

Super currents and tropical geometry

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Received: 10 September 2010 / Accepted: 16 December 2010 / Published online: 18 January 2011
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Abstract We introduce the formalism of positive super currents on \mathbb{R}^n , in strong analogy with the theory of positive currents in \mathbb{C}^n . We consider intersection of currents and Lelong numbers, and as an application we show that the formalism can be used to describe tropical varieties. This is similar in spirit to the fact that in complex analysis the current of integration of an analytic variety can be identified with a closed, positive current.

Mathematics Subject Classification (2000) 32U · 14T · 52

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1 Introduction

In complex analysis, the natural counterpart of convexity in the real setting, is that of plurisubharmonicity, and there are many similarities between convex- and plurisubharmonic functions. For instance, a smooth function defined on \mathbb{R}^n is convex if and only if its Hessian is positive definite, and a smooth function defined on \mathbb{C}^n is plurisubharmonic iff its (complex) Hessian is positive definite. On the complex side, a natural way of studying plurisubharmonicity is provided by the framework of so called positive currents. In fact, a closed $(1, 1)$ -current is positive iff it locally can be represented as $i\partial\bar{\partial}\varphi$ for a plurisubharmonic

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function φ . However, one could argue that from the point of view of geometry, the study of positive currents rather than that of plurisubharmonic functions, is in a sense more natural. For instance, a variety of higher co-dimension than 1, relates to closed, positive currents of higher bi-degree. The aim of this paper is to introduce a notion of positive currents corresponding to convex functions defined on \mathbb{R}^n . This is carried out by letting ourselves be inspired by the complex setting (indeed, many of the results and ideas will probably be familiar to the mathematician knowledgeable in pluripotential theory). The ideas can be seen as a continuation of those developed in [3], and they are also influenced by the paper [14]. Another paper which treat convex analysis with tools from complex analysis is [11], with which the present article share some similarities. We will consider the framework of positive super currents in the context of tropical geometry, proving, amongst other things, that every tropical hypersurface corresponds to a positive super current satisfying certain criterions. This is similar in spirit to the fact that in complex analysis, a complex hypersurface can be represented by a positive current satisfying certain hypotheses. The epithet “super” of “super current” comes from the field of super geometry and super spaces, where the Berezin integral (cf. [2]) share many properties with the integration of super forms considered in this article. However, in order to not to burden the language more than necessary, we will refer to super currents merely as currents, throughout the article. Our hope is for this work to provide a useful tool for attacking problems within tropical geometry, and for it to serve as a gateway between complex analysis and tropical geometry.

2 Positive super forms and currents in \mathbb{R}^n

Let \mathbb{V} and \mathbb{W} denote real vector spaces of dimension n , with coordinates $x = (x_1, \dots, x_n)$ and $\xi = (\xi_1, \dots, \xi_n)$ respectively, for which we fix an isomorphism $J : \mathbb{V} \rightarrow \mathbb{W}$, such that $J(x) = \xi$. We denote its inverse by J as well, so that $J(\xi) = x$, if $x \in \mathbb{V}$ is the element for which $J(x) = \xi$. One should think of \mathbb{V} and \mathbb{W} as two copies of \mathbb{R}^n identified via the isomorphism J . Let $\mathbb{E} = \mathbb{V} \times \mathbb{W} = \{(x, \xi), x \in \mathbb{V}, \xi \in \mathbb{W}\}$. The map J extends to \mathbb{E} by letting $J(x, \xi) = (J(\xi), J(x))$, so that $J^2 = id$. We consider the space \mathcal{E} of smooth differential forms on \mathbb{E} whose coefficients only depend on x . Thus, for $x \mapsto \alpha_{KL}(x)$ smooth functions,

$$\alpha = \sum_{K,L} \alpha_{K,L}(x) dx_K \wedge d\xi_L \tag{2.1}$$

is such a form, where we use the notation $dx_K = dx_{k_1} \wedge \dots \wedge dx_{k_p}$ if $K = (k_1, \dots, k_p)$, $|K|$ denotes the length of the vector K , and similarly for $|L| = q$. We use the convention that we only sum over indices K and L such that if $K = (k_1, \dots, k_p)$ then $k_1 < \dots < k_p$, and similarly for the index L . If α is of the form (2.1), we say that α is a form of bi-degree (p, q) , and write $\alpha \in \mathcal{E}^{p,q}$ where $0 \leq p, q \leq n$. A form α of degree (p, p) is called symmetric if $\alpha_{KL} = \alpha_{LK}$ for all indices K, L . We identify the isomorphism J with J^* , which is to say $J(dx_i) = d\xi_i$, and we extend J to an arbitrary (p, q) -form by the rule

$$J \left(\sum_{K,L} \alpha_{K,L}(x) dx_K \wedge d\xi_L \right) = \sum_{K,L} \alpha_{K,L}(x) d\xi_K \wedge dx_L.$$

Thus a (p, p) -form α is symmetric iff $J(\alpha) = (-1)^p \alpha$. Note that we use the letter J to denote several, slightly different maps, but we hope that no confusion will arise. Finally we put $\omega = \sum_{i=1}^n dx_i \wedge d\xi_i$ and $\omega_n = \frac{1}{n!} \omega^n$. In this article, we will consider three different notions of positivity for forms:

Definition 2.1 A (n, n) -form v is positive if $v = g\omega_n$ for some function $g \geq 0$. Let v be a symmetric (p, p) -form.

- (i) The (p, p) -form v is *weakly positive* if

$$v \wedge \alpha_1 \wedge J(\alpha_1) \wedge \cdots \wedge \alpha_{n-p} \wedge J(\alpha_{n-p})$$

is a positive (n, n) -form for every choice of $(1, 0)$ -forms $\alpha_1, \dots, \alpha_{n-p}$.

- (ii) We say that the (p, p) -form v is *positive*, if

$$v \wedge (\sigma_{n-p})\alpha \wedge J(\alpha) \geq 0$$

for every $(n - p, 0)$ -form α , where

$$\sigma_k = (-1)^{\frac{k(k-1)}{2}}.$$

- (iii) Finally, the (p, p) -form v is *strongly positive* if

$$v = \sum_s c_s \alpha_{1,s} \wedge J(\alpha_{1,s}) \wedge \cdots \wedge \alpha_{p,s} \wedge J(\alpha_{p,s})$$

where $c_s \geq 0$, and $\alpha_{j,s}$ are $(1, 0)$ -forms.

In the following elementary lemma, which we shall state without proof, we collect some properties concerning positive forms.

Lemma 2.1 *The following properties hold:*

- (1) For $(1, 0)$ -forms α_i ,

$$\alpha_1 \wedge J(\alpha_1) \wedge \cdots \wedge \alpha_p \wedge J(\alpha_p) = \sigma_p \alpha_1 \wedge \alpha_2 \wedge \cdots \wedge \alpha_p \wedge J(\alpha_1) \wedge \cdots \wedge J(\alpha_p).$$

- (2) If w_i are strongly positive forms for $i = 1, \dots, s$ and v is a weakly positive form, then $v \wedge w_1 \wedge \cdots \wedge w_s$ is weakly positive.

- (3) The wedge product of finitely many symmetric forms is symmetric.

- (4) We have the following inclusions:

$$\{\text{strongly positive forms}\} \subset \{\text{positive forms}\} \subset \{\text{weakly positive forms}\}.$$

Another elementary observation is that the property of a form being positive is reflected in the associated matrix of the form, and we shall again omit the proof:

Proposition 2.1 *Let $\alpha = \sum_{K,L} \alpha_{KL}(\sigma_p \cdot dx_K \wedge d\xi_L)$ be a symmetric (p, p) -form. Then α is positive iff the matrix $(\alpha_{KL})_{K,L}$ is positive definite. Moreover, if α is positive, we can find (p, p) -forms γ_k for which*

$$\alpha = \sum_K \alpha_{KK}(\sigma_p \cdot \gamma_K \wedge J(\gamma_K)),$$

where $\alpha_{KK} \geq 0$, for each K .

The set of weakly positive and strongly positive forms are convex cones, by definition dual under the pairing

$$(v, w) \mapsto v \wedge w,$$

where v is a weakly positive (p, p) -form and w is a strongly positive (q, q) -form, and $p + q = n$: By definition, v is weakly positive iff $v \wedge w$ is positive for each strongly positive

w , and since the bidual of a convex cone is equal to the closure of the cone, we see that w is strongly positive iff $w \wedge v$ is positive for every weakly positive v . Moreover, one can show that the cone of positive forms is self-dual. As an easy corollary we obtain:

Lemma 2.2 *A symmetric, weakly positive $(n - 1, n - 1)$ -form is strongly positive. The same applies for symmetric $(1, 1)$ -forms.*

In particular, the Lemma implies that the three different notions of positivity coincides for forms of bi-degree $(0, 0)$, $(1, 1)$, $(n - 1, n - 1)$ and (n, n) . Thus, for such forms we will usually only use the epithet “positive”.

Let \mathbb{V}', \mathbb{W}' be real vector spaces of dimension m , between which we fix an isomorphism J' as above, and let $\psi : \mathbb{V} \rightarrow \mathbb{V}'$ be an affine map. We let $\mathbb{E}' = \mathbb{V}' \times \mathbb{W}'$. Then ψ extends, uniquely, to an affine map from \mathbb{E} to \mathbb{E}' , which we denote by $\tilde{\psi}$, by demanding that $\tilde{\psi} \circ J = J' \circ \psi$. We can now define the pull-back operator $\tilde{\psi}^* : \mathcal{E}(\mathbb{E}') \rightarrow \mathcal{E}(\mathbb{E})$, by letting

$$\tilde{\psi}^* \left(\sum_{I,J} \alpha_{IJ} dx_I \wedge d\xi_J \right) = \sum_{I,J} (\alpha_{IJ} \circ \psi) \psi^*(dx_I) \wedge \psi^*(d\xi_J)$$

where, if $I = (i_1, \dots, i_p)$, we as usual let $\psi^*(dx_I) = \psi^*(dx_{i_1}) \wedge \dots \wedge \psi^*(dx_{i_p})$, and analogously for $\psi^*(d\xi_J)$. Observe that the pull-back operator commutes with the operator J , and also with the operator d . When no confusion seems likely to arise, we will denote the extension $\tilde{\psi}$ by ψ as well. Note that if we did not demand ψ to be affine, the pullback of a form in \mathcal{E} could have coefficients depending on the variable ξ , and thus not be a map from \mathcal{E} to \mathcal{E} . If $\psi : \mathbb{E} \rightarrow \mathbb{E}$ is an affine map, an easy computation shows that

$$\tilde{\psi}^*(\omega_n) = |\det(\psi)|^2 \omega_n. \tag{2.2}$$

This implies that if ψ corresponds to a change of coordinates, $\tilde{\psi}^* \omega_n = c \omega_n$ for some constant $c > 0$. Thus, positivity does not depend on the form ω which we use as a reference. If ψ corresponds to an inclusion of a subspace $\mathbb{V} \subset \mathbb{V}'$, we call $\tilde{\psi}^*(\alpha)$ the restriction of the form α to the subspace \mathbb{V} .

Proposition 2.2 *With the above notation the following holds: α is a weakly positive (p, p) -form on \mathbb{E}' , iff the restriction of α to every p -dimensional subspace is positive, that is, if $\tilde{\psi}^*(\alpha)$ is positive for every inclusion map $\psi : \mathbb{V} \rightarrow \mathbb{V}'$, where \mathbb{V} is a p -dimensional subspace of \mathbb{V}' .*

The proof is almost identical in the complex case, and such a proof can be found in [8, p. 102].

We want to define the integral of an (n, n) -form over the space \mathbb{E} . For this, we assume that the vector space \mathbb{V} is endowed with an inner product (\cdot, \cdot) , and choose an orthonormal basis $\{e_1, \dots, e_n\}$, with corresponding coordinates (x_1, \dots, x_n) . We endow \mathbb{W} with the same structure via the isomorphism J . Then $dx_1 \wedge \dots \wedge dx_n$ is a $(n, 0)$ form on \mathbb{V} , $d\xi_1 \wedge \dots \wedge d\xi_n$ is an $(0, n)$ -form on \mathbb{W} , and every (n, n) -form α can be written as

$$\alpha = \alpha_0(x) dx_1 \wedge \dots \wedge dx_n \wedge d\xi_1 \wedge \dots \wedge \xi_n,$$

for some function α_0 on \mathbb{V} .

Definition 2.2 The integral of an (n, n) -form α as above is given by

$$\int_{\mathbb{E}} \alpha = \int_{\mathbb{V}} \alpha_0(x) dx_1 \wedge \dots \wedge dx_n.$$

The above definition depends only on the inner product chosen. Indeed, if we choose a different orthonormal basis, say $\{e'_1, \dots, e'_n\}$, the map ψ which sends e_i to e'_i has determinant 1 or -1 . In either case, by formula (2.2) this means that

$$\int_{\mathbb{E}} \tilde{\psi}^*(\alpha) = (\pm 1)^2 \int_{\mathbb{E}} \alpha = \int_{\mathbb{E}} \alpha.$$

Thus the definition is independent of which orthonormal basis we choose to work with. In particular, it does not depend on any orientation of \mathbb{V} . One can also understand the definition as follows: the choice of inner product allows us, as above, to choose a volume-element $d\xi_1 \wedge \dots \wedge d\xi_n$ on \mathbb{W} , the total volume of which we define to be

$$\int_{\mathbb{W}} d\xi_1 \wedge \dots \wedge d\xi_n = 1.$$

Then, by formally applying Fubini’s theorem,

$$\begin{aligned} \int_{\mathbb{E}} \alpha_0(x) dx_1 \wedge \dots \wedge dx_n \wedge d\xi_1 \wedge \dots \wedge \xi_n &= \int_{\mathbb{V}} \alpha_0(x) dx_1 \wedge \dots \wedge dx_n \cdot \int_{\mathbb{W}} d\xi_1 \wedge \dots \wedge \xi_n \\ &= \int_{\mathbb{V}} \alpha_0(x) dx_1 \wedge \dots \wedge dx_n. \end{aligned}$$

If $\Omega \subset \mathbb{V}$ is open, we define

$$\int_{\Omega \times \mathbb{W}} \alpha = \int_{\mathbb{E}} \chi_{\Omega}(x) \cdot \alpha_0(x) dx_1 \wedge \dots \wedge dx_n \wedge d\xi_1 \wedge \dots \wedge \xi_n,$$

where χ_{Ω} denotes the characteristic function of the set Ω . By formula (2.2) we have the following change of variable formula for a non-singular, affine map $\psi : \mathbb{V} \rightarrow \mathbb{V}'$:

$$\int_{\mathbb{E}} \tilde{\psi}^* \alpha = \int_{\mathbb{V}} |\det(\psi)|^2 \alpha_0(\psi(x)) dx_1 \wedge \dots \wedge dx_n = |\det(\psi)| \int_{\mathbb{E}} \alpha. \tag{2.3}$$

If $L \subset \mathbb{V}$ is an oriented submanifold of dimension k , and if

$$\alpha = \sum_{|I|=k} \alpha_I(x) dx_I \wedge d\xi_1 \wedge \dots \wedge d\xi_n$$

is an arbitrary (k, n) -form, we define the integral of α over $L \times \mathbb{W}$ by

$$\int_{L \times \mathbb{W}} \left(\sum_{|I|=k} \alpha_I(x) dx_I \wedge d\xi_1 \wedge \dots \wedge d\xi_n \right) = \sum_{|I|=k} \int_L \alpha_I(x) dx_I.$$

Remark 2.1 It might be interesting at this point to compare with the complex setting. First of all, the map J could be compared with the usual complex structure which identifies \mathbb{C}^n with $\mathbb{R}^n + i\mathbb{R}^n$. Under this identification we compactify the imaginary directions by considering $\mathbb{R}^n + i\mathbb{R}^n / \mathbb{Z}^n$. A convex function f on \mathbb{R}^n can then be regarded as a plurisubharmonic function on $\mathbb{R}^n + i\mathbb{R}^n / \mathbb{Z}^n$, by demanding the extension to be independent of the imaginary

directions. In a similar way, we can consider a real n -form α as a complex (n, n) -form $\tilde{\alpha}$, and in a natural way,

$$\int_{\mathbb{R}^n + i\mathbb{R}^n / \mathbb{Z}^n} \tilde{\alpha} = \int_{\mathbb{R}^n} \alpha,$$

which could be seen as an analogue of our definition,

$$\int_{\mathbb{R}^n} d\xi_1 \wedge \cdots \wedge d\xi_n = 1.$$

Generalizing this slightly, we use another lattice in order to compactify the imaginary directions. Thus, if Γ is any lattice, we consider $\mathbb{R}^n + i\mathbb{R}^n / \Gamma$. Such a lattice induces an inner product on \mathbb{R}^n . Indeed, Γ is isomorphic to \mathbb{Z}^n via an affine map, and there is a one-to-one correspondence between affine maps and inner products. In this sense, we can say that choosing an inner product on \mathbb{R}^n (thereby defining integration of (p, p) -forms along p -dimensional subspaces of \mathbb{E}) corresponds to compactifying using a lattice as above, and vice versa.

