

Lemma 8.4 *The mean curvature vector satisfies*

$$\frac{\partial}{\partial s}\boldsymbol{H} = (\Delta V^\alpha + V^\beta h_{ij}^\beta h_{ij}^\alpha)e_\alpha + H^\alpha\nabla V^\alpha + H^\alpha V^\beta\langle\bar{\nabla}_{e_i}e_\beta, e_\alpha\rangle e_i. \tag{8.6}$$

Proof By Lemma 8.1 and Lemma 8.3, we have

$$\begin{aligned}\frac{\partial}{\partial s}H^\alpha &= \frac{\partial}{\partial s}\left(g^{ij}h^\alpha_{ij}\right) = \left(\frac{\partial}{\partial s}g^{ij}\right)h^\alpha_{ij} + g^{ij}\frac{\partial}{\partial s}h^\alpha_{ij} \\ &= -2V^\beta h^\beta_{ij}h^\alpha_{ij} + (-\Delta V^\alpha + V^\beta h^\beta_{ij}h^\alpha_{ij} + H^\beta\langle e_\beta, \bar\nabla_{\mathbf{V}} e_\alpha\rangle) \\ &= -\Delta V^\alpha - V^\beta h^\beta_{ij}h^\alpha_{ij} + H^\beta\langle e_\beta, \bar\nabla_{\mathbf{V}} e_\alpha\rangle. \end{aligned} \tag{8.7}$$

Combining with Lemma 8.2, we obtain

$$\begin{aligned}\frac{\partial}{\partial s}\mathbf{H} &= \frac{\partial}{\partial s}\left(-H^\alpha e_\alpha\right) = -\left(\frac{\partial}{\partial s}H^\alpha\right)e_\alpha - H^\alpha\frac{\partial}{\partial s}e_\alpha \\ &= \left(\Delta V^\alpha + V^\beta h^\beta_{ij}h^\alpha_{ij} - H^\beta\langle e_\beta, \bar\nabla_{\mathbf{V}} e_\alpha\rangle\right)e_\alpha \\ &\quad + H^\alpha\nabla V^\alpha + H^\alpha V^\beta\langle\bar\nabla_{e_i}e_\beta, e_\alpha\rangle e_i - H^\alpha b^\beta_\alpha e_\beta.\end{aligned}$$

Note that

$$-H^\beta\langle e_\beta, \bar\nabla_{\mathbf{V}} e_\alpha\rangle e_\alpha = -H^\beta b^\beta_\alpha e_\alpha = -H^\alpha b^\alpha_\beta e_\beta = H^\alpha b^\beta_\alpha e_\beta.$$

Thus we have

$$\frac{\partial}{\partial s}\mathbf{H} = (\Delta V^\alpha + V^\beta h^\beta_{ij}h^\alpha_{ij})e_\alpha + H^\alpha\nabla V^\alpha + H^\alpha V^\beta\langle\bar\nabla_{e_i}e_\beta, e_\alpha\rangle e_i.$$

□

Appendix B

In this appendix, we will give another two geometric identities satisfied on self-shrinkers with arbitrary dimension and codimension. These results generalized Theorem 5.2 and Lemma 10.8 of [8].

Suppose $\Sigma^n \subset \mathbf{R}^{n+k}$ is a self-shrinker. We choose a frame $\{e_A\}_{A=1}^{n+k}$ on $\mathbf{R}^{n+k}$ along Σ such that $\{e_i\}_{i=1}^n$ are tangent to Σ and $\{e_\alpha\}_{\alpha=n+1}^{n+p}$ are in the normal bundle. We will compute pointwise. So we will always choose the frame $\{e_i\}_{i=1}^n$ such that $\bar\nabla^T_{e_i}e_j(p) = 0$, i.e., at p, $\bar\nabla_{e_i}e_j = -h^\alpha_{ij}e_\alpha$.

