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Prime representations from a homological perspective

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Abstract We explore the relation between self extensions of simple representations of quantum affine algebras and the property of a simple representation being prime. We show that every nontrivial simple representation has a nontrivial self extension. Conversely, we prove that if a simple representation has a unique nontrivial self extension up to isomorphism, then its Drinfeld polynomial is a power of the Drinfeld polynomial of a prime representation. It turns out that, in the $\mathfrak{sl}_2$-case, a simple module is prime if and only if it has a unique nontrivial self extension up to isomorphism. It is tempting to conjecture that this is true in general and we present a large class of prime representations satisfying this homological property.

Keywords Quantum affine algebras · Extentions · Prime

Mathematics Subject Classification 17B67 · 81R50

1 Introduction

The study of finite-dimensional representations of quantum affine algebras has been an active field of research for at least two decades. The abstract classification of the simple representations was given in [12,13], and much of the subsequent work has focused on

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understanding the structure of these representations. This has proved to be a difficult task and a complete understanding outside the case of $\mathfrak{sl}_2$ is still some distance away. A number of important methods have been developed: for instance, the work of [22,23] on q-characters has resulted in a deeper combinatorial understanding of these representations. The geometric approach of H. Nakajima and the theory of crystal bases of M. Kashiwara have also been very fruitful. Another powerful tool is the T-system [24,31,38], which was recently shown [35] to extend beyond Kirillov–Reshetikhin modules to wider classes of representations. A connection with the theory of cluster algebras has been established recently in [26,39].

The study of the structure of the irreducible representations can be reduced to the so called prime ones, namely those simple representations which cannot be written as a tensor product of two non-trivial simple representations. Clearly any finite-dimensional simple representation can be written as a tensor product of simple prime representations and one could then focus on understanding the prime representations. This was the approach used in [12] for the $\mathfrak{sl}_2$-case, but generalizing this approach is very difficult. However, many examples of prime representations are known in general, for example the Kirillov–Reshethikhin modules are prime and, more generally, the minimal affinizations are also prime and other examples may be found for instance in [26]. However, except in the $\mathfrak{sl}_2$-case where the simple prime representations are precisely the Kirillov–Reshetikhin modules (which are also the evaluation modules), the classification of the prime representations is not known.

This paper is motivated by an effort to understand the simple prime representations via homological properties. Thus, let $\hat{\mathcal{F}}$ be the category of finite-dimensional representations of the quantum affine algebra and denote by $V(\boldsymbol{\pi})$ the irreducible representation associated to the Drinfeld polynomial $\boldsymbol{\pi}$. We construct in a natural way a non-trivial self-extension of any object V of $\hat{\mathcal{F}}$ which motivates the natural question of characterizing the simple objects which satisfy

$$\dim \operatorname{Ext}^1_{\hat{\mathcal{F}}}(V, V) = 1. \tag{1.1}$$

Our first result shows that any simple V satisfying (1.1) is of the form $V(\boldsymbol{\pi}_0^s)$ for some $s \geq 1$ where $\boldsymbol{\pi}_0$ is such that $V(\boldsymbol{\pi}_0)$ is prime. Hence, if $V(\boldsymbol{\pi}_0)$ is a real prime in the sense of [26], then using [25] we see that V is a tensor power of $V(\boldsymbol{\pi}_0)$.

In the case of $\mathfrak{sl}_2$ we prove the stronger result that a simple object V satisfies (1.1) if and only if V is prime. It is natural and now obviously interesting to ask if such a result remains true for general $\mathfrak{g}$. Our next result provides partial evidence for this to be true. Namely, we prove for a large family of simple prime representations including the minimal affinizations that the space of self-extensions is one-dimensional. Our results go beyond minimal affinizations and we prove that the representations $S(\beta)$ defined in [26] have a one-dimensional space of extensions as long as β is a positive root in which every simple root occurs with multiplicity one. It is worth comparing the results of this paper with their non-quantum counter parts. One can define in a similar way the notion of prime representations for the category of finite-dimensional representations of an affine Lie algebra $\hat{\mathfrak{g}}$. It is known through the work of [9,29] that if V, V' are irreducible finite-dimensional representations of $\hat{\mathfrak{g}}$, then

$$\operatorname{Ext}^1_{\hat{\mathfrak{g}}}(V, V') \cong \operatorname{Hom}_{\mathfrak{g}}(\mathfrak{g} \otimes V, V').$$

It is now easily seen that there exist examples of simple prime representations V such that $\operatorname{Ext}^1_{\hat{\mathfrak{g}}}(V, V)$ has dimension at least two. In Sect. 3, we give an example of a simple representation of the quantum affine algebra which has a one-dimensional space of self-extensions but whose classical limit, although also prime and simple, has a two dimensional space of self-extensions.

The paper is organized as follows. In Sect. 2 we recall the definition and some standard results on quantum affine algebras. In Sect. 3 we review some results on finite-dimensional representations of the quantum affine algebra and state the main results of the paper. In Sect. 4 we construct a self extension of any given module of the quantum affine algebra. We prove that if the module is simple and finite-dimensional then the extension is nontrivial. We also give the condition (mentioned earlier in the introduction) on the Drinfeld polynomials of a simple representation which satisfy (1.1).

In Sect. 5 we first review results on local and global Weyl modules. We then study the relationship between these modules and self extensions of simple representations. In particular, we compute the dimension of the space of self extensions of the local Weyl modules. In Sect. 6 we prove that the space of self extensions of a simple module is a subspace of the self-extensions of the corresponding Weyl module. This allows us to prove that a self-extension of a simple module is "determined" by its top weight space and this plays a crucial role in proving the remaining results of the paper. It allows us to study the relationship between self extensions and tensor products of simple modules which in turn establishes the condition for a simple representation to have a one-dimensional space of self-extensions. In Sect. 7 we prove that, in the $\mathfrak{sl}_2$ case, a simple module V satisfies (1.1) if and only if V is prime. The last section is dedicated to the proof that a certain class of simple modules satisfy (1.1). Our results show that this implies that these modules are prime. The latter fact that the modules are prime can also be proved by other methods as well, see [26] for the modules of type $S(\beta)$ and [10] for remarks on minimal affinizations. Our goal here is really to provide evidence towards the conjecture for general $\mathfrak{g}$, that V satisfies (1.1) iff V is prime.

2 Preliminaries

Throughout the paper $\mathbf{C}$ (resp. $\mathbf{Z}$, $\mathbf{Z}_+$) denotes the set of complex numbers (resp. integers, non-negative integers) and $\mathbf{C}^\times$ (resp. $\mathbf{Z}^\times$) is the set of non-zero complex numbers (resp. non-zero integers).

2.1

Let $I = \{1, \ldots, n\}$ be the index set for the set of simple roots $\{\alpha_i : i \in I\}$ of an irreducible reduced root system R in a real vector space. Let R^+ be the corresponding set of positive roots. Fix a set of fundamental weights $\{\omega_i : i \in I\}$ and let Q, P be the associated root and weight lattice respectively and recall that $Q \subset P$. If R^+ is the set of positive roots then we let Q^+ be the $\mathbf{Z}_+$-span of R^+ and P^+ the $\mathbf{Z}_+$-span of the fundamental weights.

We assume that the nodes of the Dynkin diagram are numbered as in [5] and we follow the conventions of that labeling. Let $\hat{I} = I \cup \{0\}$ be the nodes of the corresponding (untwisted) extended Dynkin diagram and denote by $A = (a_{i,j})_{i,j \in I}$ (resp. $\hat{A} = (a_{i,j})_{i,j \in \hat{I}}$) the associated Cartan (resp. untwisted affine Cartan) matrix. Finally, fix non-negative integers $\{d_i : i \in \hat{I}\}$ such that the matrix $(d_i a_{i,j})_{i,j \in \hat{I}}$ is symmetric.

From now on, we fix $q \in \mathbf{C}^\times$ and assume that q is not a root of unity. For $m \in \mathbf{Z}$, $\ell, r \in \mathbf{Z}_+$ and $i \in \hat{I}$, set $q_i = q^{d_i}$ and define,

$$[m]_i = \frac{q_i^m - q_i^{-m}}{q_i - q_i^{-1}} \qquad [0]_i! = 1, \qquad [\ell]_i! = [\ell]_i[\ell-1]_i \ldots [1]_i,$$

$$\begin{bmatrix} \ell \\ r \end{bmatrix}_i = \frac{[\ell]_i!}{[\ell-r]_i![r]_i!}.$$

2.2

Let $\hat{\mathbf{U}}_q$ (resp. $\mathbf{U}_q$) be the associative algebra over $\mathbf{C}$ with generators $x_i^{\pm}, k_i^{\pm 1}, i \in \hat{I}$, (resp. $x_i^{\pm}, k_i^{\pm 1}, i \in I$) satisfying the following defining relations: for $i, j \in \hat{I}$ (resp. $i, j \in I$), we have

$$k_i k_i^{-1} = 1, \quad k_i k_j = k_j k_i,$$

$$k_i x_j^{\pm} k_i^{-1} = q_i^{\pm a_{i,j}} x_j^{\pm},$$

$$\left[x_i^+, x_j^- \right] = \delta_{ij} \frac{k_i - k_i^{-1}}{q_i - q_i^{-1}}$$

$$\sum_{m=0}^{1-a_{ij}} (-1)^m \begin{bmatrix} 1 - a_{ij} \\ m \end{bmatrix}_i (x_i^{\pm})^{1-a_{i,j}-m} x_j^{\pm} (x_i^{\pm})^m = 0, \quad i \neq j.$$

It is well-known that $\hat{\mathbf{U}}_q$ and $\mathbf{U}_q$ are Hopf algebras with counit, comultiplication, and antipode given as follows: for $i, j \in \hat{I}$ (resp. $i, j \in I$),

$$\epsilon(k_i) = 1, \qquad \epsilon(x_i^{\pm}) = 0,$$

$$\Delta(k_i) = k_i \otimes k_i,$$

$$\Delta(x_i^+) = x_i^+ \otimes 1 + k_i \otimes x_i^+, \qquad \Delta(x_i^-) = x_i^- \otimes k_i^{-1} + 1 \otimes x_i^-,$$

$$S(k_i) = k_i^{-1}, \qquad S(x_i^+) = -k_i^{-1} x_i^+, \quad S(x_i^-) = -x_i^- k_i.$$

It is known that $\mathbf{U}_q$ can be canonically identified with the subalgebra of $\hat{\mathbf{U}}_q$ generated by the elements $x_i^{\pm}, k_i, i \in I$ and that $\mathbf{U}_q$ is a Hopf subalgebra of $\hat{\mathbf{U}}_q$. The algebra $\hat{\mathbf{U}}_q$ is naturally $\mathbf{Z}$-graded by requiring

$$\mathrm{gr} x_i^{\pm} = 0, \quad i \in I, \qquad \mathrm{gr} x_0^{\pm} = \pm 1.$$

2.3

The quantum loop algebra is the quotient of $\hat{\mathbf{U}}_q$ by two sided ideal generated by

$$k_0 \prod_{i \in I} k_i^{r_i} - 1,$$

where $r_i \in \mathbf{Z}_+$ is the coefficient of α_i in the highest root of R^+. This ideal both $\mathbf{Z}$-graded and a Hopf ideal and hence the quantum loop algebra also acquires a $\mathbf{Z}$-grading and the structure of a Hopf algebra.

From now on, we shall only be concerned with the quantum loop algebra and hence we shall by abuse of notation write $\hat{\mathbf{U}}_q$ for the quantum loop algebra.

The algebra $\hat{\mathbf{U}}_q$ has an alternative presentation given in [2,20]. It is the algebra with generators $x_{i,r}^{\pm}, h_{i,s}, k_i^{\pm 1}$, where $i \in I, r \in \mathbf{Z}, s \in \mathbf{Z}^{\times}$, and defining relations: for $i, j \in I, r, \ell \in \mathbf{Z}, s \in \mathbf{Z}^{\times}$,

$$k_i k_i^{-1} = 1, \quad k_i k_j = k_j k_i, \quad k_i h_{j,s} = h_{j,s} k_i, \quad h_{i,r} h_{j,s} = h_{j,s} h_{i,s},$$

$$\left[h_{i,s}, x_{j,r}^{\pm} \right] = \pm \frac{1}{s} \left[s a_{i,j} \right]_{q_i} x_{j,r+s}^{\pm}, \qquad k_i x_{j,r}^{\pm} k_i^{-1} = q_i^{\pm a_{i,j}} x_{j,r}^{\pm},$$

$$x_{i,r}^{\pm} x_{j,\ell}^{\pm} - q_i^{\pm a_{i,j}} x_{j,\ell}^{\pm} x_{i,r}^{\pm} = q_i^{\pm a_{i,j}} x_{i,r-1}^{\pm} x_{j,\ell+1}^{\pm} - x_{j,\ell+1}^{\pm} x_{i,r-1}^{\pm},$$

$$\left[x_{i,r}^+, x_{j,s}^-\right] = \delta_{ij} \frac{\phi_{i,r+s}^+ - \phi_{i,r+s}^-}{q_i - q_i^{-1}},$$

where $\phi_{i,\mp m}^\pm = 0$, $m > 0$, and the elements $\phi_{i,\pm m}^\pm$, $m \geq 0$, are defined by the following equality of power series in u:

$$\sum_{m \geq 0} \phi_{i,\pm m}^\pm u^m = k_i^{\pm 1} \exp\left(\pm(q_i - q_i^{-1}) \sum_{s>0} h_{i,\pm s} u^s\right).$$

Finally, for $i \neq j$ and given $r_m \in \mathbf{Z}$ for $1 \leq m \leq 1 - a_{i,j}$, we have

$$\sum_{\sigma \in S_{1-a_{i,j}}} \sum_{m=0}^{1-a_{i,j}} (-1)^m \begin{bmatrix} 1 - a_{i,j} \\ m \end{bmatrix}_i x_{i,r_{\sigma(1)}}^\pm \cdots x_{i,r_{\sigma(m)}}^\pm x_{j,s}^\pm x_{i,r_{\sigma(m+1)}}^\pm \cdots x_{i,r_{\sigma(1-a_{i,j})}}^\pm = 0,$$

where for any $k \in \mathbf{Z}_+$ we denote by S_k the symmetric group on k letters.

The $\mathbf{Z}$-grading on $\hat{\mathbf{U}}_q$ is the same as the one given by setting:

$$\mathrm{gr}x_{i,r}^\pm = r, \quad \mathrm{gr}h_{i,s} = s, \quad \mathrm{gr}\phi_{i,\pm m}^\pm = \pm m,$$

where $i \in I$, $r, s \in \mathbf{Z}$, $s \neq 0$ and $m \in \mathbf{Z}_+$.

2.4

Let $\hat{\mathbf{U}}_q^\pm$ be the subalgebra of $\hat{\mathbf{U}}_q$ generated by the elements $\{x_{i,r}^\pm : i \in I, r \in \mathbf{Z}\}$, $\hat{\mathbf{U}}_q(0)$ the subalgebra generated by $\{k_i^{\pm 1}, h_{i,s} : i \in I, s \in \mathbf{Z}^\times\}$, and $\hat{\mathbf{U}}_q^0$ the subalgebra of $\hat{\mathbf{U}}_q(0)$ generated by $\{h_{i,s} : i \in I, s \in \mathbf{Z}^\times\}$. Then we have an isomorphism of vector spaces,

$$\hat{\mathbf{U}}_q = \hat{\mathbf{U}}_q^- \hat{\mathbf{U}}_q(0)\hat{\mathbf{U}}_q^+. \tag{2.1}$$

The algebra $\hat{\mathbf{U}}_q(0)$ (resp. $\hat{\mathbf{U}}_q^0$) is also generated by the elements $\phi_{i,m}^\pm$, $i \in I$, $m \in \mathbf{Z}$ (resp. $m \in \mathbf{Z}^\times$). We shall also need a third set of generators $\Lambda_{i,r}$, $i \in I$, $r \in \mathbf{Z}$ for $\hat{\mathbf{U}}_q^0$. These were defined in [12] and are given by the following equality of power series,

$$\sum_{r=0}^\infty \Lambda_{i,\pm r} u^r = \exp\left(-\sum_{s=1}^\infty \frac{h_{i,\pm s}}{[s]_i} u^s\right). \tag{2.2}$$

In particular, $\Lambda_{i,0} = 1$ for all $i \in I$ and $\mathrm{gr}\Lambda_{i,r} = r$. We conclude the section with the following result established in [2,3].