We define the operator $d : \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p+1,q}$ by the formula

$$d \left(\sum_{|I|=p, |J|=q} \alpha_{IJ} dx_I \wedge d\xi_J \right) = \sum_{|I|=p, |J|=q} \left(\sum_{i=1}^n \frac{\partial \alpha_{IJ}(x)}{\partial x_i} dx_i \wedge dx_I \wedge d\xi_J \right).$$

In our setting, we have the following version of Stokes' formula.

Proposition 2.3 *For $\Omega \subset \mathbb{V}$ a smoothly bounded, open subset, and α an $(n - 1, n)$ -form on \mathbb{E} ,*

$$\int_{\Omega \times \mathbb{W}} d\alpha = \int_{\partial\Omega \times \mathbb{W}} \alpha.$$

Proof By the usual Stokes' formula, we have that

$$\begin{aligned} \int_{\Omega \times \mathbb{W}} d\alpha &= \sum \int_{\Omega \times \mathbb{W}} d(\alpha_{IJ}(x) dx_I) \wedge d\xi_J = \sum \int_{\Omega} d(\alpha_{IJ}(x) dx_I) \int_{\mathbb{W}} d\xi_J \\ &= \sum \int_{\partial\Omega} (\alpha_{IJ}(x) dx_I) \int_{\mathbb{W}} d\xi_J = \int_{\partial\Omega \times \mathbb{W}} \alpha. \end{aligned}$$

We define the operator $d^\# : \mathcal{E}^{p,q} \rightarrow \mathcal{E}^{p,q+1}$ by

$$d^\# = J \circ d \circ J,$$

which in coordinates is equivalent to

$$d^\# = \sum_{j=1}^n \partial_{x_j} \wedge d\xi_j.$$

As immediately follows from the definition, $d^2 = (d^\#)^2 = 0$, and moreover, $d^\# \circ J = J \circ d$.

2.1 Currents and positivity

In this section, we assume the reader to be familiar with the basic theory of currents, but we include some proofs to illustrate the setting in which we work. The precise definition of a (super) current is tedious and almost identical to the complex case, so we refer to [4] for the details. The basic idea is that by introducing a topology on $\mathcal{D}^{p,q} = \{\alpha \in \mathcal{E}^{p,q}; \alpha \text{ has compact support}\}$, we can consider the topological dual of $\mathcal{D}^{p,q}$, which we define to be the space of currents $\mathcal{D}_{n-p,n-q}$. Suffice it so say that an element of $\mathcal{D}_{n-p,n-q}$ can be viewed as a $(n - p, n - q)$ -form whose coefficients are distributions which only acts on \mathbb{V} , that is, the coefficients are “independent of ξ ”. Thus every $T \in \mathcal{D}_{p,q}$ can be written as

$$T = \sum_{|I|=p, |J|=q} T_{IJ} dx_I \wedge d\xi_J,$$

where T_{IJ} are uniquely defined distributions on \mathbb{V} . We denote the pairing between an element $\alpha \in \mathcal{D}^{p,q}$ and $T \in \mathcal{D}_{n-p,n-q}$ by $\langle T, \alpha \rangle$, and we use the convention that

$$\langle T_{IJ} dx_I \wedge d\xi_J, \alpha_0 dx_{I^c} \wedge d\xi_{J^c} \rangle = \pm \langle T_{IJ}, \alpha_0 \rangle,$$

where the sign is determined by the sign of the permutation sending $dx_I \wedge d\xi_J \wedge dx_{I^c} \wedge d\xi_{J^c}$ to ω_n . For convenience, when a current acts on an element of \mathcal{E} , we always assume this element to have compact support, without explicitly stating so. As usual we can define $dT, d^\#T, dd^\#T$ by $\langle dT, \alpha \rangle = \pm \langle T, d\alpha \rangle$ and so forth. Thus we say that a current T is d -closed if $dT = 0$, and similarly for $d^\#$. These operators so defined, act continuously on the space of currents.

Now, let ρ be a smooth, radial function with support in the unit ball, satisfying $\int \rho = 1$, and put $\rho_\epsilon(x) = \frac{1}{\epsilon^n} \rho(\frac{x}{\epsilon})$, for $\epsilon > 0$. If we consider the convolution of a current $T = \sum_{|I|=p, |J|=q} T_{IJ} dx_I \wedge d\xi_J$ with this function ρ_ϵ , defined in the usual way as

$$T \star \rho_\epsilon = \sum_{|I|=p, |J|=q} (T_{IJ} \star \rho_\epsilon) dx_I \wedge d\xi_J, \tag{2.4}$$

then $\{T \star \rho_\epsilon\}_{\epsilon>0}$ defines a family in $\mathcal{E}^{p,q}$ converging weakly to the current T , as $\epsilon \rightarrow 0$. We call this family a regularization of the current T , and it is easy to see that if $dT = 0$, then $d(T \star \rho_\epsilon) = 0$, so regularization preserves the property of being closed.

Lemma 2.3 *Let $\Omega \subset \mathbb{V}$ be an open set. For a d -closed current $T \in \mathcal{D}_{p,q}(\Omega \times \mathbb{W})$ there exists a $T' \in \mathcal{E}^{p,q}(\Omega \times \mathbb{W})$ such that $T - T' = dR$ for some $R \in \mathcal{D}_{p-1,q}(\Omega \times \mathbb{W})$.*

Proof Let

$$T = \sum_{|I|=p, |J|=q} T_{IJ} dx_I \wedge d\xi_J.$$

If we denote by S_J the $(p, 0)$ -currents $\sum_{|I|=p} T_{IJ} dx_I$, then

$$T = \sum_{|J|=q} S_J \wedge d\xi_J,$$

and by the hypothesis $dS_J = 0$. It is well known from theory of currents in \mathbb{V} , that for every such S_J there is a smooth $(p, 0)$ -form S'_J such that $S_J - S'_J = dR_J$ for some $(p - 1, 0)$ -current R_J (where we identify $(p, 0)$ -currents on \mathbb{E} with p -currents on \mathbb{V}). Thus, if we let $T' = \sum_{|J|=q} S'_J \wedge d\xi_J$, and $R = \sum_{|J|=q} R_J \wedge d\xi_J$ we have that T' is smooth and

$$T - T' = \sum_{|J|=q} (S_J - S'_J) \wedge d\xi_J = \sum_{|J|=q} dR_J \wedge d\xi_J = d \sum_{|J|=q} R_J d\xi_J = dR,$$

as required.

Proposition 2.4 *Let $\Omega \subset \mathbb{V}$ be a star shaped open subset. If $T \in \mathcal{D}_{p,q}(\Omega \times \mathbb{W})$ is d -closed and $q \geq 1$, then there exists an element $\tilde{T} \in \mathcal{D}_{p-1,q}(\Omega \times \mathbb{W})$ such that $d\tilde{T} = T$. An analogous statement holds if T is instead $d^\#$ -closed.*

Proof By Lemma 2.3, it suffices to show the proposition in the case where T has smooth coefficients, that is, $T \in \mathcal{E}^{p,q}$. To this end assume that

$$T = \sum_{|I|=p, |J|=q} T_{IJ} dx_I \wedge d\xi_J,$$

with T_{IJ} smooth. If we, as above, denote by S_J the $(p, 0)$ -forms $\sum_{|I|=p} T_{IJ} dx_I$, then $T = \sum_{|J|=q} S_J \wedge d\xi_J$, and since $dT = \sum_{|J|=q} (dS_J) \wedge d\xi_J = 0$ by the hypothesis, we see that $dS_J = 0$. Since Ω is star shaped, the Poincaré lemma tells us that there exists $(p - 1, 0)$ -forms, \tilde{S}_J such that $d\tilde{S}_J = S_J$. Thus, if we consider the $(p - 1, q)$ -form $\tilde{T} = \sum_{|J|=q} \tilde{S}_J \wedge d\xi_J$, we see that it satisfies

$$d\tilde{T} = \sum_{|J|=q} d\tilde{S}_J \wedge d\xi_J = \sum_{|J|=q} S_J \wedge d\xi_J = T.$$

If T is instead $d^\#$ -closed, the same argument as above still applies with the obvious changes.

We now come to the corresponding notions of positivity for currents.

Definition 2.3 We say that a symmetric (p, p) -current T is *weakly positive* if

$$\langle T, \alpha \rangle \geq 0$$

for each smooth *strongly positive* $(n - p, n - p)$ -form α with compact support. T is *positive* if

$$\langle T, \sigma_{n-p} \beta \wedge J(\beta) \rangle \geq 0,$$

for every smooth $(p, 0)$ -form β with compact support. T is *strongly positive*, if

$$\langle T, \gamma \rangle \geq 0$$

for each smooth, weakly positive form γ of compact support.

Proposition 2.5 *A continuous function $f : \mathbb{V} \rightarrow \mathbb{R}$ is convex iff $dd^\# f$ is a positive $(1, 1)$ -current.*

Proof This is clear if f is smooth since the matrix associated to $dd^\# f$ is the Hessian of f , so we can apply Proposition 2.1. The general case follows by approximation: if f is convex but not smooth, we can find a family $\{f_\epsilon\}_\epsilon$ of smooth, convex functions such that $f_\epsilon \rightarrow f$. Using the definition of currents, we see that $dd^\# f_\epsilon \rightarrow dd^\# f$ in the weak sense, and thus, $\langle dd^\# f, \alpha \rangle = \lim_{\epsilon \rightarrow 0} \langle dd^\# f_\epsilon, \alpha \rangle \geq 0$, for every positive $(n - 1, n - 1)$ -form α . Hence $dd^\# f \geq 0$. Conversely, if $dd^\# f \geq 0$, we put $f_\epsilon(x) = \int f(y) \rho_\epsilon(x - y)$ where ρ_ϵ is the regularizing kernel from above. One easily verifies that $dd^\# f_\epsilon \geq 0$ and hence f_ϵ is convex. Moreover, f_ϵ is smooth, $f_\epsilon \geq f$ and $f_\epsilon \rightarrow f$ uniformly on compacts. Thus f is convex, as desired.

Of course, if f is also smooth, then $dd^\# f$ is a positive, closed $(1, 1) - form$. The following proposition is fundamental for what is to come; it is the counterpart in our setting of the so called dd^c -lemma from complex analysis:

Proposition 2.6 ($dd^\#$ -lemma) *Let T be a closed, positive $(1,1)$ -current on \mathbb{E} . Then there exists a convex function $f : \mathbb{V} \rightarrow \mathbb{R}$ for which*

$$T = dd^\# f.$$

Proof By regularization, we can assume T to be smooth. Since T is also closed, by Proposition 2.4 we can find a smooth $(0, 1)$ -current

$$S = \sum_{i=1}^n S_i d\xi_i$$

such that $T = dS$. Moreover, T being symmetric implies $\frac{\partial}{\partial x_j} S_i = T_{ij} = T_{ji} = \frac{\partial}{\partial x_i} S_j$, and so we see that

$$d^\# S = \sum_{i,j=1}^n \frac{\partial}{\partial x_j} S_i d\xi_j \wedge d\xi_i = 0.$$

Another application of Proposition 2.4 provides us with a function f such that $S = d^\# f$. We conclude that

$$T = dd^\# f,$$

and since T is positive, f must be convex.

Using this proposition we can show the following:

Proposition 2.7 *If T is a closed, positive $(1,1)$ -current, then each component of $(SuppT)^c$ is convex.*

Proof By the $dd^\#$ -lemma, we know that $T = dd^\# f$, for some convex function f . Then f is affine on $(SuppT)^c$, and thus, by a standard argument, the convexity of f implies the proposition.

3 Intersection theory of currents

Let M_p denote the space of (p, p) - forms on \mathbb{E} whose coefficients are measures, endowed with the following topology: if $T_i, T \in M_p$ then $T_i \rightarrow T$ iff $T_i(\alpha) \rightarrow T(\alpha)$ for every $(n - p, n - p)$ -form α with compact support and whose coefficients are continuous functions. Note that $M_p \subset \mathcal{D}_{p,p}$ as a set, but the topology of M_p is stronger than that induced by $\mathcal{D}_{p,p}$. However, a standard proposition in the setting of currents with measure coefficients, which carries over to our case, is the following (cf. [8]):

Proposition 3.1 *Let $T_i, T \in M_p$. Then $T_i \rightarrow T$ in M_p if and only if*

$$T_i(\alpha) \rightarrow T(\alpha),$$

for every compactly supported, smooth $(n - p, n - p)$ -form α , and if for every compact subset $L \subset \mathbb{R}^n$ we have,

$$\sup_i \max_{I,J} |(T_i)_{IJ}|(L) < +\infty. \tag{3.1}$$

Here, if μ is a measure $|\mu|$ denotes the total variation of μ . Now, let us consider the map given by

$$\Psi_p(f_1, \dots, f_p) = dd^\# f_1 \wedge \dots \wedge dd^\# f_p,$$

where f_i is smooth and convex for each index i ; let us denote the set of such functions by K . We consider Ψ_p as a map from K^p to M_p . Our aim is to show that this map extends, in a natural way, to a map defined on p -tuples of convex functions (that need not be smooth). By the inclusion $M_p \subset \mathcal{D}_{p,p}$, this extension can be considered as a (p, p) -current, which we will call the intersection product. The scheme to prove this extension property is the following: first we prove that Ψ_p maps bounded subsets of K^p to bounded subsets of M_p . By the Banach–Alaoglu theorem, this implies that for every bounded subset $A \subset K^p$, the family $\{\Psi_p(x), x \in A\}$ contains a weakly convergent subsequence. Thus there exists at least one accumulation point of $\{\Psi_p(x), x \in A\}$ in M_p , and we then show that, in fact, there exists only one, unique accumulation point. Before we turn to the details, we need an important property of positive currents:

Proposition 3.2 *If*

$$T = \sum_{|I|=|J|=p} T_{IJ}(\sigma_p \cdot dx_I \wedge d\xi_J)$$

is a symmetric, positive (p, p) -current, then each coefficient T_{IJ} satisfies

$$|\langle T_{IJ}, \phi \rangle| \leq C \cdot \sum_{|I|=p} \langle T_{II}, \phi \rangle,$$

for each smooth, non-negative function with compact support, ϕ . In particular, T_{II} is a positive measure, and T_{IJ} is a signed measure, for each multi-indices I, J .

Proof The argument is clearest when T is smooth, so let us first assume this is the case. By Proposition 2.1 the $p^2 \times p^2$ matrix (T_{IJ}) , is positive definite and symmetric at every point of \mathbb{R}^n , and thus defines a metric g on \mathbb{R}^{p^2} . Let $(e_I)_{|I|=p}$ be an orthonormal basis of \mathbb{R}^{p^2} such that $g(e_I, e_J) = T_{IJ}$. Then the Cauchy–Schwartz inequality gives us

$$T_{IJ} = g(e_I, e_J) \leq \sqrt{g(e_I, e_I)} \cdot \sqrt{g(e_J, e_J)} = \sqrt{T_{II}} \cdot \sqrt{T_{JJ}} \leq \frac{T_{II} + T_{JJ}}{2}, \tag{3.2}$$

where we used the inequality between geometric and arithmetic mean in the last inequality. By exchanging e_I for $-e_I$ in (3.2) we have established the inequality

$$|T_{IJ}(x)| \leq C \cdot \sum_{|I|=p} T_{II}(x) \tag{3.3}$$

for some constant $C > 0$, which proves the proposition when T is smooth. If T is not smooth, we can still define an associated metric as follows: for each smooth, non-negative function with compact support, ϕ , we define

$$g(e_I, e_J) = \langle T, \sigma_{n-p} \phi dx_I \wedge d\xi_J \rangle,$$

and extend by linearity. Then g is a positive definite, symmetric form on \mathbb{R}^{p^2} , since, if $v = \sum_I v_I e_I \in \mathbb{R}^{p^2}$, then

$$g(v, v) = \left\langle T, \sigma_{n-p} \phi \left(\sum_I v_I dx_I \right) \wedge J \left(\sum_I v_I dx_I \right) \right\rangle = \langle T, \sigma_{n-p} \tilde{v} \wedge J(\tilde{v}) \rangle \geq 0,$$

where $\tilde{v} = \sqrt{\phi} \sum_I v_I dx_I$. Thus, $g(e_I, e_I) = \langle T_{II}, \phi \rangle \geq 0$, which implies that T_{II} is a positive measure. Moreover, $g(e_I, e_J) = \langle T_{IJ}, \phi \rangle$, and by the argument used above,

$$|\langle T_{IJ}, \phi \rangle| \leq C \cdot \sum_{|I|=p} \langle T_{II}, \phi \rangle,$$

for some constant $C > 0$. The proposition follows.

A subset $A \subset K^p$ is bounded if for every compact subset L , and every element $(f_1, \dots, f_p) \in A$, we have that

$$\max_i \sup_{x \in L} |f_i(x)| \leq C_L,$$

for some constant C_L (these norms, indexed by L , define the topology of K^p).

