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Unstable directions and fractal dimension for skew products with overlaps in fibers

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Abstract A unique feature of smooth hyperbolic non-invertible maps is that of having different unstable directions corresponding to different prehistories of the same point. In this paper we construct a new class of examples of non-invertible hyperbolic skew products with thick fibers for which we prove that there exist uncountably many points in the locally maximal invariant set Λ (actually a Cantor set in each fiber), having different unstable directions corresponding to different prehistories; also we estimate the angle between such unstable directions. We discuss then the Hausdorff dimension of the fibers of Λ for these maps by employing the thickness of Cantor sets, the inverse pressure, and also by use of continuous bounds for the preimage counting function. We prove that in certain examples, there are uncountably many points in Λ with two preimages belonging to Λ, as well as uncountably many points having only one preimage in Λ. In the end we give examples which, also from the point of view of Hausdorff dimension, are far from being homeomorphisms on Λ, as well as far from being constant-to-1 maps on Λ.

Keywords Chaotic dynamics of hyperbolic non-invertible skew products · Cantor sets · Unstable manifolds for smooth endomorphisms · Hausdorff dimension of fractals

Mathematics Subject Classification (2000) Primary 37D20 · 28A80; Secondary 37C45 · 34D45

1 Introduction

In this paper we study the chaotic dynamics of a class of non-invertible maps which are hyperbolic on their basic set of saddle type. The dynamics of non-invertible maps (or *endomorphisms*) is different than that of diffeomorphisms due to possible complicated overlappings and to the fact that the number of preimages of a given point that remain in the

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respective basic set may vary. Hyperbolicity over a compact invariant set Λ can be defined for non-invertible smooth maps $f : M \to M$ (where M is a manifold), as the continuous uniform splitting [13] of the tangent space $T_{\hat{x}}M$ into a stable subspace E^s_x and an unstable subspace $E^u_{\hat{x}}$ for $\hat{x} \in \hat{\Lambda}$, where $\hat{x}$ denotes a prehistory of x (i.e. a sequence of consecutive preimages of x belonging to Λ) and $\hat{\Lambda}$ is the space of all such sequences. This implies that, unlike in the case of diffeomorphisms, where the local unstable manifolds form a foliation, in the non-invertible case there may pass a priori several unstable manifolds through a given point of Λ. However it is a difficult problem to distinguish between unstable directions corresponding to different prehistories, and there are few actual examples of such hyperbolic endomorphisms for which several unstable manifolds pass through a given point. One such example of an Anosov endomorphism was studied in [12], where also other properties of Anosov endomorphisms were given; in fact for any arbitrary fixed point x from the torus $\mathbb{T}^m$, it was proved there that there exists an endomorphism f, obtained as a perturbation of an algebraic hyperbolic endomorphism of $\mathbb{T}^m$, so that there exists infinitely many local unstable manifolds of f through x.

In our paper we propose a way of obtaining endomorphisms with a new type of strange behaviour, namely they are far from being homeomorphisms, and also far from being constant-to-one; this happens both from the point of view of preimages, and from the point of view of Hausdorff dimension. We study hyperbolic non-invertible skew products with basic sets whose fibers will be obtained by contractions followed by translations and superpositions, thus generating overlappings in fibers. Here by **basic set** for an endomorphism f we mean an invariant compact set Λ so that there exists a neighbourhood U of Λ with $\Lambda = \cap_{n\in\mathbb{Z}} f^n(U)$. We prove that, when the contraction factors on fibers are all equal to $\frac{1}{2}$ and the other parameters belong to some open set, we obtain a class of examples in (15) which are **far** from having a homeomorphism-type behaviour on their basic set Λ.

Our case is different and complements the one involving families of IFS with overlaps (see [15]), as we do not assume any transversality condition; we focus actually on some non-generic contraction parameters and show that the strong non-invertible character is preserved for a large family of perturbations. Our examples also do not present any type of Open Set Condition behaviour, as they will be proved to be far from homeomorphisms. Our study will involve some new techniques, like the use of thickness of intersections of Cantor sets in fibers, the inverse pressure, and approximating the number of preimages belonging to Λ with continuous functions.

We will show that our method gives Cantor sets in fibers, which are obtained as subsets of intersections of Cantor sets of large thickness [2,3,11]. This will guarantee the existence of uncountably many points having more than one preimage. We show that, still, there are uncountably many points with only one prehistory, giving thus an example where the number of prehistories is infinite for some points, and equal to 1 for others. Estimation (or formulas) for the Hausdorff dimension of fractal sets with the help of the zeros of pressure functions appeared in many instances, starting with the work of Bowen [1] and Ruelle [14]. We proved in [7] an extension of this type of results for the stable dimension for non-invertible maps; we showed that this stable dimension is greatly influenced by the preimage counting function. We will see how this relates to our skew product case. We will prove also that the local unstable manifolds (in fact even the unstable directions) depend on prehistories and will estimate the angle between them. We can give also the unstable dimension by using a Bowen type equation from [4].

Our method gives a class of examples of hyperbolic endomorphisms which behave **differently** than the hyperbolic diffeomorphisms, and also **differently** from constant-to-1 maps.

Outline of main results:

The main object of study is the non-invertible skew product f defined in (2), and also its generalization from (15), together with their respective locally maximal invariant set Λ.

The main results of the paper are contained in Theorems 1, 2, 3, 5 (and the remarks thereafter), and in Corollaries 1 and 2.

First we will remind the notion of natural extension, hyperbolicity for non-invertible maps and the notions of stable dimension and unstable dimension. Then in Theorem 1 we will show, by using a result of [3] (see also [2]) about intersecting two Cantor sets of large thicknesses, that there exists a Cantor set F_x of points having several preimages/prehistories, in each fiber Λ_x of Λ; we will also estimate the thickness of F_x. Moreover, this will give an example of a dynamical system f where some points have two preimages in Λ, and other points with only one preimage in Λ.

In Proposition 1 we will show that our skew product example is hyperbolic as an endomorphism, on its basic set Λ. Then in Theorem 2 we will show that the unstable directions corresponding to different prehistories in Λ do not coincide, and will estimate the angle between them.

In Theorem 4, we give estimates of the unstable dimension by using a Bowen type equation on the natural extension, from [4]. And in Theorem 5 we show that the stable dimension at any point of Λ is strictly smaller than 1 by using a result about the inverse pressure from [5]; and we obtain estimates for the stable dimension by using approximating continuous bounds for the preimage counting function, based on results of [7]. Thus we obtain information on the set of points with more than one preimage, vis-à-vis the set of points having only one preimage.

The hyperbolicity on Λ, the existence of Cantor sets in fibers of points with more than one prehistory, and the disjointness of unstable directions coresponding to different prehistories are shown similarly, also for the more general examples defined in (15), for a suifficiently small α. This property of **strong non-invertibility** is preserved for the nonlinear examples in (15), for all $\alpha > 0$ small enough, by a type of Newhouse phenomenon, involving intersecting Cantor sets of large thickness in fibers.

By combining Theorems 5 and 2 we prove in Corollary 1 that there exist points in Λ with infinitely many prehistories and infinitely many different unstable manifolds. We deduce in Corollary 2 that for the examples in (15) having contraction factors equal to $\frac{1}{2}$ (and the rest of their parameters being in an open set), there are uncountably many points with one preimage in Λ and uncountably many points with two preimages in Λ. We prove then in Corollaries 1 and 2 that, also from the point of view of Hausdorff dimension, the behaviour of this last class of examples (with contraction factors $\frac{1}{2}$ and the other parameters in an open set), is far from that of a homeomorphism on Λ, as well as far from that of a constant-to-1 map on Λ.

We notice that the domain of definition of the functions f_α [defined in (15)] varies with α and is not equal to the entire square $I \times I$; instead it is a product $K_\alpha \times I$, with K_α a Cantor set. Also it is important to remark that our invariant set $\Lambda(\alpha)$ is **not an attractor**, but instead is a **saddle basic set**.

2 Hyperbolic skew product endomorphisms: unstable directions on the basic set

Hyperbolic smooth endomorphisms appear naturally in many instances when invariant sets have self-intersections. Several aspects of their dynamics are very different than in the case of diffeomorphisms (for example [5,7,12], etc.) Consider in the sequel a smooth (say $\mathcal{C}^2$) map $f : M \to M$ on a smooth Riemannian manifold M and let Λ be a compact invariant set, i.e. $f(\Lambda) = \Lambda$. So each point of Λ has at least one f-preimage in Λ; however it may have several f-preimages in Λ.

