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Affine crystals, one-dimensional sums and parabolic Lusztig q-analogues

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Abstract This paper is concerned with one-dimensional sums in classical affine types. We prove a conjecture of Shimozono and Zabrocki (J Algebra 299:33–61, 2006) by showing they all decompose in terms of one-dimensional sums related to affine type A provided the rank of the root system considered is sufficiently large. As a consequence, any one-dimensional sum associated to a classical affine root system with sufficiently large rank can be regarded as a parabolic Lusztig q-analogue.

Contents

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1 Introduction

Consider λ and μ two partitions with at most n parts. Schur duality asserts that the Kostka number $K_{\lambda,\mu}$ counts both the dimension of the weight space μ in the irreducible $\mathfrak{sl}_n$ representation $V(\lambda)$ of highest weight λ and the multiplicity of $V(\lambda)$ in the tensor product $S^{\mu_1}(V)\otimes\cdots\otimes S^{\mu_n}(V)$ of the symmetric powers of the vector representation. Using the Weyl character formula, the Kostka numbers may be expressed in terms of the Kostant partition function. The q-deformation of this partition function gives rise to the Kostka polynomials. The Kostka polynomials are Kazhdan–Lusztig polynomials for the affine Weyl group and thus their coefficients are nonnegative integers, being dimensions of stalks of intersection cohomology sheaves on Schubert varieties in the affine flag variety. They also admit a nice combinatorial description in terms of the Lascoux-Schützenberger charge statistic on semistandard tableaux.

The Kostka polynomials also appear in the representation theory of the quantum affine algebra $U_q(\widehat{\mathfrak{sl}}_n)$. This was established by Nakayashiki and Yamada [24] by relating the charge statistic to the energy function, a fundamental grading defined on tensor products of Kashiwara crystals associated to Kirillov-Reshetikhin modules. Their result can be regarded as a quantum analogue of Schur duality. It is also worth mentioning that the energy function naturally appears in solvable lattice models in statistical physics.

The aim of this paper is to establish a generalization of the connection observed in [24]. On the weight multiplicity side, we consider parabolic Lusztig q-analogues. These are polynomials which quantize the branching coefficients given by the restriction of an irreducible representation of a simple Lie algebra $\mathfrak{g}_0$ to a Levi subalgebra. In the case that the Levi is the Cartan subalgebra, these are Lusztig's q-analogue of weight multiplicity, and in the further special case that $\mathfrak{g}_0 = \mathfrak{sl}_n$ they are Kostka polynomials. We consider *stable* parabolic Lusztig

q-analogues, which are defined when $\mathfrak{g}_0$ is of classical type and the weights λ and μ do not involve spin weights and stay away from a certain hyperplane. The stable parabolic Lusztig q-analogues have a well-defined large rank limit.

On the other side we consider tensor products of Kirillov-Reshetikhin modules, which afford the action of the quantum enveloping algebra associated to an affine algebra $\mathfrak{g}$. Their restriction to the canonical simple Lie subalgebra $\mathfrak{g}_0$ has a natural grading by the energy function, and taking isotypic components, we obtain polynomials called one-dimensional sums. A stable one-dimensional sum is one in which $\mathfrak{g}_0$ is of classical type and the tensor factors do not involve spin weights. They are so named because they are stable in the large rank limit.

Our key result is Theorem 10.1 (previously conjectured in [32]) giving the decomposition of the one-dimensional sums for any classical affine type in terms of those of affine type A. It then suffices to observe that this decomposition is the same as the decomposition of the stable parabolic Lusztig q-analogues obtained in [18].

Let us give a more detailed description of our results. For an affine Lie algebra $\mathfrak{g}$ with classical subalgebra $\mathfrak{g}_0$, there is a finite-dimensional $U'_q(\mathfrak{g})$-module with crystal graph given by the tensor product of Kirillov-Reshetikhin (KR) crystals

$$B = B^{r_1,s_1} \otimes \cdots \otimes B^{r_p,s_p}. \tag{1.1}$$

A KR crystal $B^{r,s}$ is indexed by a pair $(r,s) \in I_0 \times \mathbb{Z}_{>0}$ where $I = \{0, 1, \ldots, n\}$ is the affine Dynkin node set and $I_j = I \setminus \{j\}$ for $j \in I$. The crystal graph B has a I_0-equivariant grading by the coenergy function $\overline{D}_B : B \to \mathbb{Z}_{\geq 0}$. Given a dominant $\mathfrak{g}_0$-weight λ, the one-dimensional (1-d) sum $\overline{X}_{\lambda,B}(q)$ is the graded multiplicity of the irreducible highest weight I_0-crystal $B(\lambda)$ in B.

Throughout the paper we shall assume that $\mathfrak{g}$ belongs to one of the nonexceptional families of affine root systems. Fix the sequence $((r_1, s_1), \ldots, (r_p, s_p))$ representing B and the sequence $(d_1, d_2, \ldots, d_n)$ such that $\lambda = \sum_{i \in I_0} d_i \omega_i$ where ω_i is the ith fundamental weight of $\mathfrak{g}_0$. Throughout the paper $r \in I_0$ is called a spin node if $r = n$ when $\mathfrak{g}_0 = B_n, C_n$ and $r = n-1, n$ when $\mathfrak{g}_0 = D_n$. In order to take a large rank limit of the 1-d sum $\overline{X}_{\lambda,B}(q)$, we assume that no spin weights appear: none of the r_i are spin nodes, and $d_i = 0$ if $i \in I_0$ is a spin node. A "spinless" sequence representing B makes sense for large rank, and the sequence $(d_1, d_2, \ldots)$ for λ also makes sense provided that we append zeros as necessary. We associate with the dominant $\mathfrak{g}_0$-weight λ the partition (also denoted λ) that has d_i columns of height i for all i.

It was observed in [32] that the 1-d sum has a large rank limit which we shall call a stable 1-d sum, and moreover, that they fall into only four distinct kinds, which are labeled by the four partitions with at most two cells: $\varnothing$ (the empty partition), (1), (2), and (1, 1). We write $\overline{X}^{\diamondsuit}_{\lambda,B}(q)$ for the stable 1-d sum of kind $\diamondsuit \in \{\varnothing, (1), (2), (1,1)\}$.

We now describe the kind $\diamondsuit$ associated to each nonexceptional affine family. Let $\mathcal{P}^{\diamondsuit}$ denote the set of partitions whose diagrams can be tiled (without rotation) by the partition diagram of $\diamondsuit$. Then $\mathcal{P}^{\varnothing}$ is the singleton consisting of the empty partition, $\mathcal{P}^{(1)}$ is the set of all partitions, $\mathcal{P}^{(2)}$ is the set of partitions with even row lengths, and $\mathcal{P}^{(1,1)}$ is the set of partitions with even column lengths. Let $\mathcal{P}_n$ denote the set of partitions with at most n parts. Write $\mathcal{P}^{\diamondsuit}_n = \mathcal{P}^{\diamondsuit} \cap \mathcal{P}_n$. For $(r,s) \in I_0 \times \mathbb{Z}_{>0}$ such that n is large with respect to r ($n \geq r+2$ suffices) define $\mathcal{P}^{\diamondsuit}_n(r,s)$ to be the set of partitions $\lambda \in \mathcal{P}_n$ such that the 180° rotation of the complement of λ in the $r \times s$ rectangular partition (s^r), is in the set $\mathcal{P}^{\diamondsuit}$. We say the affine family of $\mathfrak{g}$ is of kind $\diamondsuit$ if the KR crystal $B^{r,s}$ (for n large with respect to r) has the I_0-decomposition

$$B^{r,s} \cong \bigoplus_{\lambda \in \mathcal{P}_n^{\diamondsuit}(r,s)} B(\lambda) \tag{1.2}$$

where $B(\lambda)$ is the irreducible $U_q(\mathfrak{g}_0)$-crystal of highest weight λ. All nonexceptional affine families are of one of the four kinds [32], and note that the kind depends precisely on the attachment of the affine Dynkin node 0 to the rest of the Dynkin diagram. We use the notation of [11].

$\diamondsuit$	$\mathfrak{g}$ of kind $\diamondsuit$
$\varnothing$	$A_n^{(1)}$
(1)	$D_{n+1}^{(2)}, A_{2n}^{(2)}$
(2)	$C_n^{(1)}$
$(1,1)$	$B_n^{(1)}, A_{2n-1}^{(2)}, D_n^{(1)}$

(1.3)

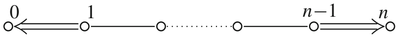
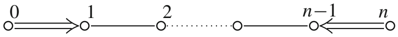
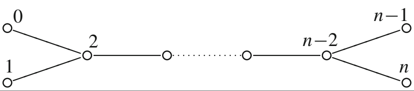
The main purpose of this paper is to establish a conjecture of [32]. To state this conjecture, we require some notation. The partition $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ (with $\lambda_{n-1} = \lambda_n = 0$ to avoid spin weights) encodes the dominant $\mathfrak{g}_0$-weight $\sum_i (\lambda_i - \lambda_{i+1})\omega_i$. For $\lambda \in \mathbb{Z}^n$ write $|\lambda| = \sum_i \lambda_i$ and $|B| := \sum_i r_i s_i$ for B as above. Finally, $c^{\nu}_{\lambda\delta}$ is the Littlewood–Richardson coefficient [22].

Conjecture 1.1 [32] *For* $\diamondsuit \in \{(1), (2), (1,1)\}$

$$\overline{X}^{\diamondsuit}_{\lambda,B}(q) = q^{\frac{|B|-|\lambda|}{|\diamondsuit|}} \sum_{\nu \in \mathcal{P}_n} \sum_{\delta \in \mathcal{P}_n^{\diamondsuit}} c^{\nu}_{\delta\lambda}\, \overline{X}^{\varnothing}_{\nu,B}(q^{\frac{2}{|\diamondsuit|}}). \tag{1.4}$$

Conjecture 1.1 gives a simple formula for all stable 1-d sums in terms of the type $A_n^{(1)}$ 1-d sums, which are fairly well-understood [29,31]. In the case that B has tensor factors of the form $B^{1,s}$, Conjecture 1.1 was proved in [30] for $\diamondsuit \in \{(1), (2)\}$ and in [20] for $\diamondsuit = (1,1)$. This is much easier than the general case: for the KR crystals $B^{1,s}$ all computations can be done explicitly.

The purpose of this paper is to prove Conjecture 1.1 in full generality (for arbitrary nonspin KR tensor factors). This is achieved in Theorem 10.1. We choose specific affine root systems $\mathfrak{g}^{\diamondsuit}$ for each $\diamondsuit \in \{(1), (2), (1,1)\}$. This choice, the classical subalgebra $\mathfrak{g}_0^{\diamondsuit}$, and the affine Dynkin diagram $X(\mathfrak{g}^{\diamondsuit})$ are given below.

$\diamondsuit$	$\mathfrak{g}^{\diamondsuit}$	$\mathfrak{g}_0^{\diamondsuit}$	$X(\mathfrak{g}^{\diamondsuit})$
(1)	$D_{n+1}^{(2)}$	B_n	
(2)	$C_n^{(1)}$	C_n	
$(1,1)$	$D_n^{(1)}$	D_n	

(1.5)

We shall call the three nonexceptional affine root systems $\mathfrak{g}^{\diamondsuit}$ *reversible*, since their affine Dynkin diagrams admit the automorphism

$$\sigma(i) = n - i \quad \text{for} \quad 0 \le i \le n. \tag{1.6}$$

Reversible root systems possess the following properties. There is an associated automorphism σ on KR crystals $B^{r,s}$ for r nonspin (Sect. 5.3). One then extends σ to tensor products of KR crystals by applying it to each factor. This map has a remarkable property: it sends all of the I_0-highest weight vertices in any tensor product B of nonspin KR crystals, into the subcrystal (called $\max(B)$) of I_0-components whose highest weights λ correspond to partitions with the maximum number of boxes (Theorem 7.1). Surprisingly, one can compute the precise change in the energy function (grading) under σ acting on I_0-highest weight vertices (Theorem 8.1). Finally, near the I_0-highest weight vertices in $\max(B)$, the crystal B looks like a similar tensor product B_A of type $A_{n-1}^{(1)}$ crystals and moreover the gradings coincide (Theorem 9.7). Along the way we make use of some I_0-crystal embeddings we call splitting maps: row splitting $B^{r,s} \to B^{r-1,s} \otimes B^{1,s}$ (Sect. 6.1) and box splitting $B^{1,s} \to (B^{1,1})^{\otimes s}$ (Sect. 6.3). These embeddings exist for any nonexceptional $\mathfrak{g}$ and nonspin $r \in I_0$. When applied to the first tensor factor in a tensor product of KR crystals, row splitting preserves energy (Theorem 11.3) and box splitting preserves coenergy. We also employ a kind of row splitting map in Sect. 4 which embeds the highest weight I_0-crystals $B(\lambda)$ of classical type, into a tensor product of I_0-crystals of the form $B(s\omega_1)$. This encoding, which we call the row tableau realization, allows us to see the shadow (that is, the image under σ) of the I_0-crystal decomposition of a KR crystal. For this purpose the well-known Kashiwara–Nakashima tableau realization [15] of $B(\lambda)$ is less illuminating.

2 Some classical multiplicity formulae

2.1 Notation on classical Lie groups

In the sequel G is one of the complex Lie groups GL_n, Sp_{2n}, SO_{2n+1}, or SO_{2n}. We follow the convention of [17] to realize G as a subgroup of GL_N and its Lie algebra $\mathfrak{g}$ as a subalgebra of $\mathfrak{gl}_N$ where

$$N = \begin{cases} n & \text{when } G = GL_n, \\ 2n & \text{when } G = Sp_{2n}, \\ 2n+1 & \text{when } G = SO_{2n+1}, \\ 2n & \text{when } G = SO_{2n}. \end{cases}$$

With this convention the maximal torus T_G of G and the Cartan subalgebra $\mathfrak{h}_G$ of $\mathfrak{g}$ coincide respectively with the subgroup and the subalgebra of diagonal matrices of G and $\mathfrak{g}$. Similarly the Borel subgroup B_G of G and the Borel subalgebra $\mathfrak{b}_G$ of $\mathfrak{g}$ coincide respectively with the subgroup and subalgebra of upper triangular matrices of G and $\mathfrak{g}$. There is an embedding of Lie algebras $\mathfrak{gl}_n \to \mathfrak{g}$ that restricts to an embedding $\mathfrak{h}_{GL_n} \to \mathfrak{h}_G$ of Cartan subalgebras and of their real forms $\mathfrak{h}_{GL_n}^{\mathbb{R}} \to \mathfrak{h}_G^{\mathbb{R}}$. Via restriction, there is an isomorphism of the real form of the weight lattice of $\mathfrak{g}$ with that of $\mathfrak{gl}_n$. For any $i \in \{1, \dots, n\}$, let $\varepsilon_i : \mathfrak{h}_{GL_n}^{\mathbb{R}} \to \mathbb{R}$ be the (i, i) matrix entry function. The functions $\{\varepsilon_i \mid i \in \{1, \dots, n\}\}$ form a $\mathbb{Z}$-basis of the weight lattice of $\mathfrak{gl}_n$, which we identify with $\mathbb{Z}^n$ via $\sum_{i=1}^n a_i \varepsilon_i \mapsto (a_1, a_2, \dots, a_n)$. In this way we may regard weights of $\mathfrak{g}$ as elements in $\mathbb{R}^n$. Let Σ_G^+ and R_G^+ be the sets of simple and positive roots of G, respectively. As usual ρ_G is the half sum of the positive roots of G. The set $\mathcal{P}_n$ is contained in the cone of dominant weights of G. Let $V^G(\lambda)$ be the finite dimensional irreducible G-module of highest weight λ. Let W_G be the Weyl group of G. Then $W_{GL_n} = S_n$ can be regarded as a subgroup of any W_G for $G = GL_n, Sp_{2n}, SO_{2n+1}$ or SO_{2n}. Given $\lambda \in \mathbb{Z}^n$ (the weight lattice of GL_n), let $\overline{\lambda} = (-\lambda_n, \dots, -\lambda_1) = -w_0^{A_{n-1}}(\lambda)$ where $w_0^{A_{n-1}} \in W_{GL_n}$ is the longest element and let $\overline{\mathcal{P}_n}$ denote the image of $\mathcal{P}_n$ under $\lambda \mapsto \overline{\lambda}$.

Note that for $\lambda \in \mathcal{P}_n$, the contragredient dual of the polynomial GL_n-module $V^{GL_n}(\lambda)$ is isomorphic to $V^{GL_n}(\overline{\lambda})$.

2.2 Decomposition of classical tensor product multiplicities

For $G = Sp_{2n}, SO_{2n+1}$, or SO_{2n}, $\diamondsuit \in \{(1), (2), (1, 1)\}$, and $\nu \in \mathcal{P}_n$, define the G-module

$$W_{\diamondsuit}^{G}(\nu) = \bigoplus_{\lambda \in \mathcal{P}_n} \bigoplus_{\delta \in \mathcal{P}_n^{\diamondsuit}} V^G(\lambda)^{\oplus c_{\delta\lambda}^{\nu}}.$$

The module $W_{\diamondsuit}^{G}(\nu)$ is defined specifically to have irreducible decomposition which mimics the decomposition of KR modules of kind $\diamondsuit$ into their classical components.

Let $\eta = (\eta_1, \dots, \eta_p)$ be a p-tuple of positive integers summing to n. Consider $\lambda \in \mathcal{P}_n$ and $(\mu^{(1)}, \dots, \mu^{(p)})$ a p-tuple of partitions such that $\mu^{(k)} \in \mathcal{P}_{\eta_k}$ for any $k = 1, \dots, p$. Define the coefficients $c_{\mu^{(1)},\dots,\mu^{(p)}}^{\lambda}$ and $\mathfrak{K}_{\mu^{(1)},\dots,\mu^{(p)}}^{\lambda,\diamondsuit}$ by

$$V^{GL_n}(\mu^{(1)}) \otimes \cdots \otimes V^{GL_n}(\mu^{(p)}) \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V^{GL_n}(\lambda)^{\oplus c_{\mu^{(1)},\dots,\mu^{(p)}}^{\lambda}} \tag{2.1}$$

$$W_{\diamondsuit}^{G}(\mu^{(1)}) \otimes \cdots \otimes W_{\diamondsuit}^{G}(\mu^{(p)}) \simeq \bigoplus_{\lambda \in \mathcal{P}_n} V^{G}(\lambda)^{\oplus \mathfrak{K}_{\mu^{(1)},\dots,\mu^{(p)}}^{\lambda,\diamondsuit}}. \tag{2.2}$$

We have the following proposition obtained by specializing at $q = 1$ Theorem 4.4.2 in [18]. It shows that the coefficients $\mathfrak{K}_{\mu^{(1)},\dots,\mu^{(p)}}^{\lambda,\diamondsuit}$ do not depend on the Lie group $G = Sp_{2n}, SO_{2n+1}$ or SO_{2n}.

Proposition 2.1 *For n sufficiently large, we have*

$$\mathfrak{K}_{\mu^{(1)},\dots,\mu^{(p)}}^{\lambda,\diamondsuit} = \sum_{\nu \in \mathcal{P}_n} \sum_{\delta \in \mathcal{P}_n^{\diamondsuit}} c_{\lambda,\delta}^{\nu}\, c_{\mu^{(1)},\dots,\mu^{(p)}}^{\lambda}.$$

We also recall Littlewood's formula [21] (see also [10]): Write $\widetilde{\mathcal{P}}_n$ for the set of pairs (γ^+, γ^-) such that $\gamma^-$and γ^+ are partitions with respectively r^+ and r^- nonzero parts, and $r^+ + r^- \leq n$. We identify each $(\gamma^+, \gamma^-) \in \widetilde{\mathcal{P}}_n$ with the GL_n-dominant weight $(\gamma_1^+, \dots, \gamma_{r^+}^+, 0^{n-r^+-r^-}, -\gamma_{r^-}^-, \dots, -\gamma_1^-) \in \mathbb{Z}^n$ and denote by $V^{GL_n}(\gamma^+, \gamma^-)$ the corresponding GL_n-module with highest weight (γ^+, γ^-). For all $\nu \in \mathcal{P}_n$ and $(\gamma^+, \gamma^-) \in \widetilde{\mathcal{P}}_n$

$$[\downarrow_{GL_n}^{G} V^G(\nu) : V^{GL_n}(\gamma^+, \gamma^-)] = \sum_{\delta \in \mathcal{P}_n^{\diamondsuit}, \kappa \in \mathcal{P}_n} c_{\gamma^+,\gamma^-}^{\kappa} c_{\delta,\kappa}^{\nu} \tag{2.3}$$

where $G = SO_{2n+1}, Sp_{2n}, SO_{2n}$ corresponds to $\diamondsuit = (1), (2), (1, 1)$ respectively, $\downarrow_{GL_n}^{G} V$ is a G-module V restricted to GL_n, and $[W : V]$ is the multiplicity of the irreducible module V in W.

Remark 2.2 For $\lambda, \mu, \nu \in \mathcal{P}_n$ with $n \geq \max(\ell(\lambda) + \ell(\mu), \ell(\nu)) + 2$, if $[V^G(\lambda) \otimes V^G(\mu) : V^G(\nu)] > 0$ then $|\nu| \leq |\lambda| + |\mu|$, and if equality occurs then the multiplicity is the LR coefficient $c_{\lambda\mu}^{\nu}$. This can be easily deduced from the following formula due to King [12]

$$[V^G(\lambda) \otimes V^G(\mu) : V^G(\nu)] = \sum_{\delta,\xi,\eta} c_{\delta,\xi}^{\nu} c_{\delta,\eta}^{\lambda} c_{\xi,\eta}^{\mu}$$

which holds in particular under the assumption $n \geq \max(\ell(\lambda) + \ell(\mu), \ell(\nu)) + 2$. The multiplicities are then independent of the group G considered.

3 Crystal generalities

3.1 Affine root systems

Let $I = \{0, 1, \dots, n\}$ be the set of nodes of the affine Dynkin diagram X with generalized Cartan matrix $(a_{ij})_{i,j\in I}$, all associated with the affine Lie algebra $\mathfrak{g}$. We use the labeling of affine Dynkin diagrams in [11]. Let $(a_0, \dots, a_n)$ and $(a_0^\vee, \dots, a_n^\vee)$ be the unique sequences of relatively prime positive integers such that

$$\sum_{j\in I} a_{ij} a_j = 0 \quad \text{for all } i \in I \tag{3.1}$$

$$\sum_{i\in I} a_i^\vee a_{ij} = 0 \quad \text{for all } j \in I. \tag{3.2}$$

Then

$$a_0^\vee = \begin{cases} 2 & \text{for } \mathfrak{g} = A_{2n}^{(2)} \\ 1 & \text{otherwise.} \end{cases} \tag{3.3}$$

Let P be the affine weight lattice, $P^* = \mathrm{Hom}_{\mathbb{Z}}(P, \mathbb{Z})$, and $\langle \cdot\,,\, \cdot \rangle : P^* \times P \to \mathbb{Z}$ the evaluation pairing. By definition P has $\mathbb{Z}$-basis denoted $\{\delta/a_0, \Lambda_0, \Lambda_1, \dots, \Lambda_n\}$ and P^* has dual $\mathbb{Z}$-basis $\{d, \alpha_0^\vee, \alpha_1^\vee, \dots, \alpha_n^\vee\}$. In particular

$$\langle \alpha_i^\vee\,,\, \Lambda_j \rangle = \chi(i = j) \quad \text{for } i, j \in I. \tag{3.4}$$

Here $\chi(P) = 1$ if P is true and $\chi(P) = 0$ otherwise. The Λ_i are called affine fundamental weights, δ is called the null root, d is called the degree derivation, and $\alpha_i^\vee$ are the simple coroots. Let $P^+ = \{\Lambda \in P \mid \langle \alpha_i\,,\, \Lambda\rangle \geq 0 \text{ for all } i \in I\}$ be the set of dominant weights. Define the elements $\alpha_j \in P$ (the simple roots) by

$$\alpha_j = \chi(j = 0)\, \delta/a_0 + \sum_{i\in I} a_{ij} \Lambda_i \quad \text{for } j \in I. \tag{3.5}$$

One may check that

$$\langle \alpha_i^\vee\,,\, \alpha_j \rangle = a_{ij} \quad \text{for all } i, j \in I \tag{3.6}$$

$$\delta = \sum_{j\in I} a_j \alpha_j \tag{3.7}$$

and that $\{\alpha_i \mid i \in I\}$ is a linearly independent set. The canonical central element $c \in P^*$ is defined by

$$c = \sum_{i\in I} a_i^\vee \alpha_i^\vee. \tag{3.8}$$

The *level* of a weight $\lambda \in P$ is defined by

$$\mathrm{lev}(\lambda) = \langle c\,,\, \lambda \rangle. \tag{3.9}$$

By (3.3) and (3.4) we have

$$\mathrm{lev}(\Lambda_i) = a_i^\vee \tag{3.10}$$

$$\mathrm{lev}(\Lambda_0) = 1. \tag{3.11}$$

Define the lattice $P' = P/(\mathbb{Z}\delta/a_0)$. For $i \in I$, write α_i' for the image of α_i under the natural projection $P \to P'$. Then $\alpha_0' = -\theta/a_0$. Since $\langle \alpha_i^\vee , \delta \rangle = 0$ for all $i \in I$, lev : $P' \to \mathbb{Z}$ is well-defined. Denote $P^0 = \{\lambda \in P' \mid \text{lev}\lambda = 0\}$.