Proposition 3.3 *If $A \subset K^p$ is bounded, then for each compact set $L \subset \mathbb{R}^n$, there exists a constant D_L such that if*

$$T = dd^\# f_1 \wedge \dots \wedge dd^\# f_p$$

where $(f_1, \dots, f_p) \in A$, the coefficients of T satisfy

$$|T_{IJ}|(L) := \int_L |T_{IJ}| \leq D_L.$$

Proof By the previous proposition, we need only to prove that $T_{II} \leq \tilde{D}_L$, for every I . Assume first that $p = 1$. Fix a compact set L and let $f \in K$. Moreover, let χ be a smooth function equal to 1 on L and 0 outside a small neighbourhood of L . Then, since $\partial_{ii}^2 f \geq 0$, by partial integration we get,

$$\begin{aligned} \left| \int_L \partial_{ii}^2 f \right| &\leq \left| \int_{\mathbb{E}} \chi(x) dd^\# f(x) \wedge \widehat{dx}_i \wedge \widehat{d\xi}_i \right| \\ &= \left| \int_{\text{Supp}\chi \times \mathbb{R}^n} f(x) dd^\# \chi(x) \wedge \widehat{dx}_i \wedge \widehat{d\xi}_i \right| \leq \sup_{x \in \text{Supp}\chi} |f(x)| \cdot C_\chi, \end{aligned}$$

using that χ has uniformly bounded second-order partial derivatives on every compact subset. Thus, there exists a constant \tilde{D}_L such that

$$|T_{ii}|(L) \leq \tilde{D}_L$$

for each i , proving the case $p = 1$. Assume now that we have proven the proposition for $p = k - 1$. We want to show that it holds for $p = k$ as well. To this end, let $S = dd^\# f_2 \wedge \dots \wedge dd^\# f_k$, and fix a multi-index I of length k . Then, using the same notation as in the case $p = 1$,

$$\begin{aligned} \left| \int_L (dd^\# f_1 \wedge S)_{II} dV \right| &\leq \left| \int_{\mathbb{E}} \chi(x) dd^\# f_1(x) \wedge S \wedge \widehat{dx}_I \wedge \widehat{d\xi}_I \right| \\ &= \left| \int_{\text{Supp}\chi \times \mathbb{R}^n} f_1(x) dd^\# \chi(x) \wedge S \wedge \widehat{dx}_I \wedge \widehat{d\xi}_I \right|. \end{aligned}$$

By the induction hypothesis, S has coefficients satisfying $|S_{KL}|(Supp\chi) \leq D_{Supp\chi}$. Thus

$$\left| \int_L (dd^\# f_1 \wedge S)_{II} dV \right| \leq \sup_{x \in Supp\chi} |f_1(x)| \cdot C_\chi,$$

and we are done.

Proposition 3.4 *Let f_1, \dots, f_p be convex (but not necessarily smooth) functions, and let, for each $i, \{f_i^k\}_k$ be a sequence of smooth, convex functions converging uniformly to f_i on compact subsets. Then the sequence $\{dd^\# f_1^k \wedge \dots \wedge dd^\# f_p^k\}_k \subset M_p$ converges to an element in M_p . If $\{g_i^k\}_k$ is another sequence of smooth convex functions converging, uniformly on compact subsets, to f_i for each i , then the limits of $dd^\# f_1^k \wedge \dots \wedge dd^\# f_p^k$ and $dd^\# g_1^k \wedge \dots \wedge dd^\# g_p^k$ are equal.*

Proof Let $A = \{(f_1^k, \dots, f_p^k), k \geq 1\}$. Obviously, the set A is bounded. Using Proposition 3.3, we see that for each compactly supported $(n - p, n - p)$ -form α with continuous coefficients, there exists a constant D_α , for which

$$(dd^\# f_1^k \wedge \dots \wedge dd^\# f_p^k)(\alpha) \leq D_\alpha \cdot \max_{IJ} \sup_{x \in Supp\alpha} |\alpha_{IJ}(x)|.$$

Thus, applying the Banach–Alaoglu theorem, we see that the sequence $\{dd^\# f_1^k \wedge \dots \wedge dd^\# f_p^k\}_k$ contains a convergent subsequence, $\{dd^\# f_1^{j_k} \wedge \dots \wedge dd^\# f_p^{j_k}\}_k$, in M_p . To prove the in fact the sequence $\{dd^\# f_1^k \wedge \dots \wedge dd^\# f_p^k\}_k$ converges, we first assume that $p = 1$ and let α be a smooth, compactly supported $(n - 1, n - 1)$ -form. Then,

$$\left| \int_{\mathbb{E}} (dd^\# f_1^k - dd^\# f_1^{j_k}) \wedge \alpha \right| = \left| \int_{\mathbb{E}} (f_1^k - f_1^{j_k}) \wedge dd^\# \alpha \right| \leq \sup_{x \in Supp\alpha} |f_1^k - f_1^{j_k}| \cdot C_\alpha$$

which tends to 0 as $k \rightarrow \infty$. This proves that $\lim_{k \rightarrow \infty} dd^\# f_1^k = \lim_{k \rightarrow \infty} dd^\# f_1^{j_k}$ converges in $\mathcal{D}^{1,1}$. However, by Proposition 3.3, both of the forms $dd^\# f_1^k$ and $dd^\# f_1^{j_k}$ satisfy (3.1), and so, by Proposition 3.1, $dd^\# f_1^k$ and $dd^\# f_1^{j_k}$ converge to the same limit in M_p as well. Now, assume the statement is proved for $p = m - 1$, and let $S^k = dd^\# f_2^k \wedge \dots \wedge dd^\# f_p^k$. Then, since S^k is closed,

$$\int_{\mathbb{E}} (dd^\# f_1^k \wedge S^k - dd^\# f_1^{j_k} \wedge S^{j_k}) \wedge \alpha = \int_{\mathbb{E}} (f_1^k S^k - f_1^{j_k} S^{j_k}) \wedge dd^\# \alpha,$$

where α is a test-form of degree $(n - p, n - p)$. Since

$$f_1^{j_k} S^{j_k} \wedge dd^\# \alpha = (f_1^{j_k} - f_1) S^{j_k} \wedge dd^\# \alpha + f_1 S^{j_k} \wedge dd^\# \alpha,$$

and since

$$\left| \lim_{k \rightarrow \infty} \int_{\mathbb{E}} (f_1^{j_k} - f_1) S^{j_k} \wedge dd^\# \alpha \right| \leq \lim_{k \rightarrow \infty} \sup_{x \in Supp\alpha} |f_1^{j_k} - f_1| \cdot \int_{\mathbb{E}} S^{j_k} \wedge dd^\# \alpha = 0$$

by the induction hypothesis, we see that

$$\lim_{k \rightarrow \infty} \int_{\mathbb{E}} (dd^\# f_1^k \wedge S^k - dd^\# f_1^{j_k} \wedge S^{j_k}) \wedge \alpha = \lim_{k \rightarrow \infty} \int_{\mathbb{E}} (f_1^k S^k - f_1 S^k) \wedge dd^\# \alpha = 0,$$

where we used that $\lim_{k \rightarrow \infty} \int_{\mathbb{E}} f_1 S^{j_k} \wedge dd^\# \alpha = \lim_{k \rightarrow \infty} \int_{\mathbb{E}} f_1 S^k \wedge dd^\# \alpha$ by assumption. Moreover, by Proposition 3.3, the coefficients of S^k satisfy

$$|S_{IJ}^k|(Supp\alpha) \leq C_\alpha.$$

Thus, yet another application of Proposition 3.3 and Proposition 3.1, shows that the sequence $\{dd^\# f_1^k \wedge \dots \wedge dd^\# f_p^k\}_k$ converges. The last statement is proved by repeating the argument above, with $f_i^{j_k}$ replaced by g_i^k .

We can now define the intersection product $dd^\# f_1 \wedge \dots \wedge dd^\# f_p$, for f_1, \dots, f_p convex functions on \mathbb{R}^n , by using the continuity of Ψ_p : it is well known that for a convex function f one can find a sequence of smooth, convex functions f^k which is monotone in k , and which converge to f pointwise. By Dini’s theorem (for its statement, see the discussion after Eq. (3.10), f^k converges uniformly to f on every compact subset of \mathbb{R}^n . Applying this for each function f_i , by using Proposition 3.4 we can define,

$$dd^\# f_1 \wedge \dots \wedge dd^\# f_p = \lim_{k \rightarrow \infty} dd^\# f_1^k \wedge \dots \wedge dd^\# f_p^k,$$

and the definition does not depend on the way we approximate the functions f_i .

Remark 3.1 A slight adaptation of the above arguments shows that if S is a (strongly) positive, closed (p, p) -current, we can, in the same way as above, define the intersection product $S \wedge dd^\# f_1 \wedge \dots \wedge dd^\# f_k$, which then is a (strongly) positive, closed, $(p + k, p + k)$ -current.

We collect some immediate observations:

Proposition 3.5 *The intersection product $dd^\# f_1 \wedge \dots \wedge dd^\# f_p$ is a strongly positive current, it is symmetric in its arguments, and its coefficients are measures. Moreover, it satisfies the relation*

$$Supp(dd^\# f_1 \wedge \dots \wedge dd^\# f_p) \subset Supp(dd^\# f_1) \cap \dots \cap Supp(dd^\# f_p). \tag{3.4}$$

We also have the following stability property:

Proposition 3.6 *Let f, g_1, \dots, g_p be convex functions, where $p < n$. If $\{f_\epsilon\}$ is a family of continuous functions converging pointwise to f , for which*

$$\sup_{\epsilon} \sup_{x \in K} f_\epsilon(x)$$

is bounded for every compact set $K \subset \mathbb{V}$, then

$$\lim_{\epsilon \rightarrow 0} dd^\# f_\epsilon \wedge dd^\# g_1 \wedge \dots \wedge dd^\# g_p = dd^\# f \wedge dd^\# g_1 \wedge \dots \wedge dd^\# g_p.$$

Proof Since the current $dd^\# g_1 \wedge \dots \wedge dd^\# g_p$ has measure coefficients when written in coordinates, we see that if α is a compactly supported, smooth $(n - p - 1, n - p - 1)$ -form, the (n, n) -current $dd^\# g_1 \wedge \dots \wedge dd^\# g_p \wedge dd^\# \alpha$ can be represented by a positive measure on \mathbb{R}^n with compact support. By the dominated convergence theorem,

$$\begin{aligned} \lim_{\epsilon \rightarrow 0} \langle dd^\# f_\epsilon \wedge dd^\# g_1 \wedge \dots \wedge dd^\# g_p, \alpha \rangle &= \lim_{\epsilon \rightarrow 0} \int_{\mathbb{E}} f_\epsilon \wedge dd^\# g_1 \wedge \dots \wedge dd^\# g_p \wedge dd^\# \alpha \\ &= \int_{\mathbb{E}} f \wedge dd^\# g_1 \wedge \dots \wedge dd^\# g_p \wedge dd^\# \alpha = \langle dd^\# f \wedge dd^\# g_1 \wedge \dots \wedge dd^\# g_p, \alpha \rangle, \end{aligned}$$

which proves the claim.

Proposition 3.7 *If f^1, \dots, f^p are convex functions, and for each $i \in \{1, \dots, p\}$ there is a family of convex functions $\{f_{\epsilon_i}^i\}_{\epsilon_i > 0}$ such that*

$$\lim_{\epsilon_i \rightarrow 0} f_{\epsilon_i}^i(x) = f^i(x),$$

for every $x \in \mathbb{V}$, and which satisfy that $\sup_{\epsilon} \sup_{x \in K} f_{\epsilon_i}^i(x)$ is bounded for every compact set $K \subset \mathbb{V}$ and for each i . Then

$$\lim_{\epsilon_1 \rightarrow 0} \dots \lim_{\epsilon_p \rightarrow 0} dd^{\#} f_{\epsilon_1}^1 \wedge \dots \wedge dd^{\#} f_{\epsilon_p}^p = dd^{\#} f^1 \wedge \dots \wedge dd^{\#} f^p, \tag{3.5}$$

for any permutation (i_1, \dots, i_p) of the n -tuple $(1, \dots, p)$.

Proof We can apply Proposition 3.6 successively to the left hand side of 3.5 to obtain the desired conclusion.

Definition 3.1 The Monge-Ampère measure of a convex function f is the positive measure defined by

$$MA(f) = (dd^{\#} f)^n := dd^{\#} f \wedge \dots \wedge dd^{\#} f,$$

where the product is taken n times.

Note that we here identify closed, positive (n, n) -currents with positive measures. If f is smooth, then

$$MA(f) = \det \left(\frac{\partial^2 f}{\partial x_i \partial x_j} \right) dx_1 \wedge d\xi_1 \wedge \dots \wedge dx_n \wedge d\xi_n.$$

A very nice paper concerning real Monge-Ampère measures is [14]. In fact, our approach in this paper could be considered as a generalization of the formalism defined there in. From that paper, it is clear that the following proposition holds:

Proposition 3.8 *Let f be a convex, 1-homogeneous function, that is, $f(\lambda x) = \lambda f(x)$ for every $x \in \mathbb{V}$ and $\lambda \in \mathbb{R}$. Then $MA(f) = 0$ at every point $x \neq 0$.*

Example 3.1 We wish to calculate the Monge-Ampère measure of $dd^{\#}|x|$. First we calculate

$$dd^{\#}|x| = d \left(\frac{1}{2|x|} d^{\#}(|x|^2) \right) = \frac{dd^{\#}|x|^2}{2|x|} - \frac{d|x|^2 \wedge d^{\#}|x|^2}{4|x|^3}. \tag{3.6}$$

But the form $\frac{d|x|^2 \wedge d^{\#}|x|^2}{4|x|^3} = 0$ on $|x| = r > 0$ and so, by using Stokes' theorem, we obtain

$$\begin{aligned} \int_{B(0,r) \times \mathbb{R}^n} (dd^{\#}|x|)^n &= \int_{\partial B(0,r) \times \mathbb{R}^n} \frac{d^{\#}|x| \wedge (dd^{\#}|x|^2)^{n-1}}{2^n |x|^n} \\ &= \frac{1}{2^n r^n} \int_{B(0,r) \times \mathbb{R}^n} \left(\sum_k 2 dx_k \wedge d\xi_k \right)^n = \frac{n!}{r^n} \int_{B(0,r) \times \mathbb{R}^n} dx_1 \wedge d\xi_1 \wedge \dots \wedge dx_n \wedge d\xi_n \\ &= n! Vol_n(B(0, 1)). \end{aligned}$$

Since the above integral is independent of r (or by using Proposition 3.8), we see that the measure $(dd^{\#}|x|)^n$ equals the Dirac measure at the origin multiplied with a dimensional constant.

A useful class of convex functions are those that grow “at most linearly at infinity”. By the similarity with the complex setting, we define the *Lelong class* to be the class of such functions:

$$\mathcal{L} = \{f : \mathbb{R}^n \rightarrow \mathbb{R} : f(x) \leq C|x| + D, f \text{ convex}, C \geq 0, D \in \mathbb{R}\}. \tag{3.7}$$

This class is useful in our context since the intersection of currents whose potentials belongs to \mathcal{L} has finite total mass. To see this, we first consider the case of the Monge–Ampère measure of functions in \mathcal{L} :

Proposition 3.9 *Let $f \in \mathcal{L}$ so that we can find a constant $c > 0$ for which $f \leq c|x|$, when $|x|$ is sufficiently large. Then f satisfies*

$$\int_{\mathbb{E}} (dd^\# f)^n < +\infty.$$

Proof Fix $\epsilon > 0$. For every $r > 0$ we can find a constant $D > 0$ such that $f \geq -D + (c + \epsilon)|x|$, if $|x| < r$ but $f \leq -D + (c + \epsilon)|x|$, if $|x| > R$, for R sufficiently large. Denote by H the function $\max\{f, -D + (c + \epsilon)|x|\}$. Then H is convex, and so $dd^\# H \geq 0$. We can exploit this as follows:

$$\begin{aligned} \int_{B(0,r) \times \mathbb{R}^n} (dd^\# f)^n &= \int_{B(0,r) \times \mathbb{R}^n} (dd^\# H)^n \leq \int_{B(0,R) \times \mathbb{R}^n} (dd^\# H)^n \\ &= \int_{\partial B(0,R) \times \mathbb{R}^n} d^\#(-D + (c + \epsilon)|x|) \wedge (dd^\#(-D + (c + \epsilon)|x|))^{n-1} \\ &= \int_{B(0,R) \times \mathbb{R}^n} (dd^\#(-D + (c + \epsilon)|x|))^n \leq (c + \epsilon)^n \int_{\mathbb{R}^n \times \mathbb{R}^n} (dd^\#|x|)^n < +\infty. \end{aligned}$$

Letting $r \rightarrow \infty$ we obtain

$$\int_{\mathbb{E}} (dd^\# f)^n < +\infty.$$

Proposition 3.10 *If $f_1, \dots, f_n \in \mathcal{L}$, then*

$$\int_{\mathbb{E}} dd^\# f_1 \wedge \dots \wedge dd^\# f_n < +\infty.$$

Proof Since $f_1 + \dots + f_n \in \mathcal{L}$ there exists a $C > 0$ such that $f_1 + \dots + f_n \leq C|x|$, when $|x|$ is large enough. By Proposition 3.9, with $f = f_1 + \dots + f_n$, we obtain

$$\int_{\mathbb{E}} (dd^\# f_1 + \dots + dd^\# f_n)^n < +\infty.$$

But $(dd^\# f_1 + \dots + dd^\# f_n)^n$ is a sum with one term equal to $dd^\# f_1 \wedge \dots \wedge dd^\# f_n$, and since every term of the sum is a positive measure, we deduce that

$$\int_{\mathbb{E}} dd^\# f_1 \wedge \dots \wedge dd^\# f_n < +\infty,$$

which concludes the proof.