We define now a **prehistory** of a point $x \in \Lambda$, as an infinite sequence $\hat{x} = (x, x_{-1}, x_{-2}, \ldots)$ of consecutive preimages, i.e. $f(x_{-1}) = x, f(x_{-2}) = x_{-1}, \ldots$, with $x_{-i} \in \Lambda, i \geq 1$. We take then the space of all these prehistories $\hat{\Lambda}$, and consider the shift homeomorphism $\hat{f} : \hat{\Lambda} \to \hat{\Lambda}$, $\hat{f}(\hat{x}) = (f(x), x, x_{-1}, x_{-2}, \ldots), \hat{x} \in \hat{\Lambda}$. The compact space $\hat{\Lambda}$ can be made a metric space in a natural way, and will be called the **natural extension** of the couple (Λ, f). One can introduce also the tangent bundle over $\hat{\Lambda}$, given by $T_{\hat{x}} := \{(\hat{x}, v), v \in T_x M\}, \hat{x} \in \hat{\Lambda}$. Now following [13], we define the notion of hyperbolicity for the endomorphism f as a continuous splitting of the tangent bundle over $\hat{\Lambda}$ into stable directions and unstable directions depending on prehistories. So we have a splitting $T_{\hat{x}} M = E^s_x \oplus E^u_{\hat{x}}, \hat{x} \in \hat{\Lambda}$ where $Df_x(E^s_x) \subset E^s_{f(x)}$ and $Df(E^u_{\hat{x}}) \subset E^u_{\hat{f}\hat{x}}$ and Df contracts uniformly the vectors from E^s_x and expands the vectors from $E^u_{\hat{x}}$. We denote also by $Df_s(x) := Df|_{E^s_x}$ and by $Df_u(\hat{x}) := Df|_{E^u_{\hat{x}}}, \hat{x} \in \hat{\Lambda}$. Associated to this splitting, one constructs local stable and local unstable manifolds of size $r > 0$ for some small r, denoted by $W^s_r(x)$, $W^u_r(\hat{x})$, for $\hat{x} \in \hat{\Lambda}$.

For hyperbolicity and SRB measures for endomorphisms see also [6,16], etc.

For a fixed $r > 0$ small enough, we define the **stable dimension** at $x \in \Lambda$, denoted by $\delta^s(x)$, as the Hausdorff dimension $HD(W^s_r(x) \cap \Lambda)$; respectively the **unstable dimension** at $\hat{x} \in \hat{\Lambda}$, denoted by $\delta^u(\hat{x})$, as $HD(W^u_r(\hat{x}) \cap \Lambda)$. We proved in [4] that the unstable dimension is equal to the unique zero of the pressure function $t \to P_{\hat{f}|_{\hat{\Lambda}}}(-t \log |Df_u(\hat{x})|)$, where the pressure is considered on the natural extension $\hat{\Lambda}$. However in the case of the stable dimension there is no simple general formula, due to the complicated foldings that may take place in Λ.

Let us also define the **preimage counting function** $d(\cdot)$ associated to f on the invariant set Λ, namely $d(x) := \text{Card}\{f^{-1}x \cap \Lambda\}$. It can be checked that $d(\cdot)$ is an upper semi-continuous function [7]; it is not necessarily constant, nor continuous. This is bringing additional difficulties in estimating the stable dimension in the non-invertible case, as noted in [7] or [8].

For a small positive α, let us take now the subintervals $I^\alpha_1, I^\alpha_2 \subset I := [0, 1]$, of small positive length, with $I^\alpha_1 = [b_1(\alpha), b_2(\alpha)]$, $I^\alpha_2 = [b_3(\alpha), b_4(\alpha)]$; assume that $b_2(\alpha) < \frac{1}{2}$, $b_2(\alpha)$ is very close to $\frac{1}{2}$, and that $b_4(\alpha)$ is very close to $1-\alpha$ and $b_4(\alpha) < 1-\alpha$; we assume that $|b_1(\alpha) - \frac{1}{2}|$ and $|b_3(\alpha) - (1-\alpha)|$ are both much smaller than α, say

$$0 < \max\left\{\left|b_1(\alpha) - \frac{1}{2}\right|, \ |b_3(\alpha) - (1-\alpha)|\right\} =: \epsilon(\alpha) < \alpha^2 \qquad (1)$$

The intervals I^α_1, I^α_2 depend on α. But may also be denoted in the sequel simply by I_1, I_2 when dependence on α is unambiguous.

Let us take now $g : I^\alpha_1 \cup I^\alpha_2 \to I$, a strictly increasing smooth map which expands both I^α_1 and I^α_2 to I, i.e. $g(I^\alpha_1) = g(I^\alpha_2) = I$. Assume that $g'(x) > \beta(\alpha) \gg 1, x \in I^\alpha_1 \cup I^\alpha_2$. From this dilation condition, we see that there exist subintervals $I^\alpha_{11}, I^\alpha_{12} \subset I^\alpha_1$ and I^α_{21}, $I^\alpha_{22} \subset I^\alpha_2$ such that $g(I^\alpha_{11}) = g(I^\alpha_{21}) = I^\alpha_1$ and $g(I^\alpha_{12}) = g(I^\alpha_{22}) = I^\alpha_2$. Let us denote by $J^\alpha := I^\alpha_{11} \cup I^\alpha_{12} \cup I^\alpha_{21} \cup I^\alpha_{22}$ and

$$J^\alpha_* := \{x \in J^\alpha, \ g^i x \in J^\alpha, i \geq 0\}$$

When the dependence on α is clear, we may denote these sets also by $J, J_*, I_{i,j}, i, j = 1, 2$.

We will define now for a small $\alpha > 0$, the skew product with overlaps in fibers $f_\alpha : J_*^\alpha \times I \to J_*^\alpha \times I$

$$f_\alpha(x, y) = (g(x), h_\alpha(x, y)), \quad \text{where}$$
$$h_\alpha(x, y) = \begin{cases} x + \frac{y}{2}, & x \in I_{11}^\alpha \\ 1 - x + \frac{y}{2}, & x \in I_{21}^\alpha \\ 1 - \frac{y}{2}, & x \in I_{12}^\alpha \\ \frac{y}{2}, & x \in I_{22}^\alpha \end{cases} \tag{2}$$

We shall denote also the function $h_\alpha(x, \cdot) : I \to I$ by $h_{x,\alpha}$ for $x \in J_*^\alpha$. From the definition of $h_\alpha(x, y)$ it can be seen that for $x \in J_*^\alpha \cap I_1$, there are two images of intervals intersecting inside $\{x\} \times I$, namely $h_{x_{-1},\alpha}(I)$ and $h_{\tilde{x}_{-1},\alpha}$, where x_{-1} denotes the g-preimage of x belonging to I_1^α, and $\tilde{x}_{-1}$ denotes the g-preimage of x belonging to I_2^α.

We denote by

$$\Lambda(\alpha) := \bigcup_{x \in J_*^\alpha} \bigcap_{n \geq 0} \bigcup_{y \in g^{-n}x \cap J_*^\alpha} h_{y,\alpha}^n(I), \tag{3}$$

where $h_{y,\alpha}^n := h_{f^{n-1}y,\alpha} \circ \cdots \circ h_{y,\alpha}$, $n \geq 0$. For $x \in J_*^\alpha$ we also denote by

$$\Lambda_x(\alpha) := \bigcap_{n \geq 0} \bigcup_{y \in g^{-n}x \cap J_*^\alpha} h_{y,\alpha}^n(I),$$

and call it **the fiber** of $\Lambda(\alpha)$ over x. It is clear that $\Lambda(\alpha)$ is a compact f-invariant set, but $\Lambda(\alpha)$ **is not necessarily totally invariant** hence the number of f_α-preimages of a point from $\Lambda(\alpha)$, belonging to $\Lambda(\alpha)$, may vary. The sets $\Lambda(\alpha)$, $\Lambda_x(\alpha)$ will be denoted simply by Λ, Λ_x when dependence on α is clear.

We shall now prove that the fibers $\Lambda_x(\alpha)$, $x \in J_*^\alpha \cap I_1$ have an interesting property, namely they contain a Cantor set of points which have two different f_α-preimages in $\Lambda(\alpha)$.

Before we proceed with the Theorem, let us remind the notion of **thickness** of a Cantor set introduced by Newhouse [10], and studied also for example in [2,3,11].

Consider a Cantor set K obtained as $I_0 \setminus \cup_{n \geq 1} U_n$, where U_n are open subintervals of I_0, called the **gaps** of K, and I_0 is the minimal interval containing K. Of course the gaps of K can be ordered in many ways, and we call such an ordering $\mathcal{U} = (U_n)_n$ a **presentation** of K. For a point $u \in \partial U_n$, let C be the connected component of $I_0 \setminus (U_1 \cup \cdots \cup U_n)$ which contains u; the component C is also called a **bridge** at u. For such a point denote by $\tau(K, \mathcal{U}, u) := \frac{\ell(C)}{\ell(U_n)}$ (see [11]), where $\ell(C)$ denotes the length of the subinterval C. Then the thickness of K is defined by

$$\tau(K) = \sup_{\mathcal{U}} \inf_{u \in K} \tau(K, \mathcal{U}, u),$$

where the infimum is taken over all the boundary points of finite gaps of K, and the supremum is taken over all different presentations of K. In fact it can be proved that the supremum in the definition of thickness is attained for a presentation with decreasing lengths of gaps, i.e. so that $\ell(U_p) \leq \ell(U_n)$ if $p \geq n$. Thickness is an important numerical invariant of a Cantor set, and it is preserved by an affine transformation of the interval. Newhouse showed that if K_1, K_2 are Cantor sets with $\tau(K_1)\tau(K_2) > 1$ and neither of them is contained in a gap of the other (i.e. they are **interleaved**), then $K_1 \cap K_2 \neq \emptyset$ (see [10]).