Lemma 9.1 *Let L be defined by (3.9). Suppose $\boldsymbol{w} \in \boldsymbol{R}^{n+k}$ is a fixed vector. Then on a self-shrinker Σ^n in $\boldsymbol{R}^{n+k}$, we have*

$$L\boldsymbol{w}^\perp = \frac{1}{2}\boldsymbol{w}^\perp. \tag{9.1}$$

Proof By definition,

$$\mathbf{w}^\perp = \langle\mathbf{w}, e_\alpha\rangle e_\alpha \equiv f^\alpha e_\alpha,$$

where

$$f^\alpha = \langle\mathbf{w}, e_\alpha\rangle. \tag{9.2}$$

Then computing at p using the above chosen frame, we have

$$\begin{aligned} f^\alpha_{,i} &= \langle \bar\nabla^N_{e_i}(f^\gamma e_\gamma), e_\alpha\rangle = e_i(f^\gamma)\langle e_\gamma, e_\alpha\rangle + f^\gamma\langle \bar\nabla^N_{e_i} e_\gamma, e_\alpha\rangle \\ &= \bar\nabla_{e_i} f^\alpha + \langle \mathbf{w}, e_\gamma\rangle\langle \bar\nabla^N_{e_i} e_\gamma, e_\alpha\rangle \\ &= \bar\nabla_{e_i}\langle \mathbf{w}, e_\alpha\rangle - \langle \mathbf{w}, e_\gamma\rangle\langle e_\gamma, \bar\nabla^N_{e_i} e_\alpha\rangle \\ &= \langle \mathbf{w}, \bar\nabla_{e_i} e_\alpha\rangle - \langle \mathbf{w}, \bar\nabla^N_{e_i} e_\alpha\rangle = h^\alpha_{ij}\langle \mathbf{w}, e_j\rangle \end{aligned} \tag{9.3}$$

and

$$\begin{aligned} f^\alpha_{,ik} &= \langle \bar\nabla^N_{e_k}\bar\nabla^N_{e_i}(f^\gamma e_\gamma), e_\alpha\rangle = \langle \bar\nabla^N_{e_k}(f^\gamma_{,i} e_\gamma), e_\alpha\rangle \\ &= e_k(f^\gamma_{,i})\langle e_\gamma, e_\alpha\rangle + f^\gamma_{,i}\langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha\rangle \\ &= e_k(f^\alpha_{,i}) + h^\gamma_{ij}\langle \mathbf{w}, e_j\rangle\langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha\rangle \\ &= e_k(h^\alpha_{ij})\langle \mathbf{w}, e_j\rangle + h^\alpha_{ij}\bar\nabla_{e_k}\langle \mathbf{w}, e_j\rangle + h^\gamma_{ij}\langle \mathbf{w}, e_j\rangle\langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha\rangle \\ &= \left(e_k(h^\alpha_{ij}) + h^\gamma_{ij}\langle \bar\nabla^N_{e_k} e_\gamma, e_\alpha\rangle\right)\langle \mathbf{w}, e_j\rangle - h^\alpha_{ij}\langle \mathbf{w}, \bar\nabla_{e_k} e_j\rangle \\ &= h^\alpha_{ik,j}\langle \mathbf{w}, e_j\rangle - h^\alpha_{ij}h^\beta_{kj}\langle \mathbf{w}, e_\beta\rangle. \end{aligned} \tag{9.4}$$

Here, we have used (6.23) and Codazzi equation. Taking trace of (9.4) and using (9.2), we obtain

$$\Delta f^\alpha = H^\alpha_{,j}\langle \mathbf{w}, e_j\rangle - h^\alpha_{ij}h^\beta_{ij}\langle \mathbf{w}, e_\beta\rangle = \langle \mathbf{w}, \nabla H^\alpha\rangle - f^\beta h^\alpha_{ij}h^\beta_{ij}. \tag{9.5}$$

By (6.21) and (9.3), we have

$$\langle \mathbf{w}, \nabla H^\alpha\rangle = H^\alpha_{,i}\langle \mathbf{w}, e_i\rangle = \frac{1}{2}h^\alpha_{ij}\langle \mathbf{x}, e_j\rangle\langle \mathbf{w}, e_i\rangle = \frac{1}{2}f^\alpha_{,j}\langle \mathbf{x}, e_j\rangle = \frac{1}{2}\langle \mathbf{x}, \nabla f^\alpha\rangle. \tag{9.6}$$

Putting (9.6) into (9.5), we obtain

$$\Delta f^\alpha + f^\beta h^\alpha_{ij}h^\beta_{ij} - \frac{1}{2}\langle \mathbf{x}, \nabla f^\alpha\rangle + \frac{1}{2}f^\alpha = \frac{1}{2}f^\alpha. \tag{9.7}$$

By definition of the operator L, this is equivalent to (9.1). □

The following result needs "flat normal bundle" assumption on the self-shrinker.