A slight modification of the proof of the above Proposition 3.9 gives us a useful comparison theorem, whose analogue in the complex setting is well known.

Proposition 3.11 *Let $f_i, g_i \in \mathcal{L}$ be such that $f_i \leq g_i + O(1)$ for each $1 \leq i \leq n$. Then*

$$\int_{\mathbb{E}} dd^\# f_1 \wedge \cdots \wedge dd^\# f_n \leq \int_{\mathbb{E}} dd^\# g_1 \wedge \cdots \wedge dd^\# g_n.$$

In particular, if $f, g \in \mathcal{L}$, satisfies $f \leq g + O(1)$, then

$$\int_{\mathbb{E}} (dd^\# f)^n \leq \int_{\mathbb{E}} (dd^\# g)^n.$$

Proof Fix $\epsilon > 0$. For every $r > 0$ we can, for each i between 1 and n , find constants $D_i > 0$ such that $f_i \geq -D_i + g_i + \epsilon|x|$, if $|x| < r$ but $f_i \leq -D_i + g_i + \epsilon|x|$, if $|x| > R_i$, for R_i a sufficiently large number. Denote by H_i the function $\max\{f_i, -D_i + g_i + \epsilon|x|\}$. Then each H_i is convex, and so $dd^\# H_i \geq 0$. Thus:

$$\begin{aligned} \int_{B(0,r) \times \mathbb{R}^n} dd^\# f_1 \wedge \cdots \wedge dd^\# f_n &= \int_{B(0,r) \times \mathbb{R}^n} dd^\# H_1 \wedge \cdots \wedge dd^\# H_n \\ &\leq \int_{B(0,R) \times \mathbb{R}^n} dd^\# H_1 \wedge \cdots \wedge dd^\# H_n. \end{aligned}$$

But for $x \in B(0, R)$, we have that $H_i(x) = -D_i + g_i + \epsilon|x|$, and thus we obtain,

$$\int_{B(0,r) \times \mathbb{R}^n} dd^\# f_1 \wedge \cdots \wedge dd^\# f_n \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} (dd^\# g_1 + \epsilon dd^\# |x|) \wedge \cdots \wedge (dd^\# g_n + \epsilon dd^\# |x|).$$

Letting $r \rightarrow \infty$ we obtain that, for every $\epsilon > 0$,

$$\int_{\mathbb{E}} dd^\# f_1 \wedge \cdots \wedge dd^\# f_n \leq \int_{\mathbb{R}^n \times \mathbb{R}^n} (dd^\# g_1 + \epsilon dd^\# |x|) \wedge \cdots \wedge (dd^\# g_n + \epsilon dd^\# |x|).$$

This last integral contains terms of the type $dd^\# g_{i_1} \wedge \cdots \wedge dd^\# g_{i_{n-k}} \wedge \epsilon^k (dd^\# |x|)^k$ with $k = 0, \dots, n$, and $1 \leq i_l \leq n$. Proposition 3.10 tells us that

$$\int_{\mathbb{E}} dd^\# g_{i_1} \wedge \cdots \wedge dd^\# g_{i_{n-k}} \wedge \epsilon^k (dd^\# |x|)^k \leq C \epsilon^k$$

and consequently there is a constant $C > 0$ (independent of ϵ) for which

$$\int_{\mathbb{R}^n \times \mathbb{R}^n} (dd^\# g_1 + \epsilon dd^\# |x|) \wedge \cdots \wedge (dd^\# g_n + \epsilon dd^\# |x|) \leq \int_{\mathbb{E}} dd^\# g_1 \wedge \cdots \wedge dd^\# g_n + \epsilon C.$$

Letting $\epsilon \rightarrow 0$ completes the proof.

Interchanging the roles of f and g in the above proposition gives us:

Corollary 3.1 *If $f, g \in \mathcal{L}$ satisfy*

$$|f - g| \leq C,$$

for some constant $C > 0$, then

$$\int_{\mathbb{E}} (dd^\# f)^n = \int_{\mathbb{E}} (dd^\# g)^n.$$

In fact, the proof gives us a slightly stronger statement, which we will find useful:

Corollary 3.2 *If $f, g \in \mathcal{L}$ satisfy*

$$|f - g| \leq C + \epsilon|x|$$

for every $\epsilon > 0$ and for some constant $C > 0$, then

$$\int_{\mathbb{E}} (dd^\# f)^n = \int_{\mathbb{E}} (dd^\# g)^n.$$

At this point, we introduce the useful notation $\widehat{dx}_i = dx_1 \wedge \dots \wedge dx_{i-1} \wedge dx_{i+1} \wedge \dots \wedge dx_n$.

Example 3.2 Let $K \subset \mathbb{R}^n$ be a convex set containing the origin, and denote by H_K its support function, that is, $H_K(x) = \sup_{\xi \in K} \{x \cdot \xi\}$. The polar of K , denoted K° , is defined by $K^\circ = \{x : H_K(x) \leq 1\}$. If ∂K is smooth and K strictly convex, it is well known that the map $x \mapsto \partial H_K(x)$ defines a diffeomorphism between ∂K and ∂K° . We can thus introduce $\frac{\partial H_K}{\partial x_j}$ as coordinates on ∂K° to obtain

$$\begin{aligned} n!Vol(K) &= \int_{K \times \mathbb{R}^n} (dd^\# |x|^2)^n = \int_{\partial K \times \mathbb{R}^n} d^\# |x|^2 \wedge (dd^\# |x|^2)^{n-1} \\ &= c_n \int_{\partial K} \sum x_i \widehat{dx}_i = c_n \int_{\partial K^\circ} \sum \frac{\partial H_K}{\partial x_i} \cdot d \widehat{\frac{\partial H_K}{\partial x_i}} = \int_{K^\circ \times \mathbb{R}^n} d^\# H_K \wedge (dd^\# H_K)^{n-1} \\ &= \int_{K^\circ \times \mathbb{R}^n} (dd^\# H_K)^n = \int_{\mathbb{E}} (dd^\# H_K)^n. \end{aligned} \tag{3.8}$$

In the last equality we used Proposition 3.8: since H_K is smooth outside the origin (thanks to ∂K being smooth) and 1-homogeneous, the support of $(dd^\# H_K)^n$ is the origin. By approximation, the same formula holds without any smoothness assumptions on ∂K .

One can generalize this example as follows: Recall that if K_1, \dots, K_n are convex sets in \mathbb{R}^n one can define the mixed volume of K_1, \dots, K_n , which we will denote by $V(K_1, \dots, K_n)$ as follows: consider the function

$$P(t_1, \dots, t_n) := Vol(t_1 K_1 + \dots + t_n K_n),$$

where

$$\sum_{j \in J} t_j K_j := \{t_{j_1} \cdot x_{j_1} + \dots + t_{j_l} \cdot x_{j_l} : x_{j_i} \in K_{j_i}, J = (j_1, \dots, j_l)\}$$

is the Minkowski sum. As will follow from the proof of Proposition 3.12, P is a n -homogeneous polynomial in n variables:

$$P(t_1, \dots, t_n) = \sum_{i_1, \dots, i_n=1}^n a_{i_1, \dots, i_n} \cdot t_{i_1} \cdot \dots \cdot t_{i_n},$$

for some coefficients a_{i_1, \dots, i_n} . The mixed volume is the coefficient in this polynomial corresponding to the monomial $t_1 \cdots t_n$, that is,

$$V(K_1, \dots, K_n) := a_{1, \dots, n}.$$

We claim the following:

Proposition 3.12 *Let K_1, \dots, K_n be convex sets in \mathbb{R}^n with corresponding support functions H_{K_i} . Then*

$$dd^\# H_{K_1} \wedge \cdots \wedge dd^\# H_{K_n} = n! \cdot V(K_1, \dots, K_n) \delta_0 \omega_n,$$

and $P(t_1, \dots, t_n) := \text{Vol}(t_1 K_1 + \cdots + t_n K_n)$ is a n -homogeneous polynomial.

Proof To begin with, we note that

$$H_{tK+sL} = tH_K + sH_L \tag{3.9}$$

if K, L are compact subsets, and t, s real numbers. Generalizing this slightly, we obtain the identity

$$MA(H_{t_1 K_1 + \dots + t_n K_n}) = MA(t_1 H_{K_1} + \cdots + t_n H_{K_n}),$$

and thus $MA(H_{t_1 K_1 + \dots + t_n K_n})$ is a n -homogeneous polynomial in (t_1, \dots, t_n) . Moreover, by (3.8) we know that

$$MA(H_{t_1 K_1 + \dots + t_n K_n}) = n! \text{Vol}(t_1 K_1 + \cdots + t_n K_n) \cdot \delta_0 \cdot \omega_n.$$

This immediately gives us that $P(t_1, \dots, t_n) = \text{Vol}(t_1 K_1 + \cdots + t_n K_n)$ is an n -homogeneous polynomial. Moreover, comparing coefficients of the two polynomials, we see that

$$dd^\# H_{K_1} \wedge \cdots \wedge dd^\# H_{K_n} = n! \cdot V(K_1, \dots, K_n) \cdot \delta_0 \cdot \omega_n$$

as desired.

Example 3.3 Let $K_i, L_i \subset \mathbb{R}^n$ be convex sets, such that $K_i \subset L_i$ for i between 1 and n . Then $H_{K_i} \leq H_{L_i}$, and an application of Proposition 3.11 together with the previous proposition, tells us that

$$V(K_1, \dots, K_n) \leq V(L_1, \dots, L_n).$$

We use this observation to prove that if $K \subset L$ are two convex, smoothly bounded sets, then the surface area of K is smaller than that of L . Indeed, let us consider the function $v(t) = \text{Vol}(K + tB)$ where B is the unit ball. Then, it is not difficult to realize that

$$v'(0) = |\partial K|,$$

that is, the derivative of v at $t = 0$ is equal to the surface area of K . On the other hand, by the description of $v(t) = \text{Vol}(K + tB)$ as a polynomial in t , we see that, in fact,

$$v'(0) = V(B, K, \dots, K).$$

Thus $|\partial K| = V(B, K, \dots, K)$, and by the above we obtain,

$$|\partial K| = V(B, K, \dots, K) \leq V(B, L, \dots, L) = |\partial L|,$$

as promised.

Remark 3.2 These results should be compared with the results already obtained in [11]. Moreover, the setting presented here seems to share similarities with that of [7].

Let f be a convex function on \mathbb{R}^n , belonging to the Lelong class \mathcal{L} . To this f we associate the function

$$\tilde{f}(x) = \lim_{t \rightarrow \infty} \frac{f(tx)}{t},$$

where the limits exists thanks to the assumption on linear growth at infinity. This function \tilde{f} is easily seen to be convex and one-homogeneous. Moreover, we claim that

$$|f - \tilde{f}| \leq C + \epsilon|x|, \tag{3.10}$$

for every $\epsilon > 0$. This is readily seen as follows: By standard properties of convex functions, the expression $\frac{f(tx)-f(0)}{t}$ is increasing in t , for every x . We recall Dini’s theorem which says the following: if f is continuous on a compact set K , and f_t is a monotone sequence of continuous functions which converge to \tilde{f} pointwise, then the convergence is in fact uniform on K . Consequently, for each $\epsilon > 0$ we have that

$$\sup_{|x|=1} \left| \frac{f(tx) - f(0)}{t} - \tilde{f}(x) \right| < \epsilon,$$

if $t > T$, for some $T > 0$. Multiplying through by t we obtain

$$\sup_{|x|=1} |f(tx) - f(0) - \tilde{f}(tx)| < t\epsilon,$$

if t is sufficiently large, which implies (3.10). An application of Corollary 3.2 shows that the total Monge-Ampère mass of f equals that of \tilde{f} :

Proposition 3.13 *With f and \tilde{f} as above*

$$\int_{\mathbb{E}} (dd^\# f)^n = \int_{\mathbb{E}} (dd^\# \tilde{f})^n.$$

As we will see in Sect. 5.3, the above Proposition is essentially Bezout’s theorem in tropical geometry.

4 Lelong numbers, trace measures and push forwards of currents

Definition 4.1 The *trace measure* of a (p, p) -current T is defined as

$$\Theta_T(U) = \frac{1}{2^{n-p}(n-p)!} \int_{U \times \mathbb{R}^n} T \wedge (dd^\#|x|^2)^{n-p},$$

for each Borel-set $U \subset \mathbb{R}^n$.

Proposition 4.1 *If T is a positive (p, p) -current, then Θ_T is a positive measure, and*

$$|T_{I,J}| \leq C \cdot \Theta_T, \tag{4.1}$$

for some $C > 0$.

Proof This follows immediately from Proposition 3.2.

Remark 4.1 Let us compare with the complex setting: if S is a complex, weakly positive (p, p) -current, then S always satisfies a trace measure inequality of the type (4.1). However, in our setting the form α , on $\mathbb{E} = \mathbb{R}^4 \times \mathbb{R}^4$, given by

$$\begin{aligned} \alpha &= dx_1 \wedge dx_2 \wedge d\xi_3 \wedge d\xi_4 + dx_2 \wedge dx_3 \wedge d\xi_1 \wedge d\xi_4 - dx_1 \wedge dx_3 \wedge d\xi_2 \wedge d\xi_4 \\ &\quad + dx_3 \wedge dx_4 \wedge d\xi_1 \wedge d\xi_2 + dx_1 \wedge dx_4 \wedge d\xi_2 \wedge d\xi_3 - dx_2 \wedge dx_4 \wedge d\xi_1 \wedge d\xi_3, \end{aligned}$$

satisfies

$$\alpha \wedge v \wedge J(v) = 0 \tag{4.2}$$

for every $(1, 0)$ -form v . Thus, α is weakly positive, but all the diagonal elements are 0, and consequently α does not satisfy an inequality of the type (4.1). This implies a significant difference between our setting and the complex setting. In fact, in the complex case, the strongly positive forms constitute a basis for the space of all forms. Equation (4.2) tells us that this is not the case in our setting.

Proposition 4.2 *For fixed $x \in \mathbb{R}^n$, if T is a weakly positive (p, p) -current, then the function*

$$r \mapsto \frac{\Theta_T}{r^{n-p}}(B(x, r))$$

is increasing on \mathbb{R}_+ .

Proof We can assume that $x = 0$. By Eq. (3.6) we have,

$$dd^\#|x| = \frac{dd^\#|x|^2}{2|x|} - \frac{d|x|^2 \wedge d^\#|x|^2}{4|x|^3},$$

and $\frac{d|x|^2 \wedge d^\#|x|^2}{4|x|^3} = 0$ on the sphere $|x| = r$. Moreover, $d^\#|x| = \frac{d^\#|x|^2}{2|x|}$, so that $d^\#|x| = \frac{d^\#|x|^2}{2r}$ if $|x| = r$. Combining these observations, we find that

$$\begin{aligned} \int_{\{|x|=r\} \times \mathbb{R}^n} T \wedge d^\#|x| \wedge (dd^\#|x|)^{n-p-1} &= \int_{\{|x|=r\} \times \mathbb{R}^n} T \wedge \frac{d^\#|x|^2}{2|x|} \wedge \left(\frac{dd^\#|x|^2}{2|x|} \right)^{n-p-1} \\ &= \frac{1}{(2r)^{n-p}} \int_{\{|x|=r\} \times \mathbb{R}^n} T \wedge d^\#|x|^2 \wedge (dd^\#|x|^2)^{n-p-1}. \end{aligned}$$

Thus, by Stokes' theorem we obtain

$$\begin{aligned} \int_{B(0,r) \times \mathbb{R}^n} T \wedge (dd^\#|x|^2)^{n-p} &= \int_{\{|x|=r\} \times \mathbb{R}^n} T \wedge d^\#|x|^2 \wedge (dd^\#|x|^2)^{n-p-1} \\ &= (2r)^{n-p} \int_{\{|x|=r\} \times \mathbb{R}^n} T \wedge d^\#|x| \wedge (dd^\#|x|)^{n-p-1} \\ &= (2r)^{n-p} \int_{B(0,r) \times \mathbb{R}^n} T \wedge (dd^\#|x|)^{n-p}. \end{aligned}$$

Since T is weakly positive, $T \wedge (dd^\#|x|)^{n-p}$ is a positive measure. Thus the function

$$r \mapsto \frac{\Theta_T}{r^{n-p}}(B(0, r))$$

is increasing.

Corollary 4.1 *Let T be a closed, positive (p, p) -current. If K is a compact set with vanishing $(n - p)$ -dimensional Hausdorff measure, then T vanishes on K . In particular, if $\text{Supp}(T)$ has vanishing $(n - p)$ -dimensional Hausdorff measure, then $T = 0$.*

Proof Let K be a compact set, satisfying the assumptions of the hypothesis. The condition $\mathcal{H}^{n-p}(K) = 0$, means that we can, for every $\epsilon > 0$, find a finite number of balls $B(a_j, r_j)$ for which $K \subset \cup_j B(a_j, r_j)$ and

$$\sum r_j^{n-p} \leq \epsilon.$$

We can assume that each $r_j \leq 1$. By Proposition 4.2 we see that

$$\frac{\Theta_T}{r_j^{n-p}}(B(a_j, r_j)) \leq \Theta_T(B(a_j, 1)) \leq \Theta_T(K'),$$

where K' is a compact set such that $K \subset \cup_j B(a_j, 1) \subset K'$ and thus, with $C = \Theta_T(K')$ we obtain the inequality: $\Theta_T(B(a_j, r_j)) \leq Cr_j^{n-p}$ for all j . We conclude that

$$\Theta_T(K) \leq \sum_j \Theta_T(B(a_j, r_j)) \leq C \sum_j r_j^{n-p} \leq C\epsilon,$$

and thus, $T|_K = 0$, since $|T|_K(K) \leq \Theta_T(K)$ by Proposition 4.1.