Let $\mathfrak{g}_0$ be the simple Lie algebra obtained from $\mathfrak{g}$ by "omitting the 0 node". Let P_0 be the weight lattice of $\mathfrak{g}_0$. There is a natural projection $P \to P_0$ with kernel $\mathbb{Z}(\delta/a_0) \oplus \mathbb{Z}\Lambda_0$. Let $\omega_i = \pi(\Lambda_i)$ for $i \in I$ (so that $\omega_0 = 0$ by convention). Then $P_0 = \bigoplus_{i\in I_0} \mathbb{Z}\omega_i$. The dual lattice $P_0^* = \text{Hom}_{\mathbb{Z}}(P_0, \mathbb{Z})$ has dual $\mathbb{Z}$-basis denoted $\alpha_i^\vee$ for $i \in I_0$. There is a natural inclusion $P_0^* \to P^*$ defined by $\alpha_i^\vee \mapsto \alpha_i^\vee$ for $i \in I_0$. There is a natural projection $P' \to P_0$ with section

$$\begin{aligned} P_0 &\to P' \\ \omega_i &\mapsto \Lambda_i - \text{lev}(\Lambda_i)\Lambda_0 = \Lambda_i - a_i^\vee \Lambda_0 \quad \text{for } i \in I_0. \end{aligned} \tag{3.12}$$

The image of this section is P^0.

Let $P_0^+ = \{\lambda \in P_0 \mid \langle \alpha_i^\vee , \lambda \rangle \geq 0 \text{ for all } i \in I_0\}$ be the dominant weights in P_0. Let $Q_0 = \bigoplus_{i\in I_0} \mathbb{Z}\alpha_i$ be the sublattice of P_0 given by the root lattice.

3.2 The extended affine Weyl group and Dynkin automorphisms

The affine Weyl group W is the subgroup of the group $\text{Aut}(P)$ of linear automorphisms of P generated by the maps

$$s_i\lambda = \lambda - \langle \alpha_i^\vee , \alpha \rangle \alpha_i \quad \text{for } \lambda \in P \quad \text{and} \quad i \in I.$$

The action of W on P^* is defined by either of the equivalent formulae:

$$\begin{aligned} \langle w \cdot \mu , \ w \cdot \lambda \rangle &= \langle \mu , \lambda \rangle \quad \text{for } w \in W, \lambda \in P, \mu \in P^* \\ s_i\mu &= \mu - \langle \mu , \alpha_i \rangle \alpha_i^\vee \quad \text{for } \mu \in P^*, i \in I. \end{aligned}$$

We write W_0 for the Weyl group of $\mathfrak{g}_0$, which is the subgroup of W generated by s_i for $i \in I_0$. W_0 acts on P_0 and P_0^*.

Let $\text{Aut}(X)$ be the group of automorphisms of the affine Dynkin diagram X. Let $\tau \in \text{Aut}(X)$. By definition τ is a permutation of the Dynkin node set I of X such that there is a bond of multiplicity m from $i \in I$ to $j \in I$ if and only if there is a bond of multiplicity m from $\tau(i)$ to $\tau(j)$, for all $i, j \in I$. In particular,

$$a_{\tau(i)} = a_i \quad \text{and} \tag{3.13}$$

$$a_{\tau(i)}^\vee = a_i^\vee \quad \text{for } i \in I \tag{3.14}$$

$$a_{\tau(i),\tau(j)} = a_{ij} \quad \text{for } i, j \in I. \tag{3.15}$$

$\tau \in \text{Aut}(X)$ induces $\tau \in \text{Aut}(P)$ by $\tau(\delta/a_0) = \delta/a_0$ and $\tau(\Lambda_i) = \Lambda_{\tau(i)}$ for all $i \in I$. This satisfies $\tau(\alpha_i) = \alpha_{\tau(i)}$ for all $i \in I$. $\tau \in \text{Aut}(X)$ also induces $\tau \in \text{Aut}(P^*)$ by

$$\langle \tau(\mu) , \tau(\lambda) \rangle = \langle \mu , \lambda \rangle \quad \text{for all } \lambda \in P \quad \text{and} \quad \mu \in P^*. \tag{3.16}$$

It satisfies $\tau(d) = d$ and $\tau(\alpha_i^\vee) = \alpha_{\tau(i)}^\vee$ for all $i \in I$. $\tau \in \text{Aut}(X)$ induces an automorphism τ on W denoted $w \mapsto w^\tau$ where $s_i^\tau = s_{\tau(i)}$ for all $i \in I$.

Define the subset of *special nodes* $I^s \subset I$ to be the orbit of $0 \in I$ under $\text{Aut}(X)$. Every element of $\text{Aut}(X)$ is determined by its action on I^s. Let

$$\theta = \sum_{i\in I_0} a_i\alpha_i = \delta - a_0\alpha_0. \tag{3.17}$$

If $\mathfrak{g}$ is untwisted then θ is the highest root of $\mathfrak{g}_0$. Let $M \subset P_0$ be the sublattice generated by the W_0-orbit of θ/a_0:

$$M = \sum_{w \in W_0} \mathbb{Z}\, w \cdot (\theta/a_0). \tag{3.18}$$

The semidirect product $W_0 \ltimes P_0$ acts on P' by

$$(wt_\lambda) \cdot \Lambda = w(\Lambda + \mathrm{lev}(\Lambda)\lambda) \quad \text{for } w \in W_0, \lambda \in P_0, \quad \text{and} \quad \Lambda \in P' \tag{3.19}$$

where λ is regarded as an element of $P^0 \subset P'$ via (3.12) and t_λ is the translation corresponding to λ. We have

$$\begin{aligned} W &\cong W_0 \ltimes M \\ s_0 &\mapsto s_\theta t_{-\theta/a_0}. \end{aligned} \tag{3.20}$$

For each $\ell \in \mathbb{Z}$ the action of $W_0 \ltimes P_0$ on P' stabilizes the affine subspace $\ell\Lambda_0 + P^0 \subset P'$ of level ℓ weights. Therefore for each $\ell \in \mathbb{Z}$, the level ℓ action is defined by the representation $\pi_\ell : W_0 \ltimes P_0 \to \widehat{\mathrm{Aut}}(P_0)$ by affine linear automorphisms of P_0, given by

$$\begin{aligned} \pi_\ell(wt_\lambda) \cdot \beta &= -\ell\Lambda_0 + wt_\lambda(\ell\Lambda_0 + \beta) \\ &= w(\beta + \ell\lambda) \quad \text{for } w \in W_0, \lambda, \beta \in P_0. \end{aligned} \tag{3.21}$$

For $r \in I_0$ define [9]

$$c_r = \max(1, a_r/a_r^\vee). \tag{3.22}$$

Remark 3.1 We have $c_r = 1$ except that $c_r = 2$ for $\mathfrak{g} = B_n^{(1)}$ with $r = n$, $\mathfrak{g} = C_n^{(1)}$ with $1 \le r \le n-1$, $\mathfrak{g} = F_4^{(1)}$ with $r \in \{3, 4\}$, and $c_r = 3$ if $\mathfrak{g} = G_2^{(1)}$ with $r = 2$. In particular $c_i = 1$ for $i \in I^s$.

Define the sublattices of P_0 given by

$$M' = \bigoplus_{i \in I_0} \mathbb{Z} c_i \alpha_i \tag{3.23}$$

$$\tilde{M} = \bigoplus_{i \in I_0} \mathbb{Z} c_i \omega_i. \tag{3.24}$$

We have $\tilde{M} \supset M \supset M'$ with $M = M'$ except for $\mathfrak{g} = A_{2n}^{(2)}$ where $M' \subset M$ is a sublattice of index 2. We define an injective group homomorphism

$$\tilde{M}/M \hookrightarrow \mathrm{Aut}(X) \tag{3.25}$$

with image denoted Σ. First, there is a bijection $I^s \to \tilde{M}/M$ given by $i \mapsto c_i\omega_i + M$. Subtraction by $c_i\omega_i + M$ induces a permutation of $\tilde{M}/M$. The induced permutation of I^s under the above bijection, extends to $\tau^i \in \mathrm{Aut}(X)$. We define $\Sigma = \{\tau^i \mid i \in I^s\}$; it is the group of *special automorphisms*.

Define the extended affine Weyl group (in particular for twisted affine types) by

$$\tilde{W} = W \rtimes \Sigma \tag{3.26}$$

via $\tau w \tau^{-1} = w^\tau$ for $\tau \in \Sigma$ and $w \in W$. We have $\tilde{W} \cong W_0 \ltimes \tilde{M}$ with

$$\tau^i = w_0^{\omega_i} t_{-c_i\omega_i} \quad \text{for } i \in I^s, \text{ where} \tag{3.27}$$

$$w_0^\lambda \in W_0 \text{ is the shortest element such that } w_0^\lambda \lambda = w_0\lambda. \tag{3.28}$$

Remark 3.2 In untwisted type one may identify M with the coroot lattice $Q_0^\vee$ and $\tilde{M}$ with the coweight lattice $P_0^\vee$, although these identifications may involve some uniform dilation.

Example 3.3

$$\begin{array}{|c||c|c|c|c|c|c|c|}\hline \mathfrak{g} & A_n^{(1)} & B_n^{(1)} & C_n^{(1)} & D_n^{(1)} & A_{2n-1}^{(2)} & A_{2n}^{(2)} & D_{n+1}^{(2)} \\ \hline I^s & \{0,1,\dots,n\} & \{0,1\} & \{0,n\} & \{0,1,n-1,n\} & \{0,1\} & \{0\} & \{0,n\} \\ \hline \end{array} \tag{3.29}$$

For $A_n^{(1)}$ and $i \in I^s$, τ^i subtracts i mod $n+1$.

For $D_n^{(1)}$, in terms of permutations of I^s, are defined as follows. τ^0 is the identity and $\tau^1 = (0,1)(n-1,n)$. If n is odd, $\tau^{n-1} = (0,n,1,n-1)$ and $\tau^n = (0,n-1,1,n)$ and if n is even, $\tau^{n-1} = (0,n-1)(1,n)$ and $\tau^n = (0,n)(1,n-1)$.

Note that $\tilde{M}/M$ admits an involution given by negation. The corresponding affine Dynkin involution is given as follows. Let $w_0 \in W_0$ be the longest element. The map $\alpha \mapsto -w_0\alpha$ is an involution on the set of positive roots of $\mathfrak{g}_0$ that sends sums to sums, and therefore restricts to an involution on the set of simple roots. So there is an involutive automorphism of the Dynkin diagram of $\mathfrak{g}_0$ denoted $i \mapsto i^*$, defined by

$$-w_0\alpha_i = \alpha_{i^*} \quad \text{for } i \in I_0. \tag{3.30}$$

This extends to an element denoted $* \in \mathrm{Aut}(X)$ by defining $0^* = 0$. The induced automorphism of P is given by

$$\lambda \mapsto -w_0\lambda \quad \text{for } \lambda \in P. \tag{3.31}$$

In particular

$$-w_0\omega_i = \omega_{i^*} \qquad \text{for } i \in I_0. \tag{3.32}$$

By (3.13), (3.14), and (3.22), we see that

$$c_{i^*} = c_i \quad \text{for } i \in I. \tag{3.33}$$

Therefore $-w_0c_i\omega_i = c_{i^*}\omega_{i^*}$. Since $w_0c_i\omega_i + M = c_i\omega_i + M$, we have $c_{i^*}\omega_{i^*} + M = -c_i\omega_i + M$ in the group $\tilde{M}/M \cong \Sigma$. It follows that for all $i \in I^s$, negation in $\tilde{M}/M$ corresponds to the involution $i \mapsto i^*$ on I^s, and that

$$\begin{aligned} \tau^i(0) &= i^* \\ (w_0^{\omega_i})^{-1} &= w_0^{\omega_{i^*}}. \end{aligned} \tag{3.34}$$

Example 3.4 We have $i^* = i$ except in the following cases. For A_{n-1} we have $i^* = n-i$. For D_n and n odd, $(n-1)^* = n$ and $n^* = n-1$. For E_6 $i \mapsto i^*$ is the unique nontrivial automorphism.

3.3 Crystals

Let $\mathfrak{g}$ be an affine Lie algebra. We consider the following categories of crystal graphs of modules over a quantum affine algebra $U_q'(\mathfrak{g})$: $\mathcal{C}_h(\mathfrak{g})$, direct sums of affine highest weight crystals, and $\mathcal{C}(\mathfrak{g})$, tensor products of Kirillov–Reshetikhin (KR) crystals. For KR crystals we refer to [3]. Let $\mathcal{C}_h(\mathfrak{g}_0)$ be the category of direct sums of crystal graphs of highest weight $U_q(\mathfrak{g}_0)$-modules.

Let B be a crystal in one of the above categories. B is a graph with vertex set also denoted B and directed edges labeled by the elements of the set K of Dynkin nodes of $\mathfrak{g}$. We call

B a K-crystal. For $K' \subset K$ write $B_{K'}(b)$ for the K'-connected component of $b \in B$, that is, the connected component of the graph in which all directed edges are removed except those labeled by K'. For $i \in K$, each $\{i\}$-connected component is a finite directed path called an i-string. Then for $b \in B$, $f_i(b)$ (resp. $e_i(b)$) is the next (resp. previous) vertex on the i-string of b if it exists, and is declared to be the special symbol 0 otherwise. Let $\varphi_i(b)$ and $\varepsilon_i(b)$ denote the number of steps to the end (resp. start) of the i-string of b. For a sequence $\mathbf{a} = (i_1, \dots, i_p)$ of indices in K define

$$e_{\mathbf{a}}(b) = e_{i_1}(e_{i_2}(\cdots e_{i_p}(b)\cdots))$$

and $f_{\mathbf{a}}(b)$ similarly.

For $B \in \mathcal{C}(\mathfrak{g})$ or $B \in \mathcal{C}_h(\mathfrak{g})$, let $K = I$ and define the functions $\varphi, \varepsilon : B \to P'$ by

$$\varphi(b) = \sum_{i \in I} \varphi_i(b)\Lambda_i \tag{3.35}$$

$$\varepsilon(b) = \sum_{i \in I} \varepsilon_i(b)\Lambda_i. \tag{3.36}$$

For $B \in \mathcal{C}_h(\mathfrak{g}_0)$ we have $\varphi, \varepsilon : B \to P_0$ with I replaced by I_0 and Λ_i replaced by ω_i.

For $B \in \mathcal{C}(\mathfrak{g})$ or $B \in \mathcal{C}_h(\mathfrak{g})$ we define the weight function $\mathrm{wt} : B \to P'$ by

$$\mathrm{wt}(b) = \varphi(b) - \varepsilon(b). \tag{3.37}$$

For $B \in \mathcal{C}(\mathfrak{g})$ the values of wt lie in the level zero sublattice $P^0 \subset P'$. For $B \in \mathcal{C}_h(\mathfrak{g}_0)$ we have $\mathrm{wt} : B \to P_0$ defined by (3.37).

For $B \in \mathcal{C}(\mathfrak{g})$ or $B \in \mathcal{C}_h(\mathfrak{g})$ we have

$$\mathrm{wt}(e_i(b)) = \mathrm{wt}(b) + \alpha'_i \quad \text{for } i \in I \text{ if } e_i(b) \neq 0 \tag{3.38}$$

$$\mathrm{wt}(f_i(b)) = \mathrm{wt}(b) - \alpha'_i \quad \text{for } i \in I \text{ if } f_i(b) \neq 0. \tag{3.39}$$

For $B \in \mathcal{C}_h(\mathfrak{g}_0)$ the same conditions hold with $\alpha'_i = \alpha_i$ and $i \in I_0$.

For $K' \subset K$, the set of K'-highest weight vertices in the K-crystal B is defined by

$$\mathrm{hw}_{K'}(B) = \{b \in B \mid e_i(b) = 0 \text{ for all } i \in K'\}.$$

Let $\mathrm{hw}_{K'}(b)$ denote the unique K'-highest weight vertex in the K'-component of b.

If λ is a K'-dominant weight then define

$$\mathrm{hw}^{\lambda}_{K'}(B) = \{b \in \mathrm{hw}_{K'}(B) \mid \mathrm{wt}_{K'}(b) = \lambda\} \tag{3.40}$$

for the subset of $\mathrm{hw}_{K'}(B)$ of vertices of weight λ and $B(\lambda) = B_{K'}(\lambda)$ for the irreducible K'-crystal of highest weight λ.

Let B_1, B_2 be K-crystals. Then $B_1 \otimes B_2$ is a K-crystal via Kashiwara's tensor convention

$$e_i(b_1 \otimes b_2) = \begin{cases} e_i(b_1) \otimes b_2 & \text{if } \varphi_i(b_1) \geq \varepsilon_i(b_2) \\ b_1 \otimes e_i(b_2) & \text{if } \varphi_i(b_1) < \varepsilon_i(b_2). \end{cases} \tag{3.41}$$

Lemma 3.5 *Let B_1, B_2 be K-crystals and $b_1, c_1 \in B_1$ and $b_2, c_2 \in B_2$ such that $c_1 \otimes c_2 \in \mathrm{hw}_K(B_1 \otimes B_2)$ and $b_1 \otimes b_2 \in B_K(c_1 \otimes c_2)$. Then $c_1 \in \mathrm{hw}_K(B_1)$ and $b_1 \in B_K(c_1)$.*

Proof $c_1 \in \mathrm{hw}_K(B_1)$ holds by (3.41). Let $\mathbf{a} = (i_1, \dots, i_m)$ be a sequence of elements in K such that $e_{\mathbf{a}}(b_1 \otimes b_2) = c_1 \otimes c_2$. By (3.41) there is a subsequence $\mathbf{b}$ of $\mathbf{a}$ such that $e_{\mathbf{b}}(b_1) = c_1$. □

3.4 KR crystal generalities

Let $\mathcal{C} = \mathcal{C}(\mathfrak{g})$ be the tensor category of tensor products of KR crystals $B^{r,s}$. An I-crystal B is *regular* if for all subsets $K \subset I$ with $|K| = 2$, the K-components of B are isomorphic to crystal graphs of $U_q(\mathfrak{g}_K)$-crystals where $\mathfrak{g}_K$ is the subalgebra of $\mathfrak{g}$ corresponding to K.

Theorem 3.6 *Let* $\mathfrak{g}$ *be of nonexceptional affine type.*

1. [26] *For every* $(r, s) \in I_0 \times \mathbb{Z}_{>0}$*, there is an irreducible* $U'_q(\mathfrak{g})$*-module* $W_s^{(r)}$ *with affine crystal basis* $B^{r,s}$*. In particular every* $B \in \mathcal{C}$ *is regular.*
2. [3] *The affine crystal structure on* $B^{r,s}$ *is determined.*

Proposition 3.7 *Let* $B_1, B_2 \in \mathcal{C}$.

(1) *There is an* I*-crystal isomorphism* $R = R_{B_1,B_2} : B_1 \otimes B_2 \to B_2 \otimes B_1$ *called the combinatorial R-matrix. By uniqueness, for* $B \in \mathcal{C}$, $R_{B,B}$ *is the identity on* $B \otimes B$.

(2) *There is a unique map* $\overline{H} = \overline{H}_{B_1,B_2} : B_1 \otimes B_2 \to \mathbb{Z}$*, called coenergy function up to additive constant, such that* $\overline{H}$ *is constant on* I_0*-components, and for* $b = b_1 \otimes b_2 \in B_1 \otimes B_2$,

$$\overline{H}(e_0(b)) = \overline{H}(b) + \begin{cases} -1 & \text{in case } LL \\ 0 & \text{in case } LR \text{ or } RL \\ 1 & \text{in case } RR \end{cases} \tag{3.42}$$

where in case LL, when e_0 *is applied to* $b_1 \otimes b_2$ *and to* $R_{B_1,B_2}(b_1 \otimes b_2) = b'_2 \otimes b'_1$ *as in* (3.41)*, it acts on the left factor both times, in case RR* e_0 *acts on the right factor both times, etc.*

Proof Arguing as in [13] one may deduce these properties from the existence of the universal R-matrix, the Yang–Baxter relation for R, and Theorem 3.6(1). □

Let B be regular. An element $b \in B$ is called an *extremal vector* of weight λ if $\mathrm{wt}(b) = \lambda$ and there exist elements $\{b_w\}_{w \in W}$ such that

- $b_w = b$ for $w = e$,
- if $\langle \alpha_i^\vee, w\lambda \rangle \geq 0$ then $e_i(b_w) = 0$ and $f_i^{\langle \alpha_i^\vee, w\lambda \rangle}(b_w) = b_{s_i w}$,
- if $\langle \alpha_i^\vee, w\lambda \rangle \leq 0$ then $f_i(b_w) = 0$ and $e_i^{-\langle \alpha_i^\vee, w\lambda \rangle}(b_w) = b_{s_i w}$.

A finite regular crystal B with weights in P^0 is called *simple* [1,23] if there exists $\lambda \in P^0$ such that the weight of any extremal vector is contained in $W\lambda$ and B contains a unique element of weight λ. Here W is the affine Weyl group, which acts on $P^0 \cong P_0$ by the level zero action.

Proposition 3.8 (1) *Every* $B \in \mathcal{C}$ *is simple. In particular* B *contains a unique extremal vector* $u(B)$ *with* $\mathrm{wt}(u(B)) \in P_0^+$*. Moreover* $u(B^{r,s}) \in B^{r,s}$ *is the unique vector of weight* $s\omega_r$ *and* $u(B_1 \otimes B_2) = u(B_1) \otimes u(B_2)$ *for* $B_1, B_2 \in \mathcal{C}$.

(2) *For every* $B \in \mathcal{C}$, B *is* I*-connected.*

Proof By [1] a simple crystal is connected and the tensor product of simple crystals is also simple. In [23, Section 4.2] Naito and Sagaki proved that a finite regular crystal B with coenergy function $\overline{H}_{B\,B}$ is simple.[1] The equality $u(B_1 \otimes B_2) = u(B_1) \otimes u(B_2)$ follows from the fact that the r.h.s is extremal. □

[1] Although they assume that B is realized as a fixed point crystal, their proof is valid under the given condition.

Remark 3.9 (1) Proposition 3.8 implies that if there is an I-crystal isomorphism $g : B \to B'$ for $B, B' \in \mathcal{C}$, then it is unique: it must satisfy $g(u(B)) = u(B')$ and the rest of its values are determined since B is I-connected.

(2) For $B_1, B_2 \in \mathcal{C}$ we normalize the coenergy function $\overline{H}$ by $\overline{H}(u(B_1 \otimes B_2)) = 0$.

The *level* of $B \in \mathcal{C}$ is defined by

$$\mathrm{lev}(B) = \min_{b \in B} \mathrm{lev}(\varphi(b)) = \min_{b\in B} \mathrm{lev}(\varepsilon(b)). \tag{3.43}$$

The subset $B_{\min} \subset B$ is defined by

$$\begin{aligned} B_{\min} &= \{b \in B \mid \mathrm{lev}(\varphi(b)) = \mathrm{lev}(B)\} \\ &= \{b \in B \mid \mathrm{lev}(\varepsilon(b)) = \mathrm{lev}(B)\}. \end{aligned}$$

The crystal B is said to be *perfect* (in the sense of [23]; compare with [13]) if B is the crystal graph of a $U'_q(\mathfrak{g})$-module, B is simple, and the maps φ and ε are bijections from $B_{\min}$ to the set of weights $\lambda \in P'$ that are dominant and have $\mathrm{lev}(\lambda) = \mathrm{lev}(B)$.

Theorem 3.10 [4] *With c_r as in (3.22),*

(1) *$lev(B^{r,s}) = \lceil \frac{s}{c_r} \rceil$.*
(2) *$B^{r,s}$ is perfect if and only if $s/c_r \in \mathbb{Z}$.*

Lemma 3.11 *Let $\mathfrak{g}$ be of nonexceptional affine type, $(r,s) \in I_0 \times \mathbb{Z}_{>0}$, $\ell = lev(B^{r,s})$ and $j \in I^s$. Then there is a unique element $u_j(r,s) \in B^{r,s}$ such that $\varepsilon(u_j(r,s)) = \ell\Lambda_j$. Moreover, writing $t_{-c_{r*}\omega_{r*}} = w\tau$ for $w \in W$ and $\tau \in \Sigma$ with $*$ as in (3.30) we have*

$$\varphi(u_j(r,s))) = \begin{cases} \ell\Lambda_{\tau(j)} & \text{if } B^{r,s} \text{ is perfect} \\ (\ell-1)\Lambda_n + \Lambda_{n-r} & \text{if } \mathfrak{g} = C_n^{(1)}, 1 \le r \le n-1, j = n \\ (\ell-1)\Lambda_{\tau(j)} + \Lambda_r & \text{otherwise.} \end{cases} \tag{3.44}$$

Proof Suppose first that $B^{r,s}$ is perfect. Then $c_r\ell = s$ and $u_j(r,s)$ is unique. Moreover the value of $\varphi(u)$ is verified by [4]. Explicitly:

(1) $\mathfrak{g} = A_n^{(1)}$. $u_j(r,s)$ consists of s copies of the same column that consists of the elements $j+1, j+2, \dots, j+r \pmod{n+1}$, sorted into increasing order.
(2) $\mathfrak{g} = A_{2n}^{(2)}$. $u_0(r,s) = \mathrm{hw}_{I_0}(B(0))$.
(3) $\mathfrak{g} = D_{n+1}^{(2)}$. Suppose $r \notin I^s$. $u_0(r,s) = \mathrm{hw}_{I_0}(B(0))$. For $s = 2s'$, $u_n(r,s) \in B(s\omega_r)$ is the KN tableau with s' columns $(n-r+1)\cdots(n-1)n$ and s' columns $\overline{n}\,\overline{n-1}\cdots\overline{n-r+1}$. For $s = 2s'+1$, $u_n(r,s) \in B(s\omega_r)$ has, in addition to the columns for $u_n(r,2s')$, a middle column of height r is given by $0\cdots0$. For $r = n \in I^s$, $u_0(n,s)$ (resp. $u_n(n,s)$) is the unique element of $B(s\omega_n)$ of weight $s\omega_n$ (resp. $-s\omega_n$).
(4) $\mathfrak{g} = C_n^{(1)}$. For $r \notin I^s$, since $c_r = 2$ and we are in the perfect case, s must be even (say $s = 2\ell$), and $u_j(r,2\ell)$ is given as for $D_{n+1}^{(2)}$. For $r = n \in I^s$, again $u_j(n,s)$ is given as for $D_{n+1}^{(2)}$.
(5) $\mathfrak{g} \in \{B_n^{(1)}, D_n^{(1)}, A_{2n-1}^{(2)}\}$. Recall that for $\mathfrak{g} = B_n^{(1)}, r = n B^{r,s}$ is perfect of level ℓ when $s = 2\ell$. First let $r \in I_0$ not be a type $D_n^{(1)}$ spin node. $u_0(2i,s) = \mathrm{hw}_{I_0}(B(0))$ and $u_1(2i,s) \in B(\ell\omega_2)$ has ℓ' columns $1\overline{2}$ and ℓ' columns $2\overline{1}$ for $\ell = 2\ell'$, and in addition a middle column $2\overline{2}$ for $\ell = 2\ell'+1$. $u_0(2i+1,s) = \mathrm{hw}_{I_0}(B(\ell\omega_1))$ and $u_1(2i+1,s) \in B(\ell\omega_1)$ is the tableau $\overline{1}^\ell$. $D_n^{(1)}$ has additional special nodes $j \in \{n-1,n\}$. Suppose r is even. For $s = 2s'$, $u_n(r,s) \in B(s\omega_r)$ has s' columns $(n-r+1)\cdots(n-1)n$

and s' columns $\overline{n}\overline{n-1}\cdots\overline{n-r+1}$. For $s=2s'+1$, $u_n(r,s)$ has, in addition to the columns for $u_n(r,2s')$, a middle column given by $\overline{n}n\overline{n}n\cdots$. If r is odd, replace s' columns $(n-r+1)\cdots(n-1)n$ with $(n-r+1)\cdots(n-1)\overline{n}$. $u_{n-1}(r,s)\in B(s\omega_r)$ is given from $u_n(r,s)$ above by interchanging n and $\overline{n}$. Now let us set $r=n$ for type $D_n^{(1)}$. $u_j(n,s)$ for $j=0,1,n-1,n$ is given by the unique element of $B(s\omega_n)$ of weight $s\omega_n, s(\omega_{n-1}-\omega_1), s((1-\gamma)\omega_1-\omega_{n-1}), s(\gamma\omega_1-\omega_n)$ where $\gamma=0,1$, $\gamma\equiv n \pmod 2$. If $r=n-1$, we interchange ω_n and ω_{n-1} in the above description.