Lemma *The algebra $\hat{\mathbf{U}}_q^0$ is the polynomial algebra in the variables $\{h_{i,r} : i \in I, r \in \mathbf{Z}^\times\}$. Analogous statements hold for the generators $\{\phi_{i,m}^\pm : i \in I, m \in \mathbf{Z}^\times\}$ and $\{\Lambda_{i,r} : i \in I, r \in \mathbf{Z}^\times\}$.* □

3 The main results

3.1

Let $\mathfrak{g}$ be a simple Lie algebra with root system R^+. Let $\mathfrak{h}$ be the Cartan subalgebra and $\mathbf{U}(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$. In this case we can regard Q and P as lattices in the vector space dual of $\mathfrak{h}$. Any finite-dimensional representation of $\mathfrak{g}$ is semi-simple. i.e.,

is isomorphic to a direct sum of irreducible representations. Further, the irreducible finite-dimensional representations are indexed by P^+. For $\mu \in P^+$, let $\overline{V}(\mu)$ be an irreducible module associated to μ. Then

$$\overline{V}(\mu) = \bigoplus_{\nu \in P} \overline{V}(\mu)_\nu, \quad \overline{V}(\mu)_\nu = \{v \in \overline{V}(\mu) : hv = \nu(h)v, \ h \in \mathfrak{h}\}.$$

3.2

A representation V of $\hat{\mathbf{U}}_q$ is said to be of type 1 if

$$V = \bigoplus_{\mu \in P} V_\mu, \qquad V_\mu = \{v \in V : k_i v = q_i^{\mu_i} v \text{ for all } i \in \hat{I}\}. \tag{3.1}$$

where we write $\mu = \sum_{i \in I} \mu_i \omega_i$. Set $\text{wt}(V) = \{\mu \in P : V_\mu \neq 0\}$. Let $\hat{\mathcal{F}}$ be the category of type 1 finite-dimensional representations of $\hat{\mathbf{U}}_q$. Specifically, the objects of $\hat{\mathcal{F}}$ are type 1 representations of $\hat{\mathbf{U}}_q$ and the morphisms in the category are just $\hat{\mathbf{U}}_q$-module maps. The category $\mathcal{F}$ of finite-dimensional representations of $\mathbf{U}_q$ is defined similarly. Any $\hat{\mathbf{U}}_q$-module can be regarded by restriction as a module for $\mathbf{U}_q$ and we shall use this fact repeatedly without mention. Since $\hat{\mathbf{U}}_q$ and $\mathbf{U}_q$ are Hopf algebras the categories $\hat{\mathcal{F}}$ and $\mathcal{F}$ contain the trivial one-dimensional representation and are closed under taking tensor products and duals.

Definition We shall say that $V \in \text{Ob}\hat{\mathcal{F}}$ is prime if either V is trivial or if there does not exist nontrivial $V_j \in \text{Ob}\hat{\mathcal{F}}$, $j = 1, 2$, with $V \cong V_1 \otimes V_2$. □

Clearly any $V \in \text{Ob}\hat{\mathcal{F}}$ can be written as a tensor product of prime representations. Following [26] we shall say that a simple object V in $\hat{\mathcal{F}}$ is a real prime if V is prime and $V^{\otimes 2}$ is irreducible. It is known through the work of [25] that for a real prime V the object $V^{\otimes r}$ is irreducible for all $r \geq 1$. One can define prime objects in $\mathcal{F}$ in a similar way but this is not interesting as we shall now see.

3.3

The following was proved in [33].

Proposition *The category $\mathcal{F}$ is semisimple. Given any $\mu \in P^+$, the $\mathbf{U}_q$-module $V(\mu)$ generated by an element v_μ with relations:*

$$x_i^+ v_\mu = 0, \ \ k_i v_\mu = q_i^{\mu_i} v_\mu, \ \ (x_i^-)^{\mu_i+1} v_\mu = 0,$$

is a simple object of $\mathcal{F}$. Further, any simple object in $\mathcal{F}$ is isomorphic to $V(\mu)$ for some $\mu \in P^+$ and

$$\dim V(\mu)_\nu = \dim \overline{V}(\mu)_\nu, \ \ \nu \in P.$$

Moreover, given $\lambda, \mu, \nu \in P^+$, we have

$$\dim Hom_{\mathfrak{g}}(\overline{V}(\mu) \otimes \overline{V}(\lambda), \ \overline{V}(\nu)) = \dim Hom_{\mathcal{F}}(V(\mu) \otimes V(\lambda), V(\nu)).$$

□

The following is now a consequence of the corresponding result for simple Lie algebras.

Corollary *The representations $V(\mu)$, $\mu \in P^+$ are prime.* □

3.4

Given any type 1 module V for $\hat{\mathbf{U}}_q$, set

$$V^+ = \{v \in V : x_{i,r}^+ v = 0 \ i \in I, r \in \mathbf{Z}\}, \qquad V_\lambda^+ = V^+ \cap V_\lambda.$$

If $v \in V_\lambda^+$, then it follows from (2.1) that wt $\hat{\mathbf{U}}_q v \subset \lambda - Q^+$. An element $v \in V$ is said to be an ℓ-weight vector if it is a joint eigenvector for the $h_{i,r}$, $i \in I$, $r \in \mathbf{Z}^\times$. An ℓ-weight vector contained in V_λ^+ is called a highest-ℓ-weight vector. Notice that this is equivalent to requiring it to be a joint eigenvector for the $\phi_{i,m}^\pm$, $i \in I$, $m \in \mathbf{Z}$. A $\hat{\mathbf{U}}_q$-module V is said to be highest-ℓ-weight if it is generated by a highest-ℓ-weight vector. The following is well-known and easily proved.

Lemma *Let V be a highest-ℓ-weight $\hat{\mathbf{U}}_q$-module. Then V has a unique irreducible quotient. Any simple object in $\hat{\mathcal{F}}$ is highest-ℓ-weight and in fact the space of highest-ℓ-weight vectors is one-dimensional.* □

If $V \in \mathrm{Ob}\hat{\mathcal{F}}$ is a highest-ℓ-weight module generated by a highest-ℓ-weight element v then there are constraints imposed on the eigenvalues of $h_{i,r}$ on v. We now explain these constraints

3.5

Let u be an indeterminate, $\mathbf{C}[u]$ the algebra of polynomials in u with coefficients in $\mathbf{C}$ and $\mathbf{C}(u)$ the field of quotients. Let $\mathcal{P}^+$ be the the multiplicative monoid consisting of all I-tuples of the form $\boldsymbol{\pi} = (\pi_i)_{i \in I}$ where π_i is a polynomial in $\mathbf{C}[u]$ with constant term 1. The I-tuple consisting of the constant polynomial 1 is called the trivial element of $\mathcal{P}^+$.

The following was proved in [12,13].

Proposition *Suppose that $V \in \mathrm{Ob}\hat{\mathcal{F}}$ is highest-ℓ-weight with generator $v \in V_\lambda^+$ and assume that $\phi_{i,m}^\pm v = d_{i,m} v$, $i \in I$, $m \in \mathbf{Z}$. There exists an element $\boldsymbol{\pi} \in \mathcal{P}^+$ such that,*

$$\sum_{m \geq 0} d_{i,m}^+ u^m = q^{\pm \deg \pi_i} \frac{\pi_i(q_i^{-1}u)}{\pi_i(q_i u)} = \sum_{m \geq 0} d_{i,-m}^- u^{-m}, \qquad \lambda_i = \deg \pi_i, \qquad (3.2)$$

in the sense that the left- and right-hand terms are the Laurent expansions of the middle term about 0 and ∞, respectively. Equivalently, one has an equality of power series,

$$\pi_i^\pm(u) = \exp\left(-\sum_{s=1}^\infty \frac{h_{i,\pm s}(\boldsymbol{\pi})}{[s]_{q_i}} u^s\right) = \sum_{r \geq 0} \Lambda_{i,\pm r}(\boldsymbol{\pi}) u^r, \qquad (3.3)$$

where $\pi_i^+(u) = \pi_i(u)$ and $\pi_i^-(u) = u^{\deg \pi_i} \pi_i(u^{-1}) / (u^{\deg \pi_i} \pi_i(u^{-1})) (0)$.

Conversely, given $\boldsymbol{\pi} \in \mathcal{P}^+$ there exists a unique (up to isomorphism) irreducible highest-ℓ-weight object $V(\boldsymbol{\pi}) \in \mathrm{Ob}\hat{\mathcal{F}}$ which is generated by a highest-ℓ-weight vector $v(\boldsymbol{\pi})$ with ℓ-weight given by (3.2). □

We remark that the trivial representation corresponds to taking the trivial n-tuple.

From now on, we shall use the convention that given $\boldsymbol{\varpi} \in \mathcal{P}^+$, the eigenvalue of $\phi_{i,m}^\pm$ on an ℓ-weight vector with ℓ-weight $\boldsymbol{\varpi}$ is denoted $\phi_{i,m}^\pm(\boldsymbol{\varpi})$, and $h_{i,r}(\boldsymbol{\varpi})$ and $\Lambda_{i,r}(\boldsymbol{\varpi})$ are defined similarly.

3.6

The category $\hat{\mathcal{F}}$ unlike $\mathcal{F}$ is not semisimple and we shall be interested in understanding extensions in the category. Our focus in this paper is the space of self extensions of the simple objects of $\hat{\mathcal{F}}$. The trivial object $\mathbf{C}$ satisfies $\mathrm{Ext}^1_{\hat{\mathcal{F}}}(\mathbf{C}, \mathbf{C}) = 0$. Since any $V \in \mathrm{Ob}\hat{\mathcal{F}}$ has a Jordan–Holder series, it follows that if all the Jordan–Holder factors of $V \in \mathrm{Ob}\hat{\mathcal{F}}$ are trivial, then V is isomorphic to a direct sum of copies of the trivial representation. Our first main result is:

Theorem 1 *Let $\pi \in \mathcal{P}^+$ be non-trivial. We have,*

(i) $\dim \mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) \geq 1$.

(ii) *Suppose that* $\dim \mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) = 1$. *Then there exists $\pi_0 \in \mathcal{P}^+$ and $s \in \mathbf{Z}_+$ such that $V(\pi_0)$ is prime and $\pi = \pi_0^s$. In addition, there exists a partition $s_1 \geq \cdots \geq s_k > 0$ of s such that $V(\pi_0^{s_r})$ is prime for all $1 \leq r \leq k$ and*

$$V(\pi) \cong V(\pi_0^{s_1}) \otimes \cdots \otimes V(\pi_0^{s_k}).$$

In particular if $V(\pi_0)$ is a real prime, this is equivalent to saying that $V(\pi)$ is a tensor power of $V(\pi_0)$.

(iii) *Let $\pi_1, \pi_2 \in \mathcal{P}^+$ and assume that $V(\pi_1) \otimes V(\pi_2)$ is irreducible. We have*

$$\dim \mathrm{Ext}_{\hat{\mathcal{F}}}(V(\pi_1) \otimes V(\pi_2), \ V(\pi_1) \otimes V(\pi_2)) \geq \dim \mathrm{Ext}_{\hat{\mathcal{F}}}(V(\pi_1), \ V(\pi_1)).$$

(iv) *If R is a root system of type A_1 then $\dim \mathrm{Ext}_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) = 1$ iff $V(\pi)$ is prime.*

Let $\hat{\mathcal{F}}_1$ be the full subcategory of $\hat{\mathcal{F}}$ consisting of objects V satisfying the following: if $V(\pi)$ is a nontrivial Jordan–Holder factor of V then $\dim \mathrm{Ext}_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) = 1$. The category $\hat{\mathcal{F}}_1$ is not closed under taking tensor products. However, we do have the following:

Corollary *If $V(\pi) \in \mathrm{Ob}\hat{\mathcal{F}}_1$ then all the prime factors of $V(\pi)$ are also in $\hat{\mathcal{F}}_1$.*

Remark There is an important family of objects of $\hat{\mathcal{F}}$ known as local Weyl modules or standard modules which are known to be generically irreducible. The reader is referred to Sect. 5 of this paper where we determine the dimension of the space of self extensions of these modules. In particular, if W is a local Weyl module then $\dim \mathrm{Ext}_{\hat{\mathcal{F}}}(W, W) = 1$ if and only if W is prime.

3.7

It is tempting to conjecture that part (iv) of Theorem 1 is true in general. The proof of (iv) uses very special properties of the category $\hat{\mathcal{F}}$ in the case of A_1 and these properties are known to be false in general. However, there are various well known families of representations of quantum affine algebras which have many nice properties (such as the Kirillov–Reshethikhin modules and minimal affinizations) which are either known to be or easily proved to be prime (see [10] for a discussion). The next main result of this paper shows that for many of these families it is true that the space of self extensions is one-dimensional.

3.8

For $\pi \in \mathcal{P}^+$ set

$$\mathrm{supp}\, \pi = \{i \in I : \pi_i \neq 1\}.$$

Given $i, j \in I$, let $[i, j]$ be the minimal connected subset of I containing i and j and let $(i, j) = [i, j] \setminus \{i, j\}$. We shall prove,

Theorem 2 *Let $\pi \in \mathcal{P}^+$ and assume that,*

(i) *for $i \in \operatorname{supp} \pi$ there exists $a_i \in \mathbf{C}^\times$ such that*

$$\pi_i = (1 - a_i u q_i^{m_i-1})(1 - a_i u q_i^{m_i-3}) \cdots (1 - a_i u q_i^{-m_i+1}), \quad \deg \pi_i = m_i,$$

(ii) *for $i, j \in \operatorname{supp} \pi$ with $(i, j) \cap \operatorname{supp} \pi = \emptyset$, we have either*

$$a_i = a_j q^{r_{i,j}},$$

or, there exists $k \in \operatorname{supp} \pi$ with $(j, k) \cap \operatorname{supp} \pi = \emptyset = (i, k) \cap \operatorname{supp} \pi$ and

$$a_j = a_k q^{r_{j,k}}, \quad a_i = a_k q^{r_{i,k}},$$

where $r_{i,j} = \pm \left(d_i m_i - \left(\frac{1}{2} \sum_{k \neq \ell \in [i,j]} d_k a_{k,\ell} \right) + d_j m_j \right)$.

Then

$$\dim \operatorname{Ext}_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) = 1.$$

Remark Observe that the second assumption in (ii) is only possible when R is of type D or E. Notice that it follows from Theorem 1 that the simple representations satisfying the conditions of Theorem 2 are prime.