As a consequence of the proposition, we can define the *Lelong number* of a weakly positive, closed (p, p) -current T at a point x by

$$v_x(T) = \lim_{r \rightarrow 0} \frac{\Theta_T(B(x, r))}{\text{Vol}(B^{n-p})r^{n-p}},$$

where $\text{Vol}(B^{n-p})$ is the volume of the $(n - p)$ -dimensional unit ball. We define the Lelong number of a convex function $f : \mathbb{R}^n \rightarrow \mathbb{R}$ by

$$v_x(f) = v_x(dd^\# f).$$

Example 4.1 We calculate the Lelong number of the function $x \mapsto |x|$. We begin with considering the behaviour at the origin. By (3.6) and Stokes',

$$\begin{aligned} & \int_{B(0,r) \times \mathbb{R}^n} dd^\# |x| \wedge (dd^\# |x|^2)^{n-1} \\ &= \frac{1}{2r} \int_{B(0,r) \times \mathbb{R}^n} (dd^\# |x|^2)^n = 2^{n-1} \cdot n! \cdot r^{n-1} \cdot \text{Vol}_n(B(0, 1)), \end{aligned}$$

and thus

$$v_0(|x|) = \lim_{r \rightarrow 0} \frac{\Theta(B(0, r))}{\text{Vol}_{n-1}(B(0, r))} = n \cdot \frac{\text{Vol}_n(B(0, 1))}{\text{Vol}_{n-1}(B(0, 1))}.$$

On the other hand, at a point x_0 away from the origin our function is smooth, and so the form $dd^\# |x|$ is locally smooth. But if g is a smooth function in a neighbourhood around x , then the trace measure of $dd^\# g(x)$ is just the Laplacian of g at x and thus there is a constant $C > 0$ such that

$$\Theta_{dd^\# g}(B(x, r)) \leq Cr^n,$$

since every coefficient $\frac{\partial^2 g}{\partial x_i \partial x_j}$ is uniformly bounded in some neighbourhood around x . Thus

$$\nu_x(g) \leq \lim_{r \rightarrow 0} \frac{Cr^n}{r^{n-1} \text{Vol}_{n-1}(B(0, 1))} = 0,$$

and so $\nu_{x_0}(|x|) = 0$ if $x_0 \neq 0$.

The above argument displays the following expected behaviour of the Lelong number, showing that the Lelong number is a measurement of the singularity at a point:

Proposition 4.3 *If a $(1, 1)$ -current T is locally smooth around a point x , then $\nu(T, x) = 0$.*

Remark 4.2 In complex analysis, it is a well known theorem due to Y.T. Siu, which states that the set $\{x : \nu(T, x) \geq c\}$ constitutes an analytic variety, for each $c > 0$ (in this remark, $\nu(T, x)$ denotes the complex version of the Lelong number). It would be interesting to know if there is a corresponding result in our setting. One could perhaps hope that one would obtain tropical varieties (which we define in Sect. 5), but this is not the case as the example $f = \max(|x|, 1)$ shows: $\{x \in \mathbb{V} : \nu(dd^\# \max(|x|, 1), x) \geq c\} = \{x : |x| = 1\}$ which is not a tropical variety.

4.1 Push forwards of currents

Let $f : \mathbb{V} \rightarrow \mathbb{V}'$ be an affine map, with $\dim(\mathbb{V}) = \dim(\mathbb{V}') = n$, inducing a map $\tilde{f} : \mathbb{E} \rightarrow \mathbb{E}'$. Then we can define the push-forward f_*T of a current $T \in \mathcal{D}_{n-p, n-p}(\mathbb{E})$ via the formula

$$\langle f_*T, \alpha \rangle = \langle T, f^*\alpha \rangle, \tag{4.3}$$

where $\alpha \in \mathcal{D}^{p,p}$. This formula only makes sense if $f^*\alpha$ has compact support on $\text{Supp}(T)$, and so we first demand that f is such that $f^{-1}(K) \cap \text{Supp}(T)$ is compact for every compact $K \subset \mathbb{V}'$, or in other words, the restriction of f to $\text{Supp}(T)$ is proper. For such f the induced map $f^* : \mathcal{E}^{p,p}(\mathbb{E}') \rightarrow \mathcal{E}^{p,p}(\mathbb{E})$ is continuous and thus the above formula defines an element in $\mathcal{D}_{n-p, n-p}(\mathbb{E}')$. If T is weakly positive, then

$$\langle f_*T, \alpha_1 \wedge J(\alpha_1) \wedge \dots \wedge \alpha_p \wedge J(\alpha_p) \rangle = \langle T, f^*\alpha_1 \wedge J(f^*\alpha_1) \wedge \dots \wedge f^*\alpha_p \wedge J(f^*\alpha_p) \rangle \geq 0.$$

Thus we have the following proposition.

Proposition 4.4 *If $T \in \mathcal{D}_{n-p, n-p}(\mathbb{E})$ is weakly positive and $f : \mathbb{V} \rightarrow \mathbb{V}'$ is an (non-constant) affine function, then \tilde{f}_*T is a weakly positive current in $\mathcal{D}_{n-p, n-p}(\mathbb{E}')$.*

Example 4.2 Every form $\beta \in \mathcal{E}^{p,p}$ can be considered as a current acting on compactly supported forms of complementary degree. Now, if $f : \mathbb{V} \rightarrow \mathbb{V}$ is a non-singular affine map, then

$$\begin{aligned} \langle \alpha, f_*\beta \rangle &= \langle f^*\alpha, \beta \rangle = \int_{\mathbb{E}} f^*\alpha \wedge \beta = \frac{1}{|\det(f^{-1})|} \int_{\mathbb{E}} (f^{-1})^*(f^*\alpha \wedge \beta) \\ &= |\det(f)| \left| \int_{\mathbb{E}} \alpha \wedge (f^{-1})^*\beta \right| = |\det(f)| \left| \langle \alpha, (f^{-1})^*\beta \rangle \right|, \end{aligned}$$

where we used formula (2.3) in the third equality. Thus we see that,

$$f_*\beta = |\det f| (f^{-1})^*\beta, \beta \in \mathcal{E}^{p,p}. \tag{4.4}$$

Now let us consider the projection (here we write $\mathbb{V}, \mathbb{W} = \mathbb{R}^n$)

$$\pi : \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}, \pi(x_1, \dots, x_n) = (x_1, \dots, x_{n-k}),$$

where $k \geq 0$ and take a (p, p) -form α on \mathbb{R}^n , with locally integrable coefficients, which we will consider as a current. Assume this form α is such that π is proper on its support, and let ω be a $(n - p, n - p)$ -form on $\mathbb{R}^{n-k} = \{(x_1, \dots, x_{n-k})\}$. Observe that if $n - p > n - k$, the form ω is 0, so we assume this is not the case. We regard, for each $x \in \mathbb{R}^{n-k}$, the set $\pi^{-1}(x)$ as \mathbb{R}^k with coordinates (x_{n-k+1}, \dots, x_n) . Thus ω only contains differentials dx_i and $d\xi_i$ with $1 \leq i \leq n - k$, and $\pi^*\omega = \omega$. Let us write $\alpha = \sum_{|I|=|J|=p} \alpha_{IJ} dx_I \wedge d\xi_J$, and $\omega = \sum_{|K|=|L|=n-p} \omega_{KL} dx_K \wedge d\xi_L$ where each K and L only contain indices between 1 and $n - k$. The (n, n) -form $\alpha \wedge \pi^*\omega$ is a sum of terms of the form $\alpha_{IJ} \omega_{KL} dx_I \wedge d\xi_J \wedge dx_K \wedge d\xi_L$ and such a term vanishes if I and K , or if J and L , contain the same indices. Since $n - p \leq n - k$, this implies that the only terms in the expression defining α that will contribute to the push-forward are those for which $dx_I \wedge d\xi_J$ contains the differential $dx_{n-k+1} \wedge \dots \wedge dx_n \wedge d\xi_{n-k+1} \wedge \dots \wedge d\xi_n$. In effect, we can write α as $\alpha = \{\sum_{|I'|=|J'|=p-k} \alpha_{I'J'} dx_{I'} \wedge d\xi_{J'}\} \wedge dx_{n-k+1} \wedge \dots \wedge dx_n \wedge d\xi_{n-k+1} \wedge \dots \wedge d\xi_n + R$, where R is such that $R \wedge \omega' = 0$ for every $(n - p, n - p)$ -form ω' on \mathbb{R}^{n-k} , and hence will not contribute to the push forward. The definition of push-forward tells us that

$$\begin{aligned} \langle \pi_*\alpha, \omega \rangle &= \langle \alpha, \pi^*\omega \rangle \\ &= \int_{\mathbb{W}} \int_{(x_1, \dots, x_{n-k}) \in \mathbb{R}^{n-k}} \int_{(x_{n-k+1}, \dots, x_n) \in \pi^{-1}(x)} \alpha(x_1, \dots, x_n) \wedge \omega(x_1, \dots, x_{n-k}), \end{aligned}$$

and from this we deduce that the push forward of α under π is given by the $(p - k, p - k)$ -form

$$\begin{aligned} (\pi_*\alpha)(x_1, \dots, x_{n-k}) &= \int_{\pi^{-1}(x_1, \dots, x_{n-k}) \times \mathbb{W}} \left\{ \sum_{|I'|=|J'|=p-k} \alpha_{I'J'} dx_{I'} \wedge d\xi_{J'} \right\} \wedge dx_{n-k+1} \wedge \dots \wedge d\xi_n \\ &= \sum_{|I'|=|J'|=p-k} \tilde{\alpha}_{I'J'}(x_1, \dots, x_{n-k}) dx_{I'} \wedge d\xi_{J'}, \end{aligned}$$

where

$$\tilde{\alpha}_{I'J'}(x_1, \dots, x_{n-k}) = \int_{\pi^{-1}(x_1, \dots, x_{n-k})} \alpha_{I'J'}(x_1, \dots, x_n) dV_k(x_{n-k+1}, \dots, x_n) \tag{4.5}$$

and dV_k is the volume measure induced by the chosen inner product on \mathbb{R}^n . We have hence obtained an explicit formula for the push-forward of a form α under a projection. By using an approximation argument, the discussion above still holds true if we assume α to be a *current*. Now, since $\pi^*d\omega = d\pi^*\omega$, we see that

$$\langle d\pi_*\alpha, \omega \rangle = \pm \langle \pi_*\alpha, d\omega \rangle = \pm \langle \alpha, \pi^*d\omega \rangle = \pm \langle \pi_*\alpha, d\omega \rangle = \langle \pi_*d\alpha, \omega \rangle,$$

which implies that $\pi_*d\alpha = d\pi_*\alpha$. Thus, if α is closed, then $\pi_*\alpha$ is closed as well. Since strong positivity of forms is preserved under pullbacks, we find that if α is a weakly positive form, then $\pi_*\alpha$ is weakly positive. Moreover, the formula

$$\langle f_*\alpha, \sigma_{(n-k)-(p-k)}\beta \wedge J(\beta) \rangle = \langle \alpha, \sigma_{n-p}f^*\beta \wedge J(f^*\beta) \rangle,$$

tells us that if α is positive, then so is $f_*\alpha$. Thus we have:

Proposition 4.5 *Let π be a projection from \mathbb{R}^n onto a $(n - k)$ -dimensional subspace, and let α be a (weakly) positive (p, p) -current on $\mathbb{R}^n \times \mathbb{R}^n$ such that π is proper on the support of α . Then the push-forward $\pi_*\alpha$ is a well-defined $(p - k, p - k)$ -current. Moreover, $\pi_*\alpha$ is (weakly) positive, and if α is closed then $\pi_*\alpha$ is closed as well.*

Let S be a strongly positive, closed, (p, p) -current such that

$$\int_{\mathbb{E}} S \wedge (dd^\#|x|)^{n-p} < +\infty. \tag{4.6}$$

We call the left hand side of (4.6) the *degree* of S . Let π be a projection from \mathbb{R}^n onto a $(n - k)$ -dimensional subspace L as above. Since the restriction of π to $Supp(S)$ might not be proper, the expression π_*S has no meaning as of yet. However, if χ is a continuous function with compact support on $\pi^{-1}(L)$ and with values in $[0, 1]$, we can consider the positive (but not closed) current $\pi_*\chi S$ on L . We have the following lemma:

Lemma 4.1 *Let S be a strongly positive, closed, (p, p) -current of finite degree. Then, for every compact subset $K \subset L$ there exists a constant $C_K > 0$ which is independent of χ , such that the coefficients of $\pi_*\chi S$ applied to K are bounded by C_K .*

Proof We choose coordinates so that $L = \mathbb{R}^{n-k}$ as above and write $x' = (x_1, \dots, x_{n-k})$. It is enough to prove that the statement holds for every ball with center at the origin, in \mathbb{R}^{n-k} . Fix such a ball $B(0, R)$. Define a function ϕ on \mathbb{R}^{n-k} by letting $\phi(x') = |x'|^2$ for $x' \in B(0, 2R)$, and $\phi(x') = 4R|x'| - 4R^2$ otherwise. Then $\phi \in \mathcal{L}$, and $dd^\#\phi = \sum_{i=1}^{n-k} dx_i \wedge d\xi_i$ on $B(0, R)$. Thus, $(dd^\#\phi)^{n-p} = (dd^\#|x'|^2)^{n-p}$ on $B(0, R)$, which implies that

$$\begin{aligned} \Theta_{B(0,R)}(\pi_*\chi S) &= \int_{(B(0,R) \times \mathbb{R}^k) \times \mathbb{R}^n} \chi S \wedge (dd^c|x'|^2)^{n-p} = \int_{(B(0,R) \times \mathbb{R}^k) \times \mathbb{R}^n} \chi S \wedge (dd^c\phi)^{n-p} \\ &\leq \sup \chi \int_{\mathbb{E}} S \wedge (dd^c\phi)^{n-p}. \end{aligned}$$

Since $\phi(x) \leq |x| + O(1)$, a slight adaptation of Proposition 3.11 tells us that $\int_{\mathbb{E}} S \wedge (dd^c\phi)^{n-p} \leq \int_{\mathbb{E}} S \wedge (dd^c|x|)^{n-p}$. Thus, by (4.6) we see that $\Theta_{B(0,R)}(\pi_*\chi S) < +\infty$. Since $\pi_*\chi S$ is positive, we can apply Proposition 4.1 to obtain that every coefficient of $\pi_*\chi S$ when applied to $B(0, R)$ has mass bounded by the trace-measure of $\chi\pi_*S$ acting on $B(0, R)$ and is consequently less than some constant $C_R > 0$ depending on R (since ϕ depends on R) but not on χ . The proposition follows.

Now, let S satisfy the hypothesis of the above lemma. Then Lemma 4.1 together with (4.5) tells us that there is a constant $C > 0$ such that for every positive, compactly supported, continuous function χ defined on $\pi^{-1}(L) = \mathbb{R}^k$ with values in $[0, 1]$, the following inequality holds:

$$\left| \int_{\pi^{-1}(x_1, \dots, x_{n-k})} \chi(x_{n-k+1}, \dots, x_n) S_{I'J'}(x_1, \dots, x_n) dV_k(x_{n-k+1}, \dots, x_n) \right| \leq C \cdot \sup(\chi).$$

This implies that all of the integrals

$$\int_{\pi^{-1}(x_1, \dots, x_{n-k})} S_{I'J'}(x_1, \dots, x_n) dV_k(x_{n-k+1}, \dots, x_n)$$

converge. Thus, if we let χ_R be functions of the kind considered with the additional assumption that their support should exhaust $\pi^{-1}(L)$ as $R \rightarrow \infty$, we see that the weak limit of $\pi_*\chi_R S$ as $R \rightarrow \infty$ exists, and we put

$$\pi_* S = \lim_{R \rightarrow \infty} \pi_* \chi_R S.$$

It is easy to see that it does not depend on the choice of functions χ_R , and if π were to have proper support on $Supp(S)$, this definition would coincide with the previous one given above. We want to show that this current is closed and positive. For this we construct explicit choices of χ_R as follows:

Let $\tilde{\chi}_R : \mathbb{R}_+ \rightarrow \mathbb{R}$ be the piecewise linear function, equal to 1 on $[0, R]$, equal to 0 on $[2R, +\infty)$ and linear in between. Then $\tilde{\chi}'_R = R^{-1}$ on the interval $[R, 2R]$ and 0 otherwise. We now put $\chi_R(x) = \tilde{\chi}_R(|x|)$.