We will use below the thickness in order to prove that the fibers Λ_x [defined in (3)] contain "big" intersections of certain Cantor sets.

Theorem 1 *In the above setting, for all points $x \in J_*^\alpha \cap I_1$, there exists a Cantor set of points $F_x(\alpha) \subset \Lambda_x(\alpha)$, so that each point from $F_x(\alpha)$ has two different f-preimages in Λ. Also if $x \in J_*^\alpha \cap I_2^\alpha$, it follows that there is a Cantor set $F_x(\alpha) \subset \Lambda_x(\alpha)$ such that each point of $F_x(\alpha)$ has more than one prehistory in $\hat{\Lambda}(\alpha)$.*

Proof Fix a small positive α; we will work with the corresponding I_j, f, Λ, h for this fixed α, without recording their dependence on it. From the construction of the subintervals I_{ij}, $i, j = 1, 2$, we see that I_{11} and I_{12} are close to $\frac{1}{2}$, while I_{21}, I_{22} are close to $1-\alpha$ (recall that $\alpha > 0$ was taken very small).

For a point $x \in J_*$, denote by $x_{-1,1}, x_{-1,2}$ the two g-preimages of x in J_*, i.e. $x_{-1,1} \in I_1, x_{-1,2} \in I_2, g(x_{-1,1}) = g(x_{-1,2}) = x$.

At the first iteration, if $x \in I_1$, we obtain $\Lambda_x(1) := h_{x_{-1,1}}(I) \cup h_{x_{-1,2}}(I) = [\alpha, c_1]$, with $c_1 = \frac{1}{2} + x_{-1,1} \approx 1$; indeed $x_{-1,1} \in [a, c]$ and $|a - \frac{1}{2}| < \alpha$. If $x \in I_2$, then $\Lambda_x := h_{x_{-1,1}}(I) \cup h_{x_{-1,2}}(I) = [0, 1]$, where again $x_{-1,1}, x_{-1,2}$ are the two g-preimages of x, one in I_1 and the other in I_2.

For the second iteration, we obtain $\Lambda_x(2) := h_{x_{-1,1}}(\Lambda_{x_{-1,1}}(1)) \cup h_{x_{-1,2}}(\Lambda_{x_{-1,2}}(1)) = [\alpha, c_2]$, where $c_2 = x_{-1,1} + c_1/2 \approx \frac{1+c_1}{2}$. For $x \in I_2$, we have $\Lambda_x(2) := h_{x_{-1,1}}(\Lambda_{x_{-1,1}}(1)) \cup h_{x_{-1,2}}(\Lambda_{x_{-1,2}}(1)) = [0, \frac{1}{2}] \cup [1 - \frac{c_1}{2}, 1 - \frac{\alpha}{2}]$. This is hinting already on the nature of the gaps of the fiber Λ_x, which is obtained as an intersection $\Lambda_x = \cap_{n \geq 1} \Lambda_x(n)$. There will be gaps coming from the successive iterations of the gap $(\frac{1}{2}, 1 - \frac{c_1}{2})$.

At the third iteration, we see that for $x \in J_* \cap I_1$, $\Lambda_x(3) := h_{x_{-1,1}}(\Lambda_{x_{-1,1}}) \cup h_{x_{-1,2}}(\Lambda_{x_{-1,2}}) = [\alpha, c_3]$, where $c_3 = x_{-1,1} + \frac{c_2}{2}$ and $x_{-1,i}$ is the g-preimage of x belonging to I_i, $i = 1, 2$. And for $x \in J_* \cap I_2$, we have $\Lambda_x(3) = h_{x_{-1,1}}(\Lambda_{x_{-1,1}}(2)) \cup h_{x_{-1,2}}(\Lambda_{x_{-1,2}}(2)) = [0, \frac{1}{4}] \cup [\frac{1}{2} - \frac{c_1}{4}, \frac{1}{2} - \frac{\alpha}{4}] \cup [1 - \frac{c_2}{2}, 1 - \frac{\alpha}{2}]$. At this iteration, we have therefore the gaps $(\frac{1}{4}, \frac{1}{2} - \frac{c_1}{4})$ and $(\frac{1}{2} - \frac{\alpha}{4}, 1 - \frac{c_2}{2})$, having lengths $\frac{1}{4} - \frac{c_1}{4}$ and $\frac{1}{2} - \frac{c_2}{2} + \frac{\alpha}{4}$ respectively. These lengths are very small in comparison to the lengths of their associated left and right bridges C. We can say that $\ell(U_j(3)) \leq \Delta(\alpha)^{-1}\ell(J)$, where J is one of the component intervals of $\Lambda_x(3)$ and $U_j(3)$ is one of the gaps between two consecutive subintervals J of $\Lambda_x(3)$, and where $\Delta(\alpha) = O(\frac{1}{\alpha})$. So $\Delta(\alpha) \to \infty$ when $\alpha \searrow 0$.

At the fourth iteration we see gaps forming inside $\Lambda_x(4)$ for $x \in J_* \cap I_1$ as well, i.e. $\Lambda_x(4)$ contains several disjoint closed subintervals. For $x \in J_* \cap I_1$, we have $\Lambda_x(4) = h_{x_{-1,1}}(\Lambda_{x_{-1,1}}(3)) \cup h_{x_{-1,2}}(\Lambda_{x_{-1,2}}(3))$; so from the above calculations we obtain $\Lambda_x(4) = [x_{-1,1} + \alpha/2, x_{-1,1} + c_3/2] \cup [1 - x_{-1,2}, 1 - x_{-1,2} + \frac{1}{8}] \cup [1 - x_{-1,2} + \frac{1}{4} - \frac{c_1}{8}, 1 - x_{-1,2} + \frac{1}{4} - \frac{\alpha}{8}] \cup [1 - x_{-1,2} + \frac{1}{2} - \frac{c_2}{4}, 1 - x_{-1,2} + \frac{1}{2} - \frac{\alpha}{4}]$. We see thus that there are gaps forming this time in $\Lambda_x(4)$, and that still $\ell(J)/\ell(U) \geq \Delta(\alpha)$, for any component J of $\Lambda_x(4)$ and adjacent gap U between two consecutive J's. If $x \in J_* \cap I_2$, then we obtain $\Lambda_x(4)$ as before and we see more gaps forming, still having the property that $\ell(J)/\ell(U) \geq \Delta(\alpha)$ for any component subinterval (bridge) J and adjacent gap U.

Assume now that at iteration n, $\Lambda_x(n) = J_1(x, n) \cup \ldots J_{k(n)}(x, n)$, where this is an ordered union of mutually disjoint closed subintervals, i.e. the right endpoint of $J_k(x, n)$ is strictly less than the left endpoint of $J_{k+1}(x, n)$, $k = 1, \ldots, k(n) - 1$. Denote $J_k(x, n) = [a_k(x, n), c_k(x, n)]$, $k = 1, \ldots, k(n)$ if $x \in I_1$, and $J_k(x, n) = [\tilde{a}_k(x, n), \tilde{c}_k(x, n)]$, $k = 1, \ldots, k(n)$ for $x \in I_2$. Then we will show that there must exist points from Λ_x as close as we want to the endpoints of each $J_k(x, n)$. Indeed we know from above that there are gaps of Λ_y for $y \in J_* \cap I_2$, as close as we want to 0. But if $x \in I_1$ and $x_{-1,2}$ is its g-preimage in I_2, we have that $h_{x_{-1,2}}$ takes $\Lambda_{x_{-1,2}}$ into Λ_x, so there exist points of Λ_x as close as we want to the left endpoint of $J_1(x, n)$, $n > 1$. Again for $x \in I_1$, and the g-preimage $x_{-1,1} \in I_1$ of x, points z in $\Lambda_{x_{-1,1}}$ are taken by $h_{x_{-1,1}}$ into points of type $x_{-1,1} + \frac{z}{2}$; if we repeat the procedure, we see that there are points from Λ_x as close as we want to the right endpoint of $J_{k(n)}(x,n)$. Now from the fact that there are points of $\Lambda_{x_{-1,1}}$ as close as we want to the left endpoint of intervals of type $J_1(x_{-1,1},n)$, we see that by applying $g_{x_{-1,1}}$ we obtain points from Λ_x as close as we want to the right endpoint of $J_{k(n)}(x,n)$ if $x \in I_2$.

But now by recalling that all the intervals of type $J_k(x,n)$ are obtained by applying repeatedly h_y (for preimages y of x), we obtain that there are points from Λ_x as close as we want to each of the endpoints of the subintervals $J_k(x,n)$ obtained at step n. This procedure tells us that indeed, we can use the subintervals $J_k(x,n)$ as bridges in the construction of the respective Cantor set, since the gaps between them do not extend inside any of $J_k(x,n)$. Thus we can use the lengths of the subintervals of type $J_k(x,n)$, $k=1,\dots,k(n)$ and the lengths of the gaps between them, in the calculation of the thickness of Λ_x.