3.9

We now make several remarks to explain the assumptions on π. The notion of minimal affinizations was introduced in [6] and studied further in [14–16,34]. The first condition on π requires that each component define a minimal affinization for the quantum affine algebra associated to A_1. The second condition requires that if we restrict our attention to the connected subset I_s of I whose intersection with $\operatorname{supp} \pi$ is $\{i_s, i_{s+1}\}$, then the I_s-tuple $(\pi_{i_s}, 1, \ldots, 1, \pi_{i_{s+1}})$ defines a minimal affinization for the quantum affine algebra associated to I_s. For ease of exposition, we are being a bit careless here in the case of the root system of type D_n but this is taken care of later in the paper.

3.10

As an example, if R^+ is of type A_3, then $\pi = (1 - u, 1 - q^3 u, 1 - u)$ and $\pi = (1 - u, 1 - q^3 u, 1 - q^6 u)$ both satisfy the conditions of Theorem 2. The latter polynomial defines a minimal affinization while the former does not.

3.11

In the special case when $\operatorname{supp} \pi = \{i_1\}$, the associated module is called a Kirillov–Reshetikhin module and our result shows that the space of self extensions of this module is one-dimensional.

3.12

We conclude this section by comparing the statement of Theorem 2 with known results for the loop algebra of $L(\mathfrak{g})$. Let t be an indeterminate. Then,

$$L(\mathfrak{g}) = \mathfrak{g} \otimes \mathbf{C}[t, t^{-1}], \quad [x \otimes t^n, y \otimes t^m] = [x, y] \otimes t^{n+m}.$$

Let $\overline{\mathcal{F}}$ be the category of finite-dimensional representations of $L(\mathfrak{g})$. Then, the irreducible representations are again given by n-tuples of polynomials $\overline{\pi}$. The structure of the simple representations is easily described in this case and, as a consequence, one also understand the irreducible prime objects in this category. As an example, if we take $\mathfrak{g}$ to be $\mathfrak{sl}_3$, then one knows that

$$\overline{V}(1-u, 1-u) \cong_{\mathfrak{g}} \overline{V}(\omega_1 + \omega_2),$$

and hence $\overline{V}(1-u, 1-u)$ is prime. It follows from the work of [9] and [29] that

$$\dim \mathrm{Ext}_{\overline{\mathcal{F}}}(\overline{V}(1-u, 1-u), \overline{V}(1-u, 1-u)) = 2.$$

It can be shown that the classical limit (as $q \to 1$) of the representation $V(\pi)$, where $\pi = (1-u, 1-q^3 u)$, is the representation $\overline{V}(1-u, 1-u)$. Theorem 2 shows however that

$$\dim \mathrm{Ext}_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) = 1 \neq \dim \mathrm{Ext}_{\overline{\mathcal{F}}}(\overline{V}(1-u, 1-u), \overline{V}(1-u, 1-u)).$$

4 Proof of Theorem 1: parts (i) and (ii)

4.1

Let $\mathrm{wt} : \mathcal{P}^+ \to P^+$ be defined by

$$\mathrm{wt}\,\pi = \sum_{i \in I}(\deg \pi_i)\omega_i.$$

We shall say that $V \in \mathrm{Ob}\hat{\mathcal{F}}$ is a self extension of $V(\pi)$, $\pi \in \mathcal{P}^+$ if V has a Jordan–Holder series of length two with both constituents being isomorphic to $V(\pi)$, or equivalently, if we have a short exact sequence

$$0 \to V(\pi) \xrightarrow{\iota} V \xrightarrow{\tau} V(\pi) \to 0,$$

of objects of $\hat{\mathcal{F}}$. We say that V is a trivial self extension if $V \cong V(\pi) \oplus V(\pi)$ and non-trivial otherwise. Notice that

$$\dim V_{\mathrm{wt}\,\pi} = 2.$$

and that $\iota(v(\pi))$ is a highest-ℓ-weight vector in $V_{\mathrm{wt}\pi}$. Suppose that V contains another linearly independent vector v' which is also highest ℓ-weight. Then

$$\dim(\hat{\mathbf{U}}_q v')_{\mathrm{wt}\pi} = 1, \quad \text{and so} \quad \iota(v(\pi)) \notin \hat{\mathbf{U}}_q v'.$$

Hence $\iota(V(\pi)) \cap \hat{\mathbf{U}}_q v' = \{0\}$ which implies that $\hat{\mathbf{U}}_q v' \cong V(\pi)$ and that V is a trivial self extension. Summarizing, we have proved,

Lemma *Let $V \in \mathrm{Ob}\hat{\mathcal{F}}$ be a self extension of $V(\pi)$. Then V is nontrivial if and only if $V_{wt\pi}$ has a unique (up to scalars) highest-ℓ-weight vector. Moreover, if V is nontrivial and $v \in V_{wt\pi}$ is not an ℓ-weight vector, then $V = \hat{\mathbf{U}}_q v$.* $\qquad\square$

4.2

A self-extension V of $V(\pi)$ defines an element $[V]$ of $\mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\pi), V(\pi))$. Moreover by Lemma 4.1, we see that $[V] = 0$ iff V is the trivial self extension.

Lemma *Let V, V' be self extensions of $V(\pi)$ for some $\pi \in \mathcal{P}^+$. Then V and V' are isomorphic as $\hat{\mathbf{U}}_q$-modules iff there exists $c \in \mathbf{C}$ such that $[V] = c[V']$ as elements of $\mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\pi), V(\pi))$.*

Proof It is clear that if there exists $c \in \mathbf{C}$ with $[V] = c[V']$ as elements of $\mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\pi), V(\pi))$ then they must be isomorphic as $\hat{\mathbf{U}}_q$-modules. For the converse let $\eta : V \to V'$ be an isomorphism of $\hat{\mathbf{U}}_q$-modules. Lemma 4.1 shows that V is a trivial self extension of $V(\pi)$ iff V' is also the trivial self-extension and in that case $[V] = [V'] = 0$. So assume that they are both non-split i.e., the short exact sequences

$$0 \to V(\pi) \xrightarrow{\iota} V \xrightarrow{\tau} V(\pi) \to 0, \qquad 0 \to V(\pi) \xrightarrow{\iota'} V' \xrightarrow{\tau'} V(\pi) \to 0$$

are non-split. Then $\eta(\iota(v(\pi))) = a\iota'(v(\pi))$, for some $a \in \mathbf{C}^\times$. Since $V(\pi)$ is irreducible, it follows in fact that

$$\eta(\iota(v)) = a\iota'(v), \quad \text{for all } v \in V(\pi).$$

This means that the short exact sequence

$$0 \to V(\pi) \xrightarrow{\eta \circ \iota} V' \xrightarrow{\tau'} V(\pi) \to 0, \tag{4.1}$$

defines the same equivalence class as $a^{-1}[V']$. Next, choose $v \in V$ such that $\tau(v) = v(\pi)$. Since $V = \hat{\mathbf{U}}_q v$ and $\tau'(\eta(v)) = bv(\pi)$ for some $b \in \mathbf{C}^\times$ we get $\tau' \circ \eta = b\tau$, i.e., the sequence defines the same equivalence class as $b^{-1}[V]$. Therefore, $a[V] = b[V']$ as required. $\qquad \square$

4.3

Recall that $\hat{\mathbf{U}}_q$ is a $\mathbf{Z}$-graded algebra and for $r \in \mathbf{Z}$ let $\hat{\mathbf{U}}_q[r]$ be the r-th graded piece. Given any $V \in \mathrm{Ob}\hat{\mathcal{F}}$ let $\mathbf{E}(V) \in \mathrm{Ob}\hat{\mathcal{F}}$ be defined by requiring

$$\mathbf{E}(V) = V \oplus V,$$

as vector spaces and the action of $\hat{\mathbf{U}}_q$ given by extending linearly the assignment,

$$g_r(v, w) = (g_r v, rg_r v + g_r w), \qquad g_r \in \hat{\mathbf{U}}_q[r], \quad v, w \in V.$$

Clearly we have a short exact sequence of $\hat{\mathbf{U}}_q$-modules

$$0 \to V \xrightarrow{\iota} \mathbf{E}(V) \xrightarrow{\tau} V \to 0. \tag{4.2}$$

The following proves part (i) of Theorem 1.

Proposition *If $\pi \in \mathcal{P}^+$ is nontrivial, $\mathbf{E}(V(\pi))$ is a nontrivial self extension of $V(\pi)$.*

Proof By Lemma 4.1 it suffices to prove that $\mathbf{E}(V(\pi))_{\mathrm{wt}\pi}$ has a one-dimensional space of highest-ℓ-weight vectors. Clearly $\iota(v(\pi))$ is a highest-ℓ-weight vector, and for $c_1, c_2 \in \mathbf{C}$, we have,

$$\phi^{\pm}_{i,m}(c_1 v(\pi), c_2 v(\pi)) = (\phi^{\pm}_{i,m}(\pi)c_1 v(\pi), m\phi^{\pm}_{i,m}(\pi)c_1 v(\pi) + \phi^{\pm}_{i,m}(\pi)c_2 v(\pi)). \tag{4.3}$$

Choosing $i \in I$ and $m \in \mathbf{Z}$, $m > 0$ with $\phi_{i,m}(\pi) \neq 0$ we see that the right hand side of the preceding equation is a multiple of $(c_1 v_\pi, c_2 v_\pi)$ iff $c_1 = 0$ and the proposition is proved. $\quad \square$

4.4

To prove (ii) of Theorem 1, we need the following result which can be found in [12]. In the current formulation, it uses the formulae for the comultiplication given in [3,19].

Proposition *Let $V_1, V_2 \in Ob\hat{\mathcal{F}}$. Let v_1 and v_2 satisfy*

$$x_{i,r}^+ v_1 = 0 = x_{i,r}^+ v_2, \quad i \in I, \quad r \in \mathbf{Z}.$$

Then,

$$\Delta(x_{i,r}^+)(v_1 \otimes v_2) = 0,$$
$$\Delta(h_{i,s})(v_1 \otimes v_2) = (h_{i,s}v_1 \otimes v_2 + v_1 \otimes h_{i,s}v_2),$$

for all $i \in I$, $r \in \mathbf{Z}$ and $s \in \mathbf{Z}^\times$. □

Corollary *Let $\boldsymbol{\pi}_j \in \mathcal{P}^+$ for $j = 1, 2$ and let $v = v(\boldsymbol{\pi}_1) \otimes v(\boldsymbol{\pi}_2) \in V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2)$. Then, $\hat{\mathbf{U}}_q v$ is a highest-ℓ-weight module with highest ℓ-weight $\boldsymbol{\pi}_1 \boldsymbol{\pi}_2$. In particular, it has $V(\boldsymbol{\pi}_1 \boldsymbol{\pi}_2)$ as its unique irreducible quotient.* □

4.5

Lemma *Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2 \in \mathcal{P}^+$ be such that $V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2)$ is irreducible. If V is a (non-trivial) self extension of $V(\boldsymbol{\pi}_1)$, then $V \otimes V(\boldsymbol{\pi}_2)$ is a (non-trivial) self extension of $V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2)$.*

Proof It is clear that $V \otimes V(\boldsymbol{\pi}_2)$ is a self extension of $V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2)$. Let v_1, v_2 be a basis for $V_{\mathrm{wt}\boldsymbol{\pi}_1}$ and assume that v_2 is an ℓ-weight vector. By Proposition 4.4 we have

$$h_{i,r}(v_1 \otimes v(\boldsymbol{\pi}_2)) = h_{i,r}v_1 \otimes v(\boldsymbol{\pi}_2) + v_1 \otimes h_{i,r}(\boldsymbol{\pi}_2)v(\boldsymbol{\pi}_2).$$

Hence $v_1 \otimes v(\boldsymbol{\pi}_2)$ is an ℓ-weight vector only if $h_{i,r}v_1 \otimes v(\boldsymbol{\pi}_2)$ is a scalar multiple of $v_1 \otimes v(\boldsymbol{\pi}_2)$, which implies that $h_{i,r}v_1$ is a scalar multiple of v_1. This implies that $V_{\mathrm{wt}\boldsymbol{\pi}}$ has two linearly independent ℓ-weight vectors and hence V is trivial by Lemma 4.1. □

4.6

Proposition *Let $\boldsymbol{\pi}_1, \boldsymbol{\pi}_2$ be nontrivial elements of $\mathcal{P}^+$ and assume that $(\boldsymbol{\pi}_1)^{r_1} \neq (\boldsymbol{\pi}_2)^{r_2}$ for all $r_1, r_2 \in \mathbf{Z}_+$. Then $\mathbf{E}(V(\boldsymbol{\pi}_1)) \otimes V(\boldsymbol{\pi}_2)$ and $V(\boldsymbol{\pi}_1) \otimes \mathbf{E}(V(\boldsymbol{\pi}_2))$ are non-trivial and non-isomorphic self extensions of $V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2)$.*

Proof The fact that $V(\boldsymbol{\pi}_1) \otimes \mathbf{E}(V(\boldsymbol{\pi}_2))$ and $V(\boldsymbol{\pi}_1) \otimes \mathbf{E}(V(\boldsymbol{\pi}_2)))$ are non-trivial self extensions of $V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2)$ was proved in Lemma 4.5. Suppose that $\eta : \mathbf{E}(V(\boldsymbol{\pi}_1)) \otimes V(\boldsymbol{\pi}_2) \to V(\boldsymbol{\pi}_1) \otimes \mathbf{E}(V(\boldsymbol{\pi}_2))$ is an isomorphism of $\hat{\mathbf{U}}_q$-modules. Then η maps an ℓ-weight vector in $(\mathbf{E}(V(\boldsymbol{\pi}_1)) \otimes V(\boldsymbol{\pi}_2))_{\mathrm{wt}\boldsymbol{\pi}_1+\mathrm{wt}\boldsymbol{\pi}_2}$ to an ℓ-weight vector in $(V(\boldsymbol{\pi}_1) \otimes \mathbf{E}(V(\boldsymbol{\pi}_2)))_{\mathrm{wt}\boldsymbol{\pi}_1+\mathrm{wt}\boldsymbol{\pi}_2}$ and hence we have,

$$\eta\left((0, \ v(\boldsymbol{\pi}_1)) \otimes v(\boldsymbol{\pi}_2)\right) = v(\boldsymbol{\pi}_1) \otimes (0, dv(\boldsymbol{\pi}_2)),$$

for some $d \in \mathbf{C}$. Moreover since η is an isomorphism, we may and do assume without loss of generality that $d = 1$. Further, there also exist $c_1, c_2 \in \mathbf{C}$ such that

$$\eta\left((v(\boldsymbol{\pi}_1), \ 0) \otimes v(\boldsymbol{\pi}_2)\right) = v(\boldsymbol{\pi}_1) \otimes (c_1 v(\boldsymbol{\pi}_2), \ c_2 v(\boldsymbol{\pi}_2)).$$