Proposition 4.6 *Let $\pi : \mathbb{R}^n \rightarrow L$ be a projection, where L is a $(n - k)$ -dimensional subspace of \mathbb{R}^n , and assume that S is a strongly positive, closed, (p, p) -current of finite degree. Then the push-forward, $\pi_* S$, is a well-defined positive, closed $(p - k, p - k)$ -current. If $p < k$ then $\pi_* S = 0$.*

Proof We have yet to show that it is closed and positive. But positivity is clear, since, if $\alpha \in \mathcal{D}^{n-p,0}$,

$$\langle (\pi_* S), \sigma_{(n-k)-(p-k)} \alpha \wedge J(\alpha) \rangle = \lim_{R \rightarrow \infty} \int_{\mathbb{E}} \chi_R S \wedge \sigma_{n-p} \cdot \pi^* \alpha \wedge \pi^* J(\alpha) \geq 0,$$

by the positivity of S . To prove closedness, we use the specific function χ_R constructed above. Then, for $\alpha \in \mathcal{D}^{n-p-1, n-p}(L)$,

$$\langle d(\pi_* S), \alpha \rangle = \pm \lim_{R \rightarrow \infty} \langle \chi_R \cdot S, \pi^*(d\alpha) \rangle = \lim_{R \rightarrow \infty} \langle d\chi_R \wedge S, \pi^* \alpha \rangle$$

thanks to S being closed. We need to show that $\lim_{R \rightarrow \infty} \langle d\chi_R \wedge S, \pi^* \alpha \rangle = 0$. To this end we define the following bi-linear form:

$$(v, w) = \int_{\mathbb{E}} S \wedge \sigma_{n-p} v \wedge J(w),$$

where v, w are compactly supported, smooth $(n - p, 0)$ -forms on \mathbb{R}^n . Clearly $(v, v) \geq 0$, since S is positive, and thus the bi-linear form is positive definite. A variant of the Cauchy-Schwartz inequality tells us that for each $\epsilon > 0$,

$$(v, w) \leq \epsilon(v, v) + \epsilon^{-1}(w, w). \tag{4.7}$$

Let us define, for each $R > 0$,

$$\psi_R(t) = \begin{cases} \frac{2}{R}t - \frac{3}{2}, & \text{if } t \in [0, R) \\ \frac{t^2}{2R^2}, & \text{if } t \in [R, 2R] \\ \frac{4}{R}t - 6, & \text{if } t \in (2R, \infty) \end{cases}.$$

Then ψ_R is convex, $\psi_R(|x|) \in \mathcal{L}$, and $\psi''_R(t) = 1/R^2$ if $t \in [R, 2R]$ and 0 otherwise. Moreover, a direct calculation shows that

$$d d^\# \psi_R - d\chi_R \wedge d^\# \chi_R \geq 0. \tag{4.8}$$

Let v be a $(n - p - 1, 0)$ -form, and w a $(0, n - p)$ -form defined on \mathbb{R}^{n-k} , both smooth and with compact support. Then

$$|\langle d\chi_R \wedge S, \pi^*v \wedge \pi^*w \rangle| = \left| \int_{\mathbb{E}} S \wedge d\chi_R \wedge \pi^*v \wedge \pi^*w \right| = |\langle d\chi_R \wedge \pi^*v, \pi^*w \rangle| \tag{4.9}$$

which by (4.7) is dominated by

$$\epsilon \int_{\mathbb{E}} S \wedge \pi^*w \wedge J(\pi^*w) + \epsilon^{-1} \int_{\mathbb{E}} S \wedge d\chi_R \wedge \pi^*v \wedge J(d\chi_R \wedge \pi^*v) = I + II.$$

Here the form π^*w will not have compact support on \mathbb{R}^n so the first term actually has no meaning. However, we may here replace π^*w with $\chi_{3R}\pi^*w$ which has compact support on \mathbb{R}^n ; doing so will not affect (4.9), since $\chi_{3R} = 1$ on $Supp(d\chi_R)$. Using Lemma 4.1 we see that the first term, I , is bounded by ϵC_w . For the second term, II , we show that the trace measure of the strongly positive current, $S \wedge d\chi_R \wedge J(d\chi_R)$ tends to 0 as $R \rightarrow \infty$: For each multi-index I of length $n - p$, we can use the idea of Lemma 4.1 to find a function $\phi_I \in \mathcal{L}$, such that,

$$dd^\# \phi_I = dd^\# \sqrt{x_{i_1}^2 + \dots + x_{i_{n-p}}^2} = dx_I \wedge d\xi_I,$$

on $B(0, 2R)$. Thus, since $S \wedge d\chi_R \wedge J(d\chi_R)$ is strongly positive,

$$\left| \int_{B(0,2R) \times \mathbb{R}^n} S \wedge d\chi_R \wedge J(d\chi_R) \wedge dx_I \wedge d\xi_I \right| \leq \left| \int_{\mathbb{R}^n \times \mathbb{R}^n} S \wedge d\chi_R \wedge J(d\chi_R) \wedge dd^\# \phi_I \right|.$$

By (4.8) the last integral is dominated by

$$\left| \int_{\mathbb{E}} S \wedge dd^\# \psi_R \wedge dd^\# \phi_I \right|.$$

Thus, since $dd^\# \psi_R(|x|) = R^{-2}dd^\#|x|$ on $B(0, R)$ and zero otherwise, we obtain by Proposition 3.10 that

$$\epsilon^{-1} \left| \int_{B(0,R) \times \mathbb{R}^n} S \wedge d\chi_R \wedge J(d\chi_R) \wedge dx_I \wedge d\xi_I \right| \leq \epsilon^{-1} R^{-2} \int_{\mathbb{E}} S \wedge dd^\#|x| \wedge dd^\# \phi_I \leq \epsilon^{-1} R^{-2} D,$$

for some constant $D > 0$ independent of R . Thus the trace measure of the positive current $S \wedge d\chi_R \wedge J(d\chi_R)$ tends to 0 with R . Thus we find that the second term, II , tends to 0 as $R \rightarrow \infty$. In conclusion, we see that

$$\lim_{R \rightarrow \infty} \langle d\chi_R \wedge S, \pi^*(v \wedge w) \rangle = 0,$$

for every pair of forms v and w as above. Since every $(n - p - 1, n - p)$ -form α can be written as a linear combination of forms of the type $v \wedge w$ as above, this implies that

$$\lim_{R \rightarrow \infty} \langle d\chi_R \wedge S, \pi^*(\alpha) \rangle = 0.$$

By definition, this mean precisely that $d(\pi_*S) = 0$, as desired.

Corollary 4.2 *If $f_1, \dots, f_p \in \mathcal{L}$, then $\pi_*(dd^\# f_1 \wedge \dots \wedge dd^\# f_p)$ is a positive and closed $(p - k, p - k)$ -current.*

Proof The current $dd^\# f_1 \wedge \dots \wedge dd^\# f_p$ has finite degree by Proposition (3.9), and thus satisfies the hypothesis of the Proposition.

5 Tropical geometry

For a finite set A in \mathbb{Z}^n we let $P = \text{conv}(A)$, the convex hull in \mathbb{R}^n of the set A .

Definition 5.1 A tropical polynomial is a function $f(x) = \max_{\alpha \in A} (-v(\alpha) + \alpha \cdot x)$, where $v : A \rightarrow \mathbb{R}$ is some arbitrary function. For a tropical polynomial f , we define the associated tropical hypersurface, which we will denote V_f , as the set where f is not smooth.

Observe that a tropical polynomial is a convex function. Moreover, since the maximum of a finite number of affine functions is piecewise affine, we see that V_f is the set where f is not affine. This set coincides with the set where two or more of the elements which we take the maximum over obtain the maximum value at the same time. It is easy to realize that V_f thus consist of finitely many affine hyperplanes (or rather convex polyhedras), glued together at $(n - 2)$ -dimensional affine manifolds of \mathbb{R}^n . Now, let us extend the function v to all of \mathbb{R}^n by letting

$$v_\infty(x) = \begin{cases} v(x), & x \in A \\ \infty, & x \notin A \end{cases}$$

The tropical polynomial f then coincides with the Legendre transform of v_∞ . It is a well known fact of convex analysis that applying the Legendre transform twice to any function $g : \mathbb{R}^n \rightarrow \mathbb{R}$ will produce the largest convex function which is smaller than g at any point. Thus, applying the Legendre transform to the tropical polynomial f gives us the largest convex function on \mathbb{R}^n which, when restricted to A , is less than or equal to v . We will denote this function by \tilde{v} . It is not hard to realize that \tilde{v} is piecewise affine on P and equal to $+\infty$ on $\mathbb{R}^n \setminus P$. Let us assume for the time being that the dimension $n = 2$. Then the set $\Gamma \subset P$ defined as the set where the function \tilde{v} is singular, is a graph which is dual to the tropical line V_f in the following sense (cf. [15]): each edge of Γ is perpendicular to an edge of V_f and vice versa. One calls the graph Γ a *convex triangulation* of the polytope P . Similar statements hold in higher dimensions as well.

We can associate weights, normal vectors and primitive integer vectors to the facets of V_f in the following way: consider an $(n - 1)$ -dimensional facet V of V_f . The set V is the set where precisely two of the affine functions competing for the maximum in the tropical polynomial f , say $-v(\alpha_1) + \alpha_1 \cdot x$ and $-v(\alpha_2) + \alpha_2 \cdot x$, are equal and realize the maximum. The facet V has two natural normal vectors defined from the data given, namely $\alpha_1 - \alpha_2$ and $\alpha_2 - \alpha_1$. We pick one of these two, and call it the normal vector v associated to V . Note that a choice of normal vector v induces an orientation on V compatible with any fixed choice of orientation on \mathbb{R}^n . We will only be interested in the pair (V, v) where we take the orientation of V into account, and consequently it does not matter which vector we chose above to be the normal vector associated to V : If we instead had chosen $-v$, the orientation of V would have been reversed. The weight w is the absolute value of the greatest common divisor of the numbers v^1, \dots, v^n , where v^j denotes the j :th component of the vector v . We let N denote the primitive integer vector associated to $v = \alpha_1 - \alpha_2$, i.e., the vector in \mathbb{Z}^n such that $wN = v$.

A tropical hypersurface V_f thus consists of a finite number of convex polyhedras of dimension $(n - 1)$, call them V_1, \dots, V_s , which are glued together along convex polyhedras of dimension $(n - 2)$, which we denote by W_1, \dots, W_r . Now, assume that W_1 is the locus of intersection of V_1, \dots, V_k , and the sign of the corresponding normal vector v_i has been chosen such that each V_i induces the same orientation on W_1 . We explain this last condition in detail. Fix an orientation of W_1 and let $\{e_1, \dots, e_{n-2}\}$ be a basis of W_1 , compatible with the orientation chosen. Fix also one of the convex polyhedras adjacent to W_1 , say V_1 . Then there exists a unique unit normal of W_1 pointing into V_1 , which we denote by w_1 , and for which $\{w_1, e_1, \dots, e_{n-2}\}$ is a basis for V_1 . We choose the sign of v_1 so that $\{v_1, w_1, e_1, \dots, e_{n-2}\}$ is a basis compatible with the fixed orientation of \mathbb{R}^n . Under these circumstances one can show:

Proposition 5.1 (Balancing property of tropical varieties, cf. [9]) *With the above hypothesis the balancing condition holds around W :*

$$\sum_{i=1}^k v_i = 0.$$

Let us study an example as to see how tropical polynomials may arise in practice.

Consider a complex algebraic hypersurface $\{h = 0\}$ in $(\mathbb{C} \setminus \{0\})^n$, where h is a Laurent polynomial $h = \sum_{\alpha \in A} c_\alpha z^\alpha$, where multi-index notation is used. Let $P = \text{conv}(A)$ be the *Newton polytope* associated with f . We consider the function $\text{Log} : (\mathbb{C} \setminus \{0\})^n \rightarrow \mathbb{R}^n$, given by $\text{Log}(z_1, \dots, z_n) = (-\log |z_1|, \dots, -\log |z_n|)$, and define $\mathcal{A}_h = \text{Log}(\{h = 0\})$. This set $\mathcal{A}_h \subset \mathbb{R}^n$ is called the *amoeba* of the polynomial h . Tropical pictures arise when we start deforming the amoeba, and shrink its width to 0. To make this precise, we let for $t > 0$, $\log_t(x) = \log(x)/\log(t)$, and define Log_t by exchanging \log for \log_t in the definition of Log . Also we define $\mathcal{A}_h^t = \text{Log}_t(\{h = 0\})$. Then as $t \rightarrow 0$, the sets \mathcal{A}_h^t converges to a set in the Hausdorff topology (cf. [9]), which we denote by \mathcal{S}_h . This set \mathcal{S}_h can actually be seen to be piecewise affine, and all its pieces have rational slope. Thus, it constitutes a tropical variety.

Example 5.1 Let us consider the two dimensional case. We choose the polynomial h to be $1 - z^2 + w$ where (z, w) are coordinates for \mathbb{C}^2 . Its Newton polytope is then the triangle with vertices at the points $(0, 0)$, $(2, 0)$, and $(0, 1)$. We want to consider the image of $\{h = 0\} \subset (\mathbb{C} \setminus \{0\})^2$, where we denote the coordinates on \mathbb{R}^2 with (x, y) , under the map Log_t . Of course, for a fixed point $(z_0, w_0) \in \{h = 0\} \cap (\mathbb{C} \setminus \{0\})^2$, we have

$$\lim_{t \rightarrow 0} \text{Log}_t(z_0, w_0) = \lim_{t \rightarrow 0} \left(-\frac{\log |z_0|}{\log t}, -\frac{\log |w_0|}{\log t} \right) = (0, 0).$$

However, if we were allowed to insert the point $(1, 0)$ (note that while $h(1, 0) = 0$, the point does not belong to $(\mathbb{C} \setminus \{0\})^2$, we would obtain

$$\lim_{t \rightarrow 0} \text{Log}_t(1, 0) = (0, -\infty).$$

Since we can find points in $\{h = 0\} \cap (\mathbb{C} \setminus \{0\})^2$ arbitrarily close to $(1, 0)$, we see that the limit set, \mathcal{S}_h , will contain the ray starting at $(0, 0)$ and “ending in” $(0, -\infty)$. Similar behaviour will be observed for every point in $\{h = 0\}$ (as a subset of \mathbb{C}^2) where at least one of the coordinates is 0. Consequently, we start searching for those. If $w = 0$ then $z = \pm 1$, if we are to have $(z, w) \in \{h = 0\}$. This point will give rise, under the Log -map, to the ray along the y -axis starting at $(0, 0)$ and ending in $(0, -\infty)$. Similarly, if $z = 0$ then $w = -1$, and this point will give rise to the ray along the x -axis, starting at $(0, 0)$ ending in $(-\infty, 0)$. Also,

for z and w large, and $(z, w) \in \{h = 0\}$, we have that $\log |w| \approx 2 \log |z|$, that is $y \approx 2x$. Thus the amoeba will have three asymptotic lines, namely the sets $(\infty, 0] \times \{0\}$, $\{0\} \times (\infty, 0]$ and $\{y = 2x\} \cap \{x \geq 0\}$. Moreover, one can show that each component of the complement of the amoeba is convex (cf. [11]). It is not hard to realize that if we consider the limit of A_f^t as t gets closer and closer to 0, the picture is that the “deformed” amoeba converges to exactly the asymptotic lines we have found, and we obtain the tropical curve given by the tropical polynomial $\max\{0, y, 2x\}$. At this point we should also note that each of the directional vectors for the lines are in fact normal vectors to the Newton polytope. As in the above discussion, we say the S_h is dual to the polytope P .