We also notice the following property: assume that $\Lambda_x(n) = J_1(x,n) \cup \dots \cup J_{k(n)}(x,n)$ for $x \in J_*$, where these disjoint subintervals are arranged in increasing order. Then if $x \in J_* \cap I_1$, we claim that the left endpoint of $J_1(x,n)$, i.e. $a_1(x,n)$ is ϵ-close to α where for our fixed α, ϵ denotes the positive number $\epsilon(\alpha)$ defined in (1). Also if $x \in J_* \cap I_2$, we claim that the left endpoint of $J_1(x,n)$, i.e. $\tilde{a}_1(x,n)$ is 0 and the right endpoint of $J_{k(n)}(x,n)$, i.e. $\tilde{c}_{k(n)}(x,n)$ is $\epsilon(\alpha)$-close to $1-\frac{\alpha}{2}$, for every $n \ge 1$. We saw this property for iterations 1 through 4, let us prove it in general by induction. Assume it is satisfied at step n for $\Lambda_x(n)$, $\forall x \in J_*$. Then at step $n+1$, if $x \in I_1$ we have $\Lambda_x(n+1) = h_{x_{-1,1}}(\Lambda_{x_{-1,1}}(n)) \cup h_{x_{-1,2}}(\Lambda_{x_{-1,2}}(n))$; so $J_1(x,n+1) = 1 - x_{-1,2} + \frac{1}{2} \cdot J_1(x_{-1,2},n)$. Since by induction, the left endpoint of $J_1(x_{-1,2},n)$ is 0, we see that the left endpoint of $J_1(x,n+1)$ is $1-x_{-1,2}$. But $x_{-1,2} \in I_{21}$ is ϵ-close to $1-\alpha$, hence the left endpoint of $J_1(x,n+1)$ is ϵ-close to α. Now if $x \in I_2$, we see that the left endpoint $J_1(x,n+1)$ is obtained from applying $y \to y/2$ to the interval $J_1(x_{-1,2},n)$, thus it is equal to 0. And the right endpoint of $J_{k(n)}(x,n+1)$ is obtained from applying the map $y \to 1-y/2$ to $J_1(x_{-1,1},n)$; thus the right endpoint of $J_{k(n)}(x,n+1)$ is ϵ-close to $1-\frac{\alpha}{2}$.

We have to see also what is happening to the subintervals that overlap in $\Lambda_x(n+1)$ for $x \in I_1$. The overlap is between the points $x_{-1,1} + \frac{a_1(x_{-1,1},n)}{2}$ and $1 - x_{-1,2} + \frac{\tilde{c}_{k(n)}(x_{-1,2},n)}{2}$. However we saw above that $a_1(x_{-1,1},n)$ is ϵ-close to α (recall that $\epsilon = \epsilon(\alpha)$); and that $\tilde{c}_{k(n)}(x_{-1,2},n)$ is ϵ-close to $1-\frac{\alpha}{2}$. We recall also that I_1, I_2 are ϵ-close to $\frac{1}{2}$ and $1-\alpha$ respectively. Thus the overlap mentioned above is between points that are ϵ-close to $\frac{1}{2}+\frac{\alpha}{2}$ and $\alpha + \frac{1}{2} - \frac{\alpha}{4} = \frac{1}{2} + \frac{3\alpha}{4}$. As for all $\alpha > 0$ small we have $\frac{1}{2}+\frac{\alpha}{2} < \frac{1}{2}+\frac{3\alpha}{4}$, we see that indeed we have overlaps inside $\Lambda_x(n)$, $x \in J_* \cap I_1$, $n > 1$.

Denote by J_{int} this maximal overlap at the third Iteration. Then J_{int} comes from applying $h_{x_{-1,1}}$ to the intervals in the last half of $\Lambda_{x_{-1,1}}(n-1)$ and from applying $h_{x_{-1,2}}$ to the intervals in the first half of $\Lambda_{x_{-1,2}}(n-1)$. However we noticed that, when $y \in I_1 \cap J_*$, the component intervals of $\Lambda_y(n-1)$ outside $[1/2+\alpha/2, 1/2+3\alpha/4]$ contain each a Cantor set, obtained successively by eliminating a fixed proportion of the intervals at step $n-1$. Thus there exists n large enough so that, for any two points $\xi, \zeta \in \Lambda_x \cap J_{int}$, there exists a gap U_1 in $\Lambda_{x_{-1,1}}(n-1)$ and a gap U_2 in $\Lambda_{x_{-1,2}}(n-1)$ with $h_{x_{-1,1}}(U_1) \cap h_{x_{-1,2}}(U_2)$ non-empty and situated between ζ and ξ. Obviously n depends on the distance between ξ and ζ.

We saw above that there are points from Λ_x as close as we wish to the endpoints of the subintervals $J_k(x,n)$, $k=1,\dots,k(n)$ of $\Lambda_x(n)$, $n \ge 1$. Therefore these subintervals can be used as bridges J and the intervals between them as gaps U, in the Cantor set construction of Λ_x, $x \in J_*$. By induction we also see that at each step n we have

$$\ell(J_k(x,n))/\ell(U) \ge \Delta(\alpha), \quad k=1,\dots,k(n)$$

for any subinterval $J_k(x,n)$ of $\Lambda_x(n)$ and any corresponding adjacent gap U of $J_k(x,n)$ (where we say that U is an **adjacent gap** for $J_k(x,n)$ if it is immediately at the left or at the right of $J_k(x,n)$ at step n). But this ratio between a subinterval and its adjacent gap is preserved by linear transformations, like the ones we deal with in h_x, $x \in J_*$. By overlapping two subintervals (as it may happen in $\Lambda_x(n)$, $x \in J_* \cap I_1$), we can only increase the lengths of bridges J and decrease the lengths of adjacent gaps U. Thus we see by induction that at each step

$$\ell(J)/\ell(U) \geq \Delta(\alpha), \tag{4}$$

for each bridge subinterval J and adjacent gap U of $\Lambda_x(n)$. Since by applying iterations we cut in half the length of the gaps between the subintervals $J_k(x,n)$, $k = 1, \dots, k(n)$ at step n (or decrease them by an even larger factor), we obtain that the gaps are ordered decreasingly when $n \nearrow \infty$; this presentation of Λ_x will be denoted by $\mathcal{U}_{step}$. And from the observation made when we defined the thickness of a Cantor set, we have that $\tau(\Lambda_x) = \inf_{\xi \in \Lambda_x} \tau(\Lambda_x, \mathcal{U}_{step}, \xi)$. But from (4) we notice that

$$\tau(\Lambda_x) \geq \Delta(\alpha), \quad x \in J_* \tag{5}$$

Now let $x \in J_* \cap I_1$; then $\Lambda_x = h_{x_{-1,1}}(\Lambda_{x_{-1,1}}) \cup h_{x_{-1,2}}(\Lambda_{x_{-1,2}})$, where $x_{-1,1} \in g^{-1}(x) \cap I_1$ and $x_{-1,2} \in g^{-1}(x) \cap I_2$. It is easy to see that the two Cantor sets $S_1(x) := h_{x_{-1,1}}(\Lambda_{x_{-1,1}})$ and $S_2(x) := h_{x_{-1,2}}(\Lambda_{x_{-1,2}})$ are interleaved, i.e. neither set is contained in a gap of the other. Also from (5) we know that $\tau(S_1(x)) \geq \Delta(\alpha)$ and $\tau(S_2(x)) \geq \Delta(\alpha)$.

We recall that $\Delta(\alpha) \underset{\alpha \to 0}{\to} \infty$. Thus if $\tau_1 := \tau(S_1(x))$, $\tau_2 := \tau(S_2(x))$ and α is small enough, we can check that $\tau_1 \tau_2 > 1$ and in addition that $(\tau_1, \tau_2) \in B$, where:

$$B := \left\{ (\tau_1, \tau_2),\ \tau_1 > \frac{\tau_2^2 + 3\tau_2 + 1}{\tau_2^2} \text{ or } \tau_2 > \frac{\tau_1^2 + 3\tau_1 + 1}{\tau_1^2} \right\}$$
$$\cap \left\{ (\tau_1, \tau_2),\ \tau_1 > \frac{(1+2\tau_2)^2}{\tau_2^3} \text{ or } \tau_2 > \frac{(1+2\tau_1)^2}{\tau_1^3} \right\}$$

Since $S_1(x)$, $S_2(x)$ are not interleaved and we do have the above conditions satisfied for $\Delta(\alpha)$ large enough (i.e. for α small enough), we obtain from [3] that there exists indeed a Cantor set F_x in $S_1(x) \cap S_2(x)$.

Moreover from [2] (page 882), it follows that we have:

$$\tau(F_x) \geq \sqrt{\min\{\tau(S_1(x)), \tau(S_2(x))\}} \geq \sqrt{\Delta(\alpha)})$$

From the definition of the sets $S_1(x)$, $S_2(x)$ we see that for every $x \in I_1 \cap J_*$, each point from the set F_x has two different f-preimages belonging to Λ.