We enumerate the nonperfect cases [4].

(1) $\mathfrak{g}=B_n^{(1)}$, $r=n$ and $s=2\ell-1$. For n even, $u_0(n,2\ell-1)=\mathrm{hw}_{I_0}(\omega_n)$ and $u_1(n,2\ell-1)\in B(\omega_n)$ is defined by $\mathrm{wt}(u)=\omega_n-\omega_1$. For n odd, $u_0(n,2\ell-1)=\mathrm{hw}_{I_0}(B((\ell-1)\omega_1+\omega_n))$. $u_1(n,2\ell-1)$ has a half-column consisting of $23\cdots(n-1)n\overline{1}$ and $\ell-1$ columns consisting of a single $\overline{1}$.

(2) $\mathfrak{g}=C_n^{(1)}$ for $1\le r\le n-1$ and $s=2\ell-1$. $u_0(r,2\ell-1)=\mathrm{hw}_{I_0}(B(\omega_r))$. $u_n(r,2\ell-1)$ has $\ell-1$ columns $(n-r+1)\cdots(n-1)n$ and ℓ columns $\overline{n}\overline{n-1}\cdots\overline{n-r+1}$. □

By Lemma 3.11 we may define $m(B^{r,s})=u_0(r,s)\in B^{r,s}$ or equivalently

$$\varepsilon(m(B^{r,s}))=\mathrm{lev}(B^{r,s})\Lambda_0. \tag{3.45}$$

Similarly, there exists a unique element $m'(B^{r,s})\in B^{r,s}$ such that

$$\varphi(m'(B^{r,s}))=\mathrm{lev}(B^{r,s})\Lambda_0. \tag{3.46}$$

Define

$$b(r,s,\lambda)=\mathrm{hw}_{I_0}^{\lambda}(B^{r,s})\quad \text{for } \lambda\in\mathcal{P}_n^{\diamond}(r,s). \tag{3.47}$$

Remark 3.12 Suppose $\diamond\neq\varnothing$ and $r\in I_0$ is not a spin node. By (1.2) the right hand side of (3.47) is a singleton. We have

$$u(B^{r,s})=b(r,s,(s^r)) \tag{3.48}$$
$$m(B^{r,s})=b(r,s,\lambda_{\min}^{\diamond}(r,s)) \tag{3.49}$$

where $\lambda_{\min}=\lambda_{\min}^{\diamond}(r,s)\in\mathcal{P}^{\diamond}(r,s)$ is the partition with $|\lambda_{\min}|$ minimum. Explicitly

$$\lambda_{\min}^{\diamond}(r,s)=\begin{cases}(s) & \text{if } r \text{ is odd and } \diamond=(1,1)\\ (1^r) & \text{if } s \text{ is odd and } \diamond=(2)\\ \varnothing & \text{otherwise.}\end{cases} \tag{3.50}$$

3.5 Grading by intrinsic coenergy

Each $B\in\mathcal{C}$ has a canonical I_0-equivariant grading by the intrinsic coenergy function $\overline{D}: B\to\mathbb{Z}$ which is defined as follows.

(1) If $B=B^{r,s}$ is a KR crystal then define

$$\overline{D}_B(b)=\overline{H}_{B,B}(m'(B)\otimes b)-\overline{H}_{B,B}(m'(B)\otimes u(B)). \tag{3.51}$$

(2) If $B_1,B_2\in\mathcal{C}$ then

$$\overline{D}_{B_1\otimes B_2}(b_1\otimes b_2)=\overline{D}_{B_1}(b_1)+\overline{D}_{B_2}(b_2')+\overline{H}_{B_1,B_2}(b_1\otimes b_2) \tag{3.52}$$

where $R_{B_1,B_2}(b_1\otimes b_2)=b_2'\otimes b_1'$.

The resulting grading satisfies

$$\overline{D}_{(B_1\otimes B_2)\otimes B_3} = \overline{D}_{B_1\otimes(B_2\otimes B_3)}$$

for all $B_1, B_2, B_3 \in \mathcal{C}$ [27]. For $B_1, \dots, B_p \in \mathcal{C}$ one may prove by induction that

$$\overline{D}_{B_1\otimes\cdots\otimes B_p}(b) = \sum_{i=1}^{p} \overline{D}_{B_i}(b_i^{(1)}) + \sum_{1\le i<j\le p} \overline{H}_{B_i,B_j}(b_i \otimes b_j^{(i+1)}) \tag{3.53}$$

where $b = b_1 \otimes \cdots \otimes b_p$ with $b_i \in B_i$ for $1 \le i \le p$ and $b_j^{(k)}$ is the kth tensor factor of the element obtained from b by the composition of combinatorial R-matrices that swaps the jth tensor factor to the kth position. We have

$$\overline{D}_B = \overline{D}_{B'} \circ g \quad \text{for any } g : B \cong B' \text{ with } B, B' \in \mathcal{C}. \tag{3.54}$$

Lemma 3.13 *Let B be a KR crystal of level ℓ. Then*

1. *$\overline{D}_B$ is constant on I_0-components.*
2. *$\overline{D}_B(e_0(b)) = \overline{D}_B(b) + 1$ if $\varepsilon_0(b) > \ell$.*
3. *$\overline{D}_B(u(B)) = 0$.*

Proof Follows immediately from (3.51), the properties of $\overline{H}_{B,B}$, and (3.41). □

Lemma 3.14 *Let $B_1, B_2 \in \mathcal{C}$, and let $b_1 \in B_1$ and $b_2 \in B_2$ be such that $e_0(b_1 \otimes b_2) \neq 0$ and let $R_{B_1,B_2}(b_1 \otimes b_2) = b_2' \otimes b_1'$. Assume that*

$$\overline{D}(e_0(b_1)) = \overline{D}(b_1) + 1 \quad \text{if} \quad e_0(b_1) \neq 0$$
$$\overline{D}(e_0(b_2')) = \overline{D}(b_2') + 1 \quad \text{if} \quad e_0(b_2') \neq 0.$$

Then $\overline{D}(e_0(b_1 \otimes b_2)) = \overline{D}(b_1 \otimes b_2) + 1$.

Proof This follows from (3.52), computing the four cases of (3.42). □

We shall prove the following explicit formula for $\overline{D}_{B^{r,s}}$ at the end of Sect. 5.3.

Proposition 3.15 *For $\mathfrak{g}$ nonexceptional of kind $\diamondsuit \in \{(1), (2), (1, 1)\}$ and $(r, s) \in I_0 \times \mathbb{Z}_{>0}$ with r nonspin, we have*

$$\overline{D}_{B^{r,s}}(b(r, s, \lambda)) = \frac{rs - |\lambda|}{|\diamondsuit|} \quad \textit{for all } \lambda \in \mathcal{P}_n^{\diamondsuit}(r, s). \tag{3.55}$$

3.6 Affine highest weight crystals

Let $B(\Lambda)$ be the crystal graph of the irreducible integrable highest weight module of highest weight $\Lambda \in P^+$. $\mathrm{hw}_I(B(\Lambda))$ is a singleton denoted u_Λ. The enhanced weight function $\widehat{\mathrm{wt}} : B(\Lambda) \to P$ is defined by $\widehat{\mathrm{wt}}(u_\Lambda) = \Lambda$ and (3.38) and (3.39) except that $\alpha_i' \in P'$ is replaced by the affine simple root $\alpha_i \in P$. Alternatively, let $b \in B(\Lambda)$. Then there is a sequence $\mathbf{a} = (i_1, i_2, \dots, i_p)$ of elements of I such that $u_\Lambda = e_{\mathbf{a}}(b)$. Define $\widehat{D}(b)$ to be the number of times that 0 occurs in the sequence $\mathbf{a}$. This yields a well-defined $\mathbb{Z}$-grading $\widehat{D} : B(\Lambda) \to \mathbb{Z}$. Then

$$\widehat{\mathrm{wt}}(b) = (\langle d\,,\ \Lambda\rangle - \widehat{D}(b))(\delta/a_0) + \sum_{i\in I}(\varphi_i(b) - \varepsilon_i(b))\Lambda_i. \tag{3.56}$$

The following theorem is fundamental to the Kyoto path model for affine highest weight crystals.

Theorem 3.16 [13, Proposition 2.4.4] *Let $\mathfrak{g}$ be an affine algebra, $B \in \mathcal{C}(\mathfrak{g})$ the crystal graph of a $U'_q(\mathfrak{g})$-module, and $\Lambda \in P^+$ a dominant weight with $lev(\Lambda) = lev(B)$. Then there is an affine crystal isomorphism*

$$B(\Lambda) \otimes B \cong \bigoplus_u B(\varphi(u)) \tag{3.57}$$

where u runs over the elements of B such that $\varepsilon(u) = \Lambda$.

3.7 One-dimensional sums and stability

For $B \in \mathcal{C}$ and $\lambda \in P_0^+$, define the one-dimensional sum

$$\overline{X}_{\lambda,B}(q) = \sum_{b \in \mathrm{hw}_{I_0}^{\lambda}(B)} q^{\overline{D}(b)}. \tag{3.58}$$

Notation 3.17 Let

$$B = B^{r_1,s_1} \otimes B^{r_2,s_2} \otimes \cdots \otimes B^{r_p,s_p}. \tag{3.59}$$

We write $R_i = (s_i^{r_i})$, which is a rectangular partition with r_i rows and s_i columns. Let $R = (R_1, R_2, \ldots, R_p)$. We write $B = B^R$ if we wish to emphasize the indexing set of rectangles.

For nonexceptional $\mathfrak{g}$, let $n = \mathrm{rank}(\mathfrak{g}_0)$ and define

$$\mathcal{C}^\infty(\mathfrak{g}) = \{B = B^R \in \mathcal{C}(\mathfrak{g}) \mid \sum_i r_i \le n-2\} \tag{3.60}$$

$$\mathcal{P}_n^\infty = \{\lambda \in \mathcal{P}_n \mid \ell(\lambda) \le n-2\} \tag{3.61}$$

$$\mathcal{C}_h^\infty(\mathfrak{g}_0) = \{B \in \mathcal{C}_h(\mathfrak{g}_0) \mid \text{if } B_{I_0}(\nu) \text{ appears in } B \text{ then } \nu \in \mathcal{P}_n^\infty.\}. \tag{3.62}$$

These restrictions have the effect of guaranteeing that spin weights do not appear.

For $\diamond \in \{(1), (2), (1,1)\}$ and fixed R and λ define the stable 1-d sum $\overline{X}^{\diamond}_{\lambda,B^R}(q)$ to be $\overline{X}_{\lambda,B^R}(q)$ of type $\mathfrak{g}$ where $\mathfrak{g}$ is chosen such that $n = \mathrm{rank}(\mathfrak{g}_0)$ is large enough so that $B^R \in \mathcal{C}^\infty(\mathfrak{g})$ and $\lambda \in \mathcal{P}_n^\infty$. Without loss of generality we may choose $\mathfrak{g}$ to be reversible [that is, of the form $\mathfrak{g}^\diamond$; see (1.5)].

4 $\mathfrak{g}^\diamond$, I_0, and A_{n-1}-crystals

In this section we assume $\mathfrak{g}$ is one of the reversible affine algebras $\mathfrak{g}^\diamond$. Its classical subalgebra $\mathfrak{g}_0^\diamond$ [see (1.5)] contains the subalgebra $\mathfrak{sl}_n$ of type A_{n-1} given by restricting to the Dynkin node subset $I_{A_{n-1}} = \{1, 2, \ldots, n-1\}$. Using the notation of Sect. 3 we write $B(b) := B_{I_0}(b)$, $B_{A_{n-1}}(b) := B_{I_{A_{n-1}}}(b)$, and $\mathrm{hw}_{A_{n-1}}(b) := \mathrm{hw}_{I_{A_{n-1}}}(b)$. In fact $\mathfrak{gl}_n \subset \mathfrak{g}_0^\diamond$ and we use the $\mathfrak{gl}_n$ weights below.

4.1 Some subcrystals

For $\mathfrak{g}_0$ of type B_n, C_n, or D_n and $B \in \mathcal{C}_h^\infty(\mathfrak{g}_0)$, define the I_0-subcrystal

$$\max(B) = \bigcup_{\substack{b \in \mathrm{hw}_{I_0}(B) \\ |\mathrm{wt}(b)| = M(B)}} B_{I_0}(b) \tag{4.1}$$

where $M(B)$ is the maximum value of $|\nu|$ over $\nu \in \mathcal{P}_n$ such that $B_{I_0}(\nu)$ is a component of B. Define

$$\mathrm{tops}(B) = \bigcup_{b \in \mathrm{hw}_{I_0}(B)} B_{A_{n-1}}(b). \tag{4.2}$$

It is an A_{n-1}-subcrystal of B given by taking all the A_{n-1}-components of I_0-highest weight vertices in B. These A_{n-1}-components sit at the top of their respective I_0-components.

Remark 4.1 For $\nu \in \mathcal{P}_n^\infty$ we have $\mathrm{tops}(B(\nu)) \cong B_{A_{n-1}}(\nu)$. Moreover this is the only A_{n-1}-component of $B(\nu)$ of highest weight ν. Therefore there is a canonical inclusion $i_A^\nu : B_{A_{n-1}}(\nu) \to B_{I_0}(\nu)$. This isomorphism just says that a type A_{n-1} tableau can be regarded as an KN tableau for $\mathfrak{g}_0$.

For $B \in \mathcal{C}_h^\infty(\mathfrak{g}_0)$, define

$$\widehat{B} = \bigcup_{\lambda \in \mathcal{P}_n} \bigcup_{c \in \mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(B)} B_{A_{n-1}}(c). \tag{4.3}$$

$\widehat{B}$ is the A_{n-1}-subcrystal of B given by the dual polynomial part of B regarded as an A_{n-1}-crystal. The terminology "dual polynomial part" makes sense: $\mathfrak{g}_0 \supset \mathfrak{gl}_n$ so that B admits a $\mathfrak{gl}_n$ weight function.

For $\nu \in \mathcal{P}_n^\infty$, write

$$\widehat{B}(\nu) := \widehat{B(\nu)}. \tag{4.4}$$

It is an A_{n-1}-subcrystal of the irreducible highest weight I_0-crystal $B(\nu)$.

4.2 Row tableaux realization of $\widehat{B}(\nu)$

This section only concerns crystals of types B_n, C_n, and D_n, and herein we let $\diamond = (1), (2), (1,1)$ correspond to B_n, C_n, and D_n respectively; they coincide with $\mathfrak{g}_0^\diamond$ but we do not employ any affine algebra here.

In [15], the classical type crystal graph $B(\nu)$ was realized by tableaux which we will call Kashiwara–Nakashima (KN) tableaux. These tableaux are based on the unique I_0-crystal embedding

$$B(\nu) \hookrightarrow B(\omega_{\nu_1'}) \otimes B(\omega_{\nu_2'}) \otimes \cdots$$

where ν_j' is the size of the jth column of the partition ν.

However we shall use a different realization of $B(\nu)$ (which we call "row tableaux") which is better suited for the study of $\widehat{B}(\nu)$. For $\nu \in \mathcal{P}_n^\infty$, there is a unique embedding of I_0-crystals

$$\mathrm{rowtab}_\nu : B(\nu) \hookrightarrow B(\nu_1\omega_1) \otimes \cdots \otimes B(\nu_p\omega_1) \tag{4.5}$$

where $p = \ell(\nu)$. The image of rowtab_ν is the connected component

$$\mathrm{Im}(\mathrm{rowtab}_\nu) = B_{I_0}(1^{\nu_1} \otimes 2^{\nu_2} \otimes \cdots \otimes p^{\nu_p}).$$

Here a^m denotes the word consisting of m copies of the symbol a. The image of $b \in B(\nu)$ is a tensor product $\mathrm{rowtab}(b) = R_1 \otimes R_2 \otimes \cdots \otimes R_p$ with $R_i \in B(\nu_i\omega_1)$; it is called the *row tableau* associated with the element $b \in B(\nu)$ and may be depicted as a tableau of shape ν whose ith row is R_i. Each R_i is a KN tableau of the single-row shape (ν_i). In general $\mathrm{rowtab}(b)$ does not coincide with the corresponding KN tableau of shape ν. We are not aware of a simple characterization of the image of rowtab_ν. Nevertheless we characterize the image of $\widehat{B}(\nu)$ under rowtab_ν.

For a tableau c of shape ν and $D \subset \nu$ a skew shape, let $c|^D$ denote the restriction of c to the subshape D.

For $\delta \in \mathcal{P}_n^\diamondsuit$ with $\delta \subset \nu$, let $L^\diamondsuit(\nu, \delta) \subset B(\nu_1\omega_1) \otimes \cdots \otimes B(\nu_p\omega_1)$ be the set of vertices $b = R_1 \otimes \cdots \otimes R_p$ such that:

(1) $b|^{\nu\backslash\delta}$ is a skew semistandard tableau on $\{\bar{n}, \dots, \bar{1}\}$.
(2) $b|^\delta = C_\delta^\diamondsuit$, where the latter tableau is the unique tableau such that:

- For $\diamondsuit = (1)$, the ith row equals $n^a 0^{\delta_i - 2a} \bar{n}^a$ where $a = \lfloor \delta_i/2 \rfloor$.
- For $\diamondsuit = (2)$, the ith row equals $n^a \bar{n}^a$ where $a = \delta_i/2$.
- For $\diamondsuit = (1, 1)$, the jth column consists of $\delta'_j/2$ copies of $\begin{array}{|c|}\hline n \\ \hline \bar{n} \\ \hline\end{array}$.

Let $L^\diamondsuit(\nu) = \bigcup_{\delta \in \mathcal{P}_n^\diamondsuit} L^\diamondsuit(\nu, \delta)$.

Example 4.2 For $\diamondsuit = (1, 1)$, $\nu = (4, 4, 4, 2, 1, 1)$ and $\delta = (3, 3, 1, 1)$,

$\overline{n-3}$			
$\overline{n-1}$			
$\mathbf{n}$	$\overline{n-1}$		
$\bar{\mathbf{n}}$	$\bar{n}$	$\bar{n}$	$\overline{n-2}$
$\mathbf{n}$	$\mathbf{n}$	$\mathbf{n}$	$\overline{n-1}$
$\bar{\mathbf{n}}$	$\bar{\mathbf{n}}$	$\bar{\mathbf{n}}$	$\bar{n}$

is a row tableau in $\mathrm{hw}_{A_{n-1}}(L^\diamondsuit(\nu, \delta))$.

Remark 4.3

(1) Given any $b \in L^\diamondsuit(\nu)$, the unbarred letters in b determine the unique $\delta \in \mathcal{P}_n^\diamondsuit$ such that $b \in L^\diamondsuit(\nu, \delta)$, and b is determined by δ and $b|^{\nu\backslash\delta}$. By definition $b \in L^\diamondsuit(\nu)$ contains no letters in $\{1, \dots, n-1\}$.
(2) Let b^ν be the lowest weight vector of $L^\diamondsuit(\nu)$. Then $\mathrm{rowtab}(b^\nu) \in L^\diamondsuit(\nu, \emptyset)$ where $\delta = \emptyset$ is the empty partition.

Proposition 4.4 *The map* rowtab_ν *restricts to an isomorphism*

$$\widehat{B}(\nu) \cong L^\diamondsuit(\nu). \tag{4.6}$$

Proposition 4.4 will be deduced from Proposition 4.5 below.

The *reading word* of a single-rowed tableau is obtained by reading its letters from *right to left*. The reading word of a tableau obtained by reading its rows from top to bottom. A word $w = x_1 x_2 \cdots x_\ell$ with $x_i \in \{\bar{n}, \overline{n-1}, \dots, \bar{1}\}$ is *Yamanouchi* if for all j, in the subword $x_1 x_2 \cdots x_j$ there at least as many letters $\overline{i+1}$ as there are letters $\bar{i}$ for $1 \le i \le n-1$.

Proposition 4.5 *Let* $b \in L^\diamondsuit(\nu, \delta)$ *for some* $\delta \in \mathcal{P}_n^\diamondsuit$.

1. $L^\diamondsuit(\nu, \delta)$ *is an* A_{n-1}*-crystal.*
2. $b \in \mathrm{hw}_{A_{n-1}}(L^\diamondsuit(\nu, \delta))$ *if and only if the row-reading word of the skew semistandard subtableau of* b *of shape* ν/δ*, is Yamanouchi of weight* $\bar{\lambda}$ *for some* $\lambda \in \mathcal{P}_n$.
3. *If* $\varphi_n(b) > 0$ *then* $f_n(b) \in L^\diamondsuit(\nu)$.
4. *There exists a finite sequence* $\mathbf{a} = (j_1, j_2, \dots)$ *in* I_0 *such that* $b = e_{\mathbf{a}}(\mathrm{rowtab}(b^\nu))$. *In particular* $L^\diamondsuit(\nu) \subset \mathrm{Im}(\mathrm{rowtab}_\nu)$.

5. *Assume* $b \in \mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(L^{\diamondsuit}(\nu,\delta))$ *for some* $\lambda \in \mathcal{P}_n$ *and let* $\mathbf{a}$ *be as above. Then*

$$\mathrm{card}\,\{k \mid j_k = n\} = \frac{|\nu| - |\lambda|}{|\diamondsuit|}. \tag{4.7}$$

The proof of Proposition 4.5 is deferred to Appendix A.

Proof of Proposition 4.4 $\mathrm{rowtab}_\nu(\widehat{B}(\nu))$ and $L^{\diamondsuit}(\nu)$ are both A_{n-1}-subcrystals of $\mathrm{Im}(\mathrm{rowtab}_\nu)$, by definition and Proposition 4.5(4) respectively. Therefore it suffices to show they have the same A_{n-1}-highest weight vertices. All such vertices have weight of the form $\overline{\lambda}$ for some $\lambda \in \mathcal{P}_n$. For $\lambda \in \mathcal{P}_n$ and $\delta \in \mathcal{P}_n^{\diamondsuit}$, $|\mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(L^{\diamondsuit}(\nu,\delta))| = c^{\nu}_{\delta\lambda}$ by Proposition 4.5(2) and the Littlewood–Richardson Rule [6]. All of these highest weight vertices are in $\mathrm{rowtab}_\nu(\mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{B}^{\diamondsuit}(\nu)))$. The result follows by summing over $\delta \in \mathcal{P}_n^{\diamondsuit}$ and using (2.3). □

4.3 $\widehat{B}(\nu)$ when ν is a rectangle

We assume $\mathfrak{g} = \mathfrak{g}^{\diamondsuit}$ is reversible, and apply the previous results to $\max(B^{r,s}) \cong B(s\omega_r)$ for $B^{r,s} \in \mathcal{C}^{\infty}(\mathfrak{g}^{\diamondsuit})$. For the rectangular partition $\nu = (s^r) \in \mathcal{P}_n^{\infty}$ let

$$\overline{b}(r,s,\lambda) = \mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(B_{I_0}(s^r)) \quad \text{for } \lambda \in \mathcal{P}_n^{\diamondsuit}(r,s) \tag{4.8}$$

$$\overline{b}^{\diamondsuit}_{\min}(r,s) = \overline{b}(r,s,\lambda^{\diamondsuit}_{\min}(r,s)) \tag{4.9}$$

where $\lambda^{\diamondsuit}_{\min}(r,s)$ is defined in (3.50). Note that the set on the right hand side of (4.8) is a singleton, by (2.3) and the Littlewood–Richardson Rule. We regard the elements $\overline{b}(r,s,\lambda)$ as being in $B^{r,s}$ since $B^{r,s}$ contains a unique I_0-component $B_{I_0}(s^r)$. We note that

$$\mathrm{hw}_{A_{n-1}}(\widehat{B}(s^r)) = \{\overline{b}(r,s,\lambda) \mid \lambda \in \mathcal{P}_n^{\diamondsuit}(r,s)\}. \tag{4.10}$$

Remark 4.6 For $\lambda \in \mathcal{P}_n^{\diamondsuit}(r,s)$, let $\delta \in \mathcal{P}_n^{\diamondsuit}$ be the partition complementary to λ in the rectangle (s^r). Then by Propositions 4.4 and 4.5(2), $\mathrm{rowtab}_{(s^r)}(\overline{b}(r,s,\lambda))$ is explicitly given by the row tableau of shape (s^r) whose restriction to the shape δ, is the canonical tableau $C^{\diamondsuit}_{\delta}$ and whose restriction to $(s^r)/\delta$ is the unique Yamanouchi tableau of that shape in the letters $\{\overline{n},\dots,\overline{2},\overline{1}\}$; each column of the latter subtableau consists of letters $\bar{n}$, $\overline{n-1}$, etc., reading from bottom to top.