By Proposition 4.4 again we see that for all $i \in I$ and $r \in \mathbf{Z}$, $r \neq 0$, we have

$$
\begin{aligned}
\eta\left(h_{i,r}\left((v(\boldsymbol{\pi}_1), 0) \otimes v(\boldsymbol{\pi}_2)\right)\right) &= \eta\left(h_{i,r}(\boldsymbol{\pi}_1)\left(v(\boldsymbol{\pi}_1), rv(\boldsymbol{\pi}_1)\right) \otimes v(\boldsymbol{\pi}_2)\right.\\
&\qquad \left. +h_{i,r}(\boldsymbol{\pi}_2)\left(v(\boldsymbol{\pi}_1), 0\right) \otimes v(\boldsymbol{\pi}_2)\right)\\
&= \left(h_{i,r}(\boldsymbol{\pi}_1) + h_{i,r}(\boldsymbol{\pi}_2)\right)v(\boldsymbol{\pi}_1) \otimes \left(c_1 v(\boldsymbol{\pi}_2),\ c_2 v(\boldsymbol{\pi}_2)\right)\\
&\qquad + rh_{i,r}(\boldsymbol{\pi}_1)v(\boldsymbol{\pi}_1) \otimes \left(0, v(\boldsymbol{\pi}_2)\right)
\end{aligned}
$$

while,

$$
\begin{aligned}
h_{i,r}\left(v(\boldsymbol{\pi}_1) \otimes (c_1 v(\boldsymbol{\pi}_2),\ c_2 v(\boldsymbol{\pi}_2))\right) &= h_{i,r}(\boldsymbol{\pi}_1)v(\boldsymbol{\pi}_1) \otimes \left(c_1 v(\boldsymbol{\pi}_2), c_2 v(\boldsymbol{\pi}_2)\right)\\
&\ + h_{i,r}(\boldsymbol{\pi}_2)v(\boldsymbol{\pi}_1) \otimes \left(c_1 v(\boldsymbol{\pi}_2),\ c_1 rv(\boldsymbol{\pi}_2) + c_2 v(\boldsymbol{\pi}_2)\right).
\end{aligned}
$$

Equating, we get

$$
h_{i,r}(\boldsymbol{\pi}_1) = c_1 h_{i,r}(\boldsymbol{\pi}_2).
$$

Writing $\boldsymbol{\pi}_1 = (\pi_1, \ldots, \pi_n)$ and $\boldsymbol{\pi}_2 = (\pi'_1, \ldots, \pi'_n)$, as

$$
\pi_i(u) = \prod_{s=1}^{k}(1 - a_s u)^{p_s}, \quad \pi'_i(u) = \prod_{s=1}^{\ell}(1 - b_s u)^{m_s},
$$

where $a_s \neq a_r$ similarly $b_r \neq b_s$ if $r \neq s$ and $p_s > 0$, $m_s > 0$, we find by using (3.3) that for all $r > 0$, we have

$$
\frac{h_{i,r}(\boldsymbol{\pi}_1)}{[r]_i} = -\frac{1}{r}\sum_{s=1}^{k} p_s a_s^r, \qquad \frac{h_{i,r}(\boldsymbol{\pi}_2)}{[r]_i} = -\frac{1}{r}\sum_{s=1}^{\ell} m_s b_s^r.
$$

Hence we get

$$
\sum_{s=1}^{k} p_s a_s^r = c_1 \sum_{s=1}^{\ell} m_s b_s^r, \quad r \in \mathbf{Z}_+^{\times}.
$$

If $a_1 \neq b_r$ for all $1 \leq r \leq \ell$, then we find by using the invertibility of the Vandermonde matrix that $p_1 = 0$ which is a contradiction. This means that we must have $\ell = k$ and also without loss of generality $a_j = b_j$ for all $1 \leq j \leq \ell$. This gives the equation,

$$
\sum_{s=1}^{\ell}(p_s - c_1 m_s)a_s^r = 0, \quad r \in \mathbf{Z}, r \neq 0.
$$

In particular, this means that

$$
p_s = c_1 m_s, \quad 1 \leq s \leq \ell.
$$

Hence c_1 is positive and rational, say $c_1 = \frac{d'}{d}$, for some $d, d' \in \mathbf{Z}_+\backslash\{0\}$, and so $d'p_s = dm_s$ for all $1 \leq s \leq \ell$. Since c_1 is independent of i we have now proved that

$$
\boldsymbol{\pi}_1^d = \boldsymbol{\pi}_2^{d'}.
$$

But this is a contradiction and hence η is not an isomorphism. $\qquad\square$

4.7

We can now prove part (ii) of Theorem 1. Observe first that if $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are such that $\boldsymbol{\pi}_1^{r_1} = \boldsymbol{\pi}_2^{r_2}$ for some $r_1, r_2 \in \mathbf{Z}_+$, then there exists $\boldsymbol{\pi}_0$ such that $\boldsymbol{\pi}_1$ and $\boldsymbol{\pi}_2$ are powers of $\boldsymbol{\pi}_0$. Suppose that $V(\boldsymbol{\pi})$ is not a prime representation and write

$$V(\boldsymbol{\pi}) = V(\boldsymbol{\pi}_1) \otimes \cdots \otimes V(\boldsymbol{\pi}_k),$$

where $\boldsymbol{\pi}_s$ is nontrivial and $V(\boldsymbol{\pi}_s)$ is prime for $1 \le s \le k$. Then by Proposition 4.6 there exists $\boldsymbol{\pi}_0$ such that

$$\boldsymbol{\pi}_1 = \boldsymbol{\pi}_0^r, \quad \boldsymbol{\pi}_2 \cdots \boldsymbol{\pi}_k = \boldsymbol{\pi}_0^\ell,$$

(note $V(\boldsymbol{\pi}_2) \otimes \cdots \otimes V(\boldsymbol{\pi}_k) \cong V(\boldsymbol{\pi}_2 \cdots \boldsymbol{\pi}_k)$) and hence $\boldsymbol{\pi} = \boldsymbol{\pi}_0^{r+\ell}$. Since

$$V(\boldsymbol{\pi}_1) \otimes V(\boldsymbol{\pi}_2) \cong V(\boldsymbol{\pi}_2) \otimes V(\boldsymbol{\pi}_1),$$

we also get by Proposition 4.6 that

$$\boldsymbol{\pi}_2^m = \boldsymbol{\pi}_1^p \boldsymbol{\pi}_3^p \cdots \boldsymbol{\pi}_k^p,$$

for some $m, p \in \mathbf{Z}_+$. This gives

$$\boldsymbol{\pi}_2^{m+p} = \boldsymbol{\pi}^p = \boldsymbol{\pi}_0^{p(\ell+r)}.$$

Hence $\boldsymbol{\pi}_2$ and $\boldsymbol{\pi}_0$, and so also $\boldsymbol{\pi}_1$ are powers of some $\tilde{\boldsymbol{\pi}}$. Repeating we find that there exists $\tilde{\boldsymbol{\pi}}_0$ such that $\boldsymbol{\pi}_s$ is a power of $\tilde{\boldsymbol{\pi}}_0$ for all $1 \le s \le k$ as required.

5 Local and global Weyl modules

To prove the remaining results of the paper we need to recall the definition of global and local Weyl modules and summarize their important properties. We use the approach developed in [8] for loop algebras.

5.1

Given $\lambda \in P^+$ the global Weyl module $W(\lambda)$ is the $\hat{\mathbf{U}}_q$-module generated by a vector w_λ with the following defining relations,

$$k_i w_\lambda = q_i^{\lambda_i} w_\lambda, \qquad x_{i,r}^+ w_\lambda = 0,$$
$$(x_{i,0}^-)^{\lambda_i+1} w_\lambda = 0,$$

for all $i \in I$ and $r \in \mathbf{Z}$. If $\lambda \ne 0$, then $W(\lambda)$ is an infinite-dimensional type 1 module while $W(0)$ is the trivial module. Note that if $\boldsymbol{\pi} \in \mathcal{P}^+$ is such that $\mathrm{wt}\boldsymbol{\pi} = \lambda$ and $V \in \mathrm{Ob}\hat{\mathcal{F}}$ is a highest-ℓ-weight module with ℓ-weight $\boldsymbol{\pi}$, then V is a quotient of $W(\lambda)$. The global Weyl module $W(\lambda)$ was originally defined in [17] in a different way, but it is not hard to see by using Proposition 4.3 of that paper that the two definitions are equivalent. It is also proved in Proposition 4.5 of [17] that $W(\lambda)$ is an integrable module: i.e the Chevalley generators $x_i^\pm$, $i \in \hat{I}$ act locally nilpotently. Finally, we remark that it is proved in [37] that the global Weyl module is isomorphic to the extremal weight modules defined by Kashiwara in [27]. The following is a very special case of the fact that $W(\lambda)$ is integrable. For all $i \in I$, we have

$$\lambda - (\lambda_i + 1)\alpha_i \notin \mathrm{wt}W(\lambda). \tag{5.1}$$

5.2

Regard $W(\lambda)$ as a right-module for $\hat{\mathbf{U}}_q^0$ by setting $(uw_\lambda)h_{i,r} = uh_{i,r}w_\lambda$, where $u \in \hat{\mathbf{U}}_q$ and $i \in I$, $r \in \mathbf{Z}$, $r \neq 0$.

Set

$$\mathrm{Ann}_\lambda = \{x \in \hat{\mathbf{U}}_q^0 : w_\lambda x = 0\} = \{x \in \hat{\mathbf{U}}_q^0 : xw_\lambda = 0\},$$

and let $\mathbf{A}_\lambda$ be the quotient of $\hat{\mathbf{U}}_q^0$ by the ideal Ann_λ. Then, $W(\lambda)$ is a $(\hat{\mathbf{U}}_q, \mathbf{A}_\lambda)$-bimodule. For all $\mu \in P$ the subspace $W(\lambda)_\mu$ is a right $\mathbf{A}_\lambda$-module. Moreover, $W(\lambda)_\lambda$ is obviously also a left $\mathbf{A}_\lambda$-module and we have an isomorphism of $\mathbf{A}_\lambda$-bimodules

$$W(\lambda)_\lambda \cong \mathbf{A}_\lambda.$$

The structure of the ring $\mathbf{A}_\lambda$ is known. Specifically regard $\hat{\mathbf{U}}_q^0$ as the polynomial ring in the variables $\Lambda_{i,r}$, $i \in I$, $r \in \mathbf{Z}^\times$. Then $\mathbf{A}_\lambda$ is the quotient obtained by setting

$$\Lambda_{i,r} = 0, \quad |r| \geq \lambda_i + 1, \quad \Lambda_{i,\lambda_i}\Lambda_{i,-s} - \Lambda_{i,\lambda_i-s} = 0, \tag{5.2}$$

for all $i \in I$ and $0 \leq s \leq \lambda_i$. In particular, if we let $\bar{\Lambda}_{i,r}$ be the image of $\Lambda_{i,r}$ in $\mathbf{A}_\lambda$, then

$$\mathbf{A}_\lambda \cong \bigotimes_{i \in I} \mathbf{C}[\bar{\Lambda}_{i,1}, \dots \bar{\Lambda}_{i,\lambda_i}, (\bar{\Lambda}_{i,\lambda_i})^{-1}].$$

The following important result will be crucial for the paper. The result was established when R is A_1 in [18], and in general the result can be deduced from the work of [4,36]. There are also other proofs of this result through the work of [11,17,21,40].

Theorem 3 *The global Weyl module $W(\lambda)$ is a free right module of finite rank for $\mathbf{A}_\lambda$.* □

5.3

Let mod-$\mathbf{A}_\lambda$ be the category of finitely generated left $\mathbf{A}_\lambda$-modules and given an object M of mod-$\mathbf{A}_\lambda$ set

$$\mathbf{W}_\lambda M = W(\lambda) \otimes_{\mathbf{A}_\lambda} M.$$

Since $W(\lambda)_\lambda$ is an $\mathbf{A}_\lambda$-bimodule, we have an isomorphism of left $\mathbf{A}_\lambda$-modules,

$$(\mathbf{W}_\lambda M)_\lambda = W(\lambda)_\lambda \otimes_{\mathbf{A}_\lambda} M \cong M.$$

If V is any quotient of $W(\lambda)$, then

$$uV_\lambda = 0, \quad u \in \mathrm{Ann}_\lambda,$$

and hence the $\hat{\mathbf{U}}_q^0$ action on V_λ descends to $\mathbf{A}_\lambda$. It is simple to check (see [8, Proposition 3.6]) that V is a quotient of $\mathbf{W}_\lambda V_\lambda$.

5.4

Proposition *The assignment*

$$M \to \mathbf{W}_\lambda M, \quad f \to 1 \otimes f,$$

defines an exact functor from the category mod-$\mathbf{A}_\lambda$ to $\hat{\mathcal{F}}$. Moreover, M is an indecomposable object of mod-$\mathbf{A}_\lambda$ iff $\mathbf{W}_\lambda M$ is indecomposable in $\hat{\mathcal{F}}$.

Proof The first statement is clear from Theorem 3. Suppose that

$$\mathbf{W}_\lambda M = V \oplus V',$$

as objects of $\hat{\mathcal{F}}$. Then we have

$$(\mathbf{W}_\lambda M)_\lambda = V_\lambda \oplus V'_\lambda,$$

as $\hat{\mathbf{U}}_q^0$-modules. Since

$$\mathrm{Ann}_\lambda (\mathbf{W}_\lambda M)_\lambda = 0$$

it follows that V_λ and V'_λ are $\mathbf{A}_\lambda$-modules and hence

$$M = V_\lambda \oplus V'_\lambda$$

as $\mathbf{A}_\lambda$-modules. The converse implication is trivial and the proposition is established. □

5.5

Given $\boldsymbol{\pi} \in \mathcal{P}^+$, let $\mathbf{C}_{\boldsymbol{\pi}}$ be the one-dimensional representation of $\hat{\mathbf{U}}_q^0$ defined by taking the quotient by the maximal ideal $\mathbf{I}(\boldsymbol{\pi})$ generated by the elements

$$\{h_{i,r} - h_{i,r}(\boldsymbol{\pi}) : i \in I, r \in \mathbf{Z}^\times\},$$

or equivalently by the elements

$$\{\Lambda_{i,r} - \Lambda_{i,r}(\boldsymbol{\pi}) : i \in I, r \in \mathbf{Z}^\times\}.$$

It is clear from (5.2) that $\mathbf{C}_{\boldsymbol{\pi}}$ is a $\mathbf{A}_\lambda$-module. The local Weyl module $W(\boldsymbol{\pi})$ is given by,

$$W(\boldsymbol{\pi}) = \mathbf{W}_\lambda \mathbf{C}_{\boldsymbol{\pi}}, \quad w(\boldsymbol{\pi}) = w_\lambda \otimes 1.$$

An alternative definition of $W(\boldsymbol{\pi})$ is that it is the quotient of $W(\lambda)$ obtained by imposing the additional relations:

$$(h_{i,r} - h_{i,r}(\boldsymbol{\pi}))w_\lambda = 0, \qquad \text{equivalently} \ (\Lambda_{i,r} - \Lambda_{i,r}(\boldsymbol{\pi}))w_\lambda = 0, \qquad (5.3)$$

for all $i \in I$, $r \in \mathbf{Z}^\times$. Clearly $W(\boldsymbol{\pi})$ is a highest-ℓ-weight module with ℓ-weight $\boldsymbol{\pi}$ and $W(\boldsymbol{\pi})$ is universal with this property. In particular $V(\boldsymbol{\pi})$ is a quotient of $W(\boldsymbol{\pi})$. Moreover $V(\boldsymbol{\pi})$ is the quotient of $W(\boldsymbol{\pi})$ by the maximal submodule not containing $W(\boldsymbol{\pi})_{\mathrm{wt}\boldsymbol{\pi}} = \mathbf{C}w(\boldsymbol{\pi})$.

5.6

We now prove,

Proposition *Let $\lambda \in P^+$ and let V be any $\hat{\mathbf{U}}_q$-module such that $\mathrm{wt}V \subset \lambda - Q^+$ and assume that $\lambda - (\lambda_i + 1)\alpha_i \notin \mathrm{wt}V$. Then,*

$$Ext^1_{\hat{\mathbf{U}}_q}(W(\lambda), V) = 0,$$

or equivalently, any short exact sequence

$$0 \to V \xrightarrow{\iota} W \xrightarrow{\tau} W(\lambda) \to 0$$

of $\hat{\mathbf{U}}_q$-modules is split.

