

When strictly locally convex hypersurfaces are embedded

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Received: 27 February 2010 / Accepted: 2 June 2011 / Published online: 14 July 2011
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Abstract In this paper we will prove Hadamard–Stoker type theorems in the following ambient spaces: $\mathcal{M}^n \times \mathbb{R}$, where \mathcal{M}^n is a 1/4–pinched manifold, and certain Killing submersions, e.g., Berger spheres and Heisenberg spaces. That is, under the condition that the principal curvatures of an immersed hypersurface are greater than some non-negative constant (depending on the ambient space), we prove that such a hypersurface is embedded and we also study its topology.

1 Introduction

Hadamard proved a compact, strictly locally convex hypersurface immersed in \mathbb{R}^n is an embedded sphere [10]. Stoker then generalized this to complete immersed strictly convex hypersurfaces in \mathbb{R}^n : they are embedded spheres or \mathbb{R}^{n-1} [16]. Do Carmo and Warner [2] extended Hadamard’s theorem to \mathbb{S}^n and \mathbb{H}^n : such a compact hypersurface of \mathbb{S}^n is an embedded \mathbb{S}^{n-1} contained in a hemisphere of \mathbb{S}^n , and an embedded sphere in \mathbb{H}^n . Currier extended Stoker’s theorem to \mathbb{H}^n , assuming all the principal curvatures are at least one [4]. Alexander [1, Theorem 1] proved Hadamard’s theorem holds in any Hadamard manifold.

We consider convexity in other ambient spaces; distinct from the space forms. The Hadamard–Stoker theorem was proved in $\mathbb{H}^2 \times \mathbb{R}$: a complete immersed surface in $\mathbb{H}^2 \times \mathbb{R}$ of positive extrinsic curvature, is an embedded sphere or plane [8]. Also, in [9], the authors generalized the above result to Killing submersions over a strict Hadamard surface.

José M. Espinar is partially supported by Spanish MEC-FEDER Grant MTM2010-19821, and Regional J. Andalucía Grants P06-FQM-01642 and FQM325.

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For related results about locally convex hypersurfaces in non-negatively curved manifolds see [7], and [1] for locally convex hypersurfaces in non-positively curved manifolds.

In this paper we will prove Hadamard–Stoker type theorems in the following ambient spaces:

- $\mathcal{M}^n \times \mathbb{R}$, where \mathcal{M}^n is a simply connected $1/4$ -pinched manifold. We will see that the $1/4$ -pinched assumption is necessary (see Remark 2.6).
- Certain Killing submersions, e.g., Berger spheres and Heisenberg spaces.

We begin Sect. 2 by studying the embeddedness of a family of strictly convex hypersurfaces in $1/4$ -pinched simply connected manifolds \mathcal{M}^n , i.e., \mathcal{M}^n is a compact n -manifold whose sectional curvatures, K_s , are strictly positive. Also, if κ^+ and κ^- denote the maximum and minimum of the sectional curvatures of \mathcal{M}^n respectively, then, they verify $\kappa^-/\kappa^+ > 1/4$. Recall that, by Berger's Theorem, such a $1/4$ -pinched manifold is either homomorphic to a sphere or isometric to a symmetric space.

More precisely, we prove (cf. [14] for a related use of this idea):

Lemma 2.2 *Let D^n and \mathcal{M}^n be n -dimensional manifolds, assume \mathcal{M}^n is simply connected and D^n is compact with non-empty boundary Σ . Assume $g(t)$ and $h(t)$, $0 \leq t \leq 1$, are continuous families of metrics on D^n and \mathcal{M}^n , respectively, and each $h(t)$ is $1/4$ -pinched.*

Let $f_t : (D^n, g(t)) \rightarrow (\mathcal{M}^n, h(t))$ be isometric immersions, $0 \leq t \leq 1$, continuous in t . Suppose $f_t(\Sigma) := \Sigma(t)$ has positive principal curvatures for all t (w.r.t. the normal pointing into D^n).

If f_0 is an embedding, then so is f_t for all t .

The above lemma relies deeply on Eschenburg's injectivity radius estimate [7, Theorem C]. Eschenburg's estimated the hessian of the distance function to a point. We refer the reader to page 331 of the book by Petersen [12], for a discussion of this. Therefore, we can think of Lemma 2.2 as an argument in the opposite direction, that is, we deduce an embedding lemma from the injectivity radius estimate.

Using Lemma 2.2, we prove the following results in product spaces:

Theorem 2.4 *Let $\Sigma \subset \mathcal{M}^n \times \mathbb{R}$ be a locally strictly convex properly immersed connected hypersurface, where \mathcal{M}^n is a $1/4$ -pinched simply connected manifold. Then Σ is properly embedded and homeomorphic to the n -sphere or to \mathbb{R}^n . In the latter case, Σ has either a top end or a bottom end.*

Also,

Theorem 2.8 *Let $\Sigma \subset \mathcal{M}^n \times \mathbb{S}^1$ be a complete immersed hypersurface whose principal curvatures are greater than c at each point of Σ . Assume also that \mathcal{M}^n is a $1/4$ -pinched sphere, where κ^- and κ^+ denote the minimum and maximum of the sectional curvatures of \mathcal{M}^n , respectively. We normalize so that $\kappa^+ = 1$. If $c > 2$, then Σ is an embedded sphere.*

And for surfaces, we obtain:

Theorem 2.10 *Let $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$ be a complete connected surface with constant positive extrinsic curvature. Then Σ is a rotational sphere in $\mathbb{S}^2 \times \mathbb{R}$.*

We continue Sect. 3 considering strictly convex surfaces immersed in a Hadamard-Killing submersion (this is defined in 3.3). We first establish the necessary tools we will use in the proof of

Theorem 3.10 *Let $\Sigma \subset \mathcal{M}(\kappa, \tau)$ be a complete connected immersed surface with $k_i(p) > |\tau(p)|$ for all $p \in \Sigma$, where $\mathcal{M}(\kappa, \tau)$ is a Hadamard-Killing submersion. Then Σ is properly embedded. Moreover, Σ is homeomorphic to \mathbb{S}^2 or to \mathbb{R}^2 . In the later case, when Σ has no point p at which $N(p)$ is horizontal, Σ is a Killing graph over a convex domain of \mathbb{M}^2 .*

We remark that the above Theorem 3.10 gives a Hadamard–Stoker type theorem in Heisenberg space.

Section 4 is devoted to convex surfaces immersed in a Berger sphere. Here, we prove

Theorem 4.2 *Let $\Sigma \subset \mathbb{S}_B^3(\kappa, \tau)$ be a complete connected immersed surface with $|k_i(p)| \geq \left| \frac{\kappa - 4\tau^2}{4\tau} \right|$ for all $p \in \Sigma$. Then, Σ is embedded and homeomorphic to a sphere.*

Moreover, we will see how to prove Theorem 3.10 in the particular case of Heisenberg space, using the techniques developed in Sect. 4.

2 1/4-pinched manifolds

In this Section, we focus our attention on 1/4-pinched manifolds.

Definition 2.1 Let \mathcal{M}^n be a compact n -manifold whose sectional curvatures, K_s , are strictly positive. Let κ^+ and κ^- denote the maximum and minimum of the sectional curvatures on \mathcal{M}^n respectively. Then, we say that \mathcal{M} is 1/4-pinched if $\kappa^- / \kappa^+ > 1/4$.

First, we establish a Lemma about embeddedness of a family of closed strictly convex submanifolds in a 1/4-pinched manifold, which will be the key result for applications in what follows. This is of independent interest.