For $\nu = (s^r)$ we are going to see that every A_{n-1}-highest weight vertex in $\widehat{B}(\nu)$ is reachable by I_0-lowering operators, starting with a certain fixed element. This is not true for a general partition $\nu \in \mathcal{P}_n^{\infty}$.

Proposition 4.7 *Let* $(r,s) \in I_0 \times \mathbb{Z}_{>0}$ *with* $B^{r,s} \in \mathcal{C}^{\infty}(\mathfrak{g}^{\diamondsuit})$ *and* $\ell = lev(B^{r,s})$. *Then for any* $\lambda \in \mathcal{P}_n^{\diamondsuit}(r,s)$ *there exists a finite sequence* $\mathbf{b} = (j_1, j_2, \dots)$ *in* I_0 *such that*

$$u_{\ell\Lambda_n} \otimes \overline{b}(r,s,\lambda) = f_{\mathbf{b}}(u_{\ell\Lambda_n} \otimes \overline{b}^{\diamondsuit}_{\min}(r,s)) \tag{4.11}$$

$$\mathrm{card}\{k \mid j_k = n\} = \frac{|\lambda| - |\lambda^{\diamondsuit}_{\min}(r,s)|}{|\diamondsuit|}. \tag{4.12}$$

This result follows by induction using Lemma 4.9 below. For $h \geq 2$ if $\diamondsuit = (1,1)$ and $h \geq 1$ if $\diamondsuit \in \{(1),(2)\}$, define the following sequences (the semicolons are just for readability):

$$\tilde{\mathbf{a}}'(h) = \begin{cases} (n-2, n-3, \ldots, n-h+1; n-1, n-2, \ldots, n-h+2) & \text{for } \diamondsuit = (1,1) \\ (n-1, n-2, \ldots, n-h+1) & \text{for } \diamondsuit = (1) \\ ((n-1)^2, (n-2)^2, \ldots, (n-h+1)^2) & \text{for } \diamondsuit = (2) \end{cases}$$

$$\tilde{\mathbf{a}}(h) = (n; \tilde{\mathbf{a}}'(h)). \tag{4.13}$$

Notation 4.8 Given $\lambda \in \mathcal{P}_n^\diamondsuit(r,s)$ with $\lambda \neq \lambda_{\min} = \lambda_{\min}^\diamondsuit(r,s)$, we define a canonical smaller element $\lambda^- \in \mathcal{P}_n^\diamondsuit(r,s)$ obtained from λ by removing a particular copy of the shape $\diamondsuit$. Suppose the rightmost column in which λ and $\lambda_{\min}$ differ, is the pth. Let $h = \lambda'_p$ be the height of that column. Let $\lambda^- \in \mathcal{P}_n^\diamondsuit(r,s)$ be obtained from λ by removing a vertical domino from the pth column if $\diamondsuit = (1,1)$, removing a cell from the pth column if $\diamondsuit = (1)$, and removing a cell from the pth and $(p-1)$th columns if $\diamondsuit = (2)$.

We note that if $\delta \in \mathcal{P}_n^\diamondsuit$ is a nonempty partition then $\delta^- \in \mathcal{P}_n^\diamondsuit$ can be defined similarly.

Lemma 4.9 *Let $\lambda \in \mathcal{P}_n^\diamondsuit(r,s)$ with $\lambda \neq \lambda_{\min}^\diamondsuit(r,s)$. Then*

$$u_{\ell\Lambda_n} \otimes \overline{b}(r,s,\lambda) = f_{\tilde{\mathbf{a}}(h)}(u_{\ell\Lambda_n} \otimes \overline{b}(r,s,\lambda^-)). \tag{4.14}$$

The proof of Lemma 4.9 is deferred to Appendix A.

5 Affine crystals and the involution σ

In this section we summarize necessary facts on a single KR crystal $B^{r,s}$ belonging to $\mathcal{C}^\infty(\mathfrak{g}^\diamondsuit)$ and show that a tensor product B of such KR crystals has an automorphism σ, which we call the reversing crystal automorphism. This σ will be effectively used to show our main theorem (Theorem 10.1).

5.1 KR crystal $B^{r,s}$

We consider a single KR crystal $B^{r,s} \in \mathcal{C}^\infty(\mathfrak{g}^\diamondsuit)$. Note that $r \in I_0$ is nonspin. We recall the crystal structure of $B^{r,s}$. Firstly, the $U_q(\mathfrak{g}_0^\diamondsuit)$-crystal structure is described as follows. As we explained in Sect. 1, $B^{r,s}$ decomposes into a multiplicity-free direct sum of highest weight crystals $B(\lambda)$, where λ runs over $\mathcal{P}_n^\diamondsuit(r,s)$, the set of partitions obtained by removing $\diamondsuit$'s from (s^r). The action of Kashiwara operators e_i, f_i $(i \in I_0)$ on $B^{r,s}$ is given by realizing its elements by KN tableaux. Hence, we are left to describe the action of e_0 and f_0. To do this we explain the notion of $\pm$-diagrams and a certain automorphism ς on $B^{r,s}$ for $\diamondsuit = (1,1)$ introduced in [28]. From here to Lemma 5.2 we assume $\diamondsuit = (1,1)$.

A $\pm$-diagram P of shape Λ/λ is a sequence of partitions $\lambda \subset \mu \subset \Lambda$ such that Λ/μ and μ/λ are horizontal strips (i.e. every column contains at most one box). We depict this $\pm$-diagram by the skew tableau of shape Λ/λ in which the cells of μ/λ are filled with the symbol $+$ and those of Λ/μ are filled with the symbol $-$. Write $\Lambda = \mathrm{outer}(P)$ and $\lambda = \mathrm{inner}(P)$ for the outer and inner shapes of the $\pm$-diagram P. We call μ the middle shape. Set $J = \{2, 3, \ldots, n\}$. There is a bijection $\Phi : P \mapsto b$ from $\pm$-diagrams P of shape Λ/λ to the set of J-highest weight elements b of J-weight λ. For details refer to section 4.2 of [28].

Now suppose $b \in B^{r,s}$ is a J-highest weight element corresponding to a $\pm$-diagram P of shape Λ/λ. Let $c_i = c_i(\lambda)$ be the number of columns of height i in λ for all $1 \le i < r$ with $c_0 = s - \lambda_1$. If $i \equiv r-1 \pmod 2$, then in P, above each column of λ of height i, there must be a $+$ or a $-$. Interchange the number of such $+$ and $-$ symbols. If $i \equiv r \pmod 2$, then in P, above each column of λ of height i, either there are no signs or a $\mp$ pair. Suppose there are $p_i \mp$ pairs above the columns of height i. Change this to $(c_i - p_i) \mp$ pairs. The result is $\mathfrak{S}(P)$, which has the same inner shape λ as P but a possibly different outer shape. The columns of height r in P are not changed by $\mathfrak{S}$. The following map ς (called σ in [28]) is an automorphism on $B^{r,s}$ corresponding to interchanging the nodes 0 and 1 of the Dynkin diagram of $D_n^{(1)}$.

Definition 5.1 Let $b \in B^{r,s}$ and $\mathbf{a}$ be a sequence of elements of J such that $e_{\mathbf{a}}(b)$ is a J-highest weight element. Let $\mathbf{a}'$ be the reverse sequence of $\mathbf{a}$. Then

$$\varsigma(b) := f_{\mathbf{a}'} \circ \Phi \circ \mathfrak{S} \circ \Phi^{-1} \circ e_{\mathbf{a}}(b). \tag{5.1}$$

With this ς the Kashiwara operators e_0 and f_0 are given by

$$\begin{aligned} f_0 &= \varsigma \circ f_1 \circ \varsigma, \\ e_0 &= \varsigma \circ e_1 \circ \varsigma. \end{aligned} \tag{5.2}$$

By (5.1) and (5.2) e_0 and f_0 commutes with e_i or f_i for $J' = \{3, 4, \dots, n\}$. Hence, the calculation of the actions of e_0 and f_0 are reduced to J'-highest weight elements. Note that J'-highest weight elements are in one-to-one correspondence with pairs of $\pm$-diagrams (P, p), where the inner shape of P is the outer shape of p. To calculate the action of e_0 it suffices to know the action of e_1 on (P, p), that is described in [28].

(1) Successively run through all $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $+$ in P weakly to the left of it.
(2) Successively run through all $-$ in p from left to right and, if possible, pair it with the rightmost yet unpaired $-$ in P weakly to the left.
(3) Successively run through all yet unpaired $+$ in p from left to right and, if possible, pair it with the leftmost yet unpaired $-$ in p.

Lemma 5.2 [28, Lemma 5.1] *If there is an unpaired* $+$ *in* p, e_1 *moves the rightmost unpaired* $+$ *in* p *to* P. *Else, if there is an unpaired* $-$ *in* P, e_1 *moves the leftmost unpaired* $-$ *in* P *to* p. *Else* e_1 *annihilates* (P, p).

For types $\diamondsuit = (2), (1)$, we use a construction of $B^{r,s}$ in section 4.3 and 4.4 of [3] (where it is called $V^{r,s}$). As above we can assume $b \in B^{r,s}$ is J-highest. Let $p = \Phi^{-1}(b)$ and let $\hat{p}$ be p itself if $\diamondsuit = (2)$, and the $\pm$-diagram whose inner, middle and outer shapes are all doubled rowwise if $\diamondsuit = (1)$. Let c_i $(1 \le i \le r)$ be the number of columns of height i in outer$(\hat{p})$. We also set $c_0 = \gamma s - \text{outer}(\hat{p})_1$ where $\gamma = 2/|\diamondsuit|$. Note that c_i is even except when $\diamondsuit = (2), i = r$ and r is odd. There exists a unique $\pm$-diagram P such that inner$(P) = $ outer$(\hat{p})$, the length of inner$(P) \le r$ and there are equal number $c_i/2$ of columns with $\mp$ and $\cdot$ in P if $i < r, i \equiv r$ (2), with $+$ and $-$ if $i \not\equiv$ (2). Then the pair of $\pm$-diagrams $(P, \hat{p})$ can be considered to correspond to a $\{3, 4, \dots, n\}$-highest element of $B^{r,\gamma s}$ of type $\diamondsuit = (1, 1)$. We now apply $e_1 \circ \varsigma \circ e_1$ to $(P, \hat{p})$ following the procedure explained previously to get $(P', \hat{p}')$. Let p' be $\hat{p}'$ if $\diamondsuit = (2)$, and the $\pm$-diagram whose inner, middle and outer shapes are all halved rowwise. (This is possible by Lemma 4.7 (1) in [3].) Finally, setting $b' = \Phi(p')$ we obtain $e_0 b = b'$. To calculate the action of f_0 we replace $e_1 \circ \varsigma \circ e_1$ with $f_1 \circ \varsigma \circ f_1$.

5.2 The reversing crystal automorphism σ

Recall $\sigma \in \mathrm{Aut}(X)$ from (1.6).

Theorem 5.3 *For every B that is a tensor product of KR crystals in $\mathcal{C}^\infty(\mathfrak{g}^\diamondsuit)$, there is a unique map $\sigma = \sigma_B : B \to B$ such that*

$$\sigma \circ e_i = e_{\sigma(i)} \circ \sigma \tag{5.3}$$

for all $i \in I$ and $b \in B$. Moreover

$$\mathrm{wt}(\sigma(b)) = -w_0^{A_{n-1}}(\mathrm{wt}(b)) \tag{5.4}$$

$$\sigma^2 = id \tag{5.5}$$

$$\sigma_{B'} \circ g = g \circ \sigma_B \quad \text{for any } g : B \cong B' \text{ for } B, B' \in \mathcal{C}. \tag{5.6}$$

Here $w_0^{A_{n-1}} \in W$ is the longest element of the type A_{n-1} Weyl group generated by s_1 through s_{n-1}.

First we assume the existence of σ satisfying (5.3) and deduce (5.4), (5.5), and (5.6).

For (5.4) we recall the discussion of the weight function on KR crystals (and therefore on B) in Sect. 3.4 and associated notation. By (5.3) and (3.37) we have $\sigma(\mathrm{wt}(b)) = \mathrm{wt}(\sigma(b))$, computing in the lattice P'. Now wt takes values in $P^0 \cong P$ and one may check that the action of σ on P^0 agrees with that of $-w_0^{A_{n-1}}$ on P.

For (5.5), σ^2 is an I-crystal isomorphism $B \to B$. By connectedness and the fact that B contains a unique element $u(B)$ of its weight, there is only one such isomorphism, namely, the identity.

For (5.6), by the connectedness of B the proof reduces to verifying the relation for a single value. However the value of both sides on $u(B)$ must agree, for the answer must be the unique element of B' whose weight is $-w_0^{A_{n-1}}(\mathrm{wt}(u(B)))$.

Next, we prove the uniqueness of σ assuming its existence. Since $B \in \mathcal{C}$ is connected we need only show that (5.3) uniquely specifies some single value of σ. The vertex $u(B)$ is the only element of its weight in B. The weight $w_0^{A_{n-1}}(u(B))$ occurs in B since B is an A_{n-1}-crystal. Since B is an I_0-crystal (of classical type B_n, C_n, or D_n) with no spin weight, it is self-dual, so its weights are closed under negation. In particular the weight $-w_0^{A_{n-1}}(u(B))$ must also occur in B. Since $\mathrm{wt}(u(B))$ occurs exactly once, the weight $-w_0^{A_{n-1}}(\mathrm{wt}(u(B)))$ also occurs exactly once. By (5.4) $\sigma(u(B))$ must be the unique element of B of weight $-w_0^{A_{n-1}}(\mathrm{wt}(u(B)))$. It follows that σ is unique.

It only remains to prove the existence of σ. By (3.41) we may reduce to the case $B = B^{r,s}$. The existence of σ on $B^{r,s}$ is proved in the next several subsections.

5.3 Definition of σ on KR crystals

Define the sequences

$$\begin{aligned}
\mathbf{a}'(h) &= \begin{cases} (2,3,\dots,h-1;1,2,\dots,h-2) & \text{if } \diamondsuit = (1,1) \\ (1,2,\dots,h-1) & \text{if } \diamondsuit = (1) \\ (1^2,2^2,\dots,(h-1)^2) & \text{if } \diamondsuit = (2) \end{cases} \\
\mathbf{a}(h) &= (0;\mathbf{a}'(h)).
\end{aligned} \tag{5.7}$$

Recalling $\tilde{\mathbf{a}}(h)$ from (4.13) we have

$$\sigma(\mathbf{a}(h)) = \tilde{\mathbf{a}}(h). \tag{5.8}$$

Lemma 5.4 *Let* $\lambda \in \mathcal{P}(r,s)$ *and* $\lambda \neq \lambda^{\diamond}_{\min}(r,s)$. *Let* $\ell = lev(B^{r,s})$ *and* λ^- *be as in Notation* 4.8. *Then*

$$u_{\ell\Lambda_0} \otimes b(r,s,\lambda) = f_{\mathbf{a}(h)}(u_{\ell\Lambda_0} \otimes b(r,s,\lambda^-)). \tag{5.9}$$

Proof We first treat the case $\diamond = (1,1)$. Suppose r is even. We apply $f_{\mathbf{a}'(h)}$. Then $b(r,s,\lambda^-)$ changes to the KN tableau t_1 of shape λ^- whose columns are filled with $123\cdots$, except the rightmost, which is filled with $34\cdots$ instead. Now we want to apply f_0 to $u_{s\Lambda_0}\otimes t_1$. To do this we first go to the J-highest element $e_{(h-1,\dots,3,2)}(t_1)$ of t_1, where we have set $J = \{2,3,\dots,n\}$. Then we have $P = \Phi^{-1}(e_{(h-1,\dots,3,2)}(t_1))$ is the $\pm$-diagram such that there is no sign in the rightmost column and only $+$ in the other ones. Hence $\mathfrak{S}(P)$ is the $\pm$-diagram described as follows. Denote the position of the rightmost column of λ by a. The height of the outer shape from the 1st to the $(a-1)$th column is the same as P, but from the ath to the sth column the height is larger than P by 2. There is only $-$ from the 1st to the $(a-1)$th column, and $\mp$ from the ath to the sth column. Now we have $\varsigma(t_1) = f_{(2,3,\dots,h-1)}\Phi(\mathfrak{S}(P))$ described as follows. The shape of $\varsigma(t_1)$ is the same as the outer shape of $\mathfrak{S}(P)$. To get contents we first place the string $23\cdots k\bar{1}$ in each column and then reading from left to right, top to bottom we change $\bar{1}$ to $\bar{2}$ and 2 to $1(s-a+1)$ times. Note that $\varepsilon_1(\varsigma(t_1)) = s+a-1$. One finds $f_1\varsigma(t_1)$ is a J-highest element corresponding to the $\pm$-diagram that differs from $\mathfrak{S}(P)$ only in the ath column where there is only $-$. Hence we have $f_0 t_1 = b(r,s,\lambda)$ by definition. Since $\varepsilon_0(t_1) = s+a-1 \geq s$, we also have $f_0(u_{s\Lambda_0}\otimes t_1) = u_{s\Lambda_0}\otimes f_0 t_1 = u_{s\Lambda_0}\otimes b(r,s,\lambda)$.

Next suppose r is odd. In this case the first row of λ has s nodes. Denote the position of the rightmost column with height greater than 1 by a. The calculation goes similarly to the r even case. The $\pm$-diagram P is given as follows. The outer shape is the same as λ^-. There is no sign in the ath column and only $+$ in the other columns. Applying $f_{(2,3,\dots,h-1)}\circ\Phi\circ\mathfrak{S}$, one obtains $\varsigma(t_1)$ described as follows. The shape of $\varsigma(t_1)$ is the same as t_1 except in the ath column where the height of $\varsigma(t_1)$ is larger than that of t_1 by 2. To get contents we place the string $23\cdots k\bar{1}$ ($\bar{1}$ in the column of height 1) in each column. Only in the leftmost column we put $\bar{2}$ instead of $\bar{1}$. Note that $\varepsilon_1(\varsigma(t_1)) = s+a-1$. We obtain $f_0 t_1 = b(r,s,\lambda)$, and since $\varepsilon_0(t_1)\geq s$, we again have $f_0(u_{s\Lambda_0}\otimes t_1) = u_{s\Lambda_0}\otimes b(r,s,\lambda)$.

Next we treat the case $\diamond = (2)$. (Since the case $\diamond = (1)$ is similar, we omit its proof.) Applying $f_{\mathbf{a}'(h)}$ makes $b(r,s,\lambda^-)$ change to the KN tableau t_2 of shape λ^- whose columns are filled with $123\cdots$, except the rightmost two, which is filled with $23\cdots$ instead. Note that t_2 is J-highest. $p = \Phi^{-1}(t_2)$ is the $\pm$-diagram such that there is no sign in the rightmost two columns and only $+$ in the other ones. From this p construct P as prescribed in the previous subsection. We want to apply $f_1\circ\varsigma\circ f_1$ to this pair (P,p) of $\pm$-diagrams. Denote the position of the rightmost column of λ by a. By Lemma 5.2 the application of f_1 changes (P,p) as follows. In the $(a-1)$th column there is $+$ (resp. $\mp$) when $h\equiv r\ (2)$ (resp. $h\not\equiv r\ (2)$) in P and no sign in p. f_1 moves $+$ in P to p. Denote this new pair by (P',p'). Next ς changes P' as follows. In the columns of P' of height h, the number of columns with $\mp$ (resp. $+$) increases by 1 while the number of those with $\cdot$ (resp. $-$) decreases by 1 when $h\equiv r\ (2)$ (resp. $h\not\equiv r\ (2)$). By applying f_1 again, we obtain (P'',p'') described as follows. p'' differs from p only at the $(a-1)$th and ath positions. outer(p'') is of height h there with $+$'s. P'' is a unique $\pm$-diagram determined from p'' as in the previous subsection. To show (5.9) we still need to check $\varepsilon_0(b(r,s,\lambda^-))\geq\ell$. Since the application of $e_0(=e_1\circ\varsigma\circ e_1)$ is similar to above, we only give its value. Let c_i $(1 \leq i \leq r)$ be the number of columns of λ of height i and set $c_0 = s - \lambda_1$. Then we have

$$\varepsilon_0(b(r, s, \lambda^-)) = c_r + c_{r-1} + \cdots + c_h - 1 + c_0/2.$$

Noting that $(c_r + c_{r-1} + \cdots + c_h + c_0 + \bar{r})/2 = \ell$ ($\bar{r} = 0$ or $1, \bar{r} \equiv r$ (2)) and $c_h \geq 2$, we obtain $\varepsilon_0(b(r, s, \lambda^-)) \geq \ell$. □

For a KR crystal B of level ℓ, say that the i-arrow $b \to b' = f_i(b)$ is *good* if either $i \in I_0$ or $i = 0$ and $\varepsilon_0(b) \geq \ell$. Traversing the above edge backwards (using a raising operator), going from b' to $e_i(b') = b$ is good if $i \in I_0$ or $i = 0$ and $\varepsilon_0(b') > \ell$.

Lemma 5.5 *Let $B^{r,s}$ be a KR crystal of level ℓ. Then for every $b \in B^{r,s}$ there is a sequence of good arrows from b to $m(B^{r,s})$.*

Proof Noting that from (3.45) $u_{\ell\Lambda_0} \otimes m(B^{r,s})$ is an affine highest weight vector in $B(\ell\Lambda_0) \otimes B^{r,s} \simeq B(\varphi(m(B^{r,s})))$, the lemma is clear from the previous one. □

We obtain the following for KR crystals $B^{r,s}$ for $\mathfrak{g}$ of kind $(1, 1)$, (2), (1) where $r \in I_0$ is nonspin.

Corollary 5.6 *For a KR crystal B of level ℓ, there is a unique function $\overline{D}_B$ satisfying the conditions of Lemma 3.13. Moreover, identifying elements of $u_{\ell\Lambda_0} \otimes B$ with their images in $B(\varphi(m(B)))$ under the isomorphism (3.57), we have*

$$\widehat{D}_{B(\varphi(m(B)))}(u_{\ell\Lambda_0} \otimes b) + \overline{D}_B(b) = \overline{D}_B(m(B)) \tag{5.10}$$

where $m(B)$ is defined in Lemma 3.11.

Proof By Lemma 5.5 B is connected by good arrows. But properties (1) and (2) of Lemma 3.13 specify how $\overline{D}_B$ must change across good arrows. Therefore a single value completely specifies $\overline{D}_B$. This is furnished by property (3) of Lemma 3.13. The left hand side of (5.10), viewed as a function of $b \in B$, is invariant under good arrows in B. But B is connected by good arrows so this function is constant, and its value is obtained by setting $b = m(B)$ and using that $\widehat{D} = 0$ on the affine highest weight vector. □

Let $\ell = \mathrm{lev}(B^{r,s})$ and let $u \in B^{r,s}$ be as in Lemma 3.11 using $j = 0$. From Theorem 3.16 there are bijections

$$B(\ell\Lambda_0) \otimes B^{r,s} \cong B(\varphi(u)) \longrightarrow B(\sigma(\varphi(u))) \cong B(\ell\Lambda_n) \otimes B^{r,s}. \tag{5.11}$$

The first and third maps are isomorphisms given by Theorem 3.16 and the middle maps are the unique automorphism in highest weight crystals induced by relabeling everything according to $\sigma \in \mathrm{Aut}(X)$.

Lemma 5.7 *Let $\overline{b}(r, s, \lambda)$ be as in (4.8). For $\lambda \in \mathcal{P}^{\diamond}(r, s)$, $u_{\ell\Lambda_0} \otimes b(r, s, \lambda)$ is sent to $u_{\ell\Lambda_n} \otimes \overline{b}(r, s, \lambda)$ under the previous bijection.*

Proof The proof proceeds by induction on $\mathcal{P}^{\diamond}(r, s)$. The claim holds for $\lambda_{\min} (= \lambda^{\diamond}_{\min}(r, s))$ since these elements are the unique affine highest weight elements of both sides of (5.11). For $\lambda \in \mathcal{P}^{\diamond}(r, s)$ with $\lambda \neq \lambda_{\min}$ the claim follows from Lemmas 4.9, 5.4, (5.8) and induction. □

Proposition 5.8 *For $B^{r,s} \in \mathcal{C}^\infty(\mathfrak{g}^\diamondsuit)$ there is a unique map $\sigma : B^{r,s} \to B^{r,s}$ such that*

1. *Equation* (5.3) *holds for good arrows.*
2. $\sigma(m(B^{r,s})) = \overline{b}^\diamondsuit_{\min}(r,s)$.

Proof Such a map σ is necessarily unique. Assertion 2 specifies one value of σ. By Lemma 5.5 $B^{r,s}$ is connected by good arrows, so Assertion 1 determines all other values of σ. So it suffices to prove existence. Consider the bijection (5.11). For an element $b \in B^{r,s}$ the image of $u_{\ell\Lambda_0} \otimes b$ by the bijection should belong to $u_{\ell\Lambda_n} \otimes B^{r,s}$ by Lemma 5.7. Denote this image by $u_{\ell\Lambda_n} \otimes \sigma(b)$. This map σ satisfies the two conditions. □

Proposition 5.9 *The map σ of Proposition 5.8 satisfies* (5.3) *for all $i \in I$ and $b \in B^{r,s}$.*

The proof of Proposition 5.9 for $\diamondsuit = (1,1)$ is deferred to Appendix B. For $\diamondsuit = (1), (2)$ the map σ constructed in Theorem 7.1 of [3] is the one we need.