Lemma 9.2 *If we extend the operator L to tensors, then on a self-shrinker Σ^n in $\boldsymbol{R}^{n+k}$ with flat normal bundle, we have*

$$LA = A. \tag{9.8}$$

Proof We will show that

$$(L\mathbf{A})^\alpha_{ij} = h^\alpha_{ij}. \tag{9.9}$$

In general, we have the following Simons' equality for the second fundamental form [12,26]:

$$\Delta h^\alpha_{ij} = \nabla_i\nabla_j H^\alpha + H^\beta h^\beta_{il}h^\alpha_{lj} - h^\beta_{ik}h^\beta_{kl}h^\alpha_{lj} + 2h^\beta_{ik}h^\alpha_{kl}h^\beta_{lj} - h^\beta_{ij}h^\beta_{kl}h^\alpha_{kl} - h^\alpha_{ik}h^\beta_{kl}h^\beta_{lj}. \tag{9.10}$$

Combining with (9.1), we have

$$
\begin{aligned}
(L\mathbf{A})^{\alpha}_{ij} &= \Delta h^{\alpha}_{ij} + h^{\beta}_{ij} h^{\beta}_{kl} h^{\alpha}_{kl} - \frac{1}{2}\langle \mathbf{x}, \nabla h^{\alpha}_{ij}\rangle + \frac{1}{2} h^{\alpha}_{ij} \\
&= \frac{1}{2}\left(h^{\alpha}_{ij,k}\langle \mathbf{x}, e_k\rangle + h^{\alpha}_{ij} - h^{\alpha}_{jk} h^{\beta}_{ki}\langle \mathbf{x}, e_\beta\rangle \right) + H^{\beta} h^{\beta}_{il} h^{\alpha}_{lj} \\
&\quad - h^{\beta}_{ik} h^{\beta}_{kl} h^{\alpha}_{lj} \ + 2 h^{\beta}_{ik} h^{\alpha}_{kl} h^{\beta}_{lj} - h^{\beta}_{ij} h^{\beta}_{kl} h^{\alpha}_{kl} - h^{\alpha}_{ik} h^{\beta}_{kl} h^{\beta}_{lj} \\
&\quad + h^{\beta}_{ij} h^{\beta}_{kl} h^{\alpha}_{kl} - \frac{1}{2}\langle \mathbf{x}, \nabla h^{\alpha}_{ij}\rangle + \frac{1}{2} h^{\alpha}_{ij} \\
&= h^{\alpha}_{ij} + 2 h^{\beta}_{ik} h^{\alpha}_{kl} h^{\beta}_{lj} - h^{\beta}_{ik} h^{\beta}_{kl} h^{\alpha}_{lj} - h^{\alpha}_{ik} h^{\beta}_{kl} h^{\beta}_{lj}.
\end{aligned}
\tag{9.11}
$$

By Ricci equation,

$$
\begin{aligned}
2 h^{\beta}_{ik} h^{\alpha}_{kl} h^{\beta}_{lj} - h^{\beta}_{ik} h^{\beta}_{kl} h^{\alpha}_{lj} - h^{\alpha}_{ik} h^{\beta}_{kl} h^{\beta}_{lj} &= h^{\beta}_{ik}(h^{\alpha}_{kl} h^{\beta}_{lj} - h^{\beta}_{kl} h^{\alpha}_{lj}) + (h^{\beta}_{ik} h^{\alpha}_{kl} - h^{\alpha}_{ik} h^{\beta}_{kl}) h^{\beta}_{lj} \\
&= h^{\beta}_{ik} R_{\alpha\beta kj} + R_{\beta\alpha il} h^{\beta}_{lj} = 0.
\end{aligned}
$$

The last equality follows from our assumption that the normal curvature is zero. Thus we obtain (9.9) from (9.11), This proves the lemma. □

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