Now if $x \in J_* \cap I_2$, there exists a g-preimage $x_{-1,1} \in I_1$ of x and then in the fiber $\Lambda_{x_{-1,1}}$ there must exist a Cantor set $F_{x_{-1,1}}$ of points having two distinct f-preimages in Λ. So we obtain that there exists a Cantor set $F_x := h_{x_{-1,1}}(\Lambda_{x_{-1,1}})$ of points with two different f^2-preimages in Λ. The Theorem is then proved. □

In fact we show in Corollary 1 that there are points with infinitely many prehistories in $\hat{\Lambda}$.

Let us prove now the hyperbolicity of f as an endomorphism on the basic set Λ. We will show that this set is of saddle type, i.e. f has both stable and unstable directions on it. Consider an expanding smooth map $g : I_1 \cup \dots \cup I_p \to I_1 \cup \dots \cup I_p$ and let $J_* := \{x \in I_1 \cup \dots \cup I_p, g^i(x) \in I_1 \cup \dots \cup I_p, i \geq 0\}$. Let also a smooth function $h(x,y) : (I_1 \cup \dots \cup I_p) \times I \to I$, which is uniformly contracting in the second coordinate, i.e. exists $\delta \in (0,1)$ with $|\partial_y h| < \delta$ everywhere. Define the skew product $f(x,y) : J_* \times I \to J_* \times$

I, $f(x,y) = (g(x), h(x,y))$, $(x,y) \in J_* \times I$. We define $\Lambda := \cup_{x \in J_*} \cap_{n \geq 0} \cup_{y \in g^{-n}x} h^n_y(I)$, where $h^n_y := h_{g^{n-1}y} \circ \cdots \circ h_y$, $y \in J_*$.

Although it may appear at a first sight that f is expanding horizontally on Λ, the calculation on derivatives shows this to be false. Indeed we have the derivative of f,

$$Df(x,y) = \begin{pmatrix} g'(x) & 0 \\ \partial_x h(x,y) & \partial_y h(x,y) \end{pmatrix},$$

where $\partial_x h(x,y)$ represents the partial derivative of h with respect to x at the point (x,y). Thus for a vector $\bar{w} = (0, v) \in \mathbb{R} \times \mathbb{R}$, we get $Df(x,y) \cdot \bar{w} = \begin{pmatrix} 0 \\ \partial_y h(x,y)v \end{pmatrix}$, so the vector space $\{(0,v) \in \mathbb{R} \times \mathbb{R}\}$ is invariant and Df is contracting on vertical lines; these vertical lines represent therefore the stable tangent subspaces.

However if we consider the horizontal vector $\bar{w} = (\zeta, 0) \in \mathbb{R} \times \mathbb{R}$, then $Df(x,y) \cdot \bar{w} = \begin{pmatrix} g'(x) \cdot \zeta \\ \partial_x h(x,y) \cdot \zeta \end{pmatrix}$, so the horizontal line $\{(\zeta, 0), \zeta \in \mathbb{R}, 0 \in \mathbb{R}\}$ is not invariated by Df, and thus the unstable spaces do not have to be equal to this line.

To prove the hyperbolicity of the non-invertible map f on Λ and to construct its unstable spaces, we use a generalization of a theorem of Newhouse [9] to this endomorphism case.

Let $f : M \to M$ be a smooth, not necessarily invertible map and suppose that Λ is a compact f-invariant set in M. Assume that there exists a field of cones in the tangent space, $\mathcal{C} = \{C_{\hat{z}}\}_{\hat{z} \in \hat{\Lambda}}$, so that the dimension of the core linear space of $C_{\hat{z}}$ is constant on $\hat{\Lambda}$; but the cone field $\mathcal{C}$ is not necessarily assumed to be Df-invariant. Let us say that a function f is **expanding and co-expanding on the cone field** $\mathcal{C}$, if given the notations:

$$m_{\mathcal{C},\hat{z}}(f) := \inf_{v \in C_{\hat{z}}, v \neq 0} \frac{|Df_z v|}{|v|}, \quad \text{and} \quad m'_{\mathcal{C},\hat{z}}(f) := \inf_{v \notin C_{\hat{f}\hat{z}}} \frac{|Df^{-1}_{fz} v|}{|v|}, \quad \hat{z} \in \hat{\Lambda},$$

we have that $\inf_{\hat{z} \in \hat{\Lambda}} m_{\mathcal{C},\hat{z}}(f) > 1$, and $\inf_{\hat{z} \in \hat{\Lambda}} m'_{\mathcal{C},\hat{z}}(f) > 1$.

Theorem (Newhouse) *In the above setting, assume that there exists an integer $N \geq 1$ such that f^N is expanding and co-expanding on $\mathcal{C}$; then it follows that f is hyperbolic on Λ.*

The proof is similar to the one given in [9]. We can prove consequently the following result of hyperbolicity for our skew product:

Proposition 1 *In the above setting, i.e. with $g : J_* \to J_*$ expanding and $h : I_* \times I \to I$ uniformly contracting in the second coordinate, we have that $f(x,y) = (g(x), h(x,y))$ is hyperbolic as an endomorphism on Λ.*

Proof Let a continuous positive function γ defined on $\hat{\Lambda}$, and the cone $C^u_{\hat{z}} := \{(v,w) \in \mathbb{R}^2, |w| \leq \gamma(\hat{z}) \cdot |v|\}$, $z = (x,y) \in \Lambda$, $\hat{z} \in \hat{\Lambda}$. The dimension of the core real linear space of this cone is 1. Our cone field will be then $\mathcal{C}^u = \{C^u_{\hat{z}}\}_{\hat{z} \in \hat{\Lambda}}$ and we do not know a priori whether it is Df-invariant. We have $Df_z(v,w) = \begin{pmatrix} g'(x) \cdot v \\ \partial_x h(z) \cdot v + \partial_y h(z) \cdot w \end{pmatrix}$. So in order to have an f-expanding field of cones, it is enough to take

$$|g'(x)|^2 > 1 + \gamma^2(\hat{z}), \quad z \in \Lambda$$

If we assume $|g'(x)| > \beta > 1$, $x \in X$, then it would be enough to have

$$0 < \gamma(\hat{z}) \leq \sqrt{\beta^2 - 1} \quad \text{or} \quad 0 < \gamma(\hat{z}) \leq \sqrt{\beta^{2N} - 1}, \tag{6}$$

where the second inequality is needed if we work with f^N instead of f. So in this last case, f^N is expanding on the cone field $\mathcal{C}^u$.

Now we estimate the co-expansion coefficient. If $N \geq 1$ is an integer and if $(v, w) \notin C^u_{\hat{f}^N \hat{z}}$, then $|w| > \gamma(\hat{f}^N \hat{z}) \cdot |v|$. Denote also $f^N z = (g^N(x), h(f^{N-1}x, h_{N-1}(x, y)))$, where h_{N-1} is given by: $f^{N-1}z = (g^{N-1}(x), h_{N-1}(x, y))$, $z = (x, y) \in \Lambda$. So

$$\begin{aligned} \partial_x h_N(x, y) &= \partial_x h(f^{N-1}x, h_{N-1}(x, y)) \cdot \partial_x g^{N-1}(x) \\ &\quad + \partial_y h(f^{N-1}x, h_{N-1}(x, y)) \cdot \partial_x h_{N-1}(x, y) \\ &= \partial_x h(f^{N-1}x, h_{N-1}(x, y)) \cdot \partial_x g^{N-1}(x) \\ &\quad + \partial_y h(f^{N-1}x, h_{N-1}(x, y)) \cdot \ \partial_x h(f^{N-2}x, h_{N-2}(x, y)) \cdot \partial_x g^{N-2}(x) \\ &\quad + \partial_y h(f^{N-1}x, h_{N-1}(x, y)) \cdot \partial_y h(f^{N-2}x, h_{N-2}(x, y)) \cdot \partial_x h_{N-2}(x, y) \end{aligned} \tag{7}$$

Denote by $K := \sup_\Lambda |\partial_x h|$ and $K' := K \cdot \frac{1}{1-\delta/\beta}$, where $\delta \in (0, 1)$ is a contraction factor, such that $|\partial_y h| < \delta < 1$ on Λ. Therefore by induction in (7) we have:

$$|\partial_x h_N(x, y)| \leq K \cdot |(g^{N-1})'x| + \delta K \cdot |(g^{N-2})'x| + \cdots \leq K' \cdot |(g^{N-1})'x| \tag{8}$$

But $D(f^N)^{-1}_{f^N z} \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} \frac{v}{(g^N)'(x)} \\ \frac{-\partial_x h_N(z) v}{(g^N)'(x) \cdot \partial_y h_N(z)} + \frac{w}{\partial_y h_N(z)} \end{pmatrix}$.