Proof of Theorem 5.3 As noted at the end of Sect. 5.2, it suffices to establish the case of a single KR crystal. The map σ in Proposition 5.8 works by Proposition 5.9. □

The following lemma is used later.

Lemma 5.10 *For any $\lambda \in \mathcal{P}^\diamondsuit(r,s)$, there is a sequence $\mathbf{a} = (i_1, \dots, i_m)$ of indices in I_n such that*

$$e_{\mathbf{a}}(u_{\ell\Lambda_0} \otimes b(r,s,\lambda)) = u_{\ell\Lambda_0} \otimes m(B^{r,s}), \tag{5.12}$$

where $\ell = lev(B^{r,s})$. Moreover

$$\operatorname{card}\{j \mid i_j = 0\} = \frac{|\lambda| - |\lambda^\diamondsuit_{\min}(r,s)|}{|\diamondsuit|}. \tag{5.13}$$

Proof This follows from Proposition 4.7, (5.3), and Lemma 5.7. □

Proof of Proposition 3.15 Equation (5.13) yields

$$\widehat{D}_{B(\varphi(u_0(r,s)))}(u_{\ell\Lambda_0} \otimes b(r,s,\lambda)) = \frac{|\lambda| - |\lambda^\diamondsuit_{\min}(r,s)|}{|\diamondsuit|}.$$

By Corollary 5.6 we have

$$\overline{D}_{B^{r,s}}(b(r,s,\lambda)) = \overline{D}_{B^{r,s}}(m(B^{r,s})) - \frac{|\lambda| - |\lambda^\diamondsuit_{\min}(r,s)|}{|\diamondsuit|}. \tag{5.14}$$

Applying this for $\lambda = (s^r)$ we have

$$\overline{D}_{B^{r,s}}(b(r,s,(s^r))) = \overline{D}_{B^{r,s}}(m(B^{r,s})) - \frac{rs - |\lambda^\diamondsuit_{\min}(r,s)|}{|\diamondsuit|}. \tag{5.15}$$

Subtracting (5.15) from (5.14) and using Lemma 3.13(3) and the fact that $u(B^{r,s}) = b(r,s,(s^r))$, we obtain (3.55) as required. □

6 Splittings

In this section we define maps that embed a KR crystal into the tensor product of KR crystals. These maps are I_0-crystal embeddings which are compatible with the grading. These results hold for any nonexceptional affine algebra $\mathfrak{g}$ and any $r \in I_0$ with $r \neq 1$ and r nonspin.

6.1 Row splitting

In this section we construct a map which we call row splitting, because in type A, the map simply splits off the top row of a rectangular tableau.

Proposition 6.1 *For $\mathfrak{g}$ nonexceptional, $r \in I_0$ not a spin node and $r \neq 1$, there exists a unique map*

$$\mathrm{S} : B^{r,s} \longrightarrow B^{r-1,s} \otimes B^{1,s}$$

satisfying

$$\mathrm{S}(e_i(b)) = e_i(\mathrm{S}(b)) \quad \text{for any good arrow } b \to e_i(b). \tag{6.1}$$

Proof By Lemma 5.5, $B^{r,s}$ is connected by good arrows. By (6.1) it follows that S is completely determined by any single value. Again by (6.1), S is an I_0-crystal embedding. But $\mathrm{S}(u(B^{r,s})) = u(B^{r-1,s}) \otimes u'$ where u' is the unique element in $B^{1,s}$ of weight $s(\omega_r - \omega_{r-1})$, since these elements are the only ones in their respective crystals that are I_0-highest weight vertices of weight $s\omega_r$. So it remains to show existence.

Let $\ell = \mathrm{lev}(B^{r,s})$ be the common level of $B^{i,s}$ for $i \in I_0$ nonspin. By Lemma 3.11 and Theorem 3.16 there are isomorphisms

$$B(\ell\Lambda_0) \otimes B^{r,s} \cong B(\varphi(m(B^{r,s}))) \tag{6.2}$$

$$B(\ell\Lambda_0) \otimes B^{r-1,s} \otimes B^{1,s} \cong \bigoplus_{u'} B(\varphi(u')) \tag{6.3}$$

where $u' \in B^{1,s}$ satisfies

$$\varepsilon(u') = \varphi(m(B^{r-1,s})). \tag{6.4}$$

In the nonperfect case there may be more than one such u'. However there is a unique $u' \in B^{1,s}$ such that (6.4) holds and also

$$\varphi(u') = \varphi(m(B^{r,s})). \tag{6.5}$$

First suppose $B^{i,s}$ is perfect for $i \in I_0$ nonspin. Since $m(B^{r-1,s}) \in B^{r-1,s}_{\min}$, u' satisfying (6.4) is unique, in which case we must show this u' satisfies (6.5).

For every $i \in I_0$ define $t_{-c_{i^*}\omega_{i^*}} = w_i \tau_i$ where $w_i \in W$ and $\tau_i \in \Sigma$. One may verify that $\tau_r = \tau_{r-1}\tau_1$. Perfectness yields the isomorphism

$$B(\ell\Lambda_0) \otimes B^{r,s} \cong B(\ell\Lambda_{\tau_r(0)}) \cong B(\ell\Lambda_0) \otimes B^{r-1,s} \otimes B^{1,s} \tag{6.6}$$

with $u_{\ell\Lambda_0} \otimes m(B^{r,s}) \mapsto u_{\ell\Lambda_0} \otimes m(B^{r-1,s}) \otimes u'$. Equation (6.5) follows by applying φ to these highest weight vectors.

Suppose $B^{1,s}$ is not perfect. Then $\mathfrak{g} = C_n^{(1)}$, $s = 2\ell - 1$ and $\mathrm{lev}(B^{1,s}) = \ell$. In this case $\varphi(m(B^{r-1,s})) = (\ell-1)\Lambda_0 + \Lambda_{r-1}$ and $\varphi(m(B^{r,s})) = (\ell-1)\Lambda_0 + \Lambda_r$. There are exactly three elements $u' \in B^{1,s}$ with $\varepsilon(u') = \varphi(m(B^{r-1,s})) = (\ell-1)\Lambda_0 + \Lambda_{r-1}$. Namely, $r, \overline{r-1} \in B(\omega_1)$ and $1(r-1)\overline{r-1} \in B(3\omega_1)$. (If $s = 1$, neglect the last one.) The values of φ are $(\ell-1)\Lambda_0 + \Lambda_r$, $(\ell-1)\Lambda_0 + \Lambda_{r-2}$, $(\ell-2)\Lambda_0 + \Lambda_1 + \Lambda_{r-1}$, respectively. Let $u' = r$. $B(\Lambda') \cong B(\ell\Lambda_0) \otimes B^{r,s}$ in $B(\ell\Lambda_0) \otimes B^{r-1,s} \otimes B^{1,s}$ such that $u_{\ell\Lambda_0} \otimes m(B^{r,s}) \mapsto u_{\ell\Lambda_0} \otimes m(B^{r-1,s}) \otimes u'$.

Since $B^{r,s}$ is connected by good arrows, we may define S by

$$\mathrm{S}(b) = b_1 \otimes b_2 \quad \text{where } u_{\ell\Lambda_0} \otimes b \mapsto u_{\ell\Lambda_0} \otimes b_1 \otimes b_2 \text{ under (6.6).} \tag{6.7}$$

Equation (6.1) follows immediately. □

6.2 Splitting $B \in \mathcal{C}$ into rows

Let $\mathfrak{g}$ be nonexceptional. We use Notation 3.17 for $B = B^R$. Let $B^{\mathrm{rows}(R)} \in \mathcal{C}^\infty(\mathfrak{g})$ be defined by replacing each $B^{r,s}$ in B by $(B^{1,s})^{\otimes r}$. We define a map

$$\mathbb{S} = \mathbb{S}_R : B^R \to B^{\mathrm{rows}(R)} \tag{6.8}$$

as follows. Starting B^R we define a sequence of maps that go through various crystals in $\mathcal{C}$, ending with $B^{\mathrm{rows}(R)}$. We repeat the following step. We locate the leftmost tensor factor of the form $B^{r,s}$ with $r > 1$, apply a sequence of combinatorial R-matrices to swap it to the left, and apply $\mathrm{S} \otimes \mathrm{id}$ (which we will sometimes by abuse of notation also denote S), which trades in $B^{r,s}$ for $B^{r-1,s} \otimes B^{1,s}$. Eventually the current crystal consists tensor factors of the form $B^{1,s}$, and we apply a sequence of combinatorial R-matrices to reorder the tensor factors, obtaining $B^{\mathrm{rows}(R)}$. Call the composite map $\mathbb{S}_R$. It is an I_0-crystal morphism, being the composition of such [see (6.1)].

Remark 6.2 One can apply splitting of the *first* tensor factor and combinatorial R-matrices in any order until $B^{\mathrm{rows}(R)}$ is reached. We conjecture that the resulting map is independent of the order that these steps were taken.

Proposition 6.3 *For $\mathfrak{g} = \mathfrak{g}^\diamond$ reversible, $B^R \in \mathcal{C}^\infty(\mathfrak{g}^\diamond)$, and $b \in \mathrm{tops}(B^R)$ we have*

$$\mathbb{S}_R \circ \sigma_{B^R}(b) = \sigma_{B^{\mathrm{rows}(R)}} \circ \mathbb{S}_R(b). \tag{6.9}$$

Proof By (5.6) we may reduce to the case $B = B^{r,s}$ and $\mathbb{S}_R = S$. Let $\ell = \mathrm{lev}(B^{r,s})$. By (5.3) and the fact that S is an I_0-crystal morphism, we may assume $b \in \mathrm{hw}_{I_0}(\mathrm{tops}(B^{r,s}))$. By Lemma 5.10, there is a sequence $\mathbf{a} = (i_1, \dots, i_p)$ of indices in I_n such that $e_{\mathbf{a}}(u_{\ell\Lambda_0} \otimes b) = u_{\ell\Lambda_0} \otimes m(B^{r,s})$. Therefore we have

$$e_{\mathbf{a}}(b) = m(B^{r,s}) \tag{6.10}$$

and moreover this sequence consists of good arrows. Applying $\sigma\mathrm{S}$ we obtain

$$\begin{aligned}\sigma(\mathrm{S}(m(B^{r,s}))) &= \sigma(\mathrm{S}(e_{\mathbf{a}}(b)))\\ &= e_{\sigma(\mathbf{a})}\sigma(\mathrm{S}(b))\end{aligned}$$

using (6.1) and (5.3). Applying $\mathrm{S}\sigma$ to (6.10) we have

$$\begin{aligned}\mathrm{S}(\sigma(m(B^{r,s})) &= \mathrm{S}(\sigma(e_{\mathbf{a}}(b)))\\ &= e_{\sigma(\mathbf{a})}\mathrm{S}(\sigma(b))\end{aligned}$$

using (5.3), the fact that $\sigma(\mathbf{a})$ has indices in I_0, and (6.1). Since $e_{\sigma(\mathbf{a})}$ has a left inverse, we may assume that $b = m(B^{r,s})$. We have $\mathrm{S}(m(B^{r,s})) = m(B^{r-1,s}) \otimes u'$ for some $u' \in B^{1,s}$. By Proposition 5.8(2) we reduce to the equality

$$S(\overline{b}^\diamond_{\min}(r,s)) = \overline{b}^\diamond_{\min}(r-1,s) \otimes \sigma(u'). \tag{6.11}$$

Since $\overline{b}^\diamond_{\min}(r,s) \in B_{I_0}(s\omega_r)$, we may apply $\mathrm{rowtab} = \mathrm{rowtab}_{(s^r)}$ and similarly for $\overline{b}^\diamond_{\min}(r-1,s)$. By definition $(\mathrm{rowtab}_{(s^{r-1})} \otimes 1_{B(s\omega_1)})(S(\overline{b}^\diamond_{\min}(r,s))) = \mathrm{rowtab}_{(s^r)}(\overline{b}^\diamond_{\min}(r,s)) = \mathrm{rowtab}_{(s^{r-1})}(\overline{b}^\diamond_{\min}(r-1,s)) \otimes u''$ where $u'' \in B_{I_0}(s\omega_1)$ is the last row of $\mathrm{rowtab}_{(s^r)}(\overline{b}^\diamond_{\min}(r,s))$. So it remains to show $u'' = \sigma(u')$. Using the explicit form of $\mathrm{rowtab}(\overline{b}^\diamond_{\min}(r,s))$ given in Remark 4.6 one has

$$\varepsilon(u'') = \begin{cases} \ell\Lambda_{n-1} & (\diamondsuit = (1,1), r : \text{even}) \\ (\ell-1)\Lambda_n + \Lambda_{n-r+1} & (\diamondsuit = (2), s : \text{odd}) \\ \ell\Lambda_n & (\text{otherwise}) \end{cases}$$

$$\varphi(u'') \quad = \begin{cases} \ell\Lambda_{n-1} & (\diamondsuit = (1,1), r : \text{odd}) \\ (\ell-1)\Lambda_n + \Lambda_{n-r} & (\diamondsuit = (2), s : \text{odd}) \\ \ell\Lambda_n & (\text{otherwise}). \end{cases}$$

Therefore $\varepsilon(\sigma(u''))$ and $\varphi(\sigma(u''))$ are given by replacing every Λ_j with Λ_{n-j} in the above table. But $\varepsilon(\sigma(u'')) = \varphi(m(B^{r-1,s}))$ and $\varphi(\sigma(u'')) = \varphi(m(B^{r,s}))$. Therefore by (6.7) $u' = \sigma(u'')$ for there is a unique element in $B^{1,s}$ having such values of ε and φ, and we are done since σ is an involution. □

6.3 Box splitting

Let $\mathfrak{g}$ be of affine type such that $\mathfrak{g}_0$ is of type B_n, C_n, or D_n.

Define a map $B^{1,s} \hookrightarrow B^{1,s-1} \otimes B^{1,1}$ as follows. For $b = x_1 \cdots x_p \in B(p\omega_1) \subset B^{1,s}$,

$$b \mapsto \begin{cases} 1b \otimes \bar{1} & \text{if } s \geq p+2 \\ b \otimes \varnothing & \text{if } s = p+1 \\ x_2 \cdots x_p \otimes x_1 & \text{if } s = p. \end{cases} \tag{6.12}$$

Here $\varnothing$ denotes the element of $B(0) \subset B^{1,1}$ for $\mathfrak{g}$ of kind (1). This map is evidently an I_0-crystal embedding. Iterating this map on the first tensor factor, we obtain the following I_0-crystal embedding $S_\square : B^{1,s} \hookrightarrow (B^{1,1})^{\otimes s}$:

$$S_\square(b) = x_p \otimes \cdots \otimes x_2 \otimes x_1 \otimes \underbrace{1 \otimes \cdots \otimes 1}_{m \text{ times}} \otimes \underbrace{\varnothing}_{k \text{ times}} \otimes \underbrace{\bar{1} \otimes \cdots \otimes \bar{1}}_{m \text{ times}} \tag{6.13}$$

where $m = \lfloor \frac{s-p}{2} \rfloor$ and k is 0 or 1 according as $s-p$ is even or odd.

Define a map $\mathbb{S}_\square : B^R \to (B^{1,1})^{\otimes |R|}$ as follows. First apply $\mathbb{S} : B^R \to B^{\mathrm{rows}(R)}$. Then do the following repeatedly until $(B^{1,1})^{\otimes |R|}$ is reached. Find the leftmost factor of the form $B^{1,s}$ with $s > 1$ and swap it to the left end using combinatorial R-matrices and then apply $S_\square \otimes \mathrm{id}$ to replace this $B^{1,s}$ with $(B^{1,1})^{\otimes s}$. Write $\mathbb{S}_\square$ for the composite map. We have

$$\mathbb{S}_\square \circ e_i = e_i \circ \mathbb{S}_\square \qquad \text{for } i \in I_0 \tag{6.14}$$

since $\mathbb{S}_\square$ is the composition of I_0-crystal morphisms $\mathbb{S}$ and $S_\square \otimes 1$.

Remark 6.4 If R consists of tensor factors of the form $B^{1,s}$ then $\overline{D}_{B^R} = \overline{D}_{(B^{1,1})^{\otimes |R|}} \circ \mathbb{S}_\square$.

Proposition 6.5 *For $\mathfrak{g} = \mathfrak{g}^\diamondsuit$ reversible and $b \in \mathrm{tops}(B^R)$,*

$$\mathbb{S}_\square(\sigma(b)) = \sigma(\mathbb{S}_\square(b)). \tag{6.15}$$

Proof By Proposition 6.3, (5.6), (6.14), and (5.3) it suffices to prove (6.15) for $b \in \mathrm{tops}(B^{1,s})$. Consider the case $\diamondsuit = (1)$ where $\mathrm{tops}(B^{1,s})$ consists of elements $1^p = \mathrm{hw}_{I_0}(B(p\omega_1)) \subset B^{1,s}$ for $0 \leq p \leq s$. With notation as in (6.13) we have

$$\begin{aligned} \sigma(S_\square(1^p)) &= \sigma(1^{\otimes p+m} \otimes \varnothing^{\otimes k} \otimes \bar{1}^m) \\ &= \bar{n}^{\otimes p+m} \otimes 0^k \otimes n^{\otimes m} \\ &= S_\square(n^m 0^k \bar{n}^{p+m}) \\ &= S_\square(\sigma(1^p)). \end{aligned}$$

The cases $\diamondsuit \in \{(1,1),(2)\}$ are easier. □

7 Correspondence on A_{n-1}-highest weight vertices

Again we assume that $\mathfrak{g} = \mathfrak{g}^\diamond$ is reversible.

Let $\widehat{\max}(B) = \widehat{\max(B)}$ in the notation of Sect. 4.1. The goal of this section is to prove the following theorem.

Theorem 7.1 *For $B \in \mathcal{C}^\infty(\mathfrak{g}^\diamond)$ and every $\lambda \in \mathcal{P}_n$, $\sigma : B \to B$ restricts to a bijection*

$$\mathrm{hw}^{\lambda}_{A_{n-1}}(\mathrm{tops}(B)) \overset{\sigma}{\simeq} \mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{\max}(B)). \tag{7.1}$$

Lemma 7.2

$$\mathrm{card}\,\mathrm{hw}^{\lambda}_{A_{n-1}}(\mathrm{tops}(B)) = \mathrm{card}\,\mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{\max}(B)). \tag{7.2}$$

Proof There is an I_0-crystal isomorphism

$$\max(B^R) \simeq \bigoplus_{\nu \in \mathcal{P}_n} B(\nu)^{\oplus c^{\nu}_{R_1,\dots,R_p}}.$$

By (2.3) we have

$$\mathrm{card}\,\mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{\max}(B^R)) = \sum_{\nu \in \mathcal{P}_n} c^{\nu}_{R_1,\dots,R_p} \sum_{\delta \in \mathcal{P}_n^\diamond} c^{\nu}_{\delta\lambda} = \mathfrak{K}^{\lambda,\diamond}_{R_1,\dots,R_p}$$

where the last equality follows from Proposition 2.1. We have $X_{\lambda,B^R}(1) = \mathfrak{K}^{\lambda,\diamond}_{R_1,\dots,R_p}$ by (3.58), (1.2), and (2.2). Therefore (7.2) holds. □

Proposition 7.3 *The map $\sigma : B \to B$ sends* $\mathrm{tops}(B)$ *into* $\max(B)$.

Proof Let $b \in \mathrm{tops}(B)$. By (6.14) $\mathbb{S}_\square(b) \in \mathrm{tops}((B^{1,1})^{\otimes|R|})$. Assuming the Proposition holds for tensor powers of $B^{1,1}$ and using Proposition 6.5 we have $\mathbb{S}_\square(\sigma(b)) \in \max((B^{1,1})^{\otimes|R|})$. By (6.14), we deduce that $\sigma(b) \in \max(B^R)$.

We now assume $B = (B^{1,1})^{\otimes m}$ and $b \in \mathrm{tops}(B)$. We may assume that $b \in \mathrm{hw}_{A_{n-1}}(\mathrm{tops}(B))$. By induction on m, the letters of b lie in the set $\{1, 2, \dots, m\} \cup \{\bar{m}, \dots, \bar{1}\}$. Thus the letters of $\sigma(b)$ belong to $\{n-m+1, \dots, n, \bar{n}, \dots, \overline{n-m+1}\}$. When n is sufficiently large, this implies that $\sigma(b) \in \max(B)$. This can either be proved by induction on m or more directly by using the insertion procedure described in [19]. □

Proof of Theorem 7.1 By Proposition 7.3 σ sends $\mathrm{tops}(B)$ into $\max(B)$. Since $\mathrm{tops}(B)$ is an A_{n-1}-crystal whose weights lie in $\mathbb{Z}^n_{\geq 0}$ and σ sends such weights to $\mathbb{Z}^n_{\leq 0}$ by (5.4), σ must send $\mathrm{tops}(B)$ into $\widehat{\max}(B)$. By (5.3) and (5.4) σ sends $\mathrm{hw}^{\lambda}_{A_{n-1}}(\mathrm{tops}(B))$ into $\mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{\max}(B))$. Theorem 7.1 follows due to Lemma 7.2 and the injectivity of σ [which holds by (5.5)]. □

8 A relation between $\overline{D}$ and $\overline{D} \circ \sigma$

In this section we assume $\mathfrak{g} = \mathfrak{g}^\diamond$ is reversible. Define the map $B \to \mathcal{P}_n$ by $b \mapsto \lambda(b)$ where

$$B_{I_0}(b) \cong B(\lambda(b)). \tag{8.1}$$

The goal of this section is to prove the following theorem.

Theorem 8.1 *For $B^R \in \mathcal{C}(\mathfrak{g}^\diamond)$ and $b \in \mathrm{tops}(B^R)$*

$$\overline{D}(b) = \overline{D}(\sigma(b)) + \frac{|R| - |\lambda(b)|}{|\diamond|}. \tag{8.2}$$

We use Notation 3.17. For $b \in \mathcal{C}(\mathfrak{g}^\diamond)$ set $\nu(b) = \mathrm{wt}(b) = (\nu_1(b), \nu_2(b), \ldots)$ and $|\nu(b)| = \sum_i \nu_i(b)$. Note that some $\nu_i(b)$'s may be negative. Hence $|\nu(b)| = \sum_i \nu_i(b)$ may also become negative. We prepare a lemma.

Lemma 8.2 *Let $B_1, B_2 \in \mathcal{C}(\mathfrak{g}^\diamond)$. Let $b_1 \otimes b_2$ be an element of $B_1 \otimes B_2$ and suppose it is mapped to $b_2' \otimes b_1'$ by the combinatorial R-matrix. Then we have*

$$\overline{H}(b_1 \otimes b_2) - \overline{H}(\sigma(b_1) \otimes \sigma(b_2)) = \frac{|\nu(b_2')| - |\nu(b_2)|}{|\diamond|}. \tag{8.3}$$

Proof Since $B_1 \otimes B_2$ is connected, it is sufficient to show

(i) if $b_1 = u(B_1), b_2 = u(B_2)$, (8.3) holds, and
(ii) (8.3) with $b_1 \otimes b_2$ replaced by $e_i(b_1 \otimes b_2)$ holds, provided that (8.3) holds and $e_i(b_1 \otimes b_2) \neq 0$.

For (i) recall $b_1' = b_1, b_2' = b_2$ if $b_1 = u(B_1), b_2 = u(B_2)$. Since $u(B_1) \otimes u(B_2)$ can be reached from $\sigma(u(B_1)) \otimes \sigma(u(B_2))$ by applying e_i $(i \neq 0)$, we have $\overline{H}(u(B_1) \otimes u(B_2)) = \overline{H}(\sigma(u(B_1)) \otimes \sigma(u(B_2))) = 0$. Hence (i) is verified.

For (ii) recall $|\nu(e_i b)| - |\nu(b)| = -|\diamond|\,(i = 0), = |\diamond|\,(i = n), = 0$ (otherwise). If $i \neq 0, n$, both sides do not change when we replace $b_1 \otimes b_2$ with $e_i(b_1 \otimes b_2)$. If $i = 0$, the first term of the l.h.s decreases by one in case LL, increases by one in case RR, and does not change in case LR or RL. (For the meaning of LL, etc, see Proposition 3.7(2).) The second term does not change, while the r.h.s varies in the same way as the first term of the l.h.s. The $i = n$ case is similar. □

Proof of Theorem 8.1 We may reduce to the case that $b \in \mathrm{hw}_{A_{n-1}}(\mathrm{tops}(B))$ since $\mathrm{tops}(B)$ is an A_{n-1}-crystal and the entire Eq. (8.2) is invariant under A_{n-1}-arrows.

We proceed by induction on the number p of tensor factors in B^R. When $p = 1$ we have $B = B^{r,s}$. By (1.2) $b = b(r, s, \lambda)$ for some $\lambda \in \mathcal{P}(r, s)$. By Theorem 7.1, $\sigma(b) \in \widehat{\max}(B) \subset \max(B) = B((s^r)) \subset B^{r,s}$. But $\overline{D}$ is 0 on $B((s^r))$ by the definition of $\overline{D}_{B^{r,s}}$. Therefore $\overline{D}(\sigma(b)) = 0$. Then (8.2) holds by Proposition 3.15.