Lemma 2.2 *Let D^n and \mathcal{M}^n be n -dimensional manifolds, assume \mathcal{M}^n is simply connected and D^n is compact with non-empty boundary Σ . Assume $g(t)$ and $h(t)$, $0 \leq t \leq 1$, are continuous families of metrics on D^n and \mathcal{M}^n respectively, and each $h(t)$ is 1/4-pinched.*

Let $f_t : (D^n, g(t)) \rightarrow (\mathcal{M}^n, h(t))$ be isometric immersions, $0 \leq t \leq 1$, continuous in t . Suppose $f_t(\Sigma) := \Sigma(t)$ has positive principal curvatures for all t (w.r.t. the normal pointing into D^n).

If f_0 is an embedding, then so is f_t for all t .

Proof Since D^n is compact, there exists $\delta > 0$ such that f_t is an embedding for $0 \leq t < \delta$. It suffices to show f_δ is an embedding as well.

Suppose not, let $x, y \in D^n$ be distinct points such that $f_\delta(x) = f_\delta(y)$. If one of the points $\{x, y\}$ is not on Σ , then one can find open neighborhoods of x and y , U_x and V_y , such that $U_x \cap V_y = \emptyset$ and $f_\delta(U_x) \cap f_\delta(V_y)$ contains an open set of \mathcal{M}^n . But then f_t would not be an embedding for $t < \delta$, t close to δ ; a contradiction. Thus, both x and y are on Σ and $f_{\delta|_{\text{int}D^n}}$ is an embedding.

Consider D^n with the metric $h(\delta) = f_\delta^*(g(\delta))$. Let β be a length-minimizing curve for $(D^n, h(\delta))$ joining x to y . By strict local convexity of Σ , β lies in the interior of D^n except at its endpoints, hence is a geodesic of $h(\delta)$. Set $l = \text{Length}(\beta)$.

On the one hand, the injectivity radius of $\mathcal{M}^n(\delta)$, $\text{inj}(\mathcal{M}^n(\delta))$, bounds l from below as (see [7, Theorem C])

$$l/2 \geq \text{inj}(\mathcal{M}^n(\delta)) \geq \frac{\pi}{\sqrt{\kappa^+(\delta)}}.$$

On the other hand, the second-variation-of-length argument of Bonnet–Myers gives an upper bound of l as

$$l \leq \frac{\pi}{\sqrt{\kappa^-(\delta)}}.$$

Thus, joining the above inequalities, we obtain

$$2 \leq \frac{\sqrt{\kappa^+(\delta)}}{\sqrt{\kappa^-(\delta)}},$$

that is

$$\frac{\kappa^-(\delta)}{\kappa^+(\delta)} \leq 1/4,$$

which contradicts the $1/4$ -pinched assumption. This proves the Lemma.

Remark 2.3 The above $1/4$ -pinched assumption is necessary as the next example shows. Let $C(l)$ be the right cylinder of height l and radius 1 endowed with a flat metric. Close it up with two spherical caps S_i , $i = 1, 2$, (one on the top and another on the bottom) of radius 1 endowed with its standard metric. Now, smooth the surface $\mathcal{M}^2 = C \cup S_1 \cup S_2$ so that it is almost flat on the cylinder and almost close to 1 on the spherical caps, and has positive curvature.

Then, if l is large enough, it is not hard to see that we can consider a one parameter family of strictly convex compact curves $\alpha(t)$ that are embedded for $0 < t < t_0$ and they become immersed for $t > t_0$. One only has to consider how a family of concentric circles in \mathbb{R}^2 becomes immersed on a cylinder as the radius increases.

2.1 Applications

First we consider strictly convex hypersurfaces Σ , i.e., all the principal curvatures of Σ are positive for a choice of a unit normal to Σ , properly immersed in a product space $\mathcal{M}^n \times \mathbb{R}$, where \mathcal{M}^n is a $1/4$ -pinched simply connected manifold.

Theorem 2.4 *Let $\Sigma \subset \mathcal{M}^n \times \mathbb{R}$ be a locally strictly convex properly immersed connected hypersurface, where \mathcal{M}^n is a $1/4$ -pinched simply connected manifold. Then Σ is properly embedded and homeomorphic to the n -sphere or to \mathbb{R}^n . In the latter case, Σ has either a top end or a bottom end.*

First we define a top or bottom end. Let $\mathcal{M}^n \times \mathbb{R}$ be a product space and Σ a hypersurface in $\mathcal{M}^n \times \mathbb{R}$. Let $\pi_{\mathbb{R}} : \mathcal{M}^n \times \mathbb{R} \rightarrow \mathbb{R}$ be the usual projection. We denote by $h : \Sigma \rightarrow \mathbb{R}$ the *height function*, that is, $h := (\pi_{\mathbb{R}})|_{\Sigma}$.

Definition 2.5 Let $\Sigma \subset \mathcal{M}^n \times \mathbb{R}$ be a complete hypersurface. We say that Σ has a top end E (resp. bottom end) if for any divergent sequence $\{p_n\} \subset E$ the height function goes to $+\infty$ (resp. $-\infty$).

Proof of Theorem 2.4 Since Σ is locally strictly convex, the Gauss equation says that all the sectional curvatures of Σ at any point are positive. Thus, from Perelman's Soul Theorem [11], Σ is either compact or homeomorphic to \mathbb{R}^n . In the latter case, Σ has one topological end E . \mathcal{M}^n is compact and Σ is properly immersed so E must go up or down, otherwise $\Sigma \cap (\mathcal{M}^n \times \{0\})$ would not be compact; so E is a top or bottom end.

In a product space the leaves $\mathcal{M}^n \times \{t\}$ are totally geodesic, hence each connected component of $\Sigma \cap (\mathcal{M}^n \times \{t\})$ is compact and strictly convex when the intersection is transverse. Now, consider the foliation by horizontal *hyperplanes* given by $P(t) := \mathcal{M}^n \times \{t\}$ for $t \in \mathbb{R}$. Since Σ is either compact or has a top or bottom end, up to an isometry, we can assume that $\Sigma \subset \mathcal{M}^n \times [0, +\infty)$ and $P(0)$ is the horizontal hyperplane with a first contact point q , with Σ . At this point q , since Σ is strictly convex, Σ lies on one side of $P(0)$ and it is (locally) a graph over a domain of \mathcal{M}^n . Thus, there is $\epsilon > 0$ so that the hypersurfaces $C(t) := P(t) \cap U$ are embedded strictly convex hypersurfaces in \mathcal{M}^n for $0 < t < \epsilon$, where U is a neighborhood of q in Σ containing the first contact point that can be expressed as a graph. Perhaps, $P(t) \cap \Sigma$ has other components distinct from $C(t)$ for $0 < t < \epsilon$, but we only care how $C(t)$ varies as t increases. We also denote by $C(t)$ the continuous variation of the submanifolds $P(t) \cap \Sigma$ when $t > \epsilon$.

Then, it is easy to see that $C(t)$ either remains compact (non-empty) and embedded for all $t > 0$, or there exists \bar{t} such that $C(t)$ are compact for all $0 < t < \bar{t}$, the component $C(t)$ disappears for $t > \bar{t}$ and $C(\bar{t})$ is a point. $C(t)$ remains embedded by Lemma 2.2.