Hence $||D(f^N)^{-1}_{f^N z} \begin{pmatrix} v \\ w \end{pmatrix}||^2 \geq \frac{v^2}{|(g^N)'(x)|^2} \left(1 + \frac{|\partial_x h_N|^2(z)}{|\partial_y h_N|^2(z)}\right) + \frac{w^2}{|\partial_y h_N|^2(z)} \cdot \left(1 - \frac{2|\partial_x h_N(z)|}{|(g^N)'(x) \cdot \gamma(\hat{f}^N \hat{z})|}\right)$, for any $N \geq 1$. But then since $|\partial_x h_N(z)| \leq K' \cdot |(g^{N-1})'(x)|$, and K' depends only on g, h, there must exist N sufficiently large such that $\left|\frac{2K' \cdot \frac{1}{g'(g^{N-1}x)}}{\gamma(\hat{f}^N(\hat{z}))}\right| \leq \frac{2K'}{\beta \cdot \sqrt{\beta^{2N}-1}} < \frac{1}{2}$, if we take the map $\gamma(\cdot)$ to be constant on $\hat{\Lambda}$ and close to $\sqrt{\beta^{2N} - 1}$ (although smaller than $\sqrt{\beta^{2N} - 1}$). So:

$$\left\| D(f^N)^{-1}_{f^N z} \begin{pmatrix} v \\ w \end{pmatrix} \right\|^2 \geq \frac{|w|^2}{2|\partial_y h_N(z)|^2} \tag{9}$$

But we had $|w| > \gamma(\hat{f}^N \hat{z}) \cdot |v|$, hence:

$$\frac{|w|^2}{2|\partial_y h_N(z)|^2} \geq \frac{|w|^2}{2\delta^{2N}} > \frac{|v|^2 + |w|^2}{\delta},$$

for N sufficiently large, since $\gamma(\cdot)$ is bounded on $\hat{\Lambda}$. Hence $\inf_{\hat{z} \in \hat{\Lambda}} m'_{\mathcal{C}^u, \hat{z}}(f^N) > 1$ for some large integer N. So f^N is both expanding and co-expanding on the cone field $\mathcal{C}^u$ over Λ, so the map f is hyperbolic according to the previous result. The unstable space corresponding to the arbitrary prehistory $\hat{z} \in \hat{\Lambda}$ is obtained then as $E^u_{\hat{z}} = \cap_{n \geq 0} Df^n(C^u_{\hat{f}^{-n}\hat{z}})$. □

We will show now that there correspond different unstable tangent directions $E^u_{\hat{z}} \neq E^u_{\hat{z}'}$, to two different prehistories of z, $\hat{z}, \hat{z}' \in \hat{\Lambda}$.

Theorem 2 *Let a fixed small $\alpha > 0$ and consider the skew product $f : J_* \times I \to J_* \times I$ defined in (2). Then if $\hat{z}$ and $\hat{z}'$ are two different prehistories of z from $\hat{\Lambda}$, it follows that $E^u_{\hat{z}} \neq E^u_{\hat{z}'}$; in particular the local unstable manifolds $W^u_r(\hat{z})$, $W^u_r(\hat{z}')$ are not tangent to each other.*

Proof We have that $f(x, y) = (g(x), h(x, y)), (x, y) \in J_* \times I$ as defined in (2) (for notational simplicity, we do not record here the dependence of f on α). Let the point $z = (x, y) \in \Lambda_x, x \in J_*$. For a tangent vector (v, w) we have

$$Df_z \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} g'(x) \cdot v \\ \partial_x h(z) \cdot v + \partial_y h(z) \cdot w \end{pmatrix}$$

Assume that the unstable tangent space corresponding to the prehistory $\hat{z} \in \hat{\Lambda}$ is given by

$$E^u_{\hat{z}} := \left\{(v, \omega(\hat{z}) \cdot v), v \in \mathbb{R}\right\}, \quad \hat{z} \in \hat{\Lambda}, \tag{10}$$

where $\omega(\cdot)$ is a bounded function on $\hat{\Lambda}$, since the unstable spaces must be transversal to the stable (vertical) ones. Thus from the above formula, we obtain

$$Df_z \begin{pmatrix} v \\ w \end{pmatrix} = \begin{pmatrix} g'(x) \cdot v \\ \partial_x h(z) \cdot v + \partial_y h(z) \cdot \omega(\hat{z}) v \end{pmatrix}$$

Now we know from the construction of unstable spaces on $\hat{\Lambda}$ as intersections of iterates of unstable cones (Proposition 1) that $Df_z(E^u_{\hat{z}}) \subset E^u_{\hat{f}\hat{z}}$. Therefore $\partial_x h(z) + \partial_y h(z) \cdot \omega(\hat{z}) = \omega(\hat{f}\hat{z}) \cdot g'(x), \hat{z} \in \hat{\Lambda}$. So if z_{-1} denotes an f-preimage of z belonging to Λ and x_{-1} denotes an g-preimage of x belonging to J_*, we obtain the following recurrence formula for $\omega(\cdot)$:

$$\omega(\hat{f}\hat{z}) = \frac{1}{g'(x)} \partial_x h(z) + \frac{1}{g'(x)} \partial_y h(z) \cdot \omega(\hat{z}) \tag{11}$$

By iterating (11) and by recalling that ω is a bounded function on $\hat{\Lambda}$ (since the stable and unstable directions must be transversal to each other), we obtain:

$$\begin{aligned} \omega(\hat{f}\hat{z}) &= \frac{1}{g'(x)} \partial_x h(z) + \frac{1}{g'(x)} \cdot \partial_y h(z) \left(\frac{1}{g'(x_{-1})} \cdot \partial_x h(z_{-1}) \right. \\ &\quad \left. + \frac{1}{g'(x_{-1})} \cdot \partial_y h(z_{-1}) \cdot \omega(\hat{z}_{-1}) \right) \\ &= \frac{1}{g'(x)} \partial_x h(z) + \frac{1}{g'(x) g'(x_{-1})} \partial_y h(z) \partial_x h(z_{-1}) \\ &\quad + \frac{1}{g'(x) g'(x_{-1})} \partial_y h(z) \partial_y h(z_{-1}) \omega(\hat{z}_{-1}) \\ &= \cdots = \frac{1}{g'(x)} \partial_x h(z) + \sum_{i=1}^{\infty} \frac{1}{g'(x) \cdot \cdots \cdot g'(x_{-i})} \\ &\quad \cdot \partial_x h(z_{-i}) \partial_y h(z_{-i+1}) \cdots \partial_y h(z) \end{aligned} \tag{12}$$

Now if $\hat{z}' = (z, z'_{-1}, z'_{-2}, \ldots)$ is another prehistory of z from $\hat{\Lambda}$, say with $z'_{-1} \neq z_{-1}$, we have from above that

$$\omega(\hat{z}') = \frac{1}{g'(x'_{-1})} \partial_x h(z'_{-1}) + \sum_{i=2}^{\infty} \frac{1}{g'(x'_{-1}) \cdots g'(x'_{-i})} \cdot \partial_x h(z'_{-i}) \partial_y h(z'_{-i+1}) \cdots \partial_y h(z'_{-1})$$

Therefore

$$\omega(\hat{z}) - \omega(\hat{z}') = \frac{1}{g'(x_{-1})}\partial_x h(z_{-1}) - \frac{1}{g'(x'_{-1})}\partial_x h(z'_{-1})$$
$$+ \left[\sum_{i=2}^{\infty} \frac{\partial_x h(z_{-i}) \cdot \partial_y h(z_{-i+1}) \cdots \partial_y h(z_{-1})}{g'(x_{-1}) \cdots g'(x_{-i})} - \sum_{i=2}^{\infty} \frac{\partial_x h(z'_{-i}) \cdot \partial_y h(z'_{-i+1}) \cdots \partial_y h(z'_{-1})}{g'(x'_{-1}) \cdots g'(x'_{-i})}\right] \tag{13}$$

Let us assume that

$$\beta^2 > g'(x) > \beta \gg 1, \quad x \in J_*,$$

for some large β which depends on α; this holds since the map g expands the small intervals I_1, I_2 to the whole $I = [0, 1]$, and g was assumed increasing.

We assume that $|\partial_y h| < \delta < 1$ on Λ. Now the expression in the straight brackets in (13), is less than $\frac{1}{\beta^2}(1 + \frac{\delta}{\beta} + \cdots) = \frac{1}{\beta^2} \cdot \frac{1}{1-\frac{\delta}{\beta}} < \frac{1.2}{\beta^2}$, if β is large enough.