Proof Observe that wt $W \subset \lambda - Q^+$ and also that $\lambda - (\lambda_i + 1)\alpha_i \notin \text{wt} W$. Hence if $w \in W_\lambda$ is such that $\tau(w) = w_\lambda$, we have

$$x_{i,r}^+ w = 0, \quad (x_{i,0}^-)^{\lambda_i + 1} w = 0.$$

It follows there exists a $\hat{\mathbf{U}}_q$-module map $\eta : W(\lambda) \to \hat{\mathbf{U}}_q w$ with $\eta(w_\lambda) = w$. The composite map $\tau.\eta : W(\lambda) \to W(\lambda)$ then satisfies $\tau.\eta(w_\lambda) = w_\lambda$ and hence is the identity map and the Lemma is established. $\qquad\square$

5.7

One can compute $\text{Ext}^r_{\hat{\mathbf{U}}_q}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi}))$ for all $\boldsymbol{\pi} \in \mathcal{P}^+$ using the Koszul complex for the $\mathbf{A}_\lambda$-module $\mathbf{C}_{\boldsymbol{\pi}}$ and using Theorem 3. Consider the the special case when $r = 1$ in which case we have

$$\dim \text{Ext}^1_{\mathbf{A}_\lambda}(\mathbf{C}_{\boldsymbol{\pi}}, \mathbf{C}_{\boldsymbol{\pi}}) = \sum_{i \in I} \lambda_i.$$

Proposition *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ such that wt $\boldsymbol{\pi} = \lambda$. Then*

$$Ext^1_{\hat{\mathcal{F}}}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi})) \cong Ext^1_{\mathbf{A}_\lambda}(\mathbf{C}_{\boldsymbol{\pi}}, \mathbf{C}_{\boldsymbol{\pi}}),$$

and hence

$$\dim Ext^1_{\hat{\mathcal{F}}}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi})) = \sum_{i \in I} \lambda_i.$$

Proof Let $\eta : W(\lambda) \to W(\boldsymbol{\pi})$ be the map of $\hat{\mathbf{U}}_q$-modules such that $\eta(w_\lambda) = w(\boldsymbol{\pi})$. Since $\dim W(\boldsymbol{\pi})_\lambda = 1$, we have

$$\dim \text{Hom}_{\hat{\mathbf{U}}_q}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi})) = 1 = \dim \text{Hom}_{\hat{\mathbf{U}}_q}(W(\lambda), W(\boldsymbol{\pi})).$$

Applying $\text{Hom}_{\hat{\mathcal{F}}}(-, W(\boldsymbol{\pi}))$ to the short exact sequence $0 \to \ker \eta \to W(\lambda) \to W(\boldsymbol{\pi}) \to 0$ and using Proposition 5.6 we get

$$\text{Hom}_{\hat{\mathcal{F}}}(\ker \eta, W(\boldsymbol{\pi})) \cong \text{Ext}^1_{\hat{\mathcal{F}}}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi})).$$

Indeed, Proposition 5.6 implies that we have an exact sequence

$$0 \to \text{Hom}_{\hat{\mathcal{F}}}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi}) \to \text{Hom}_{\hat{\mathcal{F}}}(W(\lambda), W(\boldsymbol{\pi})) \to \text{Hom}_{\hat{\mathcal{F}}}(\ker \eta, W(\boldsymbol{\pi}))$$
$$\to \text{Ext}^1_{\hat{\mathcal{F}}}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi})) \to 0.$$

Since

$$\dim \text{Hom}_{\hat{\mathcal{F}}}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi})) = \dim \text{Hom}_{\hat{\mathcal{F}}}(W(\lambda), W(\boldsymbol{\pi})) = 1,$$

the claim follows.

It follows from (5.3) that $\ker \eta$ is generated as a $\hat{\mathbf{U}}_q$-module by the elements $\{h_{i,r} - h_{i,r}(\boldsymbol{\pi}) : i \in I, r \in \mathbf{Z}.\}$ Since $h_{i,r}(\boldsymbol{\pi})$ is determined by the values of $h_{i,s}(\boldsymbol{\pi})$ for $1 \leq s \leq \lambda_i$, we see that

$$\dim \text{Ext}^1_{\hat{\mathbf{U}}_q}(W(\boldsymbol{\pi}), W(\boldsymbol{\pi})) = \dim \text{Hom}_{\hat{\mathbf{U}}_q}(\ker \eta, W(\boldsymbol{\pi})) \leq \sum_{i \in I} \lambda_i.$$

Finally, this bound must be achieved since Proposition 5.4 defines a map

$$\text{Ext}^1_{\mathbf{A}_\lambda}(\mathbf{C}_{\boldsymbol{\pi}}, \mathbf{C}_{\boldsymbol{\pi}}) \to \text{Ext}^1_{\hat{\mathcal{F}}}(\mathbf{W}_\lambda \mathbf{C}_{\boldsymbol{\pi}}, \mathbf{W}_\lambda \mathbf{C}_{\boldsymbol{\pi}})$$

which is injective since the functor exact. $\qquad\square$

5.8

The structure of the local Weyl modules is known through the work of [1,7,41]. To describe the result, we recall the definition of fundamental ℓ-weights. For $i \in I$ and $a \in \mathbf{C}^\times$, let $\pi_{i,a} = (\pi_1, \dots, \pi_n) \in \mathcal{P}^+$ be such that $\pi_i = 1 - au$ and $\pi_j = 1$ otherwise.

Proposition *Let $\pi \in \mathcal{P}^+$. Then there exists $k \in \mathbf{Z}_+$, $i_s \in I$, $a_s \in \mathbf{C}^\times$, $1 \le s \le k$, such that*

$$W(\pi) \cong V(\pi_{i_1,a_1}) \otimes \cdots \otimes V(\pi_{i_k,a_k}).$$

Moreover $W(\pi)$ is irreducible if $a_p/a_\ell \notin q^{\mathbf{Z}}$ for all $1 \le p, \ell \le s$.   □

In fact it was shown in [7] that $W(\pi)$ is irreducible if $a_p = a_\ell$ for all $1 \le p, \ell \le s$. Combining this with Proposition 5.7 we get, that

$$\dim \mathrm{Ext}_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) = 1, \quad \pi \text{ is generic} \implies V(\pi) \text{ is prime.} \tag{5.4}$$

Here, by generic we mean π such that $W(\pi) \cong V(\pi)$. Notice however that a generic simple module $V(\pi)$ is most often not prime.

6 Proof of Theorem 1 (iii)

In addition to proving part (iii) of Theorem 1, we also prove some results on self-extensions which are needed later in the paper and use the study of global and local Weyl modules.

6.1

Proposition *Let $\pi \in \mathcal{P}^+$ and let V be any self-extension of $V(\pi)$. The restriction $V \to V_{wt\pi}$ induces an injective map of vector spaces*

$$\mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) \to \mathrm{Ext}^1_{\mathbf{A}_\lambda}(\mathbf{C}_\pi, \mathbf{C}_\pi) \cong \mathrm{Ext}^1_{\hat{\mathcal{F}}}(W(\pi), W(\pi)).$$

Proof Let $K(\pi)$ (resp. $\tilde{K}(\pi)$) be the kernel of the map $W(\lambda) \to W(\pi)$ (resp. $W(\lambda) \to V(\pi)$)). Clearly $K(\pi) \subset \tilde{K}(\pi)$ and moreover, since

$$W(\pi)_{wt\pi} \cong V(\pi)_{wt\pi} \cong \mathbf{C}_\pi,$$

it follows that

$$\tilde{K}(\pi)_{wt\pi} = K(\pi)_{wt\pi}, \quad \text{and} \ \ \mathrm{Hom}_{\hat{\mathcal{F}}}(\tilde{K}(\pi)/K(\pi), V(\pi)) = 0. \tag{6.1}$$

Using Proposition 5.6 and the fact that

$$\dim \mathrm{Hom}_{\hat{\mathcal{F}}}(W(\pi), W(\pi)) = \dim \mathrm{Hom}_{\hat{\mathcal{F}}}(W(\lambda), W(\pi))$$
$$= \dim \mathrm{Hom}_{\hat{\mathcal{F}}}(W(\lambda), V(\pi)) = \dim \mathrm{Hom}_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) = 1,$$

we see that

$$\mathrm{Ext}^1_{\hat{\mathcal{F}}}(W(\pi), W(\pi)) \cong \mathrm{Hom}_{\hat{\mathcal{F}}}(K(\pi), W(\pi)), \tag{6.2}$$
$$\mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\pi), V(\pi)) \cong \mathrm{Hom}_{\hat{\mathcal{F}}}(\tilde{K}(\pi), V(\pi)). \tag{6.3}$$

Applying $\mathrm{Hom}_{\hat{\mathcal{F}}}(-, V(\pi))$ to the short exact sequence

$$0 \to K(\pi) \to \tilde{K}(\pi) \to \tilde{K}(\pi)/K(\pi) \to 0,$$

and using (6.1) we get an inclusion,

$$\mathrm{Hom}_{\hat{\mathcal{F}}}(\tilde{K}(\boldsymbol{\pi}), V(\boldsymbol{\pi})) \hookrightarrow \mathrm{Hom}_{\hat{\mathcal{F}}}(K(\boldsymbol{\pi}), V(\boldsymbol{\pi})).$$

The Proposition now follows from (6.2), (6.3) and Proposition 5.7. □

6.2

We now determine the image of the inclusion given in Proposition 6.1. Equivalently, we answer the following question: when does a self-extension of $W(\boldsymbol{\pi})$ determine a self-extension of $V(\boldsymbol{\pi})$. It is convenient to introduce the following notation: given any $\hat{\mathbf{U}}_q$-module W with $\mathrm{wt} W \subset \lambda - Q^+$, let W^λ be the unique maximal submodule of W such that

$$W^\lambda \cap W_\lambda = 0.$$

It is clear that W^λ exists and is unique—one just takes the sum of all submodules U of W such that $U_\lambda = 0$. Recall that

$$W(\boldsymbol{\pi})/W(\boldsymbol{\pi})^\lambda \cong V(\boldsymbol{\pi}).$$

Lemma *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and assume that W is a (non-split) self extension of $W(\boldsymbol{\pi})$,*

$$0 \to W(\boldsymbol{\pi}) \xrightarrow{\iota} W \xrightarrow{\tau} W(\boldsymbol{\pi}) \to 0.$$

If $\tau(W^{\mathrm{wt}\boldsymbol{\pi}}) = W(\boldsymbol{\pi})^{\mathrm{wt}\boldsymbol{\pi}}$, there exists a (non-split) self-extension V of $V(\boldsymbol{\pi})$ with

$$V_{\mathrm{wt}\boldsymbol{\pi}} \cong_{\mathbf{A}_\lambda} W_{\mathrm{wt}\boldsymbol{\pi}}.$$

Proof Since $\iota(W(\boldsymbol{\pi})^{\mathrm{wt}\boldsymbol{\pi}}) \subset W^{\mathrm{wt}\boldsymbol{\pi}}$ we see that the restrictions of ι and τ give a short exact sequence

$$0 \to W(\boldsymbol{\pi})^{\mathrm{wt}\boldsymbol{\pi}} \xrightarrow{\iota} W^{\mathrm{wt}\boldsymbol{\pi}} \xrightarrow{\tau} W(\boldsymbol{\pi})^{\mathrm{wt}\boldsymbol{\pi}} \to 0.$$

Setting $V = W/W^{\mathrm{wt}\boldsymbol{\pi}}$ it follows that V is a self-extension of $V(\boldsymbol{\pi})$. □

6.3

Proposition *Suppose that V is a nontrivial self-extension of $V(\boldsymbol{\pi})$ and $\mathrm{wt}\boldsymbol{\pi} = \lambda$. Then*

$$V \cong \mathbf{W}_\lambda V_\lambda/(\mathbf{W}_\lambda V_\lambda)^\lambda.$$

Proof If V is a non-trivial self extension, then it follows from Lemma 4.1 that there exists $v \in V_\lambda$ such that $V = \hat{\mathbf{U}}_q v$. Since $\mathrm{wt}\, V \subset \lambda - Q^+$ we see that V is a quotient of $W(\lambda)$ and hence also of $\mathbf{W}_\lambda V_\lambda$. Since

$$\dim V_\lambda = \dim(\mathbf{W}_\lambda V_\lambda)_\lambda = 2,$$

it follows that $\mathbf{W}_\lambda V_\lambda/(\mathbf{W}_\lambda V_\lambda)^\lambda$ must be a quotient of V and hence is either isomorphic to V or to $V(\boldsymbol{\pi})$. But the latter is impossible since $\dim V(\boldsymbol{\pi})_\lambda = 1$ and the proof is complete. □

The following is now immediate.

Corollary *Suppose that V and V' are self-extensions of $V(\boldsymbol{\pi})$ and $\mathrm{wt}\boldsymbol{\pi} = \lambda$. Then*

$$V \cong_{\hat{\mathcal{F}}} V' \iff V_\lambda \cong_{\mathbf{A}_\lambda} V'_\lambda.$$

6.4

The next proposition along with Lemma 4.2 proves part (iii) of Theorem 1.

Proposition *Let V_1 and V_2 be nontrivial self extensions of $V(\boldsymbol{\pi})$ for some $\boldsymbol{\pi} \in \mathcal{P}^+$. Then for all $\boldsymbol{\pi}_1 \in \mathcal{P}^+$ with $V(\boldsymbol{\pi}) \otimes V(\boldsymbol{\pi}_1)$ irreducible, we have*

$$V_1 \otimes V(\boldsymbol{\pi}_1) \cong V_2 \otimes V(\boldsymbol{\pi}_1) \iff V_1 \cong V_2.$$

Proof Let $\eta : V_1 \otimes V(\boldsymbol{\pi}_1) \to V_2 \otimes V(\boldsymbol{\pi}_1)$ be an isomorphism of $\hat{\mathbf{U}}_q$-modules. By Lemma 4.5 we know that $V_j \otimes V(\boldsymbol{\pi}_1)$ is a nontrivial extension of $V(\boldsymbol{\pi}) \otimes V(\boldsymbol{\pi}_1)$. Let $\iota_j : V(\boldsymbol{\pi}) \to V_j$ be the inclusion. Since $\iota_1(v(\boldsymbol{\pi})) \otimes v(\boldsymbol{\pi}_1)$ is a highest-ℓ-weight vector in $V_1 \otimes V(\boldsymbol{\pi}_1)$, we may assume without loss of generality that

$$\eta(\iota_1(v(\boldsymbol{\pi})) \otimes v(\boldsymbol{\pi}_1)) = \iota_2(v(\boldsymbol{\pi})) \otimes v(\boldsymbol{\pi}_1).$$

Let $v_1 \in (V_1)_{\mathrm{wt}\boldsymbol{\pi}}$ be linearly independent from $\iota_1(v(\boldsymbol{\pi}))$. Writing

$$\eta(v_1 \otimes v(\boldsymbol{\pi}_1)) = v_2 \otimes v(\boldsymbol{\pi}_1), \tag{6.4}$$

we see that v_2 and $\iota_2(v(\boldsymbol{\pi}_2))$ are linearly independent elements of $(V_2)_{\mathrm{wt}\boldsymbol{\pi}}$. Applying $h_{i,r}$ to both side of (6.4), we get

$$\eta(h_{i,r}v_1 \otimes v(\boldsymbol{\pi}_1)) = h_{i,r}v_2 \otimes v(\boldsymbol{\pi}_1),$$

for all $i \in I$, $r \in \mathbf{Z}^\times$. Writing

$$h_{i,r}v_j = h_{i,r}(\boldsymbol{\pi})v_j + c_{i,r}^j \iota_j(v(\boldsymbol{\pi})),$$

we find now that $c_{i,r}^1 = c_{i,r}^2$ for all $i \in I$, $r \in \mathbf{Z}^\times$. Hence the map $v_1 \to v_2$, $\iota_1(v(\boldsymbol{\pi})) \to \iota_2(v(\boldsymbol{\pi}))$ defines an isomorphism

$$(V_1)_{\mathrm{wt}\boldsymbol{\pi}} \cong (V_2)_{\mathrm{wt}\boldsymbol{\pi}}$$

of $\mathbf{A}_{\mathrm{wt}\boldsymbol{\pi}}$-modules. By Corollary 6.3 we see that this implies $V_1 \cong V_2$ as $\hat{\mathbf{U}}_q$-modules. The converse statement is trivial and the proof is complete.   □

7 Proof of Theorem 1 (iv)

Throughout this section we shall be concerned with R being of type A_1. In this case, $I = \{1\}$ and for ease of notation, we denote the elements $x_{i,r}^\pm$ as just $x_r^\pm$. Since we will not be using the Chevalley generators in this section there should be no confusion. We also remark that $d_1 = 1$ and hence we just denote by $[r]$ the quantum number $[r]_1$. Finally, we identify P with $\mathbf{Z}$ and Q with $2\mathbf{Z}$ and denote the modules $W(\lambda)$ by $W(n)$ etc.