Thus, Σ is either a properly embedded \mathbb{R}^n with a top end or Σ is an embedded n -sphere. □

Remark 2.6 Actually, the 1/4-pinchd assumption is necessary. consider the surface $\mathcal{M}^2 = C(l) \cup S_1 \cup S_2$ given in Remark 2.3, with l large enough so that a family of concentric geodesic circles $S(r)$, $0 < r < r_0$, in \mathbb{R}^2 become immersed when we put them on the cylinder. That is, $S(r) \rightarrow$ point as $r \rightarrow 0$ and $S(r_0)$ is immersed and strictly convex in $C(l)$. Consider the product space $\mathcal{M}^2 \times \mathbb{R}$ and let $\Sigma := \bigcup_{0 \leq t \leq r_0} (S(t), t) \cup \bigcup_{r_0 \leq t \leq 2r_0} (S(2r_0 - t), t)$. Then, Σ is a strictly convex immersed surface in $\mathcal{M}^2 \times \mathbb{R}$.

Moreover, for strictly convex surfaces, from the Bonnet and Gauss–Bonnet Theorems, we get

Corollary 2.7 *Let Σ be a complete connected surface immersed in $\mathcal{M}^2 \times \mathbb{R}$ with extrinsic curvature bounded below by a positive constant, where \mathcal{M}^2 is a 1/4-pinchd sphere. Then Σ is an embedded sphere.*

Another application of Theorem 2.4 is the following

Theorem 2.8 *Let $\Sigma \subset \mathcal{M}^n \times \mathbb{S}^1$ be a complete immersed hypersurface whose principal curvatures are greater than c at each point of Σ . Assume also that \mathcal{M}^n is a 1/4-pinchd sphere. We normalize so that $\kappa^+ = 1$. If $c > 2$, then Σ is an embedded sphere.*

Proof First, since Σ is complete and its principal curvatures are greater than a positive constant, Σ is compact by Bonnet’s Theorem.

Now, lift Σ to a compact hypersurface $\tilde{\Sigma}$ in the universal covering space of $\mathcal{M}^n \times \mathbb{S}^1$, i.e., $\tilde{\Sigma} \subset \mathcal{M}^n \times \mathbb{R}$ is a compact hypersurface whose principal curvatures are greater than a positive constant. Thus, from Theorem 2.4, $\tilde{\Sigma}$ is an embedded sphere in $\mathcal{M}^n \times \mathbb{R}$.

Therefore, we can assume, up to an isometry, that $\tilde{\Sigma} \subset \mathcal{M}^n \times [0, +\infty)$ and $\mathcal{M}^n \times \{0\}$ has a first contact point $p \in \tilde{\Sigma} \cap \mathcal{M}^n \times \{0\}$.

Let D be the geodesic disk in \mathcal{M}^n centered at p of radius $r := \frac{\pi}{2\sqrt{\kappa^+}} - \epsilon$, $\epsilon > 0$ small enough to be chosen. Note that D is (topologically) a n -ball and $S := \partial D$ is strictly convex in \mathcal{M}^n with respect to the inward orientation. We claim that $\tilde{\Sigma} \subset D \times [0, +\infty)$.

Set $C(t) := \tilde{\Sigma} \cap (\mathcal{M}^n \times \{t\})$, $t > 0$. Then, $C(t)$ is an embedded strictly convex n -sphere for $0 < t < t_0$. For t close to 0, $C(t)$ is contained in D . Assume there exists $\bar{t} \in (0, t_0)$ so

that $C(\bar{i}) \cap S \neq \emptyset$. Set $q \in C(\bar{i}) \cap S$, then $d(p, q) \geq r$, where $d(p, q)$ denotes the distance in $\tilde{\Sigma}$.

Now, from the Gauss equation, the sectional curvatures \tilde{K}_s of $\tilde{\Sigma}$ are bounded below by c^2 . So, the Bonnet Theorem bounds the diameter of $\tilde{\Sigma}$ from above as

$$\text{diam}(\tilde{\Sigma}) < \pi/c.$$

Thus,

$$\frac{\pi}{2\sqrt{\kappa^+}} - \epsilon = r \leq d(p, q) \leq \text{diam}(\tilde{\Sigma}) < \pi/c,$$

but, since $\kappa^+ = 1$ and $c > 2$, we can choose ϵ small enough to contradict the above inequality. Thus, $\tilde{\Sigma} \subset D \times [0, +\infty)$.

Since $\tilde{\Sigma} \subset D \times \mathbb{R}$, we claim:

Claim 1 *For any geodesic $\gamma \subset D$ joining two points in the boundary $q_0, q_1 \in S$, if the geodesic plane (note that it is not complete since γ is not a complete geodesic) $P := \gamma \times \mathbb{R}$ and $\tilde{\Sigma}$ intersect transversally, then $\alpha := \tilde{\Sigma} \cap P$ is a strictly convex embedded Jordan curve in P . Moreover, α has geodesic curvature greater than c .*

Proof of Claim 1 Assume $P \cap \tilde{\Sigma}$ has two components (or more). Let C_1 and C_2 denote such components. Since $\tilde{\Sigma}$ is an embedded sphere, $C_i, i = 1, 2$, is a strictly convex embedded Jordan curve in P . Let $p_1 \in \Omega_1$ and $p_2 \in \Omega_2$ be points in the convex domains determined by C_1 and C_2 in P respectively. Let $\beta \subset \mathcal{M}^n \times \mathbb{R}$ be the geodesic joining p_1 and p_2 , that is, β is nothing but the straight line in P joining p_1 and p_2 (recall they are in the same vertical). Thus, β intersects C_1 and C_2 (note that P is totally geodesic and flat in $\mathcal{M}^n \times \mathbb{R}$), which is a contradiction since $\tilde{\Sigma}$ is a strictly convex embedded n -sphere.

Now, α has geodesic curvature greater than c , since the principal curvatures of $\tilde{\Sigma}$ are greater than c and P is totally geodesic. This proves Claim 1. □

Now, we claim that $\Pi(\tilde{\Sigma}) = \Sigma$ is embedded; here $\Pi : \mathcal{M}^n \times \mathbb{R} \rightarrow \mathcal{M}^n \times \mathbb{S}^1$ is the covering map. Assume Σ is not embedded, then, there exist two distinct points $p, q \in \tilde{\Sigma}$ that project to the same point. Also, p and q are contained in the same fiber in $\mathcal{M}^n \times \mathbb{R}$ and their distance (along the fiber) has to be an integer multiple of 1. Now, let γ be a geodesic in D passing through $\tilde{p} = \tilde{q}$, where \tilde{p} and \tilde{q} are the projections of p and q into \mathcal{M}^n respectively, so that $P := \gamma \times \mathbb{R}$ is always transverse to $\tilde{\Sigma}$. Such a geodesic clearly exists. To see this last assertion suppose $\gamma \times \mathbb{R}$ is tangent at some (finite) set of points to $\tilde{\Sigma}$ (note that the intersection, if tangential, has to be points since $\tilde{\Sigma}$ is strictly locally convex and $\gamma \times \mathbb{R}$ is totally geodesic). Therefore, at these tangential points, $\tilde{\Sigma}$ is locally a graph over $\gamma \times \mathbb{R}$. So, since the number of these points is finite ($\tilde{\Sigma}$ is compact), we can consider another geodesic γ' passing through p such that $\gamma' \times \mathbb{R}$ meets $\tilde{\Sigma}$ transversally.