Let $B = B' \otimes B^{r_p,s_p}$ and $b_1 \otimes b_2 \in B' \otimes B^{r_p,s_p}$ is mapped to $b_2' \otimes b_1' \in B^{r_p,s_p} \otimes B'$ by the affine crystal isomorphism. Then $\sigma(b_1) \otimes \sigma(b_2)$ should be mapped to $\sigma(b_2') \otimes \sigma(b_1')$. Using (3.52) we have

$$\overline{D}(b) = \overline{D}(b_1) + \overline{D}(b_2') + \overline{H}(b_1 \otimes b_2),$$
$$\overline{D}(\sigma(b)) = \overline{D}(\sigma(b_1)) + \overline{D}(\sigma(b_2')) + \overline{H}(\sigma(b_1) \otimes \sigma(b_2)).$$

On the other hand, by the previous lemma we have

$$\overline{H}(b_1 \otimes b_2) - \overline{H}(\sigma(b_1) \otimes \sigma(b_2)) = \frac{|\lambda(b_2')| - |\lambda(b_2)|}{|\diamond|}.$$

Using the induction hypothesis we obtain

$$\begin{aligned}\overline{D}(b) - \overline{D}(\sigma(b)) &= \frac{|B'| - |\lambda(b_1)|}{|\diamond|} + \frac{|B^{r_p,s_p}| - |\lambda(b_2')|}{|\diamond|} + \frac{|\lambda(b_2')| - |\lambda(b_2)|}{|\diamond|} \\ &= \frac{|B| - |\lambda(b)|}{|\diamond|}\end{aligned}$$

as desired. □

9 Energy function on max elements

9.1 Highest elements in $\max(B^{r_1,s_1} \otimes B^{r_2,s_2})$

Proposition 9.1 *Let $b_1 \otimes b_2 \in \mathrm{hw}_{I_0}(\max(B^{r_1,s_1} \otimes B^{r_2,s_2}))$ and $r = \min(r_1, r_2)$.*

(1) *Then $b_1 = b(r_1, s_1, (s_1^{r_1}))$ and there exists a partition $\lambda \subset (s_2^{r_2})$ such that $\ell(\lambda) \leq r$ and $\lambda_r \geq s_2 - s_1$, and $b_2 \in B(s_2^{r_2})$ is the tableau whose entries are i in the ith row in λ and $r_1 + 1, r_1 + 2, \ldots$ from bottom to top outside of λ.*
(2) *Let λ be as in (1). Suppose $b_1 \otimes b_2$ is sent to $b_2' \otimes b_1'$ by the combinatorial R. Then the corresponding partition μ of b_1' is obtained from λ by adding $s_1 - s_2$ (resp. removing $s_2 - s_1$) columns of height r if $s_1 \geq s_2$ (resp. $s_1 \leq s_2$).*

Proof (1) is immediate from (3.41). For (2) note that the combinatorial R preserves the weight. Given a highest element $b_1 \otimes b_2$ as in (1), there is a unique highest element in $\max(B^{r_2,s_2} \otimes B^{r_1,s_1})$ of the same weight. □

Example 9.2

$$\begin{array}{|c|c|c|c|}\hline 4&4&4&4\\\hline 3&3&3&3\\\hline 2&2&2&2\\\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 5&6&6&7&9\\\hline \mathbf{4}&5&5&6&8\\\hline \mathbf{3}&\mathbf{3}&\mathbf{3}&5&7\\\hline \mathbf{2}&\mathbf{2}&\mathbf{2}&\mathbf{2}&6\\\hline \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&5\\\hline\end{array}$$

is the unique element of $\mathrm{hw}_{I_0}(\max(B^{4,4} \otimes B^{5,5}))$ with associated $\lambda = (4, 4, 3, 1)$. By the combinatorial R it is sent to

$$\begin{array}{|c|c|c|c|c|}\hline 5&5&5&5&5\\\hline 4&4&4&4&4\\\hline 3&3&3&3&3\\\hline 2&2&2&2&2\\\hline 1&1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|}\hline 6&6&7&9\\\hline \mathbf{3}&\mathbf{3}&6&8\\\hline \mathbf{2}&\mathbf{2}&\mathbf{2}&7\\\hline \mathbf{1}&\mathbf{1}&\mathbf{1}&6\\\hline\end{array}$$

Our goal in this section is to prove the following proposition.

Proposition 9.3 *Assume $s_1 \geq s_2$ and let $r = \min(r_1, r_2)$. Let $b_1 \otimes b_2 \in \mathrm{hw}_{I_0}(\max(B^{r_1,s_1} \otimes B^{r_2,s_2}))$ whose partition is λ. Then*

$$\overline{H}(b_1 \otimes b_2) = \frac{2}{|\diamondsuit|}(rs_2 - |\lambda|).$$

Let $e_i^{\max}(b) = e_i^{\varepsilon_i(b)}(b)$.

Lemma 9.4 *Let $B^{r,s}$ be a KR crystal of type $D_n^{(1)}$. Let $\alpha, \beta, \gamma \in \mathbb{Z}_{\geq 0}$ sum to s and let b be the element of $\max(B^{r,s})$ with α columns whose entries are $1, 2, \ldots, r$ from bottom to top, β columns with $2, 3, \ldots, r + 1$ and γ columns with $3, 4, \ldots, r + 2$. Then*

(1) *$\varepsilon_0(b) = 2\alpha + \beta$, $\varphi_0(b) = 0$, and*
(2) *$e_0^{\max}(b)$ is the tableau with γ columns with $3, 4, \ldots, r+2$, β columns with $3, 4, \ldots, r+1, \overline{1}$ and α columns with $3, 4, \ldots, r, \overline{2}, \overline{1}$.*

Proof b is a $\{3, 4, \dots, n\}$-highest weight vertex. As is explained in section 4.2 of [3], such elements are in one-to-one correspondence with pairs of $\pm$-diagrams (P, p), where the inner shape of P is the outer shape of p. b corresponds to (P, p), where P has the outer shape (s^r) and the inner shape $(s^{r-1}, s-\alpha)$, and p has the inner shape $(s^{r-2}, s-\alpha, s-\alpha-\beta)$. The signs in P and p are all $+$. Once we have the corresponding pair of $\pm$-diagrams, it is easy to see ε_0, φ_0, and the action of e_0. As a result we see $e_0^{\max}(b)$ corresponds to the pair of $\pm$-diagrams with all $+$ in (P, p) being replaced with $-$. In turn this yields the above tableau. □

Lemma 9.5 *Let b and λ be as in Proposition 9.1(1). Let $r = \min(r_1, r_2)$ and let w_0^r be the longest element of the symmetric group $S_r \subset W$ generated by s_1 through s_{r-1}. Then $\mathrm{hw}_{I_0}(e_0^{\max}(b^{w_0^r}))$ has associated partition λ^- obtained by adding $\min(2, r-\lambda'_j)$ (resp. $\min(1, r-\lambda'_j)$) boxes to the jth column for $1 \le j \le s_2$ for $\diamondsuit = (1,1)$ (resp. $\diamondsuit = (1), (2)$).*

Example 9.6 Let b be the element of $\max(B^{4,4} \otimes B^{5,5})$ of Example 9.2.

$$b = \begin{array}{|c|c|c|c|}\hline 4&4&4&4\\\hline 3&3&3&3\\\hline 2&2&2&2\\\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 5&6&6&7&9\\\hline \mathbf{4}&5&5&6&8\\\hline \mathbf{3}&\mathbf{3}&\mathbf{3}&5&7\\\hline \mathbf{2}&\mathbf{2}&\mathbf{2}&\mathbf{2}&6\\\hline \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&5\\\hline\end{array} \xrightarrow{w_0^r} \begin{array}{|c|c|c|c|}\hline 4&4&4&4\\\hline 3&3&3&3\\\hline 2&2&2&2\\\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 5&6&6&7&9\\\hline 4&5&5&6&8\\\hline 3&4&4&5&7\\\hline 2&3&3&4&6\\\hline 1&2&2&3&5\\\hline\end{array}$$

$$\xrightarrow{e_0^{\max}} \begin{array}{|c|c|c|c|}\hline \overline{1}&\overline{1}&\overline{1}&\overline{1}\\\hline \overline{2}&\overline{2}&\overline{2}&\overline{2}\\\hline 4&4&4&4\\\hline 3&3&3&3\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 7&7&\overline{1}&\overline{1}&\overline{1}\\\hline 6&6&6&8&\overline{2}\\\hline 5&5&5&5&7\\\hline 4&4&4&4&6\\\hline 3&3&3&3&5\\\hline\end{array} \xrightarrow{\mathrm{hw}_{I_0}} \begin{array}{|c|c|c|c|}\hline 4&4&4&4\\\hline 3&3&3&3\\\hline 2&2&2&2\\\hline 1&1&1&1\\\hline\end{array} \otimes \begin{array}{|c|c|c|c|c|}\hline 5&5&5&5&7\\\hline \mathbf{4}&\mathbf{4}&\mathbf{4}&\mathbf{4}&6\\\hline \mathbf{3}&\mathbf{3}&\mathbf{3}&\mathbf{3}&5\\\hline \mathbf{2}&\mathbf{2}&\mathbf{2}&\mathbf{2}&\mathbf{2}\\\hline \mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}&\mathbf{1}\\\hline\end{array}$$

We indicate λ and λ^- by boldface entries.

Proof of Lemma 9.5 We first treat the case of $\diamondsuit = (1,1)$. $b^{w_0^r}$ is obtained from b by modifying the λ-part of the second component of b as follows. The column of entries $1, 2, \dots, h$ $(h \le r)$ reading from bottom to top is replaced by $r-h+1, r-h+2, \dots, r$.

Next we want to apply $e_0^{\max}$. Suppose $r_1 \le r_2$. (The other case is similar.) Write $b^{w_0^r} = \tilde{b}_1 \otimes \tilde{b}_2$. From Lemma 9.4 we have $\varphi_0(\tilde{b}_1) = 0$, and $e_0^{\max}(\tilde{b}_1)$ is the tableau with s_1 columns of entries $3, 4, \dots, r_1, \overline{2}, \overline{1}$. To calculate $e_0^{\max}(\tilde{b}_2)$ define a sequence $\boldsymbol{a} = \boldsymbol{a}_{r_2} \sqcup \cdots \sqcup \boldsymbol{a}_2 \sqcup \boldsymbol{a}_1$ where

$$\boldsymbol{a}_j = ((j+2)^{s_2-\lambda_{r_1-2}}, \dots, (r_1+j-2)^{s_2-\lambda_2}, (r_1+j-1)^{s_2-\lambda_1})$$

for $j = 1, 2, \dots, r_2$, and set $\tilde{b}'_2 = e_{\boldsymbol{a}}(\tilde{b}_2)$. Then $\tilde{b}'_2$ is the tableau with λ_{r_1} columns of entries $1, 2, \dots, r_2$, $\lambda_{r_1-1} - \lambda_{r_1}$ columns of entries $2, 3, \dots, r_2+1$ and $s_2 - \lambda_{r_1-1}$ columns of entries $3, 4, \dots, r_2+2$. Again from Lemma 9.4 $e_0^{\max}(\tilde{b}'_2)$ is the tableau with $s_2 - \lambda_{r_1-1}$ columns of $3, 4, \dots, r_2+2$, $\lambda_{r_1-1} - \lambda_{r_1}$ columns of $3, 4, \dots, r_2+1, \overline{1}$ and λ_{r_1} columns of $3, 4, \dots, r_2, \overline{2}, \overline{1}$. Since e_i, f_i for $3 \le i \le n$ commutes with e_0, we have $\tilde{b}''_2 = e_0^{\max}(\tilde{b}_2) = f_{\mathrm{Rev}(\boldsymbol{a})} e_0^{\max}(\tilde{b}'_2)$, where $\mathrm{Rev}(\boldsymbol{a})$ is the reverse sequence of $\boldsymbol{a}$. The jth row of $\tilde{b}''_2$ from bottom $(1 \le j \le r_2)$ is given by

$$\begin{array}{ll} (j+2)^{\lambda_{r_1-2}}(j+3)^{\lambda_{r_1-3}-\lambda_{r_1-2}} \cdots (r_1+j-1)^{\lambda_1-\lambda_2}(r_1+j)^{s_2-\lambda_1} & \text{for } 1 \le j \le r_2-2 \\ (r_2+1)^{\lambda_{r_1-2}-\lambda_{r_1}}(r_2+2)^{\lambda_{r_1-3}-\lambda_{r_1-2}} \cdots (r_1+r_2-1)^{s_2-\lambda_1}\overline{2}^{\lambda_{r_1}} & \text{for } j = r_2-1 \\ (r_2+2)^{\lambda_{r_1-2}-\lambda_{r_1-1}}(r_2+3)^{\lambda_{r_1-3}-\lambda_{r_1-2}} \cdots (r_1+r_2)^{s_2-\lambda_1}\overline{1}^{\lambda_{r_1-1}} & \text{for } j = r_2. \end{array}$$

Thus we have $e_0^{\max} b^{w_0^r} = e_0^{\max}(\tilde{b}_1) \otimes \tilde{b}_2''$.

Finally, we want to calculate the I_0-highest vertex of $e_0^{\max} b^{w_0^r}$. This calculation is long but not difficult, and it is checked that the statement is true.

For the proof for $\diamondsuit = (1), (2)$ we use the construction of a KR crystal in [3, §4.3&§4.4]. Namely, $B^{r,s}$ is realized as a suitable subset of an $A_{2n+1}^{(2)}$-KR crystal where 0 actions are defined in the same way as $D_n^{(1)}$. Hence we do not repeat the proof. □

Proof of Proposition 9.3 Let $b_2' \otimes b_1'$ be the image of $b_1 \otimes b_2$ by the combinatorial R. We apply w_0^r to both $b_1 \otimes b_2$ and $b_2' \otimes b_1'$ as prescribed in Lemma 9.5. Noting that e_0 commutes with e_j and f_j for $j \geq 3$ we find the 0-signature of these elements are $-^{2s_1} \otimes -^{\lambda_r + \lambda_{r-1}}$ and $-^{2s_2} \otimes -^{\lambda_r+\lambda_{r-1}+2s_1-2s_2}$ for $\diamondsuit = (1,1)$, $-^{s_1} \otimes -^{\lambda_r}$ and $-^{s_2} \otimes -^{\lambda_r+s_1-s_2}$ for $\diamondsuit = (2)$, $-^{2s_1} \otimes -^{2\lambda_r}$ and $-^{2s_2} \otimes -^{2\lambda_r+2s_1-2s_2}$ for $\diamondsuit = (1)$ by Lemma 9.4. Setting $b_1^\circ \otimes b_2^\circ = \mathrm{hw}_{I_0}(e_0^{\max}((b_1 \otimes b_2)^{w_0^r}))$ and recalling (3.42) we have

$$\overline{H}(b_1^\circ \otimes b_2^\circ) = \overline{H}(b_1 \otimes b_2) + \begin{cases} (\lambda_r + \lambda_{r-1}) - 2s_2 & \text{for } \diamondsuit = (1,1) \\ \lambda_r - s_2 & \text{for } \diamondsuit = (2) \\ 2\lambda_r - 2s_2 & \text{for } \diamondsuit = (1). \end{cases}$$

This formula implies the desired result. □

9.2 The general case

In this section let $\mathfrak{g}$ be an affine algebra such that $\mathfrak{g}_0$ is of type B_n, C_n, or D_n. Using Remark 4.1 with $\nu = (s^r) \in \mathcal{P}_n^\infty$ there is a unique embedding of A_{n-1}-crystals

$$B_A^{r,s} \cong B_{A_{n-1}}(s^r) \xrightarrow{i_A} B_{I_0}(s^r) \subset B^{r,s}, \tag{9.1}$$

which yields an A_{n-1}-crystal isomorphism

$$B_A^{r,s} \cong \mathrm{tops}(\max(B^{r,s})). \tag{9.2}$$

We use Notation 3.17. Define

$$B_A = B_A^R = B_A^{r_1,s_1} \otimes \cdots \otimes B_A^{r_p,s_p} \tag{9.3}$$

where $B_A^{r,s}$ is the type $A_{n-1}^{(1)}$ KR crystal. There is an embedding

$$i_A^R : B_A^R \to B^R \tag{9.4}$$

given by the tensor product of embeddings (9.1), inducing the isomorphism of A_{n-1}-crystals

$$B_A^R \cong \mathrm{tops}(\max(B^R)). \tag{9.5}$$

Theorem 9.7 *For $\diamondsuit \in \{(1), (2), (1,1)\}$, $B^R \in \mathcal{C}^\infty(\mathfrak{g})$ and $\nu \in \mathcal{P}_n^\infty$ such that $|\nu| = |R|$ we have*

$$\overline{X}^{\diamondsuit}_{\nu,B^R}(q) = \overline{X}^{\emptyset}_{\nu,B_A^R}(q^{\frac{2}{|\diamondsuit|}}). \tag{9.6}$$

Proof Immediate from (9.5) and Proposition 9.10 below. □

Lemma 9.8 *Let R and R' be sequences of rectangles that are reorderings of each other with $B^R, B^{R'} \in \mathcal{C}^\infty(\mathfrak{g})$ and let $g : B^R \to B^{R'}$ be the unique isomorphism of I-crystals. Denote by $g_A : B_A^R \to B_A^{R'}$ the corresponding isomorphism of crystals of type $A_{n-1}^{(1)}$. Then on B_A^R we have*

$$g \circ i_A^R = i_A^{R'} \circ g_A \tag{9.7}$$

Proof One may reduce to the case that $R = (R_1, R_2)$ and $R' = (R_2, R_1)$ and further to considering only A_{n-1}-highest weight vertices. But then the two sides must agree since $B_A^{R_1} \otimes B_A^{R_2}$ is A_{n-1}-multiplicity-free. □

Lemma 9.9 *For $B^{R_1} \otimes B^{R_2} \in \mathcal{C}^\infty(\mathfrak{g})$, we have $\overline{H}_{B_A^{R_1} \otimes B_A^{R_2}} = \overline{H}_{B^{R_1} \otimes B^{R_2}} \circ i_A^{R_1,R_2}$.*

Proof This follows from Proposition 9.3 and the analogous type $A_{n-1}^{(1)}$ result [29,31]. □

Proposition 9.10 $\overline{D}_A = \overline{D} \circ i_A$ *on* B_A.

Proof By (3.52), induction, and Lemmata 9.8 and 9.9, we may reduce to the case of a single tensor factor $B^{r,s}$. Since $B_A^{r,s} \cong B_{A_{n-1}}(s^r)$ as A_{n-1}-crystals [14], $\overline{D}_{B_A^{r,s}} = 0$. But i_A sends the A_{n-1}-highest weight vertex of $B_A^{r,s}$ to $b(r, s, (s^r)) = u(B^{r,s})$, on which $\overline{D}_{B^{r,s}}$ has value 0 by definition. □

10 Main results

10.1 The decomposition theorem

We prove Conjecture 1.1 and any tensor product of KR crystals.

Theorem 10.1 *Let $B^R \in \mathcal{C}^\infty(\mathfrak{g})$ where $\mathfrak{g}$ is of kind $\diamond \in \{(1), (2), (1,1)\}$. Then for any $\lambda \in \mathcal{P}_n$ we have*

$$\overline{X}^{\diamond}_{\lambda, B^R}(q) = q^{\frac{|R|-|\lambda|}{|\diamond|}} \sum_{\nu \in \mathcal{P}_n} \sum_{\delta \in \mathcal{P}_n^{\diamond}} c^{\nu}_{\lambda,\delta}\, \overline{X}^{\emptyset}_{\nu, B_A^R}(q^{\frac{2}{|\diamond|}}).$$

Proof We have

$$\begin{aligned}\overline{X}^{\diamond}_{\lambda, B^R}(q) &= q^{\frac{|R|-|\lambda|}{|\diamond|}} \sum_{b \in \mathrm{hw}^{\lambda}_{I_0}(B^R)} q^{\overline{D}(\sigma(b))} \\ &= q^{\frac{|R|-|\lambda|}{|\diamond|}} \sum_{b \in \mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{\max}(B^R))} q^{\overline{D}(b)}\end{aligned}$$

by (3.58) and Theorems 8.1 and 7.1. $\max(B^R)$ has I_0-decomposition

$$\max(B^R) = \bigoplus_{\substack{\nu \in \mathcal{P}_n \\ |\nu| = |R|}} \bigoplus_{c \in \mathrm{hw}^{\nu}_{I_0}(B^R)} B(c).$$

For $c \in \mathrm{hw}_{I_0}(\max(B))$, let $\widehat{B}(c) := \widehat{B(c)}$ for the dual polynomial part of $B(c)$; see Sect. 4.1. Taking the dual polynomial part, we have

$$\widehat{\max}(B^R) = \bigoplus_{\substack{\nu \in \mathcal{P}_n \\ |\nu| = |R|}} \bigoplus_{c \in \mathrm{hw}^{\nu}_{I_0}(B^R)} \widehat{B}(c).$$

Taking $\mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}$, we have

$$\mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{\max}(B^R)) = \bigsqcup_{\substack{\nu \in \mathcal{P}_n \\ |\nu| = |R|}} \bigsqcup_{c \in \mathrm{hw}^{\nu}_{I_0}(B^R)} \mathrm{hw}^{\overline{\lambda}}_{A_{n-1}}(\widehat{B}(c)).$$

For $c \in \mathrm{hw}_{I_0}^{\nu}(B^R)$, observe that $\overline{D}(b) = \overline{D}(c)$ for $b \in \mathrm{hw}_{A_{n-1}}^{\overline{\lambda}}(\widehat{B}(c))$ since these vertices b all belong to the same classical component $B(c)$. This gives

$$\overline{X}_{\lambda,B^R}^{\diamond}(q) = q^{\frac{|R|-|\lambda|}{|\diamond|}} \sum_{\substack{\nu\in\mathcal{P}_n\\ |\nu|=|R|}} \sum_{c\in \mathrm{hw}_{I_0}^{\nu}(B^R)} q^{\overline{D}(c)} \operatorname{card} \mathrm{hw}_{A_{n-1}}^{\overline{\lambda}}(\widehat{B}(c)).$$

But by (2.3) we have

$$\operatorname{card} \mathrm{hw}_{A_{n-1}}^{\overline{\lambda}}(\widehat{B}(c)) = \sum_{\delta\in\mathcal{P}_n^{\diamond}} c_{\lambda,\delta}^{\nu}.$$

By Theorem 9.7 we have

$$\begin{aligned}\overline{X}_{\lambda,B^R}^{\diamond}(q) &= q^{\frac{|R|-|\lambda|}{|\diamond|}} \sum_{\substack{\nu\in\mathcal{P}_n\\ |\nu|=|R|}} \sum_{\delta\in\mathcal{P}_n^{\diamond}} c_{\lambda,\delta}^{\nu} \sum_{c\in \mathrm{hw}_{I_0}^{\nu}(B^R)} q^{\overline{D}(c)}\\ &= q^{\frac{|R|-|\lambda|}{|\diamond|}} \sum_{\nu\in\mathcal{P}_n} \sum_{\delta\in\mathcal{P}_n^{\diamond}} c_{\lambda\delta}^{\nu}\, \overline{X}_{\nu,B^R}^{\emptyset}(q^{\frac{2}{|\diamond|}}).\end{aligned}$$

□

10.2 Link with parabolic Lusztig q-analogues

We now give a brief overview on parabolic Lusztig q-analogues in type A_{n-1}, B_n, C_n and D_n. Assume $G = GL_n, SO_{2n+1}, SP_{2n}$ or SO_{2n}. Consider U a subset of Σ_G^+ and denote by π_U the standard parabolic subgroup of G (that is, containing the Borel subgroup B_G) defined by U. Write L_U for the Levi subgroup of the parabolic π_U and $\mathfrak{l}_U$ its corresponding Lie algebra. Let R_U be the subsystem of roots spanned by U and R_U^+ the subset of positive roots in R_U. Then R_U and R_U^+ are respectively the set of roots and the set of positive roots of $\mathfrak{l}_U$.

The Levi subgroup L_U corresponds to the removal, in the Dynkin diagram of G, of the nodes which are not associated to a simple root belonging to U. When $U \neq \Sigma_G^+$, write

$$V = \Sigma_G^+ \setminus U = \{\alpha_{j_1}, \dots, \alpha_{j_p}\}$$

where for any $k = 1, \dots, p, \alpha_{j_k}$ is a simple root of Σ_G^+ and $j_1 < \cdots < j_p$. Then set $l_1 = j_1, l_k = j_k - j_{k-1}, k = 2, \dots, p$ and $l_{p+1} = n - j_p$. The Levi group L_U is isomorphic to a direct product of classical Lie groups determined by the $(p+1)$-tuple $l_U = (l_1, \dots, l_{p+1})$ of nonnegative integers summing to n. Namely, we have

$$L_U \simeq \begin{cases} GL_{l_1} \times \cdots \times GL_{l_p} & \text{if } G = GL_n\\ GL_{l_1} \times \cdots \times GL_{l_p} \times SO_{2l_{p+1}+1} & \text{if } G = SO_{2n+1}\\ GL_{l_1} \times \cdots \times GL_{l_p} \times Sp_{2l_{p+1}} & \text{if } G = Sp_{2n}\\ GL_{l_1} \times \cdots \times GL_{l_p} \times SO_{2l_{p+1}} & \text{if } G = SO_{2n}.\end{cases}$$

Let $\mathcal{P}_U = \mathcal{P}_{l_1} \times \cdots \times \mathcal{P}_{l_{p+1}}$. Then each $(p+1)$-partition of $\mathcal{P}_U$ can be regarded as a dominant weight for L_U. For any $\mu \in \mathcal{P}_U$, let $V^{L_U}(\mu)$ be the finite dimensional irreducible representation of L_U with highest weight μ. We denote by $\mu \in \mathbb{N}^n$ the concatenation of the parts of the partitions $\mu^{(k)}, k = 1, \dots, p$.