Tropical geometry can also be seen as algebraic geometry over a non-archimedean field, \mathbb{K} . The attribute non-archimedean means that the field has a norm which satisfy a stronger condition than the triangle inequality, namely that

$$|x + y| \leq \max\{|x|, |y|\}.$$

Here we let \mathbb{K} be the field of Puiseux series, namely the set of all formal power series $\sum_{q \in \mathbb{Q}} a_q t^q$, where we demand that the set of all q such that $a_q \neq 0$ is bounded from below. We can equip \mathbb{K} with a valuation map $v : \mathbb{K} \rightarrow \mathbb{R}$, by demanding that $v(\sum_{q \in \mathbb{Q}} a_q t^q)$ is the infimum of all q such that $a_q \neq 0$. Let us now consider the polynomial ring $\mathbb{K}[z_1, \dots, z_n]$, and an element in it, G . Thus $G = \sum_{\alpha_i \in A} r_{\alpha_i} z^{\alpha_i}$, for A some finite subset of \mathbb{Z}^n , and $r_{\alpha_i} \in \mathbb{K}$. To this G we associate its *tropicalization*

$$trop(G)(x) = \max_{\alpha_i \in A} \{\alpha_i \cdot x - v(r_{\alpha_i})\},$$

where $x \in \mathbb{R}^n$, and $\alpha \cdot x$ is the scalar product between x and α . For instance, $trop(t^{-2}zw + tw^2)(x, y) = \max\{-2 + x + y, 1 + 2y\}$. Similar as for the function *Log* defined above, we put $Val(z_1, \dots, z_n) = (val(z_1), \dots, val(z_n))$. We now come to an important point: One can show ([5]) that the closure of the set $Val(\{G = 0\})$ is equal to the set in \mathbb{R}^n where the maximum $tropG(x) = \max_{\alpha_i \in A} \{\alpha_i \cdot x - v(r_{\alpha_i})\}$ is obtained by two or more of the α_i . Thus the closure of the set $Val(\{G = 0\})$ defines a tropical variety. Of course, by letting the function v in Definition 5.1 be equal to $\alpha_i \mapsto v(r_{\alpha_i})$, the tropical polynomial will be just $trop(G)$, and so we could equally well take the following as a definition of a tropical variety:

Proposition 5.2 ([5]) *A tropical variety is given by the closure of $Val(\{G = 0\})$ in \mathbb{R}^n , where $G \in \mathbb{K}[z_1, \dots, z_n]$.*

Remark 5.1 Let $h = \sum_{\alpha \in A} c_{\alpha} z^{\alpha}$ be a complex polynomial with $h(0) = 0$, where A is a finite subset of \mathbb{Z}^n . In the complex setting, there is a generalization of the Lelong number, called Kiselman’s directed Lelong number, denoted $\gamma_{z, \varphi}(x)$, which gives more precise information concerning the singularities of φ . It depends on three parameters: a plurisubharmonic function φ on \mathbb{C}^n , a point $z \in \mathbb{C}^n$, and a vector in $x \in \mathbb{R}_+^n$, and if $x = (1, \dots, 1)$ it reduces to the ordinary (complex) Lelong number. It is well known that $x \mapsto -\gamma_{z, \varphi}(x)$ is a convex function, for $x \in \mathbb{R}_+^n$. For our discussion, it suffices to say that, with $\varphi(z) = \log(h(z))$,

$$\gamma_{0, \varphi}(x) = \min_{\alpha \in A} \{\alpha \cdot x\}.$$

This is closely related to choosing $\mathbb{K} = \mathbb{C}$ and endowing \mathbb{K} with the trivial valuation $v(w) = 0$ for every $w \in \mathbb{C}$. Indeed, then we would have

$$trop(h)(-x) = \max_{\alpha \in A} \{-\alpha \cdot x\} = -\gamma_{0, \log(h(z))}(x).$$

For ν a valuation on \mathbb{K} , we put

$$\text{trop}_\nu(\log(h))(x) = \max_{\alpha_i \in A} \{\alpha_i \cdot x - \nu(r_{\alpha_i})\}.$$

Thus, if we restrict ourselves to complex polynomials, trop_ν could be considered as a generalization of Kiselman’s Lelong number. It would be interesting to know if one could extend trop_ν to act on arbitrary plurisubharmonic functions.

5.1 Tropical geometry and super currents

Since a tropical polynomial f is the maximum of a finite number of \mathbb{Z} -affine functions on \mathbb{V} and since $dd^\# f = 0$ at points where f is affine, we must have that $\text{Supp}(dd^\# f) = V_f$. Thus the support of the current $dd^\# f$ coincides as a point set with the tropical hypersurface V_f . We make the following definition.

Definition 5.2 The support of a current T is of dimension k if $\text{Supp}(T)$ is a piecewise smooth manifold of dimension k , that is, $\text{Supp}(T)$ consists of a finite number of smooth manifolds of dimension k , glued together along manifolds of lower dimension.

We can now prove the fundamental result of this paper.

Proposition 5.3 *There is a one to one correspondence between tropical hypersurfaces V_f , and closed, positive $(1,1)$ -currents T whose support is of dimension $n - 1$, and whose normal vectors (see below) are integral.*

Proof Let T be as in the hypothesis, and denote by A the support of T . We first show that A is a piecewise affine manifold. Fix a point $x \in A$, and a small ball B centered at x , such that $B \setminus A$ consists of precisely two components; call them C_1 and C_2 . By Proposition 2.7 both C_1 and C_2 are convex, which implies that each $(n - 1)$ -dimensional piece of A must be affine. Thus A is piecewise affine. Now, by Proposition 2.6, we can find a convex function f such that $T = dd^\# f$. Let us denote by V_1, \dots, V_N the $(n - 1)$ -dimensional (affine) pieces of A . For each $i = 1, \dots, N$, there is a vector $v_i \in \mathbb{V}$ and a real number c_i such that, for $x \in V_i$,

$$f(x) = -c_i + v_i \cdot x.$$

The convexity of f implies that, in fact,

$$f(x) = \max_{i=1, \dots, N} (-c_i + v_i \cdot x),$$

for $x \in \mathbb{V}$. The condition that the currents *normal vectors* are integral, means that each vector v_i belongs to \mathbb{Z}^n (under the identification of \mathbb{V} with \mathbb{R}^n). If this is the case, then f is actually a tropical polynomial, and thus we can conclude that $\text{Supp}(T)$ coincides with a tropical hypersurface. This establishes one part of the correspondence. The second part is easier: for each tropical hypersurface V_f we let $T = dd^\# f$; then T satisfies the hypothesis of the proposition.

The arguments used above immediately give:

Corollary 5.1 *If the support $\text{Supp}(T)$ is of dimension $n - 1$, for a closed, positive $(1, 1)$ -current T , then $\text{Supp}(T)$ is piecewise affine.*

By the previous discussion, we know that $\text{Supp}(T) = V_f$ consists of a finite number of convex polyhedras V_i , whose affine hull is of dimension $n - 1$, glued together at affine convex polyhedras W_k of dimension $n - 2$. Let us recall the discussion before Proposition 5.1:

each facet V_i is the set where, for some $\alpha_1, \alpha_2 \in A$,

$$-c_{\alpha_1} + v_{\alpha_1} \cdot x = -c_{\alpha_2} + v_{\alpha_2} \cdot x$$

attains the maximum defining the tropical polynomial f , and we defined the normal vector of V_i to be, up to sign, equal to $v_i = \alpha_1 - \alpha_2$. Thus v_i is a normal vector to V_i whose length is determined by the tropical polynomial. We make the following definition:

Definition 5.3 The normal 1-form associated to V_i is defined as $v_i^* = d(v_i \cdot x)$.

Let V be a hyperplane in \mathbb{R}^n with normal v , and let δ_V denote the surface measure of V . We will consider the $(1, 1)$ -current

$$\frac{1}{|v|} \delta_V v^* \wedge J(v^*),$$

whose action on an $(n - 1, n - 1)$ -form α is defined by

$$\left\langle \frac{1}{|v|} \delta_V v^* \wedge J(v^*), \alpha \right\rangle = \frac{1}{|v|} \int_V \frac{v^* \wedge J(v^*) \wedge \alpha}{\omega_n} \delta_V.$$

Observe that this current does not depend on which sign we have chosen for the normal vector v . This current represents a tropical variety:

Proposition 5.4 Let $V \subset \mathbb{R}^n$ be a hyperplane, determined by a normal vector $v = (v_1, \dots, v_n)$, and define $f = \max(0, v \cdot x)$. Then

$$dd^\# f = \frac{1}{|v|} \delta_V v^* \wedge J(v^*).$$

Moreover, we have the following equality

$$[V] \wedge J(v^*) = \frac{1}{|v|} \delta_V v^* \wedge J(v^*), \tag{5.1}$$

where $[V]$ is the current of integration on V , with orientation determined by v .

Proof We prove Eq. (5.1) first. We have the following equality of currents on \mathbb{R}^n :

$$\frac{1}{|v|} \delta_V v^* = [V], \tag{5.2}$$

where $[V]$ is the current of integration of V , defined in a natural way as

$$\langle [V], \alpha \rangle = \int_V \alpha,$$

for α a compactly supported $\dim V$ -form on \mathbb{R}^n . To prove this we extend $\{v\}$ to an orthonormal basis of \mathbb{R}^n , compatible with the orientation chosen, which we denote $\{|v|^{-1}v, e_1, \dots, e_{n-1}\}$. For simplicity, we use the notation $e^* = e_1^* \wedge \dots \wedge e_{n-1}^*$. Then we need only to prove the formula for forms of the type $\alpha_0 e^*$, where α_0 is a function. But, since $\delta_V = e^*$, and $v^* \wedge e^* = |v| dx_1 \wedge \dots \wedge dx_n$, we see that

$$\left\langle \frac{\delta_V}{|v|} v^*, \alpha_0 e^* \right\rangle = \langle \delta_V, \alpha_0 |v|^{-1} v^* \wedge e^* \rangle = \int_V \alpha_0 e^*.$$

On the other hand,

$$\langle [V], \alpha_0 e^* \rangle = \int_V \alpha_0 e^*,$$

which proves the formula (5.2). Thus, we see that

$$\frac{1}{|v|} \delta_V v^* \wedge J(v^*) = [V] \wedge J(v^*).$$

We now proceed to prove the first formula of the proposition. Recall that if $P' \subset V$ is a submanifold of the same dimension as that of V , and with piecewise smooth boundary, then the current $T := [P'] \wedge J(v^*)$ satisfies

$$\langle dT, \alpha \rangle = \langle [\partial P'] \wedge J(v^*), \alpha \rangle,$$

which follows from Stokes' theorem (2.3).

We begin by considering the function $f = \max \{0, x_n\}$. To compute $dd^\# f$ we choose for $\epsilon > 0$, a family of smooth, one-variable functions g_ϵ satisfying $\lim_{\epsilon \rightarrow 0} g_\epsilon(t) = \max \{0, t\}$, and $\lim_{\epsilon \rightarrow 0} g''_\epsilon = \delta_0$ (the Dirac measure at 0). For such a family g_ϵ , we put

$$f_\epsilon(x_1, \dots, x_n) = g_\epsilon(x_n).$$

Then, for each $\alpha \in \mathcal{E}^{n-1, n-1}$,

$$\begin{aligned} \langle dd^\# f_\epsilon, \alpha \rangle &= \left\langle f''_\epsilon dx_n \wedge d\xi_n, \alpha \right\rangle = \left\langle f''_\epsilon dx_n \wedge d\xi_n, \alpha_{nn} \widehat{dx_n} \wedge \widehat{d\xi_n} \right\rangle \\ &= \int_{\mathbb{R}^n} f''_\epsilon \alpha_{nn} dx_1 \wedge \dots \wedge dx_n, \end{aligned}$$

where α_{nn} is the coefficient in front of $\widehat{dx_n} \wedge \widehat{d\xi_n}$ in the sum defining α . Thus we see that

$$\lim_{\epsilon \rightarrow 0} \langle dd^\# f_\epsilon, \alpha \rangle = \int_{\mathbb{R}^{n-1}} \alpha_{nn}(x_1, \dots, x_{n-1}, 0) dx_1 \wedge \dots \wedge dx_{n-1}, \tag{5.3}$$

which is the same as saying

$$dd^\# f = dd^\# \max \{0, x_n\} = [\{x_n = 0\}] \wedge d\xi_n.$$

Now, let's turn to the general case: as above, let $\{v/|v|, e_1, \dots, e_{n-1}\}$ be an orthonormal basis, and let F correspond to the matrix $(v/|v|, e_1, \dots, e_{n-1})$ in the standard basis of \mathbb{R}^n . We again use the notation $e^* = e_1^* \wedge \dots \wedge e_{n-1}^*$. Then, if we consider the action of $dd^\# f$ on the form $\alpha e^* \wedge J(e^*)$, since $\det F = 1$ we get by (4.4) and the discussion above, that

$$\begin{aligned} \langle dd^\# \max(0, v \cdot x), \alpha e^* \wedge J(e^*) \rangle &= \langle dd^\# \max(0, |v|x_n), F^*(\alpha e^* \wedge J(e^*)) \rangle \\ &= |v| \int_{\{x_n=0\}} F^*(\alpha e^*) = |v| \int_{\{v \cdot x=0\}} \alpha e^*. \end{aligned}$$

On the other hand,

$$\langle [\{v \cdot x = 0\}] \wedge J(v^*), \alpha e^* \wedge J(e^*) \rangle = |v| \int_{\{v \cdot x=0\}} \alpha e^*,$$

where we used that $\int_{\mathbb{W}} J(v^*) \wedge J(e^*) = |v|$. Since we need only to consider forms that are multiples of $e^* \wedge J(e^*)$, we have proved that

$$[\{v \cdot x = 0\}] \wedge J(v^*) = dd^\# \max(0, v \cdot x),$$

which is what we aimed for.

For a tropical hypersurface V_f consisting of $(n - 1)$ -dimensional convex polyhedras V_i as discussed before, we consider the current defined as $T' = \sum_{i=1}^N [V_i] \wedge J(v_i^*)$, and let $T = dd^\# f$. The previous proposition shows that $Supp(T - T')$ is of dimension at most $n - 2$. Moreover, $T - T'$ is closed. These hypotheses actually implies that $T - T' = 0$ as follows from the following lemma.

Lemma 5.1 *Let S be a closed (p, q) -current whose coefficients are measures and whose support is a piecewise affine manifold $M \subset \mathbb{R}^n$ of co-dimension $p + 1$. Then $S = 0$.*

Proof Let $x \in M$. Assume that, for a small neighbourhood U of x , we can choose coordinates so that $M \cap U = \{x_1 = x_2 = \dots = x_{p+1} = 0\}$. Since S has measure coefficients, it is easy to see that

$$x_1 S = x_2 S = \dots = x_p S = 0$$

on U . It follows that $d(x_1 S) = dx_1 \wedge S = 0$, thanks to S being closed. Thus, we can write $S = S' \wedge dx_1$ for a d -closed $(p - 1, q)$ -current S' . By the same means $x_2 S' = 0$ from which we see that $dx_2 \wedge S' = 0$ and so $S' = S'' \wedge dx_2$ for some $(p - 2, q)$ -current S'' . Repeating the argument, we eventually find that there is a $(0, q)$ -current S''' such that $S = S''' \wedge dx_1 \wedge \dots \wedge dx_p$. As before, this S''' satisfies the equation $S''' x_{p+1} = 0$, which implies $S''' \wedge dx_{p+1} + dS''' \cdot x_{p+1} = 0$. Thus $S''' \wedge dx_{p+1} = 0$ on M , and since $S''' = \sum_{|J|=q} S_J d\xi_J$ for some measures S_J , we see that S''' , and hence S , vanish on U . Thus we have shown that S carries no mass on the pieces of $Supp(S)$ which are of pure co-dimension $p + 1$. Thus S has support that is a piecewise affine manifold of co-dimension $p + 2$. Iterating the procedure above gives the desired result.

Concluding the discussion before the lemma, we obtain the following result:

Proposition 5.5 *With the notation above, $dd^\# f = \sum [V_i] \wedge J(v_i^*)$.*

We can also show the following proposition, shedding more light on the connection between currents and tropical hypersurfaces.

Proposition 5.6 *Let $T = \sum [V_i] \wedge J(v_i^*)$ be a tropical hypersurface. Then the condition $dT = 0$ is equivalent to the balancing condition (cf. Proposition 5.1): for each $(n - 2)$ -dimensional affine manifold W defined as the locus where m hyperplanes V_1, \dots, V_m meet, we have that $\sum_1^m v_i = 0$.*

Proof For $1 \leq j \leq n$, we let v_i^j denote the j :th component of the vector v_i . Each V_i has a boundary built up from a number of $(n - 2)$ dimensional pieces which we call P_i^r . For a fixed W where V_1, \dots, V_m meet, we then have finitely many P_i^r , say P_1^1, \dots, P_k^1 coinciding with W as sets. Fix a $(n - 2, n - 1)$ -form $\alpha = \sum_{I, j} \alpha_{I, j} dx_I \wedge d\hat{\xi}_j$ with compact support in a small neighbourhood of a point on W . We can choose the support so small that W is the only part of co-dimension two of V_f that lies in $Supp(\alpha)$. Then, by Stokes' theorem,

$$\langle dT, \alpha \rangle = \left\langle \sum_{i=1}^m [\partial V_i] \wedge J(v_i^*), \alpha \right\rangle = \left\langle \sum_{i=1}^m [P_i^1] \wedge J(v_i^*), \alpha \right\rangle.$$

Thus, since $J(v_i^*) \wedge \widehat{d\xi_j} = v_i^j d\xi_1 \wedge \cdots \wedge d\xi_n$,

$$\langle dT, \alpha \rangle = \sum_{i=1}^m \left\langle [P_i^1] \wedge J(v_i^*), \sum_{I,j} \alpha_{I,\hat{j}} dx_I \wedge \widehat{d\xi_j} \right\rangle = \pm \sum_{j,I} \left(\sum_{i=1}^n v_i^j \right) \int_W \alpha_{I,\hat{j}} dx_I.$$

Thus, if $\sum_{i=1}^m v_i = 0$, this last sum is 0, whence $\langle dT, \alpha \rangle = 0$. Since we can do this for every W , we see that $\dim(\text{Supp}(dT)) \leq n - 3$ and since dT has measure coefficients, Lemma (5.1) implies that $dT = 0$. Conversely, if $\langle dT, \alpha \rangle = 0$, then

$$\sum_{j,I} \left(\sum_{i=1}^n v_i^j \right) \int_W \alpha_{I,\hat{j}} dx_I = 0$$

and so, to see that the balancing property holds, it suffices to choose, for every fixed I_0, j_0 , a form α such that $\int_W \alpha_{I_0,\hat{j}_0} dx_{I_0} = 1$ and $\int_W \alpha_{I,\hat{j}} = 0$ for $I \neq I_0, J \neq J_0$.