Let $\alpha := \tilde{\Sigma} \cap P$ be the intersection curve, which is a simple Jordan curve in P with geodesic curvature greater than $c > 2$ (from Claim 1). As P is isometrically a strip in \mathbb{R}^2 since it is flat, α is contained in a circle of radius strictly less than $1/2$ in P . But, note that $p, q \in \alpha$ and the distance from p to q is (at least) one. This is a contradiction. Therefore, Σ is embedded. This proves the result. □

Also, by using Theorem 2.4, one can give an alternative, and more geometric, proof of [8, Theorem 7.3] when $\mathcal{M}^2 = \mathbb{S}^2$. First we shall recall the rotational surfaces in $\mathbb{S}^2 \times \mathbb{R}$ with constant extrinsic curvature (see [3]):

Proposition 2.9 *Let Σ be a complete immersed sphere of revolution (about the vertical line $\{1, 0, 0\} \times \mathbb{R}$) in $\mathbb{S}^2 \times \mathbb{R} \subset \mathbb{R}^4$ with constant extrinsic curvature $K_e > 0$, given by*

$$\psi_{K_e}(u, v) = (\cos k(u), \sin k(u) \cos v, \sin k(u) \sin v, h(u))$$

where $\alpha(u) = (\cos k(u), \sin k(u), 0, h(u))$ is the generating curve of Σ . Then,

- (i) Σ must be topologically a sphere and embedded.
- (ii) Σ stays in $\mathbb{D} \times \mathbb{R}$, where \mathbb{D} denotes the open hemisphere of \mathbb{S}^2 of center $(1, 0, 0)$.
- (iii) the generating curve is given by

$$k(u) = \cos^{-1} \exp\left(-\frac{1-u^2}{2K_e}\right)$$

$$h(u) = -\frac{1}{K_e} \int_1^u \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{K_e}\right) - 1}} du + C$$

where $-1 \leq u \leq 1$ and C is a real constant.

Also, $\mathbb{D} \times \{h_0\}$, where

$$h_0 = \frac{1}{K_e} \int_{-1}^0 \frac{\sqrt{1-u^2}}{\sqrt{\exp\left(\frac{1-u^2}{K_e}\right) - 1}} du + C,$$

divides Σ into two (upper and lower) symmetric parts.

Now, we can announce:

Theorem 2.10 *Let $\Sigma \subset \mathbb{S}^2 \times \mathbb{R}$ be a complete connected surface with constant positive extrinsic curvature. Then Σ is a rotational sphere in $\mathbb{S}^2 \times \mathbb{R}$.*

Proof From Theorem 2.4, Σ is an embedded sphere. So, we can assume that $\Sigma \subset \mathbb{S}^2 \times (0, +\infty)$. Do Alexandrov reflection w.r.t. $P(t) = \mathbb{S}^2 \times \{t\}$, $t > 0$. Then, since Σ is an embedded sphere, there exists $t_0 > 0$ so that Σ is a bi-graph over $\mathbb{S}^2 \times \{t_0\}$. Up to an isometry we can assume Σ is a bi-graph over $\mathbb{S}^2 \times \{0\}$.

Set $\alpha = \Sigma \cap \mathbb{S}^2 \times \{0\}$, this curve is a strictly convex simple Jordan curve, so, α is contained in some open hemisphere \mathbb{D} of \mathbb{S}^2 (see [2]). Let Ω be the compact domain bounded by α . Since Σ is a bi-graph over $\bar{\Omega}$ and α is contained in an open hemisphere \mathbb{D} , Σ is contained in $\mathbb{D} \times \mathbb{R}$. Thus, [3, Corollary 5.1] implies that Σ is a rotational sphere. \square

3 Hadamard-Killing submersions

In [9], the authors studied locally strictly convex surfaces immersed in a strict Hadamard-Killing submersion. We begin this Section reviewing the basis properties of a Hadamard-Killing submersion (see [9] for details).

3.1 On basic properties

Most of this part is contained in [9], but we need to introduce some concepts and properties in order to make this paper self contained.

First, we start with Hadamard surfaces. For more details on Hadamard manifolds with non positive sectional curvature see [6].

Let \mathbb{M}^2 be a Hadamard surface, that is, \mathbb{M}^2 is a complete, simply connected surface with Gaussian curvature $\kappa \leq 0$.

It is well known that given two points $p, q \in \mathbb{M}^2$, there exists a unique geodesic γ_{pq} joining p and q . We say that two geodesics γ, β in \mathbb{M}^2 are asymptotic if there exists a constant $C > 0$ such that $d(\gamma(t), \beta(t)) \leq C$ for all $t > 0$. To be asymptotic is an equivalence relation on the oriented unit speed geodesics or on the set of unit vectors of \mathbb{M}^2 . We will denote by $\gamma(+\infty)$ and $\gamma(-\infty)$ the equivalence classes of the geodesic $t \rightarrow \gamma(t)$ as t goes to plus infinity and minus infinity respectively. An equivalence class is called a point at infinity. $\mathbb{M}^2(\infty)$ denotes the set of all points at infinity for \mathbb{M}^2 and $\mathbb{M}_*^2 = \mathbb{M}^2 \cup \mathbb{M}^2(\infty)$.

The set $\mathbb{M}_*^2 = \mathbb{M}^2 \cup \mathbb{M}^2(\infty)$ admits a natural topology, called the cone topology, which makes \mathbb{M}_*^2 homeomorphic to the closed 2-disk in \mathbb{R}^2 .

When \mathbb{M}^2 is a Hadamard surface with sectional curvature bounded above by a negative constant then any two asymptotic geodesics γ, β satisfy that the distance between the two curves $\gamma|_{[t,+\infty)}, \beta|_{[t,+\infty)}$ is zero for any $t \in \mathbb{R}$. For each point $p \in \mathbb{M}^2$ and $x \in \mathbb{M}^2(\infty)$, there is a unique geodesic γ_{px} with initial condition $\gamma_{px}(0) = p$ and it is in the equivalence class of x . For each point $p \in \mathbb{M}^2$ we may identify $\mathbb{M}^2(\infty)$ with the circle \mathbb{S}^1 of unit vectors in $T_p\mathbb{M}^2$ by means of the bijection

$$G_p : \mathbb{S}^1 \subset T_p\mathbb{M}^2 \rightarrow \mathbb{M}^2(\infty)$$

$$v \mapsto \lim_{t \rightarrow +\infty} \gamma_{p,v}(t)$$

where $\gamma_{p,v}$ is the geodesic with initial conditions $\gamma_{p,v}(0) = p$ and $\gamma'_{p,v}(0) = v$. In addition the hypothesis on the sectional curvature (it is bounded above by a negative constant) yields there is an unique geodesic joining two points of $\mathbb{M}^2(\infty)$.

Given a set $\Omega \subseteq \mathbb{M}^2$, we denote by $\partial_\infty\Omega$ the set $\partial\Omega \cap \mathbb{M}^2(\infty)$, where $\partial\Omega$ is the boundary of Ω for the cone topology. We orient \mathbb{M}^2 so that its boundary at infinity is oriented counter-clockwise.

Let α be a complete oriented geodesic in \mathbb{M}^2 , then

$$\partial_\infty\alpha = \{\alpha^-, \alpha^+\}$$

where $\alpha^- = \lim_{t \rightarrow -\infty} \alpha(t)$ and $\alpha^+ = \lim_{t \rightarrow +\infty} \alpha(t)$. Here t is arc length along α . We identify α with its boundary at infinity, writing $\alpha = \{\alpha^-, \alpha^+\}$. Here we assume the curvature of M is bounded above by a negative constant.

Definition 3.1 Let θ_1 and $\theta_2 \in \mathbb{M}^2(\infty)$, we define the oriented geodesic joining θ_1 and θ_2 , $\alpha(\theta_1, \theta_2)$, as the oriented geodesic from $\theta_1 \in \mathbb{M}^2(\infty)$ to $\theta_2 \in \mathbb{M}^2(\infty)$.