7.1

Given $m \in \mathbf{Z}_+$ and $a \in \mathbf{C}^\times$, set

$$\boldsymbol{\pi}(m, a) = (1 - aq^{m-1}u)(1 - aq^{m-3}u)\cdots(1 - aq^{-m+1}u).$$

The representation $V(\boldsymbol{\pi}(m, a))$ has the following explicit realization. It has a basis $v_m, \ldots, v_0$ and the action of the generators $x_r^\pm$ is given by,

$$x_r^+ v_j = \left(aq^{-m+2j+2}\right)^r [j+1]v_{j+1} \quad x_r^- v_j = \left(aq^{-m+2j}\right)^r [m-j+1]v_{j-1}, \tag{7.1}$$

where $0 \leq j \leq m$ and we understand that $v_{-1} = v_{m+1} = 0$. The action of the remaining generators is determined by these and we note for future use that for all $r \in \mathbf{Z}$ with $r \neq 0$, we have

$$\phi_{\pm r}^{\pm} v_m = \pm (q - q^{-1})(aq^m)^{\pm r}[m]v_m, \qquad h_r v_m = a^r \frac{[rm]}{r} v_m. \tag{7.2}$$

Clearly $V(\boldsymbol{\pi}(m, a))$ is irreducible for the subalgebra $\mathbf{U}_q$ and hence by Corollary 3.3 is a prime object of $\hat{\mathcal{F}}$. The following is a consequence of [12, Theorem 4.8].

Proposition *Any prime simple object in $\hat{\mathcal{F}}$ is isomorphic to $V(\boldsymbol{\pi}(m, a))$ for some $m \in \mathbf{Z}_+, a \in \mathbf{C}^\times$. Moreover, for all $s \in \mathbf{Z}_+$, we have*

$$V(\boldsymbol{\pi}(m, a)^s) \cong V(\boldsymbol{\pi}(m, a))^{\otimes s}.$$

$\square$

7.2

The next proposition together with part (iii) of Theorem 1 and Proposition 7.1 establishes part (iv) of Theorem 1.

Proposition *Let $m \in \mathbf{Z}_+, a \in \mathbf{C}^\times$. Then,*

(i) $\dim \mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}(m, a)), V(\boldsymbol{\pi}(m, a))) = 1$,
(ii) $\dim \mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}(m, a))^{\otimes 2}, V(\boldsymbol{\pi}(m, a))^{\otimes 2}) \geq 2$.

The rest of this section is devoted to proving the proposition.

Remark The space $\mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}(m, a)), V(\boldsymbol{\pi}))$ with $\deg(\boldsymbol{\pi}) < m$ was computed in [30, Proposition 3.2.2].
Proposition 7.2 answers positively [30, Conjecture 3.3.2] and the comment following it.

7.3

Recall from Sect. 5 that the local Weyl module $W(\boldsymbol{\pi}(m, a))$ is the module generated by an element w_m with relations:

$$x_r^+ w_m = 0, \quad h_r w_m = a^r \frac{[rm]}{r} w_m \;, \quad (x_0^-)^{m+1} w_m = 0.$$

Proposition *Let $m \in \mathbf{Z}_+, a \in \mathbf{C}^\times$.*

(i) *The module $V(\boldsymbol{\pi}(m, a))$ is the quotient of $W(\boldsymbol{\pi}(m, a))$ obtained by imposing the single additional relation*

$$(x_1^- - aq^m x_0^-)w_m = 0.$$

(ii) *The module $V(\boldsymbol{\pi}(m, a))^{\otimes 2}$ is the quotient of $W(\boldsymbol{\pi}(m, a)^2)$ obtained by imposing the single additional relation*

$$(x_2^- - 2aq^m x_1^- + a^2 q^{2m} x_0^-)w_{2m} = 0.$$

Proof To prove (i), notice that the formulae given in Sect. 7.1 imply that the element $v_m \in V(\boldsymbol{\pi}(m, a))$ satisfies,

$$(x_1^- - aq^m x_0^-)v_m = 0,$$

In particular, if

$$W = W(\pi(m, a))/\hat{\mathbf{U}}_q(x_1^- - aq^m x_0^-)w_m,$$

then $V(\pi(m, a))$ is a quotient of W. Part (i) follows if we prove that dim $W \le m+1$. For this we denote by $\bar{w}_m$ the image of w_m in W and observe that W is spanned by $\bar{w}_m$ and elements of the form $x_{s_1}^- \cdots x_{s_k}^- \bar{w}_m$ where $1 \le k \le m$. Since

$$0 = [h_r, x_1^- - aq^m x_0^-]\bar{w}_m = \frac{[2r]}{r}(x_{r+1}^- - aq^m x_r^-)\bar{w}_m,$$

we get that $x_r^- \bar{w}_m \in \mathbf{C}x_0^- \bar{w}_m$ for all $r \in \mathbf{Z}$. This implies that W is spanned by elements of the form $x_{s_1}^- \cdots x_{s_{k-1}}^- x_0^- \bar{w}_m$ where $1 \le k \le m$. Suppose that we have proved that we may take $s_2 = \cdots = s_k = 0$. If $s_1 = \pm 1$, then using the relation

$$x_0^- x_{\pm 1}^- = q^{\pm 2} x_{\pm 1}^- x_0^-$$

shows that $x_{\pm 1}^- (x_0^-)^{k-1} \bar{w}_m$ is a multiple of $(x_0^-)^k \bar{w}_m$. An obvious induction on s using the relation

$$x_s^- x_0^- - q^{-2} x_0^- x_s^- = q^{-2} x_1^- x_{s-1}^- - x_{s-1}^- x_1^-,$$

now proves that W is spanned by elements of the form $(x_0^-)^s \bar{w}_m$, $1 \le s \le m$ and hence dim $W \le m + 1$ as required.

The proof of part (ii) is very similar. We observe that we have the relation

$$(x_2^- - 2aq^m x_1^- + a^2 2q^{2m} x_0^-)(v_m \otimes v_m) = 0,$$

in $V(\pi(m, a))^{\otimes 2}$. We set

$$W = W(\pi(m, a)^2)/\hat{\mathbf{U}}_q(x_2^- - 2aq^m x_1^- + a^2 2q^{2m} x_0^-)w_{2m},$$

and let $\bar{w}_{2m}$ be the image of w_{2m} in W. We now prove exactly as before that W is spanned by $\bar{w}_{2m}$ and elements of the form $(x_1^-)^s (x_0^-)^\ell \bar{w}_m$ with $1 \le s + \ell \le 2m$. The spanning set is now of cardinality bigger than $(m + 1)^2$ if $m > 1$. To show that in fact we can choose a suitable subset of cardinality at most $(m + 1)^2$ we observe that

$$\dim W_{2r} = \dim W_{-2r},$$

and hence it is enough to determine a bound for dim W_{2r} for $0 \le r \le m$. This bound is easily seen to be $m - r + 1$ and so we now have

$$\dim W = 2 \dim W_{2m} + 2 \dim W_{2m-2} + \cdots + 2 \dim W_2 + \dim W_0$$
$$= 2(1 + 2 \cdots + m) + (m + 1) = (m + 1)^2.$$

This completes the proof of the Proposition. $\square$

7.4

We now prove Proposition 7.2(i). Consider the canonical map from the global Weyl module $\eta : W(m) \to V(\pi(m, a))$ which sends $w_m \to v_m$. We claim that ker η is generated by the element $v = (x_1^- - aq^m x_0^-)w_m$. By Proposition 7.3(i) we see that $v \in$ ker η. Moreover,

$$(q - q^{-1})x_r^+ v = \begin{cases} (\phi_{r+1}^\pm - aq^m \phi_r^\pm)w_m, & r \in \mathbf{Z}, r \ne 0, -1, \\ (\phi_1^+ - aq^m(q^m - q^{-m}))w_m, & r = 0, \\ ((q^m - q^{-m}) - aq^m \phi_{-1}^-)w_m, & r = -1. \end{cases}$$

An induction on r shows that

$$(\phi_r^\pm - (aq^m)^r (q^m - q^{-m}))w_m \in \hat{\mathbf{U}}_q v, \quad r \neq 0.$$

Setting

$$\tilde{W} = W(m)/\hat{\mathbf{U}}_q v,$$

we see that the defining relations of $W(\boldsymbol{\pi}(m, a))$ imply that $\tilde{W}$ is a quotient of $W(\boldsymbol{\pi}(m, a))$. It now follows from Proposition 7.3(i) that

$$\tilde{W} \cong V(\boldsymbol{\pi}(m, a)),$$

and the claim is established.

By Proposition 5.6, $\text{Ext}^1_{\hat{\mathcal{F}}}(W(m), V(\boldsymbol{\pi}(m, a))) = 0$. Thus, applying $\text{Hom}_{\hat{\mathcal{F}}}(-, V(\boldsymbol{\pi}(m, a)))$ to the short exact sequence

$$0 \to \ker \eta \to W(m) \to V(\boldsymbol{\pi}(m, a)) \to 0,$$

and noting also that $\text{Hom}_{\hat{\mathcal{F}}}(W(m), V(\boldsymbol{\pi}(m, a))) \cong \text{Hom}_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}(m, a)), V(\boldsymbol{\pi}(m, a)))$, one finds

$$\dim \text{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}(m, a)), V(\boldsymbol{\pi}(m, a))) = \dim \text{Hom}_{\hat{\mathcal{F}}}(\ker \eta, V(\boldsymbol{\pi}(m, a))) \leq 1.$$

By Proposition 4.3 we know $\mathbf{E}(V(\boldsymbol{\pi}(m, a))$ is a non-trivial self extension and hence part (i) of Proposition 7.2 is proved.

7.5

The proof of part (ii) proceeds as follows. We construct an ideal of $\mathbf{A}_{2m}$ of codimension two and show that it can be used to define a non-trivial self-extension of $W(\boldsymbol{\pi})$. We then show that this self-extension satisfies the conditions of Lemma 6.2 and hence defines a non-trivial self-extension V of $V(\boldsymbol{\pi})$. Finally, we prove that this extension is not isomorphic to $\mathbf{E}(V(\boldsymbol{\pi}))$. Proposition 4.2 implies that $[V]$ and $[\mathbf{E}(V(\boldsymbol{\pi}))]$ are linearly independent elements of $\text{Ext}^1_{\hat{\mathcal{F}}_q}(V(\boldsymbol{\pi}), V(\boldsymbol{\pi}))$ which proves (ii).

7.6

We recall for the reader's convenience that

$$\mathbf{A}_{2m} = \mathbf{C}[\bar{\Lambda}_1, \bar{\Lambda}_2, \ldots, \bar{\Lambda}_{2m}, \bar{\Lambda}_{2m}^{-1}],$$

and that we have an algebra homomorphism $\hat{\mathbf{U}}_q^0 \to \mathbf{A}_{2m}$ given by

$$\Lambda_r \to \begin{cases} 0, & |r| \geq 2m + 1, \\ \bar{\Lambda}_r, & 0 < r \leq 2m, \\ \bar{\Lambda}_{2m+r} \bar{\Lambda}_{2m}^{-1}, & -2m \leq r \leq 0. \end{cases}$$

Let $\mathbf{I}$ be the ideal in $\mathbf{A}_{2m}$ generated by $(\bar{\Lambda}_1 + 2a[m])^2$, and the elements:

$$[r + 2]\bar{\Lambda}_{r+2} - \left(q^{r+1}\bar{\Lambda}_1 + 2aq^m[r + 1]\right)\bar{\Lambda}_{r+1} - a^2[2m - r]\bar{\Lambda}_r, \quad 0 \leq r \leq 2m. \quad (7.3)$$

Set $\boldsymbol{\pi} = \boldsymbol{\pi}(m, a)^2$.

Lemma *The ideal* **I** *is of codimension two and we have a non-split short exact sequence of* $\mathbf{A}_{2m}$*-modules,*

$$0 \to \mathbf{C}(\boldsymbol{\pi}) \to \mathbf{A}_{2m}/\mathbf{I} \to \mathbf{C}(\boldsymbol{\pi}) \to 0.$$

Proof We first prove that $\mathbf{I} \subset \mathbf{I}(\boldsymbol{\pi})$ so that we have a surjective map $\mathbf{A}_{2m}/\mathbf{I} \to \mathbf{C}(\boldsymbol{\pi}) \to 0$ of $\mathbf{A}_{2m}$-modules. Write $\boldsymbol{\pi} = \sum_{s=0}^{2m} d_s u^s$, and using the fact that

$$(1 - aq^{-m}u)^2 \boldsymbol{\pi}(qu) = (1 - aq^m u)^2 \boldsymbol{\pi}(q^{-1}u), \tag{7.4}$$

we see that (7.3) is identically satisfied if we replace $\bar{\Lambda}_r$ by d_r and hence $\mathbf{I} \subset \mathbf{I}(\boldsymbol{\pi})$. Next, note that after an obvious change of variables of the form

$$\bar{\Lambda}_r \mapsto X_r = \bar{\Lambda}_r + p_r(\bar{\Lambda}_{r-1}, \ldots, \bar{\Lambda}_1), \qquad r > 1, \quad \bar{\Lambda}_1 = X_1,$$

we have that **I** is generated by

$$(X_1 + 2a[m])^2, \quad X_2, \ldots, X_{2m}.$$

Hence the ideal generated by these elements in $\mathbf{C}[X_1, \ldots, X_{2m}] = \mathbf{C}[\bar{\Lambda}_1, \ldots, \bar{\Lambda}_{2m}]$ is of codimension two. Since $\bar{\Lambda}_{2m} \notin \mathbf{I}$ (recall $\bar{\Lambda}_{2m} \notin \mathbf{I}(\boldsymbol{\pi})$) the conclusion does not change if we localize at Λ_{2m} and work with the ideal **I**. In particular we have proved that $\mathbf{A}_{2m}/\mathbf{I}$ is an indecomposable module of dimension two and that we have a non-split short exact sequence

$$0 \to (\bar{\Lambda}_1 + 2a[m])\mathbf{A}_{2m}/\mathbf{I} \to \mathbf{A}_{2m}/\mathbf{I} \to \mathbf{C}(\boldsymbol{\pi}) \to 0,$$

of $\mathbf{A}_{2m}$-modules, or equivalently, we have a non-split short

$$0 \to \mathbf{C}_{\boldsymbol{\pi}} \to \mathbf{A}_{2m}/\mathbf{I} \to \mathbf{C}_{\boldsymbol{\pi}} \to 0.$$