But if z_{-1}, z'_{-1} are different preimages of z, it follows that we must have $z_{-1} \in \Lambda_{x_{-1}}$ and $z'_{-1} \in \Lambda_{x'_{-1}}$ for two different g-preimages of x, say $x_{-1} \in I_1$ and $x'_{-1} \in I_2$. Then we have $\partial_x h(z_{-1}) = 1$ and $\partial_x h(z'_{-1}) = -1$. Therefore we have that

$$\left|\frac{1}{g'(x_{-1})}\partial_x h(z_{-1}) - \frac{1}{g'(x'_{-1})}\partial_x h(z'_{-1})\right| > \frac{2}{\beta^2}$$

Hence from the above estimate of the expression in the straight brackets of (13), we see that for two prehistories $\hat{z}, \hat{z}' \in \hat{\Lambda}$ of z with $z_{-1} \neq z'_{-1}$, we obtain:

$$|\omega(\hat{z}) - \omega(\hat{z}')| > \frac{0.7}{\beta^2}$$

In general, let $\hat{z}, \hat{z}'$ two different prehistories of z from $\hat{\Lambda}$. Then there exists $m \geq 1$ so that $z_{-m} \neq z'_{-m}$ and $z_{-i} = z'_{-i}$ for $i = 0, \ldots, m-1$ (where as always we denote $z = z_0$). Then similarly as above we obtain that

$$|\omega(\hat{z}) - \omega(\hat{z}')| > \frac{0.7}{\beta^{m+1}}, \tag{14}$$

where we recall that $\beta = \beta(\alpha)$. Clearly if $E^u_{\hat{z}} \neq E^u_{\hat{z}'}$ for two different prehistories of z then also the corresponding local unstable manifolds $W^u_r(\hat{z}), W^u_r(\hat{z}')$ are different, and they are not tangent to each other. □

We estimated thus in the previous Theorem the angle between unstable directions corresponding to different prehistories of the same point, by using the dilation factor β of $g : I_1 \cup I_2 \to I$.

We will give now a generalization of example (2) to a family which is nonlinear in x and linear in y, with fiber contraction factors belonging to a neighbourhood of $\frac{1}{2}$.

First let us fix a small $\alpha \in (0, 1)$. Then take the subintervals $I_1^\alpha, I_2^\alpha \subset I = [0, 1]$ so that I_1^α is contained in $[\frac{1}{2} - \epsilon(\alpha), \frac{1}{2} + \epsilon(\alpha)]$ and I_2^α is contained in $[1 - \alpha - \epsilon(\alpha), 1 - \alpha + \epsilon(\alpha)]$, for some small $\epsilon(\alpha) < \alpha^2$. We take then a strictly increasing smooth map $g : I_1^\alpha \cup I_2^\alpha \to I$ such that $g(I_1^\alpha) = g(I_2^\alpha) = I$; assume that there exists a large $\beta \gg 1$ s. t $\beta^2 > g'(x) > \beta >> 1, x \in I_1^\alpha \cup I_2^\alpha$. Then there exist subintervals $I_{11}^\alpha, I_{12}^\alpha \subset I_1^\alpha, I_{21}^\alpha, I_{22}^\alpha \subset I_2^\alpha$ such that

$g(I^\alpha_{11}) = g(I^\alpha_{21}) = I^\alpha_1$ and $g(I^\alpha_{12}) = g(I^\alpha_{22}) = I^\alpha_2$. We define again $J^\alpha := I^\alpha_{11} \cup I^\alpha_{12} \cup I^\alpha_{21} \cup I^\alpha_{22}$ and $J^\alpha_* := \{x \in J^\alpha, g^i(x) \in J^\alpha, i \geq 0\}$.

Now define $f_\alpha : J^\alpha_* \times I \to J^\alpha_* \times I$,

$$f_\alpha(x, y) = (g(x), h_\alpha(x, y)), \quad \text{with}$$
$$h_\alpha(x, y) = \begin{cases} \psi_{1,\alpha}(x) + s_{1,\alpha} y, & x \in I^\alpha_{11} \\ \psi_{2,\alpha}(x) + s_{2,\alpha} y, & x \in I^\alpha_{21} \\ \psi_{3,\alpha}(x) - s_{3,\alpha} y, & x \in I^\alpha_{12} \\ s_{4,\alpha} y, & x \in I^\alpha_{22}, \end{cases} \tag{15}$$

where for some small ε_0, we take $s_{1,\alpha}, s_{2,\alpha}, s_{3,\alpha}, s_{4,\alpha}$ to be positive numbers, ε_0-close to $\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2}$ respectively; and $\psi_{1,\alpha}(\cdot), \psi_{2,\alpha}(\cdot), \psi_{3,\alpha}(\cdot)$ are smooth (say $\mathcal{C}^2$) functions on I which are ε_0-close in the $\mathcal{C}^1$-metric, to the linear functions $x \to x$, $x \to 1 - x$ and $x \to 1$, respectively. By $|g_1 - g_2|_{\mathcal{C}^1}$ we shall denote the distance in the $\mathcal{C}^1(I)$-metric between two smooth functions on I, g_1 and g_2.

We shall denote also the function $h_\alpha(x, \cdot) : I \to I$ by $h_{x,\alpha}(\cdot)$, for $x \in J^\alpha_*$.

Again when α is unambiguous and fixed, we will not record the dependence of the above on α (but will keep it in mind).

Theorem 3 *There exists a function $\vartheta(\alpha) > 0$ defined for all positive small enough numbers α, with $\vartheta(\alpha) \underset{\alpha \to 0}{\to} 0$ and such that, if f is an arbitrary map defined in (15) whose parameters satisfy:*

$$\max\left\{|\psi_1(x) - x|_{\mathcal{C}^1}, |\psi_2(x) - 1 + x|_{\mathcal{C}^1}, |\psi_3(x) - 1|_{\mathcal{C}^1}, \left|s_1 - \frac{1}{2}\right|, \left|s_2 - \frac{1}{2}\right|, \left|s_3 - \frac{1}{2}\right|, \left|s_4 - \frac{1}{2}\right|\right\} < \vartheta(\alpha) \tag{16}$$

then we obtain:

a) *For $x \in J_* \cap I_1$, there exists a Cantor set $F_x \subset \Lambda_x$, s. t every point of F_x has two different f-preimages in Λ. And if $x \in J_* \cap I_2$, then there exists a Cantor set $F_x \subset \Lambda_x$ s. t every point of F_x has two different f^2-preimages in Λ.*

b) *f is hyperbolic on Λ.*

c) *If $\hat{z}, \hat{z}' \in \hat{\Lambda}$ are two different prehistories of an arbitrary point $z \in \Lambda$, then $E^u_{\hat{z}} \neq E^u_{\hat{z}'}$. We have the same estimate for the angle between $E^u_{\hat{z}}$ and $E^u_{\hat{z}'}$ as in (14).*

Proof The proof uses basically the same ideas as in Theorem 1, Proposition 1 and Theorem 2, with certain modifications.

a) I_1, I_2 are $\epsilon(\alpha)$-close to $\frac{1}{2}, 1 - \alpha$. We assumed in the definition (15), that $\epsilon(\alpha) < \alpha^2 \ll 1$. Like in the proof of Theorem 1, the set $\Lambda_x(n)$ is formed by $k(n)$ disjoint subintervals $J_1(x, n), \ldots, J_{k(n)}(x, n)$ arranged in an increasing order; assume also that for $x \in J_* \cap I_1$, $J_k(x, n) = [a_k(x, n), c_k(x, n)]$, $k = 1, \ldots, k(n)$ and for $x \in J_* \cap I_2$, $J_k(x, n) = [\tilde{a}_k(x, n), \tilde{c}_k(x, n)]$, $k = 1, \ldots, k(n)$. Also for a point $x \in J_*$, there exist two g-preimages of x in J_*, which will be denoted by $x_{-1,1}$ (for the g-preimage of x from I_1), and $x_{-1,2}$ (for the g-preimage of x belonging to I_2).

Then by induction we see that $\tilde{a}_1(x, n) = 0$ for all $n > 1$ and $x \in J_* \cap I_2$. This implies that for all $x \in J_* \cap I_1, n > 1$, we have $a_1(x, n) = h_{x_{-1,2}}(\tilde{a}_1(x_{-1,2}, n - 1)) = h_{x_{-1,2}}(0)$. Therefore we obtain that $a_1(x, n)$ is α^2-close to $\psi_2(1 - \alpha)$ if $x \in J_* \cap I_1$. This implies also that the right most endpoint of $\Lambda_y(n)$, $y \in I_2 \cap J_*$, namely $\tilde{c}_{k(n)}(y, n)$, is α^2-close to $\psi_3(\frac{1}{2}) - s_3 \psi_2(1 - \alpha)$.

We see now that the right endpoint of $J_{k(n)}(x,n), x \in I_1 \cap J_*$, namely $c_{k(n)}(x,n)$ is α^2-close to $\psi_1(\frac{1}{2}) + s_1 c_{k(n-1)}(x_{-1,1}, n-1)$. Thus we see that $c_{k(n)}(x,n) \nearrow c(x,\infty)$ when $n \to \infty$ and that $c(x,\infty)$ is α^2-close to $\psi_1(\frac{1}{2})(1 + s_1 + s_1^2 + \cdots) = \frac{\psi_1(\frac{1}{2})}{(1-s_1)}$. It follows as in Theorem 1 that we do have points of Λ_x as close as we want to $c_{k(n)}(x,n)$ for $n > 1$; hence there are points of Λ_x as close as we want to $c(x,\infty)$ when $x \in J_* \cap I_1$.