Definition 3.2 Let α be an oriented complete geodesic in \mathbb{M}^2 . Let J be the standard counter-clockwise rotation operator. We call exterior set of α in \mathbb{M}^2 , $ext_{\mathbb{M}^2}(\alpha)$, the connected component of $\mathbb{M}^2 \setminus \alpha$ towards which $J\alpha'$ points. The other connected component of $\mathbb{M}^2 \setminus \alpha$ is called the interior set of α in \mathbb{M}^2 and denoted by $int_{\mathbb{M}^2}(\alpha)$.

We continue with Riemannian submersions. Let \mathcal{M} be a 3-dimensional Riemannian manifold so that it is a Riemannian submersion $\pi : \mathcal{M} \rightarrow \mathbb{M}^2$ over a surface (\mathbb{M}^2, g) with Gauss curvature κ , and the *fibers*, i.e., the inverse image of a point at \mathbb{M}^2 by π , are the trajectories of a unit Killing vector field ξ , and hence geodesics. Denote by $\langle \cdot, \cdot \rangle, \bar{\nabla}, \wedge, \bar{R}$ and $[\cdot, \cdot]$ the metric, Levi-Civita connection, exterior product, Riemann curvature tensor and Lie bracket in \mathcal{M} , respectively. Moreover, associated to ξ , we consider the operator $J : \mathfrak{X}(\mathcal{M}) \rightarrow \mathfrak{X}(\mathcal{M})$ given by

$$JX := X \wedge \xi, \quad X \in \mathfrak{X}(\mathcal{M}).$$

Given $X \in \mathfrak{X}(\mathcal{M})$, X is *vertical* if it is always tangent to fibers, and *horizontal* if always orthogonal to fibers. Moreover, if $X \in \mathfrak{X}(\mathcal{M})$, we denote by X^v and X^h the projections onto the subspaces of vertical and horizontal vectors respectively.

One can see that, under these conditions, (see [9, Proposition 2.6]) there exists a function $\tau : \mathcal{M} \rightarrow \mathbb{R}$ so that

$$\bar{\nabla}_X \xi = \tau X \wedge \xi, \tag{1}$$

and then, it is natural to introduce the following definition:

Definition 3.3 A Riemannian submersion over a Hadamard surface \mathbb{M}^2 , whose fibers are the trajectories of a unit Killing vector field ξ will be called a *Hadamard-Killing submersion* and denoted by $\mathcal{M}(\kappa, \tau)$, where κ is the Gauss curvature of \mathbb{M}^2 and τ is given by (1).

Let $\Sigma \subset \mathcal{M}(\kappa, \tau)$ be an oriented immersed connected surface. We endow Σ with the induced metric (*First Fundamental Form*), $\langle \cdot, \cdot \rangle_{|\Sigma}$, in $\mathcal{M}(\kappa, \tau)$, which we still denote by $\langle \cdot, \cdot \rangle$. Denote by ∇ and R the Levi-Civita connection and the Riemann curvature tensor of Σ , respectively, and S the shape operator, i.e., $SX = -\nabla_X N$ for all $X \in \mathfrak{X}(\Sigma)$ where N is the unit normal vector field along the surface. Then $II(X, Y) = \langle SX, Y \rangle$ is the *Second Fundamental Form* of Σ . Moreover, we denote by J the (oriented) rotation of angle $\pi/2$ on $T\Sigma$.

Set $\nu = \langle N, \xi \rangle$ and $T = \xi - \nu N$, i.e., ν is the normal component of the vertical field ξ , called the *angle function*, and T is the tangent component of the vertical field.

In order to establish our result, we shall introduce some definitions and properties about some particular surfaces in $\mathcal{M}(\kappa, \tau)$.

Definition 3.4 We say that $\Sigma \subset \mathcal{M}(\kappa, \tau)$ is a vertical cylinder over α if $\Sigma := \pi^{-1}(\alpha)$, where α is a curve on (\mathbb{M}^2, g) . If α is a geodesic, $\Sigma := \pi^{-1}(\alpha)$ is called a vertical plane.

One can check that a vertical plane is minimal, isometric to \mathbb{R}^2 and its principal curvature are bounded, in absolute value, by $|\tau(p)|$ at any point $p \in \Sigma$ (see [9, Proposition 2.10]).

We introduce a definition analogous to that given for complete geodesics in a Hadamard surface since the notions of interior and exterior domains of a horizontal oriented geodesic extend naturally to vertical planes.

Definition 3.5 Let $\mathcal{M}(\kappa, \tau)$ be a Hadamard-Killing submersion. For a complete oriented geodesic α in \mathbb{M}^2 we call, respectively, interior and exterior of the vertical plane $P = \pi^{-1}(\alpha)$ the sets

$$int_{\mathcal{M}(\kappa, \tau)}(P) = \pi^{-1}(int_{\mathbb{M}^2}(\alpha)), \quad ext_{\mathcal{M}(\kappa, \tau)}(P) = \pi^{-1}(ext_{\mathbb{M}^2}(\alpha))$$

Moreover, we will often use foliations by vertical planes of $\mathcal{M}(\kappa, \tau)$. We now make this precise.

Definition 3.6 Let $\mathcal{M}(\kappa, \tau)$ be a Hadamard-Killing submersion. Let P be a vertical plane in $\mathcal{M}(\kappa, \tau)$, and let $\beta(t)$ be an oriented horizontal geodesic in \mathbb{M}^2 , with t arc length along β , $\beta(0) = p_0 \in P$, $\beta'(0)$ orthogonal to P at p_0 and $\beta(t) \in \text{ext}_{\mathcal{M}(\kappa, \tau)}(P)$ for $t > 0$. We define the oriented foliation of vertical planes along β , denoted by $P_{\beta(t)}$, to be the vertical planes orthogonal to $\beta(t)$ with $P = P_{\beta(0)}$.

To finish, we will give the definition of a particular type of curve in a vertical plane. To do so, we recall a few concepts about Killing graphs in a Killing submersion (see [13]).

Under the assumption that the fibers are complete geodesics of infinite length, it can be shown (see also [15]) that such a fibration is topologically trivial, since the fibers are homeomorphic to either \mathbb{R} or \mathbb{S}^1 . Moreover, there always exists a global section

$$s : \mathbb{M}^2 \rightarrow \mathcal{M}(\kappa, \tau),$$

so, considering the flow ϕ_t of ξ , a trivialization of the fibration is given by the diffeomorphism

$$\begin{aligned} \mathbb{M}^2 \times \mathbb{R} &\rightarrow \mathcal{M}(\kappa, \tau) \\ (p, t) &\mapsto \phi_t(s(p)) \end{aligned}$$

Definition 3.7 Let $\pi : \mathcal{M}(\kappa, \tau) \rightarrow \mathbb{M}^2$ be a Killing submersion. Let $\Omega \subset \mathbb{M}^2$ be a domain. A Killing graph over Ω is a surface $\Sigma \subset \mathcal{M}(\kappa, \tau)$ which is the image of a section $s : \Omega \rightarrow \mathcal{M}(\kappa, \tau)$, with $s \in C^2(\Omega) \cap C^0(\bar{\Omega})$. We may also consider graphs, $\Sigma \subset \mathcal{M}(\kappa, \tau)$, without boundary.

Finally, we define:

Definition 3.8 Let P be a vertical plane in $\mathcal{M}(\kappa, \tau)$ and γ a complete embedded convex curve in P . We say that α is an untilted curve in P if there exists a point $p \in \alpha$ so that $\phi_t(p)$ is contained in the convex body P bounded by α for all $t > 0$ (or $t < 0$). Otherwise, we say that α is tilted.

3.2 The result

First, note that if $\Sigma \subset \mathcal{M}(\kappa, \tau)$ is an immersed surface with positive extrinsic curvature, then we can choose a globally defined unit normal vector field N so that the principal curvatures, i.e., the eigenvalues of the shape operator, are positive. We denote them by k_i for $i = 1, 2$.