Define the partition function $\mathcal{P}^U$ by the formal identity

$$\prod_{\alpha \in R_G^+ \setminus R_U^+} \frac{1}{1-e^{\alpha}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}^U(\beta) e^{\beta}.$$

Consider $\lambda \in \mathcal{P}_n$ and $\mu \in \mathcal{P}_U$ then we have [7, Theorem 8.2.1]

$$[V^G(\lambda) : V^{L_U}(\mu)] = \sum_{w \in W_G} (-1)^{\ell(w)} \mathcal{P}^U(w \circ \lambda - \mu).$$

Here $[V^G(\lambda) : V^{L_U}(\mu)]$ is the branching multiplicity of the irreducible L_U-module $V^{L_U}(\mu)$ in the restriction of $V^G(\lambda)$ to L_U. For $GL_n, SO_{2n+1}, SP_{2n}, SO_{2n}$, we define the q-partition function $\mathcal{P}_q^U$ from the formal identity

$$\prod_{\alpha \in R_G^+ \setminus R_U^+} \frac{1}{1-qe^{\alpha}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^U(\beta) e^{\beta}. \tag{10.1}$$

In type B_n, we shall also need another partition function. Consider the weight function L on the set $R^+_{SO_{2n+1}}$ of positive roots of SO_{2n+1} such that $L(\alpha) = 2$ (resp. $L(\alpha) = 1$) on the long (resp. short) roots. The partition function $\mathcal{P}_q^{U,L}$ is defined by

$$\prod_{\alpha \in R^+_{SO_{2n+1}} \setminus R_U^+} \frac{1}{1-q^{L(\alpha)}e^{\alpha}} = \sum_{\beta \in \mathbb{Z}^n} \mathcal{P}_q^{U,L}(\beta) e^{\beta}.$$

Definition 10.2 Let λ be a partition of $\mathcal{P}_n$ and $\mu \in \mathcal{P}_U$.

1. The parabolic Lusztig q-analogue $K_{\lambda,\mu}^{G,U}(q)$ is the polynomial

$$K_{\lambda,\mu}^{G,U}(q) = \sum_{w \in W_G} (-1)^{\ell(w)} \mathcal{P}_q^U(w \circ \lambda - \mu) \tag{10.2}$$

   where $w \circ \lambda = w(\lambda + \rho_G) - \rho_G$.
2. The stable parabolic Lusztig q-analogue ${}^{\infty}K_{\lambda,\mu}^{G,U}(q)$ is the polynomial

$${}^{\infty}K_{\lambda,\mu}^{G,U}(q) = \sum_{w \in S_n} (-1)^{\ell(w)} \mathcal{P}_q^U(w \circ \lambda - \mu) \quad \text{for } G = GL_n, SP_{2n}, SO_{2n} \tag{10.3}$$

$${}^{\infty}K_{\lambda,\mu}^{G,U}(q) = \sum_{w \in S_n} (-1)^{\ell(w)} \mathcal{P}_q^{U,L}(w \circ \lambda - \mu) \quad \text{for } G = SO_{2n+1}. \tag{10.4}$$

Remark (i) When $U = \Sigma_G^+$, $\mathfrak{l}_U$ is the Cartan subalgebra of $\mathfrak{g}$ and $K_{\lambda,\mu}^{G,U}(q)$ is the usual Lusztig q-analogue.

(ii) The terminology for the polynomials ${}^{\infty}K_{\lambda,\mu}^{G,U}(q)$ is motivated by the following identities proved in [18]

$${}^{\infty}K_{\lambda,\mu}^{G,U}(q) = {}^{\infty}K_{\lambda+\kappa,\mu+\kappa}^{G,U}(q) \quad \text{for } G = GL_n, SO_{2n+1}, SP_{2n}, SO_{2n}$$

$${}^{\infty}K_{\lambda,\mu}^{G,U}(q) = K_{\lambda+k\kappa,\mu+k\kappa}^{G,U}(q) \quad \text{for } G = GL_n, SP_{2n}, SO_{2n} \text{ and } k \text{ sufficiently large}$$

where $\kappa = (1, \ldots, 1) \in \mathcal{P}_n$. In particular ${}^{\infty}K_{\lambda,\mu}^{GL_n,U}(q) = K_{\lambda,\mu}^{GL_n,U}(q)$.

The problem of the positivity of the coefficients appearing in the polynomials $K_{\lambda,\mu}^{G,U}(q)$ has been barely addressed in the literature.

Conjecture 10.3 *Let λ be partition of $\mathcal{P}_n$ and $\mu \in \mathcal{P}_U$ such that μ is a partition. Then $K^{G,U}_{\lambda,\mu}(q)$ has nonnegative coefficients.*

We have the following result due to Broer [2].

Theorem 10.4 *Let λ be a partition of $\mathcal{P}_n$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(p)})$ a dominant weight of L_U such that the $\mu^{(k)}$'s are rectangular partitions of decreasing widths with $\mu^{(p)} = 0$ when L_U is not a direct product of linear groups. Then $K^{G,U}_{\lambda,\mu}(q)$ has nonnegative coefficients.*

This theorem has been recently extended in [8]. Nevertheless, as far as we are aware, Conjecture 10.3 has not been completely proved yet.

Let $\eta = (\eta_1, \ldots, \eta_p)$ be a p-tuple of positive integers summing n. Consider $\lambda \in \mathcal{P}_n$ and $\mu = (\mu^{(1)}, \ldots, \mu^{(p)})$ a p-tuple of partitions such that $\mu^{(k)}$ belongs to $\mathcal{P}_{\eta_k}$ for any $k = 1, \ldots, p$. Recall that $\mathfrak{K}^{\lambda,\diamondsuit}_{\mu^{(1)},\ldots,\mu^{(p)}}$ is the multiplicity of $V^G(\lambda)$ in $W^G(\mu^{(1)}) \otimes \cdots \otimes W^G(\mu^{(p)})$. Write $\mu \in \mathbb{N}^n$ for the n-tuple obtained by reading successively the parts of the partitions $\mu^{(1)}, \ldots, \mu^{(p)}$ defining μ from left to right. Let a be the minimal integer such that

$$a \geq \frac{|\mu| - |\lambda|}{2} \quad \text{and} \tag{10.5}$$
$$\widehat{\lambda} = (a - \lambda_n, \ldots, a - \lambda_1) \in \mathbb{N}^n, \quad \widehat{\mu} = (a - \mu_n, \ldots, a - \mu_1) \in \mathbb{N}^n.$$

Then $\widehat{\lambda}$ is a partition of length n.

For any $k = 1, \ldots, p$, set $\widehat{\eta}_k = \eta_{p-k+1}$ and $\widehat{\eta} = (\widehat{\eta}_1, \ldots, \widehat{\eta}_p)$. Denote by $\widehat{\mu} = (\widehat{\mu}^{(1)}, \ldots, \widehat{\mu}^{(p)})$ the p-tuple of partitions such that $\widehat{\mu}^{(1)} = (\mu_1, \ldots, \mu_{\widehat{\eta}_1}) \in \mathcal{P}_{\widehat{\eta}_1}$ and $\widehat{\mu}^{(k)} = (\mu_{\widehat{\eta}_1+\cdots+\widehat{\eta}_{k-1}+1}, \ldots, \mu_{\widehat{\eta}_1+\cdots+\widehat{\eta}_k}) \in \mathcal{P}_{\widehat{\eta}_k}$ for any $k = 2, \ldots, p$. The Lie groups GL_n, SO_{2n+1},
SP_{2n}, SO_{2n} contain Levi subgroups L_U isomorphic to $GL_{\widehat{\eta}_1} \times \cdots \times GL_{\widehat{\eta}_p}$. With the above terminology, the corresponding subset of simple roots is

$$U = \{0 < \alpha_i < \widehat{\eta}_1\} \cup_{1 \leq k \leq p-1} \{\alpha_i \mid \widehat{\eta}_1 + \cdots + \widehat{\eta}_k < i < \widehat{\eta}_1 + \cdots + \widehat{\eta}_{k+1}\}. \tag{10.6}$$

In particular when $G = SO_{2n+1}, SP_{2n}$ or SO_{2n}, U never contains the simple root α_n.

Example 10.5 Consider $\mu^{(1)} = (5, 4, 4)$, $\mu^{(2)} = (6, 3, 2)$ and $\mu^{(3)} = (4, 3)$. Take $\lambda = (4, 4, 3, 2, 2, 1, 0, 0)$. Then $a = 8$, $\widehat{\mu}^{(1)} = (5, 4)$, $\widehat{\mu}^{(2)} = (6, 5, 2)$, $\widehat{\mu}^{(3)} = (4, 4, 3)$ and $\widehat{\lambda} = (8, 8, 7, 6, 6, 5, 4, 4)$.

The coefficients $\mathfrak{K}^{\lambda,\diamondsuit}_{\mu^{(1)},\ldots,\mu^{(p)}}$ defined in (2.1) can in fact be regarded as branching coefficients corresponding to the restriction to Levi subgroup isomorphic to a direct product of linear groups. The following duality was established in [18].

Proposition 10.6 *With the previous notation for $\widehat{\lambda}, \widehat{\mu}$ we have for $G = SO_{2n+1}, SP_{2n}$ and $SO_{2n} \mathfrak{K}^{\lambda,\diamondsuit}_{\mu^{(1)},\ldots,\mu^{(p)}} =^\infty K^{G,U}_{\widehat{\lambda},\widehat{\mu}}(1)$.*

We then define for $G = SO_{2n+1}, SP_{2n}$ and SO_{2n} the q-analogue $\mathfrak{K}^{\lambda,\diamondsuit}_{\mu^{(1)},\ldots,\mu^{(p)}}(q)$ of $\mathfrak{K}^{\lambda,\diamondsuit}_{\mu^{(1)},\ldots,\mu^{(p)}}$ by setting

$$\mathfrak{K}^{\lambda,\diamondsuit}_{\mu^{(1)},\ldots,\mu^{(p)}}(q) =^\infty K^{G,U}_{\widehat{\lambda},\widehat{\mu}}(q).$$

Theorem 10.7 [18]

1 *We have the decomposition*

$$\mathfrak{K}^{\lambda,\diamond}_{\mu^{(1)},\ldots,\mu^{(p)}}(q) = q^{\frac{|\mu|-|\lambda|}{|\diamond|}} \sum_{\nu\in\mathcal{P}_n} \sum_{\delta\in\mathcal{P}_n^{\diamond}} c^{\nu}_{\lambda,\delta}\, K^{GL_n,U}_{\nu,\mu}(q^{\frac{2}{|\diamond|}}). \tag{10.7}$$

2 *The polynomial $\mathfrak{K}^{\lambda,\diamond}_{\mu^{(1)},\ldots,\mu^{(p)}}(q)$ has nonnegative integer coefficients when the $\mu^{(k)}$'s are rectangular partitions of decreasing widths*

Remark (i) Assertion 2 of the previous theorem follows directly from Theorem 10.4 for $G = SP_{2n}$ and SO_{2n}. For $G = SO_{2n+1}$, we have to use Assertion 1 and Theorem 10.4 for $G = GL_n$.

(ii) Proposition 10.6 generalizes a similar duality result in type A_{n-1}. For $(\mu^{(1)}, \ldots, \mu^{(p)})$ a p -tuple of partitions, we have $K^{GL_n,U}_{\nu,\mu}(1) = c^{\nu}_{\mu^{(1)},\ldots,\mu^{(p)}}$ where $c^{\nu}_{\mu^{(1)},\ldots,\mu^{(p)}}$ is the multiplicity of $V^{GL_n}(\nu)$ in $V^{GL_n}(\mu^{(1)}) \otimes \cdots \otimes V^{GL_n}(\mu^{(p)})$. We set for completion

$$\mathfrak{K}^{\lambda,\emptyset}_{\mu^{(1)},\ldots,\mu^{(p)}}(q) = K^{GL_n,U}_{\nu,\mu}(q). \tag{10.8}$$

Recall the following theorem connecting one-dimensional sums in affine type $A_n^{(1)}$ with parabolic Lusztig q-analogues for GL_n.

Theorem 10.8 [30] *Let B be the tensor product of type $A_n^{(1)}$ KR crystals associated to the p-tuple of rectangular partitions $(R^{(1)}, \ldots, R^{(p)})$ of decreasing widths. Then for any partition λ in $\mathcal{P}_n$, we have*

$$\overline{X}^{\emptyset}_{\lambda,B}(q) = q^{||R||}\mathfrak{K}^{\lambda,\emptyset}_{R^{(1)},\ldots,R^{(p)}}(q^{-1}) = q^{||R||}\, K^{GL_n,U}_{\nu,\mu}(q^{-1})$$

where U is defined in (10.6) and

$$||R|| = \sum_{1\le i<j\le p} |R_i \cap R_j|. \tag{10.9}$$

Theorem 10.9 *Let B be a tensor product of p KR crystals. Assume the widths of the rectangles $R^{(1)}, \ldots, R^{(p)}$ associated to B are decreasing and the large rank hypothesis is satisfied. Then, for any $\lambda \in \mathcal{P}_n$*

$$\overline{X}^{\diamond}_{\lambda,B}(q) = q^{\frac{2(||R||+|R|-|\lambda|)}{|\diamond|}}\, \mathfrak{K}^{\lambda,\diamond}_{R^{(1)},\ldots,R^{(p)}}(q^{-1}) = q^{\frac{2(||R||+|R|-|\lambda|)}{|\diamond|}}\, {}^{\infty}K^{G,U}_{\widehat{\lambda},\widehat{\mu}}(q^{-1})$$

where U is defined in (10.6) and $||R||$ in (10.9).

Proof This follows from Theorems 10.8, 10.7 and 10.1. □

Theorem 10.10 *Let B be a tensor product of KR crystals. Assume the large rank hypothesis is satisfied. Then, for any $\lambda \in \mathcal{P}_n$*

$$\overline{X}^{\diamond^t}_{\lambda^t,B^t}(q) = q^{\frac{2(||R||+|R|-|\lambda|)}{|\diamond|}}\, \overline{X}^{\diamond}_{\lambda,B}(q^{-1}).$$

Proof For $\diamond = \emptyset$, the equality $\mathfrak{K}^{\lambda^t,\emptyset}_{(R^{(1)})^t,\ldots,(R^{(p)})^t}(q) = q^{\|B\|}\mathfrak{K}^{\lambda,\diamond}_{R^{(1)},\ldots,R^{(p)}}(q^{-1})$ was proved in [16]. By using Theorem 10.8, one obtains $\overline{X}^{\emptyset}_{\lambda^t,B^t}(q) = q^{\|B\|}\overline{X}^{\emptyset}_{\lambda,B}(q^{-1})$. Theorem 10.1 gives

$$\overline{X}^{\diamond^t}_{\lambda^t,B^t}(q) = q^{\frac{|B|-|\lambda|}{|\diamond|}} \sum_{\nu\in\mathcal{P}_n} \sum_{\delta\in\mathcal{P}_n^{\diamond}} c^{\nu^t}_{\lambda^t,\delta^t}\, \overline{X}^{\emptyset}_{\nu^t,B^t}(q^{\frac{2}{|\diamond|}}).$$

But $c^{\nu^t}_{\lambda^t,\delta^t} = c^{\nu}_{\lambda,\delta}$. Thus by using the previous identity for $\diamond = \emptyset$

$$\overline{X}^{\diamond^t}_{\lambda^t,B^t}(q) = q^{\frac{2\|B\|}{|\diamond|}+\frac{|B|-|\lambda|}{|\diamond|}} \sum_{\nu\in\mathcal{P}_n}\sum_{\delta\in\mathcal{P}_n^{\diamond}} c^{\nu}_{\lambda,\delta}\, \overline{X}^{\emptyset}_{\nu,B}(q^{-\frac{2}{|\diamond|}}) = q^{\frac{2(||R||+|R|-|\lambda|)}{|\diamond|}}\, \overline{X}^{\diamond}_{\lambda,B}(q^{-1}).$$

□

11 Splitting preserves energy

In this section we assume $\mathfrak{g}$ is of affine type with $\mathfrak{g}_0$ of type B_n, C_n, or D_n.

For $B \in \mathcal{C}(\mathfrak{g})$ we define the opposite grading $D : B \to \mathbb{Z}$ (the intrinsic energy) to $\overline{D}_B$. We show in Theorem 11.3 that it is invariant under the row-splitting map S. The normalization of D is somewhat subtle. For example, $\overline{D}$ is nonnegative with minimum value zero, while D may be negative. Also, if $B_1, B_2 \in \mathcal{C}$ are both tensor products of KR crystals, then the formula relating H_{B_1,B_2} and $\overline{H}_{B_1,B_2}$, requires knowledge of all the KR tensor factors in B_1 and B_2.

For this reason, instead of an inductive definition analogous to that of $\overline{D}_B$ we make the following definitions.

For $B_i = B^{R_i} = B^{r_i,s_i} \in \mathcal{C}^\infty(\mathfrak{g})$ for $i \in \{1,2\}$ we define

$$H_{B_1,B_2}(b_1 \otimes b_2) = |R_1 \cap R_2| - \overline{H}_{B_1,B_2}(b_1 \otimes b_2) \tag{11.1}$$

for $b_1 \in B_1$ and $b_2 \in B_2$, where $|R_1 \cap R_2| = \min(r_1,r_2)\min(s_1,s_2)$ is the number of cells in the rectangular partition given by the intersection of the Young diagrams of the rectangular partitions R_1 and R_2. We define

$$D_{B^{R_1}}(b) = -\overline{D}_{B^{R_1}}(b) \quad \text{for } b \in B^{R_1}. \tag{11.2}$$

Analogous to (3.53) we define

$$D_{B^R}(b) = \sum_{i=1}^{p} D_{B^{R_i}}(b_i^{(1)}) + \sum_{1\le i<j\le p} H_{B_i,B_j}(b_i \otimes b_j^{(i+1)}). \tag{11.3}$$

We make the same definitions (11.1), (11.2), and (11.3) for type $A^{(1)}_{n-1}$ also. Then (11.2) reads $D_{B_A^{R_1}} = -\overline{D}_{B_A^{R_1}} \equiv 0$. Using (3.53) we deduce that

$$D_{B^R}(b) = ||R|| - \overline{D}_{B^R}(b) \quad \text{for } b \in B^R \tag{11.4}$$

$$D_{B^R_A}(b) = ||R|| - \overline{D}_{B^R_A}(b) \quad \text{for } b \in B^R_A \tag{11.5}$$

where $||R||$ is defined in (10.9). $D_{B^R_A}$ has nonnegative values with minimum value 0 in the large rank case, while D_{B^R} has negative values in general.

Proposition 11.1 *For any sequence of rectangles R such that $B^{R_i} \in \mathcal{C}^\infty(\mathfrak{g})$,*

$$D_{B^R_A} = D_{B^R} \circ i^R_A. \tag{11.6}$$

Proof As in the proof of Proposition 9.10, we reduce to checking the case $R = (R_1)$ and

$$H_{B_A^{R_1},B_A^{R_2}} = H_{B^{R_1},B^{R_2}} \circ i_A^{R_1,R_2}. \tag{11.7}$$

For (11.6) for $R = (R_1)$ we see that both sides yield zero by definition. Equation (11.7) follows from Lemma 9.9 and the definitions. □

Let $S_A : B_A^R \to B_A^{S(R)}$ be type $A_{n-1}^{(1)}$ row-splitting of the first tensor factor and $\mathbb{S}_A : B_A^R \to B_A^{\mathrm{rows}(R)}$ the type A complete splitting into rows (split first factor if possible and use R-matrices).

Proposition 11.2 *With R such that $B^{R_i} \in \mathcal{C}^\infty(\mathfrak{g})$,*

$$\begin{aligned} i_A^{S(R)} \circ S_A &= S \circ i_A^R \\ i_A^{\mathrm{rows}(R)} \circ \mathbb{S}_A &= \mathbb{S} \circ i_A^R. \end{aligned} \tag{11.8}$$

Proof By Lemma 9.8 and the definitions, we may reduce the statement on $\mathbb{S}$ to that of S and check S only in the single tensor factor case $B = B^R = B^{r,s}$. In this case tops(max(B)) is the A_{n-1}-component of $b(r,s,(s^r)) \in B(s^r) \subset B^{r,s}$, tops(max$(B)$) consists of type B_n, C_n, or D_n KN tableaux of shape (s^r) with no barred letters, and (11.8) is easily verified. □

Theorem 11.3 *For $B = B^R \in \mathcal{C}^\infty(\mathfrak{g}^\diamond)$,*

$$D_{B^R} = D_{B^{S(R)}} \circ S \tag{11.9}$$

$$D_{B^R} = D_{B^{\mathrm{rows}(R)}} \circ \mathbb{S}. \tag{11.10}$$

Proof We need only prove (11.9). Since energy functions are constant on I_0-components, it suffices to check (11.9) on $b \in \mathrm{tops}(B^R)$. We have

$$D(b) = D(\sigma(b)) - \frac{|R| - |\lambda(b)|}{|\diamond|}$$

by Theorem 8.1 and (11.4). Since S is an embedding of I_0-crystals, $S(b) \in \mathrm{tops}(B^{S(R)})$. Applying the previous argument to $S(b)$ we have

$$D(S(b)) = D(\sigma(S(b))) - \frac{|S(R)| - |\lambda(S(b))|}{|\diamond|}.$$

But $|S(R)| = |R|$ and $|\lambda(S(b))| = |\lambda(b)|$ since S is an embedding of I_0-crystals. So it suffices to prove that $D(\sigma(b)) = D(\sigma(S(b)))$. By Proposition 6.3, this is equivalent to $D(\sigma(b)) = D(S(\sigma(b)))$. So by Theorem 7.1 we are reduced to prove the equality $D(c) = D(S(c))$ for any $c \in \widehat{\max}(B^R)$. Since D is constant on I_0-components we need only show $D(c) = D(S(c))$ for $c \in \mathrm{hw}_{I_0}(\max(B^R)) = \mathrm{hw}_{A_{n-1}}(\mathrm{tops}(\max(B^R)))$. By Proposition 11.1 applied for R and $S(R)$, the desired equality reduces to the identity $D_A(a) = D_A(S_A(a))$ for any $a \in B_A^R$ which was established in [29]. □

Remark 11.4 In the statement of Theorem 11.3, it should be unnecessary to assume that $\mathfrak{g}$ is reversible and $B^R \in \mathcal{C}^\infty(\mathfrak{g})$. However for S to make sense there cannot be spin nodes in the R_i.

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Appendix A: Proofs for Section 4

A.1. Proof of Proposition 4.5

Proof of Proposition 4.5 Let $b \in L^\diamond(\nu,\delta)$ for $\delta \in \mathcal{P}_n^\diamond$. Observe that the letters of the canonical subtableau $C_\delta^\diamond$ collectively do not affect any A_{n-1}-string. Now $b|^{\nu\backslash\delta}$ is a semistandard tableau in the alphabet $\{\bar{n}, \ldots, \bar{1}\}$. It is well-known that the set of skew tableaux of a fixed shape, form an A_{n-1}-crystal. This proves 1.

For Assertion 2, based on the above observations, b is A_{n-1}-highest weight, if and only if $b|^{\nu/\delta}$ is A_{n-1}-highest weight as an element of the type A_{n-1} skew tableau crystal. But it is well-known that such a skew tableau is A_{n-1}-highest weight if and only if its row-reading word is Yamanouchi. Finally, since the tableau has letters in $\{\bar{n}, \ldots, \bar{1}\}$, if it is A_{n-1}-highest weight, then its weight must have the form $\overline{\lambda}$ for some $\lambda \in \mathcal{P}_n$.

For Assertion 3, suppose b admits f_n.

(1) $\diamondsuit = (1, 1)$: The application of f_n to b, changes an n (which by the signature rule, must be in a corner cell of δ) to a $\overline{n-1}$. Since every n sits atop a $\bar{n}$, Assertion 3 follows.
(2) $\diamondsuit = (1)$: The application of f_n to b changes some 0 to $\bar{n}$ or some n to 0 (say in row i). The tableau $f_n(b)$ contains $C^{(1)}_{\delta^-}$ where $\delta^- \in \mathcal{P}^{(1)}_n$ is obtained from δ by removing a cell in row i. The only way that $f_n(b)$ is not in $L^{(1)}(\nu, \delta^-)$ is if $f_n(b)|^{\nu/\delta^-}$ contains two letters $\bar{n}$ in the same column, either because the changed letter became $\bar{n}$ and now lies beneath another $\bar{n}$, or because in b there was a pair of letters $\bar{n}$ atop each other but one was in δ and the other not in δ, but now in $f_n(b)$ both are outside δ^-. However the assumption that $\delta_{i+1} \le \delta_i$ and the signature rule, imply that this cannot occur.
(3) $\diamondsuit = (2)$: The application of f_n to b changes some n to $\bar{n}$ (say in the ith row). The tableau $f_n(b)$ contains $C^{(2)}_{\delta^-}$ where $\delta^- \in \mathcal{P}^{(2)}_n$ is obtained from δ by removing two cells in row i. Similarly to the case $\diamondsuit = (1)$, one may deduce Assertion 3.

We prove Assertions 4 and 5 by induction on $|\delta| = |\nu| - |\lambda|$. Equivalently we find a sequence $\mathbf{a}$ of indices in I_0 such that $f_{\mathbf{a}}(b) = \mathrm{rowtab}(b^\nu)$. By Assertion 1 we may assume b is a A_{n-1}-lowest weight vertex.