$\square$

7.7

We now set,

$$W = W(2m) \otimes_{\mathbf{A}_{2m}} \mathbf{A}_{2m}/\mathbf{I}, \qquad w = w_{2m} \otimes 1,$$

and observe that by Theorem 3 there exists a non-split short exact sequence

$$0 \to W(\boldsymbol{\pi}) \xrightarrow{\iota} W \xrightarrow{\tau} W(\boldsymbol{\pi}) \to 0.$$

Recall that W^{2m} is the unique maximal submodule of W such that $W_{2m} \cap W^{2m} = 0$, and let

$$\tilde{w} = (x_2^- - 2aq^m x_1^- + a^2 q^{2m} x_0^-)w.$$

Lemma *We have*

$$\hat{\mathbf{U}}_q \tilde{w} \subset W^{2m},$$

and hence $\tau : W^{2m} \to W(\boldsymbol{\pi})^{2m}$ *is surjective.*

Proof The subspace $(\hat{\mathbf{U}}_q \tilde{w})_{2m}$ is the $\hat{\mathbf{U}}_q^0$-submodule generated by the elements $x_r^+ \tilde{w}$, $r \in \mathbf{Z}$ and hence it suffices to prove that $x_r^+ \tilde{w} = 0$. This means we must prove that

$$
0 = x_r^+ \tilde{w} = \begin{cases} (\phi_{r+2}^{\pm} - 2aq^m \phi_{r+1}^{\pm} + a^2 q^{2m} \phi_r^{\pm})w, & r \neq -2, -1, 0 \\[2mm] (\phi_2^{\pm} - 2aq^m \phi_1^{\pm} + a^2 q^{2m}(q^{2m} - q^{-2m}))w, & r = 0, \\[2mm] (\phi_1^{\pm} - 2aq^m(q^{2m} - q^{-2m}) + a^2 q^{2m} \phi_{r-1}^{\pm})w, & r = -1, \\[2mm] ((q^{2m} - q^{-2m}) - 2aq^m \phi_{-1}^{\pm} + a^2 q^{2m} \phi_{-2}^{\pm})w, & r = -2. \end{cases}
\tag{7.5}
$$

Using the functional equation,

$$
\phi^{\pm}(u) = \frac{\Lambda^{\pm}(q^{\mp 1} u)}{\Lambda^{\pm}(q^{\pm 1} u)},
$$

we see that (7.5) is equivalent to requiring,

$$
(1 - aq^m u)^2 \Lambda^+(q^{-1} u) w = \big((1 - aq^m u)^2 - a^2(q^{2m} - q^{-2m})u^2
$$
$$
- (q - q^{-1})\Lambda_1 u\big) \Lambda^+(qu)w,
\tag{7.6}
$$
$$
(u - aq^m)^2 \Lambda^-(q^{-1} u) w = \big((u - aq^m)^2 + (q^{4m} - 1)u^2
$$
$$
+ a^2 q^{2m}(q - q^{-1})\Lambda_{-1} u\big) \Lambda^-(qu)w,
\tag{7.7}
$$
$$
(q^{2m}\Lambda_1 + 2a[2m]q^m + a^2 \Lambda_{-1})w = 0.
\tag{7.8}
$$

Since W is a quotient of $W(2m)$ we have that $\mathrm{Ann}_{2m} w = 0$ and hence it suffices to prove that the equations in (7.6), (7.7),(7.8) are satisfied in $\mathbf{A}_{2m}$. It is easily seen that (7.6) is exactly (7.3). To see that the other two equations are satisfied, one recalls that we have the relation

$$
\Lambda_{-2m}\Lambda_r = \Lambda_{-2m+r}, \ \ 0 \leq r \leq 2m.
$$

Then (7.8) follows by taking the case of $r = 2m - 1$ in (7.3), which gives

$$
(q^{2m}\Lambda_1 + 2aq^m[2m])\Lambda_{2m} + a^2 \Lambda_{2m-1} = 0.
$$

Multiplying through by Λ_{2m}^{-1} gives the result. Equation (7.7) follows similarly by using the cases when $0 \leq r \leq 2m - 2$. $\square$

7.8

As a consequence of the preceding Lemma and Lemma 6.2 we have a non-split short exact sequence

$$
0 \rightarrow V(\boldsymbol{\pi}) \rightarrow W/W^{2m} \rightarrow V(\boldsymbol{\pi}) \rightarrow 0.
$$

The final step is to show that this extension is not isomorphic to $\mathbf{E}(V(\boldsymbol{\pi}))$. For this, we observe that if $\eta : W/W^{2m} \rightarrow \mathbf{E}(V(\boldsymbol{\pi}))$ is an isomorphism, then we must have $\eta(\bar{w}) = (c_1 v(\boldsymbol{\pi}), c_2 v(\boldsymbol{\pi}))$ for some $c_1 \neq 0$, where $\bar{w}$ is the image of w in W/W^{2m}. Since

$$
\eta(\phi_2^+ - 2aq^m \phi_1^+ + a^2 q^{2m}(q^{2m} - q^{-2m}))(v(\boldsymbol{\pi}), 0) = (0, (2\phi_2^+ - 2aq^m \phi_1^+)v(\boldsymbol{\pi})) \neq 0,
$$

and $\eta(\phi_2^+ - 2aq^m \phi_1^+ + a^2 q^{2m}(q^{2m} - q^{-2m}))\bar{w} = 0$, we have a contradiction. This completes the proof of Theorem 1 (iv).

8 Proof of Theorem 2

In this section we prove Theorem 2. We begin by noting some additional consequences of the results of the preceding sections.

8.1

Given a connected subset J of I let $\hat{\mathbf{U}}_q^J$ be the subalgebra of $\hat{\mathbf{U}}_q$ generated by the elements $x_{i,r}^{\pm}$, $h_{i,s}$, $k_i^{\pm 1}$, $i \in J$, $r \in \mathbf{Z}$, $s \in \mathbf{Z}^{\times}$. If R_J is the subset of the root system spanned by the elements α_j, $j \in J$, then $\hat{\mathbf{U}}_q^J$ is the quantum loop algebra associated to R_J with parameter q_J where $q_J = q^{\min\{d_j : j \in J\}}$. In the special case when $J = \{i\}$ we write $\hat{\mathbf{U}}_q^i$ for the algebra $\hat{\mathbf{U}}_q^J$ and note that $\hat{\mathbf{U}}_q^i$ is the quantum loop algebra associated to A_1 with parameter q_i. Let $\mathcal{P}_J^+$ be the submonoid of $\mathcal{P}^+$ consisting of I-tuples $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n)$ satisfying $\pi_i = 1$ if $i \notin J$. It is also convenient to regard $\mathcal{P}_J^+$ as a quotient of $\mathcal{P}^+$ via the map which sends

$$\boldsymbol{\pi} = (\pi_i)_{i \in I} \rightarrow \boldsymbol{\pi}_J = (\pi_j)_{j \in J}.$$

The category $\hat{\mathcal{F}}_J$ is defined in the obvious way and the elements of $\mathcal{P}_J^+$ index the isomorphism classes of the simple objects of $\hat{\mathcal{F}}_J$. The following is well known.

Lemma *Let $\boldsymbol{\pi} \in \mathcal{P}^+$. The $\hat{\mathbf{U}}_q^J$-submodule of $V(\boldsymbol{\pi})$ generated by $v(\boldsymbol{\pi})$ is isomorphic to $V(\boldsymbol{\pi}_J)$.* □

8.2

Proposition *Let J be a connected subset of I. There exists a canonical map of vector spaces*

$$Ext_{\hat{\mathcal{F}}}^1(V(\boldsymbol{\pi}), V(\boldsymbol{\pi})) \rightarrow Ext_{\hat{\mathcal{F}}_J}^1(V(\boldsymbol{\pi}_J), V(\boldsymbol{\pi}_J)), \quad [V] \rightarrow [V_J].$$

Moreover

$$[V_J] = 0 \iff (h_{j,r} - h_{j,r}(\boldsymbol{\pi}))v = 0, \quad v \in (V_J)_{wt\boldsymbol{\pi}}, \quad j \in J, \quad r \in \mathbf{Z}^{\times}.$$

Proof Let V be a non-trivial self extension of $V(\boldsymbol{\pi})$. By Lemma 4.1 we may choose $v \in V_{wt\boldsymbol{\pi}}$ such that $V = \hat{\mathbf{U}}_q v$. Setting $V_J = \hat{\mathbf{U}}_q^J v$, it is clear from Lemma 8.1 that V_J is a self-extension of $V(\boldsymbol{\pi}_J)$. If V is the trivial extension, then we set $V_J = V(\boldsymbol{\pi}_J) \oplus V(\boldsymbol{\pi}_J)$. It is now easily checked that $[V] \rightarrow [V_J]$ is well defined map of vector spaces. The second statement of the proposition is immediate from Lemma 4.1. □

The following is immediate.

Corollary *Let $\boldsymbol{\pi} \in \mathcal{P}^+$ and let $J_1, \ldots, J_m$ be a family of disjoint connected subsets of I such that $I = J_1 \cup \cdots \cup J_m$. We have an injective map of vector spaces,*

$$Ext_{\hat{\mathcal{F}}}^1(V(\boldsymbol{\pi}), V(\boldsymbol{\pi})) \rightarrow \bigoplus_{s=1}^m Ext_{\hat{\mathcal{F}}_{J_s}}^1(V(\boldsymbol{\pi}_{J_s}), V(\boldsymbol{\pi}_{J_s})).$$

In particular,

$$\dim Ext_{\hat{\mathcal{F}}}^1(V(\boldsymbol{\pi}), V(\boldsymbol{\pi})) \leq \sum_{i=1}^n \dim Ext_{\hat{\mathcal{F}}^i}^1(V(\boldsymbol{\pi}_{\{i\}}), V(\boldsymbol{\pi}_{\{i\}})).$$

□

Proof The only statement that requires explanation is that the map

$$\mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}), V(\boldsymbol{\pi})) \to \bigoplus_{s=1}^{m} \mathrm{Ext}^1_{\hat{\mathcal{F}}_{J_s}}(V(\boldsymbol{\pi}_{J_s}), V(\boldsymbol{\pi}_{J_s})),$$

is injective. Let V be a non-trivial extension of $V(\boldsymbol{\pi})$ and recall that we may choose $v \in V_{\mathrm{wt}\boldsymbol{\pi}}$ with $V = \hat{\mathbf{U}}_q v$. Moreover there exist $i \in I$ and $r \in \mathbf{Z}^\times$ such that $h_{i,r} v \notin \mathbf{C}v$. Choose $1 \le s \le m$ such that $i \in J_s$. Then V_{J_s} is a non-trivial extension of $\hat{\mathbf{U}}_q^J$ and hence

$$\mathrm{Ext}^1_{\hat{\mathcal{F}}_{J_s}}(V(\boldsymbol{\pi}_{J_s}), V(\boldsymbol{\pi}_{J_s})) \neq 0,$$

as required.   □

8.3

For $\boldsymbol{\pi} \in \mathcal{P}^+$, set

$$\mathrm{supp}\,\boldsymbol{\pi} = \{i \in I : \pi_i \neq 1\}.$$

Together with Proposition 7.2 we have now established the following.

Proposition *Let* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n) \in \mathcal{P}^+$ *be such that*

$$\pi_i = (1 - q_i^{m_i-1} a_i u)(1 - q_i^{m_i-3} a_i u) \cdots (1 - q_i^{-m_i+1} a_i u),$$

for some $m_i \in \mathbf{Z}_+$ *and* $a_i \in \mathbf{C}^\times$, $1 \le i \le n$. *Then*

$$\dim \mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}), V(\boldsymbol{\pi})) \le |\mathrm{supp}\,\boldsymbol{\pi}|.$$

In particular the space of self extensions of the Kirillov–Reshethikhin modules is one-dimensional.   □

8.4

We shall prove the following proposition in the rest of the section.

Proposition *Suppose that* $\boldsymbol{\pi} = (\pi_1, \ldots, \pi_n) \in \mathcal{P}^+$ *is such that* $\mathrm{supp}\,\boldsymbol{\pi} = \{1, n\}$ *and that*

$$\pi_i = (1 - q_i^{m_i-1} c_i u)(1 - q_i^{m_i-3} c_i u) \cdots (1 - q_i^{-m_i+1} c_i u) \ \ i = 1, n,$$

and $c_1, c_n \in \mathbf{C}^\times$. *If*

$$\frac{c_1}{c_n} = q^{\pm N}, \quad N = d_1 m_1 - \frac{1}{2} \sum_{i \neq j \in [1,n]} d_i a_{i,j} + d_n m_n$$

then

$$\dim \mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}), V(\boldsymbol{\pi})) = 1.$$

8.5

Assuming Proposition 8.4 the proof of the theorem is completed as follows. Given $\boldsymbol{\pi} \in \mathcal{P}^+$, let $\mathbf{A}_{\deg \pi_i}$ be the subalgebra of $\mathbf{A}_{\mathrm{wt}\boldsymbol{\pi}}$ generated by the elements $\{\Lambda_{i,r} : 1 \le r \le \lambda_i\}$ and $\Lambda_{i,r}^{-1}$. It is clear that

$$\mathbf{A}_\lambda \cong \bigotimes_{i \in I} \mathbf{A}_{\deg \pi_i},$$

where $\lambda = \sum_{i \in I} \deg \pi_i \omega_i$. Moreover if J is any connected subdiagram of I, and $\lambda_J = \sum_{j \in J} \deg \pi_j \omega_j$, then

$$\mathbf{A}_{\lambda_J} \cong \bigotimes_{j \in J} \mathbf{A}_{\deg \pi_j}.$$

Suppose that V is a non-split self extension of $V(\pi)$ so that

$$0 \to V(\pi) \xrightarrow{\iota} V \xrightarrow{\tau} V(\pi) \to 0.$$

We shall prove that $V \cong \mathbf{E}(V(\pi))$. We first prove that: if $0 \neq v \in V_\lambda$ is such that for some $i \in \text{supp}\,\pi$ we have

$$h_{i,r} v = h_{i,r}(\pi) v, \quad r \in \mathbf{Z}^\times,$$

then $v = a\iota(v(\pi))$ for some $a \in \mathbf{C}^\times$. For this, suppose that $i = i_s \in \text{supp}\,\pi$ and consider $J = [i_s, i_{s+1}]$ or $J' = [i_{s-1}, i_s]$. If $v \neq a\iota(v(\pi))$, then $\mathbf{E}(V(\pi_J))$ and V_J are non-isomorphic extensions of $V(\pi)$. It follows from Proposition 8.4 that $[V_J] = 0$ and hence $(V_J)_{\lambda_J}$ is an eigenspace for the action of $h_{i_{s+1},r}$ as well. A similar argument works for J' and hence we find that V_λ is an eigenspace for $h_{i,r}$ for all $i \in I$, $r \in \mathbf{Z}^\times$ contradicting Lemma 4.1.