Example 5.2 For the tropical hyperplane corresponding to the polynomial $f = \max \{0, v \cdot x\}$ we have

$$v_0(V_f) = v_0(dd^\# f) = |v|.$$

To see this, let V be the singularity locus of f . Then, since $T := dd^\# f = [V] \wedge J(v^*)$, we have

$$\Theta_T(B(0, r)) = \frac{1}{2^{n-1}(n-1)!} \sum_{i=1}^n v_i \int_{\{B(0,r) \cap V\} \times \mathbb{R}^n} d\xi_i \wedge (dd^\# |x|^2)^{n-1}$$

and it easy to see that,

$$\frac{1}{2^{n-1}(n-1)!} \int_{\{B(0,r) \cap V\} \times \mathbb{R}^n} d\xi_i \wedge (dd^\# |x|^2)^{n-1} = \frac{v_i}{|v|} \text{Vol}(B^{n-1}) r^{n-1}.$$

Thus

$$\Theta_T(B(0, r)) = \text{Vol}(B^{n-1}) |v| r^{n-1}.$$

and so

$$v_x(dd^\# f) = \lim_{r \rightarrow 0} \frac{\Theta_T(B(x, r))}{r^{n-1}} = |v|,$$

as claimed.

An easy adaptation of this example shows the following:

Proposition 5.7 *Let $T = \sum [V_i] \wedge J(v_i^*)$ and let x be a point where a finite number of the polyhedra V_i meet at a convex polyhedron of co-dimension 2. Assume, after reordering, that they are V_1, \dots, V_m . Then*

$$v_x(T) = \frac{1}{2} \sum_{i=1}^m |v_i|.$$

5.2 Intersection theory and tropical varieties of higher co-dimension

Let f_1, \dots, f_p be tropical polynomials with corresponding tropical hypersurfaces V_{f_1}, \dots, V_{f_p} .

Definition 5.4 The intersection of V_{f_1}, \dots, V_{f_p} is defined as the strongly positive (p, p) -current

$$V_{f_1} \wedge \dots \wedge V_{f_p} := dd^\# f_1 \wedge \dots \wedge dd^\# f_p.$$

By Proposition 3.6 the intersection is stable in the following sense: if $V_\epsilon \rightarrow V$, then

$$V_\epsilon \wedge V_1 \wedge \dots \wedge V_p \rightarrow V \wedge V_1 \wedge \dots \wedge V_p,$$

as $\epsilon \rightarrow 0$. However, it is important to realize that this does not imply that the support of $V_\epsilon \wedge V_1 \wedge \dots \wedge V_p$ tend to the support of $V \wedge V_1 \wedge \dots \wedge V_p$ in the Hausdorff topology (see Example 5.3). We proceed by investigating properties of intersections of tropical hypersurfaces.

Proposition 5.8 Let v_i be linearly independent vectors in \mathbb{R}^n , for $1 \leq i \leq n$, and define $f_i(x) = \max\{0, v_i \cdot x\}$, where $x \in \mathbb{R}^n$. Then

$$dd^\# f_1 \wedge \dots \wedge dd^\# f_n = c\delta_0$$

where

$$c = |\det(v_1, \dots, v_n)|.$$

Proof Let F be the linear map corresponding to the inverse of the matrix (v_1, \dots, v_n) , so that $|\det F| = c^{-1}$. Then, $f_i \circ F = \max\{0, x_i\}$, and consequently

$$F^* dd^\# f_i = dd^\# \max\{0, x_i\}.$$

Using formula (4.4) we see that for a compactly supported, smooth function g ,

$$\begin{aligned} \langle dd^\# f_1 \wedge \dots \wedge dd^\# f_n, g \rangle &= c \langle F^*(dd^\# f_1 \wedge \dots \wedge dd^\# f_n), F^* g \rangle \\ &= c \langle dd^\# \max\{0, x_1\} \wedge \dots \wedge dd^\# \max\{0, x_n\}, g \circ F \rangle. \end{aligned}$$

By Fubini's theorem and Eq. (5.3), the last line of the above equation is equal to $c \cdot (g \circ F)(0) = c \cdot g(0)$, which finishes the proof.

If V_i is a tropical hyperplane of the form

$$V_i = dd^\# f_i = [\{v_i \cdot x = 0\}] \wedge J(v_i^*)$$

for each $1 \leq i \leq q$, the same argument as in the above proof gives us that

$$\begin{aligned} V_1 \wedge \dots \wedge V_q &= dd^\# f_1 \wedge \dots \wedge dd^\# f_q = [\{v_1 \cdot x = 0\}] \cap \dots \cap [\{v_q \cdot x = 0\}] \\ &\quad \wedge J(v_1^*) \wedge \dots \wedge J(v_q^*). \end{aligned}$$

Notice that if two of the q hyperplanes are parallel, then the intersection vanishes. Thus, in this case the support, $Supp(V_1 \wedge \dots \wedge V_q)$, is either of dimension $n - q$ or empty. This property holds for general tropical hypersurfaces as well:

Proposition 5.9 Let V_{f_1}, \dots, V_{f_p} be tropical hypersurfaces such that $V_{f_1} \wedge \dots \wedge V_{f_p} \neq 0$. Then,

$$\dim(Supp(V_{f_1} \wedge \dots \wedge V_{f_p})) = n - p.$$

Proof By Proposition 5.5, each tropical hypersurface V can be written as

$$V = \sum_{i=1}^k [V_i] \wedge J(v_i^*),$$

where each V_i is a convex polyhedron of dimension $n - 1$. Each $[V_i]$ can be considered as $\chi_i[\tilde{V}_i]$ where \tilde{V}_i denotes the affine hull of V_i , and χ_i is the characteristic function

$$\chi_i(x) = \begin{cases} 1, & x \in V_i \\ 0, & x \notin V_i \end{cases}.$$

Define the $(1, 1)$ -current \tilde{V} by

$$\tilde{V} = \sum_{i=1}^k [\tilde{V}_i] \wedge J(v_i^*). \tag{5.4}$$

This current is positive and closed since each summand is, and satisfies the relation

$$Supp(V) \subset Supp(\tilde{V}).$$

Thus the inclusion

$$Supp(V_{f_1} \wedge \dots \wedge V_{f_p}) \subset Supp(\tilde{V}_{f_1} \wedge \dots \wedge \tilde{V}_{f_p})$$

holds, which implies, since $dim(Supp(\tilde{V}_{f_1} \wedge \dots \wedge \tilde{V}_{f_p})) = n - p$, that

$$dim(Supp(V_{f_1} \wedge \dots \wedge V_{f_p})) \leq n - p.$$

By the assumption that $V_{f_1} \wedge \dots \wedge V_{f_p} \neq 0$ we see that in fact equality must hold, since $dim(Supp(V_{f_1} \wedge \dots \wedge V_{f_p})) < n - p$ would force $V_{f_1} \wedge \dots \wedge V_{f_p} = 0$ by Corollary 4.1.

The following result can be compared with the case of positive $(1, 1)$ -currents in Proposition 2.7:

Proposition 5.10 *Let T be a strongly positive, closed (p, p) -current of finite degree (i.e. T satisfies (4.6)) such that $Supp(T)$ is a piecewise smooth manifold of dimension $n - p$. Then $Supp(T)$ is piecewise affine.*

Proof Let L be an affine subspace of \mathbb{R}^n of dimension $n - p + 1$ and let π_L denote the projection onto L . By Proposition 4.5, the push-forward $(\pi_L)_*T$ is a closed, positive $(1, 1)$ -current on $L \times L$. Since $Supp((\pi_L)_*T) \subset \pi_L(Supp(T))$, we see that for every subspace L as above, $Supp((\pi_L)_*T)$ is of at least co-dimension 1 in L . If the co-dimension is larger than 1, then $(\pi_L)_*T = 0$ by Lemma 5.1. On the other hand, if the co-dimension is 1 we obtain by Corollary 5.1, that $Supp((\pi_L)_*T)$ is piecewise affine. This implies that $Supp(T)$ is piecewise affine.

As an immediate corollary, we obtain:

Corollary 5.2 *With the notation of Proposition 5.9, $Supp(V_{f_1} \wedge \dots \wedge V_{f_p})$ is piecewise affine.*

Example 5.3 For a simple example, assume that we are in \mathbb{R}^2 and consider the intersection $dd^\# f \wedge dd^\# g$ where $f = \max(0, v_1x + v_2y)$, and $g = \max(0, w_1x + w_2y)$ where

$(v_1, v_2) \neq (w_1, w_2)$ are vectors in \mathbb{R}^2 . This corresponds to the intersection between the lines $\{v_1x + v_2y\}$ and $\{w_1x + w_2y = 0\}$. Indeed, by Proposition 5.8,

$$dd^\# f \wedge dd^\# g = \left| \det \begin{bmatrix} w_1 & v_1 \\ w_2 & v_2 \end{bmatrix} \right| \delta_0.$$

Thus the intersection of the two lines is the origin with intersection multiplicity determined by the volume of the parallelepiped spanned by the defining vectors of the two lines. As was noticed above, we see that if the two lines coincide, the intersection vanishes.

Let us take $(v_1, v_2) = (w_1, w_2) = (1, 0)$ with associated tropical lines V and W and let us perturb W slightly by considering the tropical line W_ϵ associated to $g_\epsilon(x) = \max\{(1 + \epsilon)x_1 + \epsilon x_2, 0\}$. Then

$$W_\epsilon \cdot V = \delta_0 \left| \det \begin{bmatrix} 1 & 1 + \epsilon \\ 0 & \epsilon \end{bmatrix} \right| = \delta_0 \cdot \epsilon,$$

and

$$W_\epsilon \cdot V \rightarrow W \cdot V,$$

as $\epsilon \rightarrow 0$. But, $Supp(W \cdot V) = \emptyset$ and $Supp(W_\epsilon \cdot V) = \{0\}$, which shows that the condition $W_\epsilon \cdot V \rightarrow W \cdot V$ does not imply that $Supp(W_\epsilon \cdot V) \rightarrow Supp(W \cdot V)$.

These results motivate the following definition:

Definition 5.5 A tropical variety of co-dimension p is the support of strongly positive, closed (p, p) -current T of finite degree, and where T satisfies:

- $\dim(Supp(T)) = n - p$, and
- each of the affine pieces of $Supp(T)$ have rational slope.

A piece has rational slope if its affine hull, which is a plane of co-dimension k , is the set where k linear forms with integer coefficients vanish.

Proposition 5.11 Let V_1, \dots, V_p be tropical hypersurfaces, and let $T = V_1 \wedge \dots \wedge V_p$. Then, if $T \neq 0$, then T is a tropical variety of co-dimension p .

Proof Indeed, by Proposition 5.9 the dimension of $Supp(T)$ is $n - p$, and by Corollary 5.2 we know that $Supp(T)$ is piecewise affine. Moreover, since

$$Supp(V_1 \cdot \dots \cdot V_p) \subset Supp(V_1) \cap \dots \cap Supp(V_p),$$

by Eq. (3.4), each piece has rational slope. Thus, T is a topological variety of co-dimension $n - p$.

The set theoretic intersection of two tropical hypersurfaces need not coincide with the support of a tropical variety. Indeed, if $f = \max\{0, x, y\}$ and $g = \max\{0, x - y\}$ then the set theoretic intersection of V_f and V_g is the half-ray $\{(x, y) : y = x, x \geq 0\}$ which is not a tropical variety (for instance, it does not satisfy the balancing property). However, $V_f \wedge V_g$ is equal to $\delta_0 \omega_n$, which is a tropical variety.

Remark 5.2 The intersection theory developed here seems to fit well with the intersection theory for tropical geometry considered in [10].

5.3 Bezout’s theorem

We use the ideas we have developed to prove known theorems within tropical geometry. Recall that we associated to an element $f \in \mathcal{L}$ the function

$$\tilde{f}(x) = \lim_{t \rightarrow \infty} \frac{f(tx)}{t},$$

and that by Proposition 3.13 we have the following relation between f and \tilde{f} :

$$\int_{\mathbb{E}} (dd^{\#} f)^n = \int_{\mathbb{E}} (dd^{\#} \tilde{f})^n. \tag{5.5}$$

Let us explore the effects of this result to tropical polynomials. For $f(x) = \max_{\alpha \in A} \{c_{\alpha} + \alpha \cdot x\}$ it is easy to see that $\tilde{f}(x) = \max_{\alpha \in A} \{\alpha \cdot x\}$. Relating to the discussion in the beginning of Sect. 5, this means that if f corresponds to a convex triangulation of $\text{conv}(A)$ then \tilde{f} corresponds to “forgetting” this triangulation. In fact, since

$$\max_{\alpha \in A} \{\alpha \cdot x\} = \max_{\alpha \in \text{conv}(A)} \{\alpha \cdot x\},$$

we see that \tilde{f} is just the support function of the set $\text{conv}(A)$.

Proposition 5.12 *Let f_1, \dots, f_n be tropical polynomials defined by*

$$f_i = \max_{\alpha \in A_i} \{c_{\alpha}^i + \alpha \cdot x\},$$

where each A_i is a finite set of point of \mathbb{Z}^n , and c_{α}^i are real numbers. Let $\tilde{A}_i = \text{conv}(A_i)$. Then \tilde{f}_i is the support function of \tilde{A}_i and

$$\int_{\mathbb{E}} dd^{\#} f_1 \wedge \dots \wedge dd^{\#} f_n = n! \cdot V(\tilde{A}_1, \dots, \tilde{A}_n). \tag{5.6}$$

Proof It is clear that, if $g = f_1 + \dots + f_n$, then $\tilde{g} = \tilde{f}_1 + \dots + \tilde{f}_n$. Thus, (5.5) implies that, for every $(t_1, \dots, t_n) \in \mathbb{R}^n$,

$$\int_{\mathbb{E}} \left(dd^{\#} \left(\sum_{j \in J} t_j f_j \right) \right)^n = \int_{\mathbb{E}} \left(dd^{\#} \left(\sum_{j \in J} t_j \tilde{f}_j \right) \right)^n$$

and so, by comparing coefficients, we see that

$$\int_{\mathbb{E}} dd^{\#} f_1 \wedge \dots \wedge dd^{\#} f_n = \int_{\mathbb{E}} dd^{\#} \tilde{f}_1 \wedge \dots \wedge dd^{\#} \tilde{f}_n.$$

Since \tilde{f}_i is the support function of the set \tilde{A}_i , we obtain from Proposition 3.12 that

$$\int_{\mathbb{E}} dd^{\#} f_1 \wedge \dots \wedge dd^{\#} f_n = n! \cdot V(\tilde{A}_1, \dots, \tilde{A}_n).$$

Remark 5.3 In the complex setting, there is an analogue of the associated function \tilde{f} , called the indicator associated to a plurisubharmonic function of logarithmic growth. See for instance the article [12], and also the article [13] where relations between mixed Monge-Ampère measures (of plurisubharmonic functions of logarithmic growth) and mixed volumes are obtained.

Let us apply the above Proposition to obtain, in an easy way, two results from tropical geometry. We stress that these are not new results. Let us assume for the moment that $n = 2$. A tropical curve associated to a tropical polynomial $f = \max_{\alpha \in A} \{c_\alpha + \alpha \cdot x\}$ is of degree d if $Newt(f)$ is equal to the set $\{(x, y) : x + y \leq d, x, y \geq 0\}$. The tropical version of the Bezout theorem in 2 dimensions is the following:

Theorem 5.1 *Consider two generic tropical curves C_1 and C_2 in \mathbb{R}^2 of degree d_1 and d_2 respectively. Then the number of intersection points counted with multiplicity is equal to $d_1 d_2$.*

Proof If the curve C_i corresponds to the tropical polynomial f_i then we know that the number of intersection between the curves is equal to

$$\int_{\mathbb{R}^2 \times \mathbb{R}^2} dd^\# f_1 \wedge dd^\# f_2.$$

By Proposition 5.12 this number is equal to $n! \cdot V(Newt(f_1), Newt(f_2))$. But, if we let $S_d = \{(x, y) : x + y \leq d, x, y \geq 0\}$, then

$$\begin{aligned} V(S_{d_1}, S_{d_2}) &= Vol(S_{d_1} + S_{d_2}) - Vol(S_{d_1}) - Vol(S_{d_2}) \\ &= 2^{-1}((d_1 + d_2)^2 - d_1^2 - d_2^2) = d_1 d_2, \end{aligned}$$

and we are done.

In general, we consider Eq. (5.6) to be the general version of Bezout’s theorem. As another application, we show that the Proposition implies an interesting interplay between tropical and ordinary polynomials (cf. [16]): We begin by recalling Bernstein’s theorem.

Theorem 5.2 *Let P_1, \dots, P_n be (generic) polynomials on $(\mathbb{C} \setminus \{0\})^n$. Then the number of solutions of the system $P_1 = \dots = P_n = 0$ is equal to*

$$n! \cdot V(Newt(P_1), \dots, Newt(P_n)),$$

where $Newt(P_i)$ is the Newton polytope of P_i .

For each polynomial P_i , we associate the function $\tilde{f}_i(x) = \sup_{\xi \in Newt(P_i)} \xi \cdot x$, that is, the support function of $Newt(P_i)$. Clearly, \tilde{f}_i belongs to \mathcal{L} and is a tropical polynomial. Then Proposition 5.12 combined with Bernstein’s theorem says the following: the number (counted with multiplicities) of intersection points of the tropical hypersurfaces associated to \tilde{f}_i is equal to the number of intersection points of the varieties $\{P_i = 0\} \subset (\mathbb{C} \setminus \{0\})^n$.

Acknowledgments I would like to express my gratitude towards my advisor, Bo Berndtsson, for his invaluable help with writing this article.

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