We start with the following elementary result (see [9, Proposition 3.1]).

Proposition 3.9 *Let $\Sigma \subset \mathcal{M}(\kappa, \tau)$ be an immersed surface whose principal curvatures satisfy $k_i(p) > |\tau(p)|$ for all $p \in \Sigma$. Let P be a vertical plane. If Σ and P intersect transversally then each connected component C of $\Sigma \cap P$ is a strictly convex curve in P .*

Now, we have the necessary tools for establishing our Theorem.

Theorem 3.10 *Let $\Sigma \subset \mathcal{M}(\kappa, \tau)$ be a complete connected immersed surface with $k_i(p) > |\tau(p)|$ for all $p \in \Sigma$, where $\mathcal{M}(\kappa, \tau)$ is a Hadamard-Killing submersion. Then Σ is properly embedded. Moreover, Σ is homeomorphic to \mathbb{S}^2 or to \mathbb{R}^2 . In the later case, when Σ has no point p at which $N(p)$ is horizontal, Σ is a Killing graph over a convex domain of \mathbb{M}^2 .*

Proof As in [9, Theorem 3.3], we distinguish two cases depending on the existence of a point p in Σ where $N(p)$ is horizontal.

Case 1: *Suppose there is no point $p \in \Sigma$ where $N(p)$ is horizontal. Then, Σ is embedded and homeomorphic to the plane. Moreover, it is a Killing graph over a convex domain in \mathbb{M}^2 .*

Proof of Case 1: It is the same as Case 1 in [9, Theorem 3.3].

Case 2: *Suppose there is a point $p \in \Sigma$ so that $N(p)$ is horizontal. Then, Σ is embedded and homeomorphic to the sphere or to the plane.*

Proof of Case 2 By assumption N is horizontal at p and so, the tangent plane $T_p \Sigma$ is spanned by $\{\xi(p), X(p)\}$, where $X(p)$ is horizontal. Set $\bar{p} := \pi(p)$ and $v := d\pi_p(X(p))$. Let α be the complete geodesic in \mathbb{M}^2 with initial conditions $\alpha(0) = \bar{p}$ and $\alpha'(0) = v$. Set $P := \pi^{-1}(\alpha)$. Note that $p \in P \cap \Sigma$ and the principal curvatures of Σ at p are greater than the principal curvatures of P at p , thus Σ lies (locally around p) on one side of P . Without loss of generality we can assume that $N(p)$ points to $ext_{\mathcal{M}(\kappa, \tau)}(P)$ (see Definition 3.2), therefore, Σ lies (locally around p) in $ext_{\mathcal{M}(\kappa, \tau)}(P)$. Moreover, we parametrize the boundary at infinity by $B : [0, 2\pi] \rightarrow \mathbb{M}^2(\infty)$ so that $B(0) = \alpha^-$, $B(\pi) = \alpha^+$ and $\partial_\infty ext_{\mathcal{M}(\kappa, \tau)}(P) = B([0, \pi])$. Also, from now on, we identify the points at infinity with the points of the interval $[0, 2\pi]$.

Let N_P be the unit normal vector field along P pointing into $ext_{\mathcal{M}(\kappa, \tau)}(P)$. Then, there exists neighborhoods $V \subset P$ and $U \subset \Sigma$ so that

$$U := \{ \exp_q(f(q)N_P(q)) : q \in V \},$$

where $f : V \rightarrow \mathbb{R}$ is a smooth function and \exp is the exponential map in $\mathcal{M}(\kappa, \tau)$.

Let $P_\beta(t)$ be the foliation of vertical planes along β (see Definition 3.6). From Proposition 3.9 and the fact that locally Σ is (in exponential coordinates) a graph, there is $\epsilon > 0$ such that the curves $P_\beta(t) \cap U$ are embedded strictly convex curves (in $P_\beta(t)$) for $0 < t < \epsilon$. Perhaps, $P_\beta(t) \cap \Sigma$ has other components distinct from $C(t)$ for each $0 < t < \epsilon$, but we only care how $C(t)$ varies as t increases. We also denote by $C(t)$ the continuous variation of the curves $P_\beta(t) \cap \Sigma$ when $t < \epsilon$.

Here, we also distinguish two cases:

Case A: *If $C(t)$ remains compact for all $t > 0$, then Σ is properly embedded and homeomorphic to the sphere or to the plane.*

Proof of Case A: The proof is as Case A in [9, Theorem 3.3].

Case B: *If $C(t)$ becomes non-compact, then Σ is a properly embedded plane.*

Proof of Case B: First, note that Claims 1 and 2 in [9, Theorem 3.3] remain valid in this context with the same proof, i.e.,

Claim 1 $C(\bar{t})$ is tilted (see Definition 3.8).

Claim 2 $\partial_\infty \pi(C(\bar{t}))$ is one point.

Thus, at this point, and following the notation above, we have: Let $P_\beta(t)$ be the foliation of vertical planes along β , where $P(0)$ is the vertical plane over which Σ is locally a graph at $p \in \Sigma$. Moreover, such a graphical part of Σ is contained in $ext_{\mathcal{M}(\kappa, \tau)}(P)$. Note that $\beta(0) = \pi(p)$ and $\beta'(0) = d\pi_p(N(p))$.

Let γ_t be the complete geodesic in \mathbb{M}^2 passing through $\beta(t)$ and orthogonal to β at $\beta(t)$. Set $P_\beta(\bar{t}) = \pi^{-1}(\gamma_{\bar{t}})$, $\gamma_{\bar{t}} = \{\gamma_{\bar{t}}^-, \gamma_{\bar{t}}^+\}$, we parametrize the boundary at infinity by $B : [0, 2\pi] \rightarrow \mathbb{M}^2(\infty)$ so that $B(0) = \gamma_{\bar{t}}^-, B(\pi) = \gamma_{\bar{t}}^+$ and $\partial_\infty int_{\mathcal{M}(\kappa, \tau)}(P_\beta(\bar{t})) = B([0, \pi])$. Also, we already know that $\tilde{\Sigma}_1 := \bigcup_{0 \leq t \leq \bar{t}} C(t) \subset \Sigma$ is connected and embedded. By Claim 1, we may assume $\partial_\infty C(\bar{t}) = \{\gamma_{\bar{t}}^-\}$.

Set $\epsilon > 0$. Fix $t_\epsilon < \bar{t}$, close enough to \bar{t} , so that $\pi(C(t_\epsilon)) = \gamma_{t_\epsilon}([a, b])$ for some $a, b \in \mathbb{R}$ (recall that $C(t_\epsilon)$ is compact).

Denote by $\Gamma_\epsilon(\theta)$ the complete geodesic in \mathbb{M}^2 passing through $\gamma_{t_\epsilon}(r_\epsilon)$ and making an angle θ with γ_{t_ϵ} at $\gamma_{t_\epsilon}(r_\epsilon)$, $0 \leq \theta \leq \pi$. Fix $r_\epsilon < a$ so that $\tilde{\Sigma}_1 \subset int_{\mathcal{M}(\kappa, \tau)}(\pi^{-1}(\Gamma_\epsilon(\theta)))$ for all $0 < \theta \leq \pi/2$. We orient $\Gamma_\epsilon(\theta)$ so that $\Gamma_\epsilon(0)^- = \gamma_{t_\epsilon}^-$, i.e., so that $\Gamma_\epsilon(\theta)^-$ moves away from $\gamma_{t_\epsilon}^-$ as θ increase from 0. Also, set $Q(\theta, \epsilon) := \pi^{-1}(\Gamma_\epsilon(\theta))$.