To prove that $V \cong \mathbf{E}(V(\pi))$ we must prove that $V_\lambda \cong \mathbf{E}(V(\pi))_\lambda$ as $\mathbf{A}_\lambda$-modules. For this, note that as modules for $\mathbf{A}_{\deg \pi_{i_1}}$ we may assume that there exists a basis v_1, v_2 of V_λ such that $v_1 \mapsto (v(\pi), 0)$, $v_2 \mapsto (0, v(\pi))$ is an isomorphism. Suppose that this is not an isomorphism of $\mathbf{A}_{\deg \pi_{i_2}}$-modules. Then V_λ and $\mathbf{E}(V(\pi))_\lambda$ are not isomorphic as $\mathbf{A}_J$-modules where $J = [i_1, i_2]$ which then implies that V_J and $\mathbf{E}(V(\pi_J))$ are not isomorphic as $\hat{\mathbf{U}}_q^J$-modules. Since both extensions are non-trivial this contradicts Proposition 8.4. Iterating the argument gives the result that $V_\lambda \cong \mathbf{E}(V(\pi))_\lambda$ as $\mathbf{A}_\lambda$-modules and the proof of Theorem 2 is complete once we establish Proposition 8.4.

8.6

Notice that when π is as in Proposition 8.4, then $[1, n]$ is not of type D or E. We assume from now that $a_{i,j} \neq 0$ only if $i = j \pm 1$ and also without loss of generality that,

$$c_n = c_1 q^N, \quad N = d_1 m_1 - \frac{1}{2} \sum_{i \neq j \in [1,n]} d_i a_{i,j} + d_n m_n = d_1 m_1 + d_n m_n - \sum_{i=1}^{n-1} d_i a_{i,i+1}.$$

For $0 \leq i \leq n$, define elements $w_i \in V(\pi)$ recursively, by

$$w_0 = v(\pi), \quad w_i = x_{i,0}^- w_{i-1}. \tag{8.1}$$

Proposition *For $0 \leq i \leq n - 1$, we have an isomorphism of $\hat{\mathbf{U}}_q^i$-modules*

$$\hat{\mathbf{U}}_q^{i+1} w_i \cong \begin{cases} V(\pi(m_1, c_1)), & i = 0, \\ V(\pi(-a_{i+1,i}, c_{i+1})), & 1 \leq i \leq n - 2, \\ V(\pi(m_n - a_{n,n-1}, c_n q_n^{a_{n,n-1}})), & i = n - 1. \end{cases} \tag{8.2}$$

where $c_2 = c_1 q_1^{m_1}$ and $c_i = c_{i-1} q_{i-1}^{-a_{i-1,i-2}}$ for $2 < i < n$.

Proof To prove the proposition we see from Proposition 7.3 that we must show the following:

(i) $x_{i+1,r}^+ w_i = 0$,

(ii)

$$h_{1,r}w_0 = \frac{[rm_1]_1}{r}c_1^r w_0, \quad h_{n,r}w_{n-1} = q_n^{ra_{n,n-1}}\frac{[r(m_n - a_{n,n-1})]_n}{r}c_n^r w_{n-1} \quad (8.3)$$

$$h_{i,r}w_{i-1} = -\frac{[ra_{i,i-1}]_i}{r}c_i^r w_{i-1}, \quad 1 < i < n, \quad (8.4)$$

(iii)

$$(x_{1,1}^- - c_1 q_1^{m_1} x_{1,0}^-)w_0 = 0, \quad (x_{n,1}^- - c_n q_n^{m_n} x_{n,0}^-)w_{n-1} = 0, \quad (8.5)$$

$$(x_{j,1}^- - c_j q_j^{-a_{j,j-1}} x_{j,0}^-)w_{j-1} = 0, 1 < j < n. \quad (8.6)$$

Part (i) is trivial. We prove (ii) and (iii) simultaneously by an induction on i. Notice that induction begins at $i = 1$ by Lemma 4.1 and Proposition 7.3. Suppose that we have proved (ii) for $1 < i \le n$ and (iii) for $1 < i - 1 < n$. We prove (iii) for $i < n$ by showing that

$$x_{j,r}^+(x_{i,1}^- - c_i q_i^{-a_{i,i-1}} x_{i,0}^-)w_{i-1} = 0, \quad j \in I, r \in \mathbf{Z}_+. \quad (8.7)$$

For this, writing $\mathrm{wt}\pi = m_1\omega_1 + m_n\omega_n$, we have

$$x_{j,r}^- w_0 = 0, \quad j \ne 1, n, \quad r \in \mathbf{Z}. \quad (8.8)$$

We claim that

$$x_{j,r}^+ w_i = 0, \quad j \ne i, \quad (i, j) \ne (n, n-1), \quad r \in \mathbf{Z}^\times. \quad (8.9)$$

Since $[x_{j,r}^+, x_{k,0}^-] = 0$ if $j \ne k$, it is clear that the claim is true if $j > i$. If $j < i$, then we see that

$$x_{j,r}^+ w_i = x_{i,0}^- \cdots x_{j+1,0}^- \left(\frac{\phi_{j,r}^\pm}{q - q^{-1}}\right)x_{j-1,0}^- \cdots x_{1,0}^- w.$$

It is easily seen from the defining relations of $\hat{\mathbf{U}}_q$ that

$$[x_{j+1,0}^-, \phi_{j,r}^\pm] \in \sum_{s \in \mathbf{Z}} \hat{\mathbf{U}}_q^- x_{j+1,s}^-, \quad [x_{j+1,s}^-, x_{\ell,0}^-] = 0, \quad \ell < j + 1,$$

and (8.9) follows now by using (8.8). It is now clear that (8.7) follows from (8.9) for $i \ne j, j - 1$. For $i = j$ it holds from (8.4) and (7.2). If $i = j - 1$, then we use the same argument as the one given for establishing (8.9) to see that

$$x_{j-1,r}^+(x_{j,1}^- - c_j q_j^{-a_{j,j-1}} x_{j,0}^-)w_{j-1} \in \hat{\mathbf{U}}_q(x_{j,1}^- - c_j q_j^{-a_{j,j-1}} x_{j,0}^-)w_0 = 0.$$

This completes the proof of (iii) with $i < n$. For $i = n$ one proves using similar arguments that

$$x_{j,r}^+(x_{n,1}^- - c_n q_n^{m_n} x_{n,0}^-)w_{n-1} = 0, \quad j \in I, r \in \mathbf{Z}_+. \quad (8.10)$$

We omit the details.

It remains to prove that (ii) holds for $i + 1 \le n$. If $i + 1 < n$, then we use

$$h_{j+1,r}w_j = -\frac{[ra_{j+1,j}]_{j+1}}{r}x_{j,r}^- w_{j-1} = -\frac{[ra_{j+1,j}]_{j+1}}{r}(c_j q_j^{-a_{j,j-1}})^r w_j, \quad 1 \le j < n - 1,$$

while if $i + 1 = n$

$$
\begin{aligned}
h_{n,r} w_{n-1} &= \left(-\frac{[r a_{n,n-1}]_n}{r} x_{n-1,r}^- + x_{n-1,0}^- h_{n,r} \right) w_{n-2} \\
&= \left(-\frac{[r a_{n,n-1}]_n}{r} (c_{n-1} q_{n-1}^{-a_{n-1,n-2}})^r + \frac{[r m_n]_n}{r} c_n^r \right) w_{n-1} \\
&= \left(-\frac{[r a_{n,n-1}]_n}{r} (c_n q_n^{a_{n,n-1}} q_n^{-m_n})^r + \frac{[r m_n]_n}{r} c_n^r \right) w_{n-1}
\end{aligned}
$$

as required.   □

8.7

Suppose that V is a nontrivial self-extension of $V(\boldsymbol{\pi})$:

$$
0 \to V(\boldsymbol{\pi}) \xrightarrow{\iota} V \xrightarrow{\tau} V(\boldsymbol{\pi}) \to 0.
$$

To prove Proposition 8.4 we must show that we have an isomorphism of $\hat{\mathbf{U}}_q$-modules

$$
V \cong \mathbf{E}(V(\boldsymbol{\pi})).
$$

It is enough by Corollary 6.3 to prove that

$$
V_{\mathrm{wt}\boldsymbol{\pi}} \cong \mathbf{E}(V(\boldsymbol{\pi}))_{\mathrm{wt}\boldsymbol{\pi}},
$$

as modules for $\mathbf{A}_{\mathrm{wt}\boldsymbol{\pi}}$ and in fact it is enough to prove that they are isomorphic as $\hat{\mathbf{U}}_q^0$-modules. The proposition is a consequence of the following Lemma. Once the Lemma is proved it is clear that the map $\mathbf{E}(V(\boldsymbol{\pi}))_{\mathrm{wt}\boldsymbol{\pi}} \to V_{\boldsymbol{\pi}}$ sending $(v(\boldsymbol{\pi}), 0) \to \tilde{w}_0$ is an isomorphism of $\hat{\mathbf{U}}_q^0$-modules.

Lemma *There exists a basis* $\tilde{w}_0, w_0$ *of* $V_{\mathrm{wt}\boldsymbol{\pi}}$ *such that*

$$
h_{i,r} \tilde{w}_0 = h_{i,r}(\boldsymbol{\pi})(\tilde{w}_0 + r w_0), \quad h_{i,r} w_0 = h_{i,r}(\boldsymbol{\pi}) w_0, \quad i \in I, r \in \mathbf{Z}^\times.
$$

8.8

We shall use the following remark repeatedly in the proof of the Lemma. It is a special case of results proved elsewhere in this paper, we formulate it here in the precise form that it is used in the proof of the Lemma.

Remark Suppose that $\boldsymbol{\pi} \in \mathcal{P}^+$ is such that $\dim \mathrm{Ext}^1_{\hat{\mathcal{F}}}(V(\boldsymbol{\pi}), V(\boldsymbol{\pi})) = 1$ and let V be any self-extension of $V(\boldsymbol{\pi})$. Suppose that $\tilde{v}, v$ is a basis of $V_{\mathrm{wt}\boldsymbol{\pi}}$ such that $(h_{i,r} - h_{i,r}(\boldsymbol{\pi}))v = 0$ for all $i \in I$ and $r \in \mathbf{Z}^\times$. Then, there exists $z \in \mathbf{C}^\times$ such that

$$
h_{i,r} \tilde{v} = h_{i,r}(\boldsymbol{\pi})(\tilde{v} + z r v).
$$

Moreover, this implies that, if $V = V_1 \oplus V_2$ is a decomposition of V as a direct sum of $\mathbf{U}_q$-submodules isomorphic to $V(\boldsymbol{\pi})$ with $v \in V_1$ and $\tilde{v} \in V_2$, the projection of $x_{i,r}^- \tilde{v}$ onto V_1 is $r z x_{i,r}^- v$. Finally, V is nontrivial iff and only if $z \neq 0$.

8.9

Proof of Lemma 7.7 Let $\tilde{w}_0$ be such that $\tau(\tilde{w}_0) = v(\pi)$ and set $w_0 = \iota(v(\pi))$. If $j \neq 1, n$, then $\hat{\mathbf{U}}_q^j \tilde{w}_0$ is the trivial representation of $\hat{\mathbf{U}}_q^j$. If $j = 1, n$, then by Proposition 8.2 and the results of Sect. 7 we know that either

$$\hat{\mathbf{U}}_q^j \tilde{w}_0 \cong V(\pi(m_j, c_j)),$$

or

$$\hat{\mathbf{U}}_q^j \tilde{w}_0 \cong \mathbf{E}(V(\pi(m_j, c_j))).$$

In any case, since $\iota(v(\pi))$ is a joint eigenvector for $h_{i,r}$, $i \in I$, $r \in \mathbf{Z}^\times$, it follows from Sect. 8.8 that there exists $z_1, z_n \in \mathbf{C}^\times$ such that

$$h_{j,r}\tilde{w}_0 = h_{j,r}(\pi)(\tilde{w}_0 + rz_j w_0), \qquad x_{j,r}^- \tilde{w}_0 = (c_j q_j^{m_j})^r x_{j,0}^-(\tilde{w}_0 + rz_j w_0), \quad j = 1, n.$$

Our goal is to prove that we must have $z_1 = z_n$. Since V is non-split this means that we can assume $z_1 = z_n = 1$ which would establish the Lemma.

For $1 \leq i \leq n$, define elements

$$\tilde{w}_i = x_{i,0}^- \tilde{w}_{i-1}, \quad w_i = x_{i,0}^- w_{i-1}.$$

We now prove by induction on $1 \leq i < n$ that

$$h_{1,r}\tilde{w}_0 = \frac{[rm_1]_1}{r}c_1^r(\tilde{w}_0 + rz_1 w_0),$$

$$h_{i,r}\tilde{w}_{i-1} = -\frac{[ra_{i,i-1}]_i}{r}c_i^r(\tilde{w}_{i-1} + rz_1 w_{i-1}), \quad 1 < i < n.$$

For $i = 1$ this follows from the above, so induction starts. Suppose we have proved the above for $1 \leq i < n - 1$. In particular, if $i > 1$, it follows by applying Sect. 8.8 to $\hat{\mathbf{U}}_q^i w_{i-1}$ that,

$$x_{i,r}^- \tilde{w}_{i-1} = (c_i q_i^{-a_{i,i-1}})^r(\tilde{w}_i + rz_1 w_i) = c_{i+1}^r(\tilde{w}_i + rz_1 w_i).$$

The inductive step is completed by using the preceding equation and noting that

$$h_{i+1,r}\tilde{w}_i = h_{i+1,r}x_{i,0}^- \tilde{w}_{i-1} = -\frac{[ra_{i+1,i}]_{i+1}}{r}x_{i,r}^- \tilde{w}_{i-1}.$$

Now observe that $\hat{\mathbf{U}}_q^n$-submodule generated by $\tilde{w}_{n-1} + \hat{\mathbf{U}}_q^n w_{n-1}$ is a self extension of $V(\pi(m_n - a_{n,n-1}, c_n q_n^{a_{n,n-1}}))$. Hence, it follows from Sect. 8.8 that

$$h_{n,r}\tilde{w}_{n-1} = (c_n q_n^{a_{n,n-1}})^r \frac{[r(m_n - a_{n,n-1})]_n}{r}(\tilde{w}_{n-1} + z'r w_{n-1}) \quad \text{for some} \quad z' \in \mathbf{C}. \tag{8.11}$$

Further, since $[h_{n,r}, x_{i,s}^-] = 0$ if $i \neq n, n - 1$, we see that

$$h_{n,r}\tilde{w}_i = c_n^r \frac{[rm_n]_n}{r}(\tilde{w}_i + z_n r w_i) \quad \text{for all} \quad i < n - 1. \tag{8.12}$$

This implies

$$
\begin{aligned}
h_{n,r}\tilde{w}_{n-1} &= (-\frac{[ra_{n,n-1}]_n}{r}x^-_{n-1,r} + x^-_{n-1,0}h_{n,r})\tilde{w}_{n-2} \\
&= (-\frac{[ra_{n,n-1}]_n}{r}(c_{n-1}q_{n-1}^{-a_{n-1,n-2}})^r)(\tilde{w}_{n-1} + rz_1 w_{n-1}) \\
&\quad +\frac{[rm_n]_n}{r}c_n^r(\tilde{w}_{n-1} + rz_n w_{n-1}) \\
&= (c_n q_n^{a_{n,n-1}})^r \frac{[r(m_n - a_{n,n-1})]_n}{r}\tilde{w}_{n-1} \\
&\quad +r(-z_1\frac{[ra_{n,n-1}]_n}{r}(c_n q_n^{a_{n,n-1}}q_n^{-m_n})^r + z_n\frac{[rm_n]_n}{r}c_n^r)w_{n-1}.
\end{aligned}
$$

Comparing this with (8.11) we get

$$
-z_1[ra_{n,n-1}]_n(q_n^{r(a_{n,n-1}-m_n)})^r + z_n[rm_n]_n = z'q_n^{ra_{n,n-1}}[r(m_n - a_{n,n-1})]_n \quad (8.13)
$$

for all $r \in \mathbf{Z}$. It follows that $z_1 = z_n = z'$. ▫

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