If $|\delta| = 0$ then $b = \mathrm{rowtab}(b^\nu)$ and the empty sequence works. Suppose $\delta \neq \emptyset$. Since b is A_{n-1}-lowest weight and $\nu \in \mathcal{P}^\infty_n$ the skew tableau $b|^{\nu\backslash\delta}$ admits no A_{n-1}-lowering operator and contains letters in $\{\overline{n-2}, \ldots, \bar{2}, \bar{1}\}$. So the letters outside $b|^\delta = C^\diamondsuit_\delta$ are irrelevant for the n-signature. In the various cases we see that $f_n(b) \in L^\diamondsuit(\nu, \delta^-)$ where $\delta^- \in \mathcal{P}^\diamondsuit_n$ is obtained from δ in the same way that λ^- is obtained from λ in Lemma 4.9. Induction completes the proof. □

Example A.1 $\lambda = (4, 3, 3, 1, 1) \in \mathcal{P}(5, 4)$ since $\delta = (3, 3, 1, 1) \in \mathcal{P}^{(1,1)}$.

$$\mathrm{rowtab}(\overline{b}(5,4,\lambda)) = \begin{array}{|c|c|c|c|}\hline \bar{n} & \overline{n-2} & \overline{n-2} & \overline{n-4} \\ \hline \mathbf{n} & \overline{n-1} & \overline{n-1} & \overline{n-3} \\ \hline \bar{\mathbf{n}} & \bar{n} & \bar{n} & \overline{n-2} \\ \hline \mathbf{n} & \mathbf{n} & \mathbf{n} & \overline{n-1} \\ \hline \bar{\mathbf{n}} & \bar{\mathbf{n}} & \bar{\mathbf{n}} & \bar{n} \\ \hline \end{array} \quad \text{and } \mathrm{rowtab}(\overline{b}^{(1,1)}_{\min}(5,4)) = \begin{array}{|c|c|c|c|}\hline \bar{n} & \bar{n} & \bar{n} & \bar{n} \\ \hline n & n & n & n \\ \hline \bar{n} & \bar{n} & \bar{n} & \bar{n} \\ \hline n & n & n & n \\ \hline \bar{n} & \bar{n} & \bar{n} & \bar{n} \\ \hline \end{array}.$$

Proof of Lemma 4.9 Let $\mathrm{rowtab} = \mathrm{rowtab}_{(s^r)}$, $b' = \mathrm{rowtab}(\overline{b}(r, s, \lambda^-))$ and $b = \mathrm{rowtab}(\overline{b}(r, s, \lambda))$. It suffices to show

$$\varepsilon_n(f_{\tilde{\mathbf{a}}'(h)}(b')) \ge \ell \tag{A.1}$$

$$\varphi_n(f_{\tilde{\mathbf{a}}'(h)}(b')) > 0 \tag{A.2}$$

$$u_{\ell\Lambda_n} \otimes b = f_{\tilde{\mathbf{a}}(\mathbf{h})}(u_{\ell\Lambda_n} \otimes b'). \tag{A.3}$$

Let $\delta^- \in \mathcal{P}^{(1,1)}$ be the complementary partition to λ^- within (s^r). We will need to keep track of certain letters that may contribute to the n-signature.

Suppose $\diamondsuit = (1, 1)$. By Proposition 4.5(2), the restriction of b' to the skew shape $(s^r)\backslash\delta^-$, has the letter $\bar{n}$ at the bottom of each column and a letter $\overline{n-1}$ atop the letter $\bar{n}$ if it fits into

(s^r). We may think that every n not in the top row is paired (in the $(n-1)$-signature) with the $\bar{n}$ sitting atop it. The n's in the top row (which may occur if r is even) are unpaired and occur at the end of the rowwise n-signature reading. There are s unpaired letters $\bar{n}$ in the bottom row, and an unpaired $\overline{n-1}$ in each column of δ that is not of maximum height $2\lfloor r/2 \rfloor$. We now apply $f_{\tilde{\mathbf{a}}'(h)}$ to b'; call the result b''. It only changes letters at the top of the pth column from the right, from (reading down) $\overline{n-h+3}, \dots, \overline{n-1}, \overline{n}$, to $\overline{n-h+1}, \dots, \overline{n-2}$. The bottom row still consists of s copies of $\overline{n}$ which occur at the beginning of the n-signature, so (A.1) holds. The dominant elements in the n-signature of b'' are the unpaired letters n in the top row if r is even, and the copy of n in the active column, since the relevant letters changed from $\overline{n-1}, \overline{n}, n$ to $\overline{n-3}, \overline{n-2}, n$. Therefore (A.2) holds. Applying f_n to b'' changes the n in the active column to the letter $\overline{n-1}$, with final result $\mathrm{rowtab}(\overline{b}(r,s,\lambda))$.

Now assume $\diamond = (2)$. Similarly, the restriction of b' to the skew shape $(s^r) \setminus \delta^-$, has the letter $\bar{n}$ at the bottom of each column and a letter $\bar{n}$ or $\overline{n-1}$ atop the letter $\bar{n}$ if it fits into (s^r). Moreover, each letter n is paired with a letter $\bar{n}$ in the $(n-1)$-signature of b'. Therefore, $b'' = f_{\tilde{\mathbf{a}}'(h)}(b')$ is obtained by changing the letters at the top of the pth and $(p+1)$th columns from (reading down) $\overline{n-h+2}, \dots, \overline{n-1}, \overline{n}$, to $\overline{n-h+1}, \dots, \overline{n-2}, \overline{n-1}$. In b' and b'' the bottom contains at least $\lceil \frac{s}{2} \rceil$ letters $\bar{n}$ which are unpaired in the n-signature. Thus (A.1) holds. In the columns p and $p+1$, the letters $\overline{n}$ are changed in $\overline{n-1}$. Therefore (A.2) holds. Applying f_n to b'' changes the n in the active column p to the letter $\overline{n}$, with final result $\mathrm{rowtab}(\overline{b}(r,s,\lambda))$ as desired.

The case $\diamond = (1)$ is similar. □

Appendix B: Proofs for Section 5

In this appendix we assume $\diamond = (1,1)$ and $\mathfrak{g}^\diamond = D_n^{(1)}$.

B.1. Reduction to relation on automorphisms of $B^{r,s}$

Our first reduction for proving Proposition 5.9 in the case $\diamond = (1,1)$ is to rephrase it in terms of a relation among various automorphisms on $B^{r,s}$. Recall the automorphism ς on $B^{r,s}$ from Sect. 5.1.

Let $\varsigma' \in \mathrm{Aut}(D_n^{(1)})$ be defined by the permutation of I^s given by $(n-1,n)$. ς' is also not a special automorphism. It coincides with $* \in \mathrm{Aut}(D_n^{(1)})$ if n is odd. There is a unique bijection $\varsigma' : B^{r,s} \to B^{\varsigma'(r),s}$

$$\varsigma' e_i = e_{\varsigma'(i)} \varsigma' \quad \text{for all } i \in I. \tag{B.1}$$

It is explicitly given by exchanging n's with $\bar{n}$'s in KN tableaux. For $r \in I_0$ nonspin, ς' is an involution on $B^{r,s}$.

Lemma B.1 *If*

$$\varsigma' \sigma = \sigma \varsigma \tag{B.2}$$

holds on $B^{r,s}$, then Proposition 5.9 holds.

Proof We have

$$\varsigma' \sigma e_0 = \sigma \varsigma e_0 = \sigma e_1 \varsigma = e_{n-1} \sigma \varsigma = e_{n-1} \varsigma' \sigma = \varsigma' e_n \sigma.$$

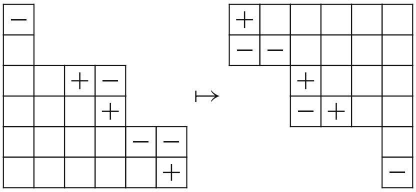
Applying the involution ς', we have $\sigma e_0 = e_n \sigma$ on $B^{r,s}$. By Proposition 5.8 it follows that σ satisfies (5.3) as required. □

B.2. Rule for rowtab($\sigma(\Phi(P))$) for a $\pm$-diagram P

We give a rule for rowtab($\sigma(\Phi(P))$) for any $\pm$-diagram P.

Rule
1. Rotate P 180 degrees and place it in the $r \times s$ rectangle so that the NE corners of the rotated P and the rectangle coincide.
2. Fill each column of the inner shape of P by sequences of the form $\overline{k}, \ldots, \overline{n-2}, \overline{n-1}$ reading from the top, place $\overline{n}$ in each node where $+$ is situated, and fill all columns from top to bottom in the rest of the rectangle by sequences of the form $n, \overline{n}, n, \overline{n}, \ldots$, always starting with n.
3. In each row perform the following substitution. Suppose there are $k_+ n$'s and $k_- \overline{n}$'s in the row. Then replace them with $(n-1)^{k_-} n^{k_+ - k_-} (\overline{n-1})^{k_-}$ if $k_+ \geq k_-$, and
$(n-1)^{k_+} \overline{n}^{k_- - k_+} (\overline{n-1})^{k_+}$ otherwise.

Example B.2 Let $n = 9$, $r = 6$, $s = 7$.



$\mapsto$

9	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$
$\overline{9}$	9	9	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$
9	$\overline{9}$	$\overline{9}$	$\overline{9}$	$\overline{8}$	$\overline{7}$	$\overline{6}$
$\overline{9}$	9	9	9	$\overline{9}$	$\overline{8}$	$\overline{7}$
9	$\overline{9}$	$\overline{9}$	$\overline{9}$	9	9	$\overline{8}$
$\overline{9}$	9	9	9	$\overline{9}$	$\overline{9}$	9

$\mapsto$

8	$\overline{8}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$	$\overline{4}$
8	9	$\overline{8}$	$\overline{8}$	$\overline{7}$	$\overline{6}$	$\overline{5}$
8	$\overline{9}$	$\overline{9}$	$\overline{8}$	$\overline{8}$	$\overline{7}$	$\overline{6}$
8	8	9	$\overline{8}$	$\overline{8}$	$\overline{8}$	$\overline{7}$
8	8	8	$\overline{8}$	$\overline{8}$	$\overline{8}$	$\overline{8}$
8	8	8	9	$\overline{8}$	$\overline{8}$	$\overline{8}$

Proposition B.3 *For any $\pm$-diagram P for $B^{r,s}$, the above rule gives* rowtab($\sigma(\Phi(P))$).

The proof of this key technical result is given in the following subsection. We use it to finish up the proofs of Sect. 5.

Proposition B.4 *For any $b \in \mathrm{hw}_J(B^{r,s})$, we have $\sigma\varsigma(b) = \varsigma'\sigma(b)$.*

Proof Compare P and $\mathfrak{S}(P)$ where $\mathfrak{S}$ is the involution on $\pm$-diagrams corresponding to the automorphism ς on $B^{r,s}$. The inner shapes are the same and in each column, if there is $+$ in P, then there is no $+$ in $\mathfrak{S}(P)$, and vice versa. Therefore, at the moment when Rule 2 is finished, the number of n's and $\overline{n}$'s in each row are switched for P and $\mathfrak{S}(P)$. Hence, we have $\sigma\Phi(\mathfrak{S}(P)) = \varsigma'\sigma\Phi(P)$. This is what we needed to show. □

Proof of Proposition 5.9 Let $b \in B^{r,s}$. Let $b^\circ = \mathrm{hw}_J(b)$ and let $\mathbf{b} = (i_1, i_2, \ldots)$ be a finite sequence in J such that $b = f_{\mathbf{b}}(b^\circ)$. Then

$$\sigma\varsigma(b) = f_{\sigma\varsigma(\mathbf{b})}\sigma\varsigma(b^\circ),$$
$$\varsigma'\sigma(b) = f_{\varsigma'\sigma(\mathbf{b})}\varsigma'\sigma(b^\circ),$$

where the Dynkin automorphisms ς, ς', and σ act on sequences of Dynkin nodes in the obvious way. Since $\sigma\varsigma(\mathbf{b}) = \varsigma'\sigma(\mathbf{b})$ and $\sigma\varsigma(b^\circ) = \varsigma'\sigma(b^\circ)$ by Lemma 5.7, we obtain $\sigma\varsigma(b) = \varsigma'\sigma(b)$. □

B.3. Proof of Proposition B.3

We need some notation. Let $r' = \lfloor r/2 \rfloor$. For $\lambda \in \mathcal{P}(r,s)$ and $0 \le j \le r'$, define c_j by

$$\lambda = \sum_{j=0}^{r'} (c_j - c_{j+1})\omega_{r-2j}$$

with $c_0 = s$ and $c_{r'+1} = 0$. Then a sequence $(c_1, c_2, \dots, c_{r'})$ such that $s \ge c_1 \ge c_2 \ge \cdots \ge c_{r'} \ge 0$ is in one-to-one correspondence with the I_0-highest element $b(r,s,\lambda) \in B^{r,s}$.

It remains to prove Proposition B.3. First we assume the $\pm$-diagram P has no column for which a $+$ can be added. Let λ be the outer shape of P, c_i^- the number of columns that has a $-$ at height i in P. Set $a_i = \sum_{j=1}^{i} c_j^-$. By [25, Prop. 2.2] we have

$$\Phi(P) = f_{(1^{a_r},\dots,(n-1)^{a_r},n^{a_r},(n-2)^{a_r},\dots,(r+1)^{a_r},r^{a_r},\dots,2^{a_2},1^{a_1})} b(r,s,\lambda).$$

(The notation a_i in [25, Prop. 2.2], is equal to $\sum_{j=1}^{r} c_j^-$.) Hence, by Lemma 5.7 and the definition of σ we obtain

$$\sigma(\Phi(P)) = f_{((n-1)^{a_r},\dots,1^{a_r},0^{a_r},2^{a_r},\dots,(n-2)^{a_2},(n-1)^{a_1})} \overline{b}(r,s,\lambda).$$

Lemma B.5 *The row tableau*

$$t_1 = f_{(2^{a_r},\dots,(n-r-1)^{a_r},(n-r)^{a_r},\dots,(n-2)^{a_2},(n-1)^{a_1})} \overline{b}(r,s,\lambda)$$

differs from $\overline{b}(r,s,\lambda)$ *only in the top row, which is given by*

$$n^{s-\lambda_1}\,\overline{n-1}^{\lambda_1-\lambda_3-c_2^-}\,\overline{n-3}^{\lambda_3-\lambda_5-c_4^-}\cdots\overline{n-r+1}^{\lambda_{r-1}-c_r^-}\,\overline{2}^{a_r}$$

for r *is even and*

$$\overline{n}^{s-\lambda_2-c_1^-}\,\overline{n-2}^{\lambda_2-\lambda_4-c_3^-}\,\overline{n-4}^{\lambda_4-\lambda_6-c_5^-}\cdots\overline{n-r+1}^{\lambda_{r-1}-c_r^-}\,\overline{2}^{a_r}$$

for r *is odd.*

Proof We consider the r even case. Consider the $(n-1)$-signature. $+$'s in the $(2j)$th row and $-$'s in the $(2j+1)$th row from bottom cancel out for any $j = 1,\dots,r-1$. Hence $f_{((n-1)^{a_1})}$ acts only on the top row. We proceed similarly. □

Let t be the row tableau constructed by the Rule 3. The following lemma allows us to calculate the action of Kashiwara operators on t before applying Rule 3.

Lemma B.6 *Let* t_- *be another row tableau obtained by putting* $n^{k_+}\overline{n}^{k_-}$ *instead of applying Rule 3 in each row. One can formally apply* e_n *and* f_n *on* t_-. *Then the action of* e_n *(resp.* f_n*) commutes with applying Rule* 3. *A similar fact hold also for* e_{n-1} *and* f_{n-1} *by replacing* $n^{k_+}\overline{n}^{k_-}$ *with* $\overline{n}^{k_-}n^{k_+}$.

Proof It suffices to prove the statement for a one-row tableau. This is done easily. □

Lemma B.7 *The row tableau* t_2 *for*

$$e_{(1^{a_r},\dots,(n-2)^{a_r},(n-1)^{a_r})} t$$

differs from t *only in the bottom row, which is given by* $1^{a_r}\overline{n}^{s-a_r}$.

Proof In view of the previous lemma we can replace t with t_-. Note that the lowest row of t is given by $n^{a_r}\overline{n}^{s-a_r}$. Since $e_{((n-1)^{a_r})}$ acts only on the lowest row, we get $(n-1)^{a_r}\overline{n}^{s-a_r}$. The application of $e_{(1^{a_r},\dots,(n-2)^{a_r})}$ is easier. □

Next we want to show $f_{(0^{a_r})}t_1 = t_2$. To do this we calculate the $\{3,4,\dots\}$-highest element of t_1 and t_2. Let $N(\alpha)$ (resp. $N'(\alpha)$) be the number of letter α in t_1 (resp. $(t_2)_-$) (see Lemma B.6 for the definition of t_-). Define a sequence $\boldsymbol{a}$ by $\boldsymbol{a} = \boldsymbol{a}_r \sqcup \cdots \sqcup \boldsymbol{a}_2 \sqcup \boldsymbol{a}_1$ where

$$\boldsymbol{a}_j = ((j+2)^{s+N(\overline{3})}, \dots, (n-2)^{s+N(\overline{n-j-1})}, (n-\delta_j^{(2)})^{s+N(\overline{n-j})})$$

for $j = 1,2,\dots,r-1$, and $\boldsymbol{a}_r = ((r+2)^{s-N(\overline{2})}, \dots, (n-2)^{s-N(\overline{2})}, (n-\delta_r^{(2)})^{s-N(\overline{2})})$. Here $\delta_j^{(2)} = 1$ if j is even, $= 0$ otherwise and $\sqcup$ means the concatenation of sequences. We also define $\boldsymbol{a}' = \boldsymbol{a}'_1\boldsymbol{a}'_2\cdots\boldsymbol{a}'_r$ by replacing $N(\overline{k})$ with $N'(\overline{k})$ for $k = 3,4,\dots,n-1$ and $N(\overline{2})$ with $N'(1)$ in $\boldsymbol{a}$. Then we have the following lemma.

Lemma B.8 *1. $e_{\boldsymbol{a}}t_1$ is a $\{3,4,\dots\}$-highest element whose jth row from bottom is given by $(j+2)^s$ for $j = 1,\dots,r-1$ and $(r+2)^{s-a_r}\overline{2}^{a_r}$ for $j = r$.*
2. $e_{\boldsymbol{a}'}t_2$ is a $\{3,4,\dots\}$ -highest element whose jth row from bottom is given by $1^{a_r}3^{s-a_r}$ for $j = 1$ and $(j+1)^{a_r}(j+2)^{s-a_r}$ for $j = 2,\dots,r$.
3. $N(\overline{k}) = N'(\overline{k})$ for $k = 3,4,\dots,n-1$ and $N(\overline{2}) = N'(1)$.

Proof For (1) and (2) simply calculate the action of $e_{\boldsymbol{a}}$ and $e_{\boldsymbol{a}'}$ using Lemma B.6. For (3) note that for $1 \le j \le n-3$ $N(\overline{n-j}) = \lambda_j - c_{j+1}^-$ if j is odd, $= \lambda_{j+1}$ otherwise when r is even, and $N(\overline{n-j}) = \lambda_{j+1}$ if j is odd, $= \lambda_j - c_{j+1}^-$ otherwise when r is odd. We also have $N(\overline{2}) = N'(1) = a_r$. For the definition of the partition λ, c_j^- or a_r see the paragraph before Lemma B.5. □

Now we can prove Proposition B.3 under the assumption that P has no column for which a $+$ can be added. Using Lemma 9.4 with $\alpha = 0, \beta = a_r, \gamma = s - a_r$ and with applying $e_1^{a_r}$, the results in Lemma B.8 show that $f_{(0^{a_r})}e_{\boldsymbol{a}}t_1 = e_{\boldsymbol{a}}t_2$. Since f_0 commutes with e_j for $3 \le j \le n$, we obtain $f_{(0^{a_r})}t_1 = t_2$, but this equality is what we wanted to show.

Finally, we prove Proposition B.3 for general $\pm$ -diagram P. We show by induction on the number of columns for which a $+$ can be added. If there is no such column, the statement is proven already. Now let P be a $\pm$-diagram with at least one column for which a $+$ can be added. Let c be the rightmost such column. Let P' be the $\pm$-diagram obtained from P by adding a $+$ in column c. Let h be the height of this added $+$. Then it is known [28] that $\Phi(P) = f_{(1,\dots,h-1,h)}\Phi(P')$. Hence, we have

$$\sigma(\Phi(P)) = f_{(n-1,\dots,n-h)}\sigma(\Phi(P')).$$

Since we know the row tableau of $\sigma(\Phi(P'))$ is given by the Rule by induction hypothesis, it suffices to calculate the right hand side and see it agrees with the row tableau of $\sigma(\Phi(P))$ given by the Rule. Careful calculation using Lemma B.6 shows that the application of $f_{(n-1,\dots,n-h)}$ changes the row tableau of $\sigma(\Phi(P'))$ only in the rightmost column with letters $\overline{n-h+1}, \overline{n-h+2}, \dots, \overline{n}, \dots$ reading from top to $\overline{n-h}, \overline{n-h+1}, \dots, \overline{n-1}, \dots$. This completes the proof of Proposition B.3.

References

1. Akasaka, T., Kashiwara, M.: Finite-dimensional representations of quantum affine algebras. Publ. Res. Inst. Math. Sci. **33**, 839–867 (1997)

2. Broer, A.: Normality of some nilpotent varieties and cohomology of lines bundles on the cotangent bundles of the flag variety. In: Brylinski, J.L., Guillemin, V., Kac, V. (eds.) Lie Theory and Geometry in Honor of Bertrand Kostant. Progess in Mathematics, vol. 123, pp. 1–18 (1994)
3. Fourier, G., Okado, M., Schilling, A.: Kirillov-Reshetikhin crystals for nonexceptional types. Adv. Math. **222**, 1080–1116 (2009)
4. Fourier, G., Okado, M., Schilling, A.: Perfectness of Kirillov-Reshetikhin crystals for nonexceptional types. Contemp. Math. **506**, 127–143 (2010)
5. Fourier, G., Schilling, A., Shimozono, M.: Demazure structure inside Kirillov-Reshetikhin crystals. J. Algebra **309**, 386–404 (2007)
6. Fulton, W.: Young Tableaux. With applications to representation theory and geometry. In: London Mathematical Society Student Texts, vol. 35. Cambridge University Press, Cambridge (1997)
7. Goodman, R., Wallach, N.R.: Representations and Invariants of the Classical Groups. Cambridge University Press, Cambridge (1998)
8. Hague, C.: Cohomology of flag varieties and the Brylinski-Kostant filtration. J. Algebra **321**, 3790–3815 (2009)
9. Hatayama, G., Kuniba, A., Okado, M., Takagi, T., Tsuboi, Z.: Paths, crystals and fermionic formulae, Math. Phys. Odyssey 2001. In: Prog. Math. Phys. vol. 23, pp. 205–272. Birkhäuser, Boston (2002)
10. Howe, R., Tan, E.-C., Willenbring, J.-F.: Stable branching rules for classical symmetric pairs. Trans. Am. Math. Soc. **357**, 1601–1626 (2005)
11. Kac, V.: Infinite dimensional Lie algebras, 3rd edn. Cambridge University Press, Cambridge (1990)
12. King, R.C.: Modification rules and products of irreducible representations for the unitary, orthogonal and symplectic groups. J. Math. Phys. **12**, 1588–1598 (1971)
13. Kang, S.-J., Kashiwara, M., Misra, K.C., Miwa, T., Nakashima, T., Nakayashiki, A.: Affine crystals and vertex models. Int. J. Mod. Phys. A **7**(suppl 1A), 449–484 (1992)
14. Kang, S.-J., Kashiwara, M., Misra, K.C., Miwa, T., Nakashima, T., Nakayashiki, A.: Perfect crystals of quantum affine Lie algebras. Duke Math. J. **68**, 499–607 (1992)
15. Kashiwara, M., Nakashima, T.: Crystal graphs for representations of the q-analogue of classical Lie algebras. J. Algebra **165**(2), 295–345 (1994)
16. Kirillov, A.N., Shimozono, M.: A generalization of the Kostka-Foulkes polynomials. J. Algebraic Combin. **15**, 27–69 (2002)
17. Koike, K., Terada, I.: Young diagrammatic methods for the restriction of representations of complex classical Lie groups to reductive subgroups of maximal rank. Adv. Math. **79**, 104–135 (1990)
18. Lecouvey, C.: Quantization of branching coefficients for classical Lie groups. J. Algebra **308**, 383–413 (2007)
19. Lecouvey, C.: Schensted-type correspondences and plactic monoids for types B_n and D_n. J. Algebraic Combin. **18**, 99–133 (2003)
20. Lecouvey, C., Shimozono, M.: Lusztig's q-analogue of weight multiplicity and one-dimensional sums for affine root systems. Adv. Math. **208**, 438–466 (2007)
21. Littlewood, D.-E.: The Theory of Group Characters and Matrix Representations of Groups, 2nd edn. Oxford University Press, Oxford (1958)
22. Macdonald, I.G.: Symmetric functions and Hall polynomials, 2nd edn. With contributions by A. Zelevinsky. Oxford Mathematical Monographs. Oxford Science Publications/The Clarendon Press/Oxford University Press, New York (1995)
23. Naito, S., Sagaki, D.: Construction of perfect crystals conjecturally corresponding to Kirillov-Reshetikhin modules over twisted quantum affine algebras. Commun. Math. Phys. **263**(3), 749–787 (2006)
24. Nakayashiki, A., Yamada, Y.: Kostka-Foulkes polynomials and energy function in sovable lattice models. Selecta Math. (N. S.) **3**, 547–599 (1997)
25. Okado, M., Sakamoto, R.: Combinatorial R-matrices for Kirillov-Reshetikhin crystals of type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$. Int. Math. Res. Not. **2010**, 559–593 (2010)
26. Okado, M., Schilling, A.: Existence of Kirillov-Reshetikhin crystals for nonexceptional types. Represent. Theory **12**, 186–207 (2008)
27. Okado, M., Schilling, A., Shimozono, M.: Virtual crystals and fermionic formulas of type $D_{n+1}^{(2)}, A_{2n}^{(2)}$, and $C_n^{(1)}$. Represent. Theory **7**, 101–163 (2003)
28. Schilling, A.: Combinatorial structure of Kirillov-Reshetikhin crystals of type $D_n^{(1)}, B_n^{(1)}, A_{2n-1}^{(2)}$. J. Algebra **319**, 2938–2962 (2008)
29. Shimozono, M.: Affine type A crystal structure on tensor products of rectangles, Demazure characters, and nilpotent varieties. J. Algebraic Combin. **15**(2), 151–187 (2002)
30. Shimozono, M.: On the $X = K$ conjecture. arXiv:math.CO/0501353

31. Schilling, A., Warnaar, S.: Inhomogeneous lattice paths, generalized Kostka polynomials and A_{n-1} supernomials. Commun. Math. Phys. **202**(2), 359–401 (1999)
32. Shimozono, M., Zabrocki, M.: Deformed universal characters for classical and affine algebras. J. Algebra **299**, 33–61 (2006)