Now, $C(t_\epsilon)$ is a connected component of $\Sigma \cap Q(\theta, \epsilon)$. Denote by $C'(\theta, \epsilon)$ the continuous variation of the curves $\Sigma \cap Q(\theta, \epsilon)$ when θ increases. Recall that $C(t_\epsilon) = C'(0, \epsilon)$. Since $C(t_\epsilon)$ is a compact embedded curve in the vertical plane $Q(0, \epsilon)$, there exists $\theta_0 > 0$ so that $C'(\theta, \epsilon)$ remains compact and embedded in $\tilde{\Sigma} \cap Q(\theta, \epsilon)$ for all $0 < \theta < \theta_0$.

Now, we have the following two possibilities:

(a) *There exists $\epsilon > 0$ so that $C'(\theta, \epsilon)$ remains compact for all θ satisfying $B^{-1}(\Gamma_\epsilon(0)^+) < B^{-1}(\Gamma_\epsilon(\theta)^+) < 2\pi$.*

If this were the case, arguing as in Case B.1 in [9, Theorem 3.3], Σ is properly embedded and homeomorphic to the plane.

(b) *For all $\epsilon > 0$ there exists θ_ϵ so that $C'(\theta_\epsilon, \epsilon)$ becomes non-compact.*

We will show that **(b)** is not possible. Letting $\epsilon \rightarrow 0$, we get the existence of two distinct points on the boundary at infinity $\eta^- < \eta^+$ so that $\Gamma_\epsilon(\theta_\epsilon)^- \rightarrow \eta^-$ and $\Gamma_\epsilon(\theta_\epsilon)^+ \rightarrow \eta^+$ as $\epsilon \rightarrow 0$. Note that $\eta^- = \gamma_{\bar{t}}^-$. Set $\eta = \{\eta^-, \eta^+\}$ (see Definition 3.1).

Let $T(s)$ be the foliation by vertical planes along a geodesic orthogonal to η so that $T(0) := \pi^{-1}(\eta)$. Take the orientation so that $int_{\mathcal{M}(\kappa, \tau)}(T(0)) = int_{\mathcal{M}(\kappa, \tau)}(\pi^{-1}(\eta))$.

By construction, $\tilde{\Sigma} \subset int_{\mathcal{M}(\kappa, \tau)}(T(0))$ where $\tilde{\Sigma} = \Sigma_1 \cup \tilde{\Sigma}_2$, here $\tilde{\Sigma}_2$ is the union of all the compact (embedded) components of $C(\theta, \epsilon)$ associated to the continuous variation of $C(t_\epsilon)$. Moreover, $T(s) \cap \tilde{\Sigma}$ is either a compact embedded strictly convex curve, or a point or empty, for all $s < 0$. Set $\tilde{C}(s)$ the continuous variation of $\tilde{\Sigma} \cap T(s)$. Thus, $\tilde{C}(0) = \lim_{s \rightarrow 0} \tilde{C}(s)$ should be an open embedded strictly convex curve in $T(0)$ so that $\partial_\infty \pi(C(0)) = \{\eta^-, \eta^+\}$. But this is impossible by Claim 2. So, **(b)** is proved.

This completes the proof of Theorem 3.10. □

4 Berger spheres

For a discussion of Berger spheres, we refer the reader to [17]. We will recall here only the necessary tools we will need, and for that, we follow [17]. A Berger sphere, denoted by $\mathbb{S}_B^3(\kappa, \tau)$, is the usual three dimensional sphere

$$\mathbb{S}^3 := \left\{ (z, w) \in \mathbb{C}^2 : |z|^2 + |w|^2 = 1 \right\},$$

endowed with the metric

$$\langle X, Y \rangle_{(\kappa, \tau)} := \frac{4}{\kappa} \left(\langle X, Y \rangle + \left(\frac{4\tau^2}{\kappa} - 1 \right) \langle X, V \rangle \langle Y, V \rangle \right),$$

here $\langle \cdot, \cdot \rangle$ denotes the standard round metric on \mathbb{S}^3 , $V : \mathbb{S}^3 \rightarrow \mathbb{S}^3$ is given by $V(z, w) := (iz, iw)$, and $\kappa > 0$ and $\tau \neq 0$ are constants.

The vertical Killing field is $\xi := \frac{\kappa}{4\tau^2} V$. Now, set $E_1(z, w) := (-\bar{w}, \bar{z})$ and $E_2(z, w) := (-i\bar{w}, i\bar{z})$. Then, $\{E_1, E_2, V\}$ is an orthonormal basis of $T\mathbb{S}_B^3(\kappa, \tau)$ which satisfies $|E_i|^2 = 4/\kappa, i = 1, 2$, and $|V|^2 = 16\tau^2/\kappa$. Moreover, the connection ∇ associated to $\langle \cdot, \cdot \rangle_{(\kappa, \tau)}$ is given by:

$$\begin{aligned} \nabla_{E_1} E_1 &= 0, & \nabla_{E_1} E_2 &= -V, & \nabla_{E_1} V &= \frac{4\tau^2}{\kappa} E_2 \\ \nabla_{E_2} E_1 &= V, & \nabla_{E_2} E_2 &= 0, & \nabla_{E_2} V &= -\frac{4\tau^2}{\kappa} E_1 \\ \nabla_V E_1 &= \left(\frac{4\tau^2}{\kappa} - 1 \right) E_2, & \nabla_V E_2 &= -\left(\frac{4\tau^2}{\kappa} - 1 \right) E_1, & \nabla_V V &= 0 \end{aligned} \tag{2}$$

First, we need to compute the principal curvatures of any equator of \mathbb{S}^3 as a submanifold of $\mathbb{S}_B^3(\kappa, \tau)$. To do so, we only need to compute the principal curvatures of the one parameter family of equators given by

$$\psi(x, y) = (\cos x \sin y, \cos x \cos y, \sin x \sin \theta, \sin x \cos \theta),$$

where $\theta \in [0, \pi/2]$ is a constant. Any other equator is a rotation and/or a translation (w.r.t. the Berger metric) of one in this family.

Proposition 4.1 *Let $\psi : [0, 2\pi] \times [0, 2\pi] \rightarrow \mathbb{S}_B^3(\kappa, \tau)$ be an equator given, for $\theta \in [0, \pi]$, by*

$$\psi(x, y) = (\cos x \sin y, \cos x \cos y, \sin x \sin \theta, \sin x \cos \theta).$$

Then, it is minimal, i.e., $H = 0$, and its extrinsic K_e curvature is

$$K_e := -\frac{4\tau^2(\kappa - 4\tau^2)^2 \cos^4 x}{(\kappa + 4\tau^2 - (\kappa - 4\tau^2) \cos 2x)^2}.$$

In particular, its principal curvatures k_i are bounded in absolute value by

$$|k_i| \leq \left| \left(\frac{\kappa}{4\tau^2} - 1 \right) \tau \right|.$$

The proof of the above Proposition 4.1 will be given in Sect. 5. Now, we have:

Theorem 4.2 *Let $\Sigma \subset \mathbb{S}_B^3(\kappa, \tau)$ be a complete connected immersed surface so that $|k_i(p)| \geq \left| \frac{\kappa - 4\tau^2}{4\tau} \right|$ for all $p \in \Sigma$. Then, Σ is embedded and homeomorphic to a sphere.*

Proof First, note that Σ is orientable by the assumption on the principal curvatures. Since the principal curvatures of the immersion are greater than or equal to those of any equator (see Proposition 4.1), Σ is locally on one side of its tangent equator at each point (note that the intersection can be more than one point, but, in any case, locally Σ is on one side). Thus, if we endow \mathbb{S}^3 with the usual round metric, this means that Σ has principal curvatures greater than or equal to zero at any point.

Claim 1 *If $\Sigma \subset \mathbb{S}_B^3(\kappa, \tau)$ is complete, then $\Sigma \subset (\mathbb{S}^3, \langle \cdot, \cdot \rangle)$ is complete.*

Proof of Claim 1 To see this, we can easily check that, for $X \in \mathfrak{X}(\mathbb{S}^3)$, we have

$$\begin{aligned} \langle X, X \rangle_{(\kappa, \tau)} &\leq \frac{4}{\kappa} \left(\|X\|^2 + \left| \frac{4\tau^2}{\kappa} - 1 \right| \langle X, V \rangle^2 \right) \\ &\leq a^2 \|X\|^2, \end{aligned}$$

where $\|\cdot\|$ denotes the norm w.r.t. $\langle \cdot, \cdot \rangle$, and

$$a^2 := \frac{4}{\kappa} \left(1 + \left| \frac{4\tau^2}{\kappa} - 1 \right| \right).$$

This proves Claim 1. □

That is, $\Sigma \subset (\mathbb{S}^3, \langle \cdot, \cdot \rangle)$ is a complete oriented connected immersed surface whose principal curvatures are non-negative at any point. Then, from [2, Theorem 1.1], Σ is embedded and homeomorphic to a sphere. Moreover, Σ has to be contained in an open hemisphere. Note that, from [2, Theorem 1.1], $\Sigma \subset (\mathbb{S}^3, \langle \cdot, \cdot \rangle)$ could be an equator, but our original surface immersed in $\mathbb{S}_B^3(\kappa, \tau)$ is not (since both of its principal curvatures are non-negative).

This finishes the proof. □

4.1 A note on Heisenberg space

One can prove Theorem 3.10 in the particular case of Heisenberg space, by using the same methods as in Theorem 4.2. Heisenberg space (see [5] for details), denoted by $\text{Nil}_3(\tau)$, is the usual 3-dimensional Euclidean space \mathbb{R}^3 endowed with the metric

$$g_N := dx^2 + dy^2 + (\tau(y dx - x dy) + dz)^2,$$

where (x, y, z) are the standard coordinates in \mathbb{R}^3 , and $\tau \neq 0$.

Then, it is not hard to see that the principal curvatures $k_i^P, i = 1, 2$, of any affine plane P , as a submanifold of $\text{Nil}_3(\tau)$, verify

$$|k_i^P| \leq \tau, \quad i = 1, 2.$$

Thus, if Σ is a complete immersed surface whose principal curvatures are greater than τ at any point, this implies that Σ is locally on one side of its tangent affine plane at that point. And so, it implies that $\Sigma \subset (\mathbb{R}^3, g_0)$, where g_0 is the standard metric in \mathbb{R}^3 , is locally strictly convex. Moreover, one can also check that a complete surface in $\text{Nil}_3(\tau)$ is complete in \mathbb{R}^3 . Thus, Stoker’s Theorem [16] implies that Σ is properly embedded and homeomorphic to the plane or to the sphere.

5 Proof of Proposition 4.1

Here, we include the proof of Proposition 4.1 for completeness. The proof is based on tedious and straightforward computations.

First, we compute the orthogonal basis $\{E_1, E_2, V\}$ along ψ . It is easy to check that

$$\begin{aligned}
 E_1 &= (-\sin x \sin \theta, \sin x \cos \theta, \cos x \sin y, -\cos x \cos y), \\
 E_2 &= (-\sin x \cos \theta, -\sin x \sin \theta, \cos x \cos y, \cos x \sin y), \\
 V &= (-\cos x \cos y, \cos x \sin y, -\sin x \cos \theta, \sin x \sin \theta)
 \end{aligned}$$

Second, we compute the partial derivatives of the immersion, which are given by:

$$\begin{aligned}
 \psi_x &= (-\sin x \sin y, -\cos y \sin x, \cos x \sin \theta, \cos \theta \cos x), \\
 \psi_y &= (\cos x \cos y, -\cos x \sin y, 0, 0).
 \end{aligned}$$

Now, we relate $\{\psi_x, \psi_y\}$ in terms of $\{E_1, E_2, V\}$, that is:

$$\begin{aligned}
 \psi_x &= -\cos(y + \theta)E_1 + \sin(y + \theta)E_2, \\
 \psi_y &= -\frac{1}{2} \sin(2x) \sin(y + \theta)E_1 - \frac{1}{2} \sin(2x) \cos(y + \theta)E_2 - \cos^2 x V.
 \end{aligned}$$

From the above equations, it is easy to see that the unit normal vector field is given by

$$N = -\alpha \left(\cos x \sin(y + \theta)E_1 + \cos x \cos(y + \theta)E_2 - \frac{\kappa}{4\tau^2} \sin x V \right),$$

where

$$\alpha = \sqrt{\frac{2\kappa\tau^2}{\kappa + 4\tau^2 - (\kappa - 4\tau^2) \cos(2x)}}$$

The next step is to compute the covariant derivatives $\nabla_{\psi_x} \psi_x, \nabla_{\psi_x} \psi_y = \nabla_{\psi_y} \psi_x$ and $\nabla_{\psi_y} \psi_y$. To do so, we use (2) and the expressions of ψ_x and ψ_y in terms of $\{E_1, E_2, V\}$. So, we get:

$$\begin{aligned}
 \nabla_{\psi_x} \psi_x &= 0 \\
 \nabla_{\psi_x} \psi_y &= \frac{(2\tau^2 - (\kappa - 2\tau^2) \cos(2x)) \sin \theta \sin(y + \theta)}{4\alpha} E_1 \\
 &\quad + \frac{(2\tau^2 - (\kappa - 2\tau^2) \cos(2x)) \sin \theta \cos(y + \theta)}{4\alpha} E_2 \\
 &\quad + \frac{\kappa}{8\alpha} \sin \theta \sin(2x) V \\
 \nabla_{\psi_y} \psi_y &= -\frac{(4\tau^2 - (\kappa - 4\tau^2) \cos(2x)) \sin \theta \sin(2x) \cos(y + \theta)}{8\alpha} E_1 \\
 &\quad + \frac{(4\tau^2 - (\kappa - 4\tau^2) \cos(2x)) \sin \theta \sin(2x) \sin(y + \theta)}{8\alpha} E_2.
 \end{aligned}$$

Thus, the coefficients of the first, I , and second, II , fundamental forms are given by:

$$\begin{aligned}
 I(\psi_x, \psi_x) &= \frac{4}{\kappa} \\
 I(\psi_x, \psi_y) &= 0 \\
 I(\psi_y, \psi_x) &= \frac{4\tau^2}{\kappa\alpha^2} \cos^2 x \\
 II(\psi_x, \psi_x) &= 0 \\
 II(\psi_x, \psi_y) &= 4\alpha(\kappa - 4\tau^2) \cos^3 x \\
 II(\psi_y, \psi_x) &= 0
 \end{aligned}$$

From the above expressions, we obtain that $H = 0$ and the extrinsic curvature K_e is given by

$$K_e = -\frac{\alpha^4 (\kappa - 4\tau^2)^2 \cos^4 x}{\tau^2 \kappa^2}.$$

Since $H = 0$ and the expression of the extrinsic curvature given above, we have

$$|k_i| \leq \left| \left(\frac{\kappa}{4\tau^2} - 1 \right) \tau \right|,$$

where k_i , $i = 1, 2$, are the principal curvatures. This finishes the proof of Proposition 4.1.

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