Now if $a_1(x,n)$ is $\epsilon(\alpha)$-close to $\psi_2(1-\alpha)$ for $x \in I_1 \cap J_*$, it follows from construction that $\tilde{c}_{k(n)}(x', n)$ is $\epsilon(\alpha)$-close to $\psi_3(\frac{1}{2}) - s_3\psi_2(1-\alpha)$ if $x' \in I_2 \cap J_*$. Recall also that $\epsilon(\alpha) < \alpha^2$. Again, since there are points of Λ_x as close as we want to $a_1(x,n)$ when $x \in I_1 \cap J_*$, we see that $a_1(x,n) = h_{x_{-1,2}}(0) \in \Lambda_x$; thus there exist points of $\Lambda_{x'}$ as close as we want to $\tilde{c}_{k(n)}(x,n) = \psi_3(\frac{1}{2}) - s_3 \cdot a_1(x'_{-1,1}, n)$ for $x' \in I_2 \cap J_*$.

Now for an iteration of order $n > 3$, we will want to have the two phenomena which gave the fractal structure of Λ_x in Theorem 1. The first desired phenomenon is the overlapping in $\Lambda_x, x \in I_1 \cap J_*$, of the first intervals of $h_{x_{-1,1}}(\Lambda_{x_{-1,1}})$ (i.e. subintervals $J_k(x,n), k < m(n)$ for some $m(n)$), with the last intervals of $h_{x_{-1,2}}(\Lambda_{x_{-1,2}})$ (i.e. with the subintervals of type $J_k(x,n), k > p(n)$). And the second desired phenomenon is the appearance of a gap of length comparable to α, in $\Lambda_x, x \in I_2 \cap J_*$, between $h_{x_{-1,2}}(\tilde{c}_{k(n-1)}(x_{-1,2}, n-1))$ and $h_{x_{-1,1}}(c(x_{-1,1},\infty))$. We use $c(x_{-1,1},\infty)$ since $c_{k(n)}(x_{-1,1}, n) \underset{n\to\infty}{\nearrow} c(x_{-1,1},\infty)$ and there are points from $\Lambda_{x_{-1,1}}$ as close as we want to $c(x_{-1,1},\infty)$; thus the gap in Λ_x is bounded above by $h_{x_{-1,1}}(c(x_{-1,1},\infty))$.

Therefore our two conditions are satisfied if:

$$\psi_1\left(\frac{1}{2}\right) + s_1 \cdot \psi_2(1-\alpha) < \psi_2(1-\alpha) + s_2 \cdot \left[\psi_3\left(\frac{1}{2}\right) - s_3\psi_2(1-\alpha)\right], \quad \text{and} \tag{17}$$

$$\frac{\alpha}{2} < \psi_3\left(\frac{1}{2}\right) - s_3 \cdot \frac{\psi_1\left(\frac{1}{2}\right)}{1-s_1} - s_4 \cdot \left[\psi_3\left(\frac{1}{2}\right) - s_3\psi_2(1-\alpha)\right] < \alpha \tag{18}$$

As we can see these two conditions are verified if there exists a sufficiently small $\vartheta(\alpha) > 0$, s.t.

$$\begin{gathered}\left|s_i - \frac{1}{2}\right| < \vartheta(\alpha), \quad i = 1, \ldots, 4 \quad \text{and} \\ \max\left\{|\psi_1(x) - x|_{\mathcal{C}^1}, |\psi_2(x) - 1 + x|_{\mathcal{C}^1}, |\psi_3(x) - 1|_{\mathcal{C}^1}\right\} < \vartheta(\alpha)\end{gathered} \tag{19}$$

It is clear that $\vartheta(\alpha) \underset{\alpha\to 0}{\to} 0$. If $\vartheta(\alpha)$ is small enough, then the thickness of the fibers Λ_x remains larger than $\Delta(\alpha)$. This permits for every $x \in I_1 \cap J_*$, to have an intersection between the images $h_{x_{-1,1}}(\Lambda_{x_{-1,1}})$ and $h_{x_{-1,2}}(\Lambda_{x_{-1,2}})$ inside Λ_x. Thus we obtain a Cantor set $F_x \subset \Lambda_x$ s.t every point of F_x has two different f-preimages inside Λ. We obtain again that

$$\tau(F_x) \geq \sqrt{\Delta(\alpha)}, \quad x \in I_1 \cap J_*,$$

where $\Delta(\alpha) = O(\frac{1}{\alpha}), \alpha > 0$.

b) The hyperbolicity follows in the same way as in Proposition 1, if the inequalities in (19) are satisfied and $\vartheta(\alpha)$ is small enough.

c) The disjointness of unstable directions corresponding to different prehistories of the same point, follows as in the proof of Theorem 2 since the derivatives of ψ_1, ψ_2 are $\vartheta(\alpha)$-close to 1, respectively -1. And similarly we obtain the same estimates (14) for the angle between two unstable directions $E^u_{\hat{z}}$ and $E^u_{\hat{z}'}$, corresponding to two different prehistories of the same point $z \in \Lambda$; we recall also that β depends on α, in (14). □

Let us study now the unstable, respectively stable dimension on Λ; from this study we shall obtain **further information** about the preimage counting function on Λ. We will see more arguments towards the idea that the skew products defined in (15) are **far** from being homeomorphisms on their respective basic set Λ, and also **far** from being 2-to-1 on Λ.

First we have the following result about the unstable dimension:

Theorem 4 *For a small fixed $\alpha > 0$, let the function $f : \Lambda \to \Lambda$ defined in (15). Then the unstable dimension $\delta^u(\hat{z}) = t^u, \forall \hat{z} \in \hat{\Lambda}$, where t^u is the unique zero of the pressure function $t \to P_{\hat{f}|_{\hat{\Lambda}}}(t\Phi^u)$, and where $\Phi^u(\hat{y}) := -\log|Df_u(\hat{y})|, \hat{y} \in \hat{\Lambda}$. Consequently if $g'(x) > \beta(\alpha) >> 1$ on J_*, we have*

$$\delta^u(\hat{z}) < \frac{\log 2}{\log \frac{\beta(\alpha)}{2}}, \quad \hat{z} \in \hat{\Lambda}$$

Proof The first part of the Theorem follows from [4] since the unstable manifolds in our case are 1-dimensional (hence conformal). So $\delta^u(\hat{z}) = t^u$, for all $\hat{z} \in \hat{\Lambda}$. We notice that t^u is the zero of a pressure function calculated on the natural extension $\hat{\Lambda}$.

Now, from the proof of Theorem 3 we know that $\omega(\hat{z}) < \frac{1}{\beta(\alpha)}, \hat{z} \in \hat{\Lambda}$. Hence $|Df_u(\hat{z})| > \frac{\beta(\alpha)}{2}, \hat{z} \in \hat{\Lambda}$ so $\Phi^u(\hat{z}) < -\log \frac{\beta(\alpha)}{2}, \hat{z} \in \hat{\Lambda}$.

Also it is easy to see that $h_{top}(f|_\Lambda) = \log 2$ since the Bowen balls of f are given mainly by the expansion of g in the horizontal direction, and $g|_{J_*}$ is conjugated to σ_2 on the one-sided Bernoulli shift Σ_2^+. Therefore $0 = P_{\hat{f}|_{\hat{\Lambda}}}(t^u\Phi^u) < -t^u \log \frac{\beta(\alpha)}{2} + \log 2$ and we obtain $\delta^u(\hat{z}) = t^u < \frac{\log 2}{\log \frac{\beta(\alpha)}{2}}$. □

Now we want to estimate the stable dimension over Λ; by contrast to Theorem 4 or to the diffeomophism case, we do not know here that $\delta^s(z)$ is constant when z ranges in Λ.

Recall that we denoted by $d(\cdot)$ the *preimage counting function* for f on Λ, defined by $d(z) := \text{Card}\{z' \in \Lambda, f(z') = z\}, z \in \Lambda$. One major difficulty is that $d(\cdot)$ **is not necessarily continuous** on Λ, and not necessarily constant. So the expression $P(t\Phi^s - \log d)$ does not make sense, since pressure was defined for continuous maps. We will overcome this obstacle in 2 different ways. The first one will be by using a notion of **inverse pressure** (see [8]). For a continuous non-invertible map $f : X \to X$ on a compact metric space X, the inverse pressure $P_f^-(\cdot)$ is a functional defined by using consecutive f-preimages of points (rather than the forward iterates like for usual pressure). It is useful in the case of estimating the stable dimension of endomorphisms that are not necessarily constant-to-1 [5,8]. For the negative function Φ^s, we proved that there exists a unique zero t_s^- of the function $t \to P_f^-(t\Phi^s)$, and that in our case

$$\delta^s(z) \le t_s^-, \quad z \in \Lambda$$

The second way is by using **continuous upper bounds** $\eta(\cdot)$ for the preimage counting function $d(\cdot)$, and then to employ the unique zero t_η of the function $t \to P(t\Phi^s - \log \eta)$.

Theorem 5 *Let a sufficiently small $\alpha > 0$ and a function f defined as in (15), and assume that the parameters of f satisfy condition (16).*

a) *Then the stable dimension $\delta^s(z) \le t_s^- < 1$, for any point $z \in \Lambda$.*
b) *If $\eta(\cdot)$ is a continuous function on Λ such that $d(z) \le \eta(z), z \in \Lambda$, it follows that $\delta^s(z) \ge t_\eta, z \in \Lambda$, where t_η is the unique zero of the function $t \to P(t\Phi^s - \log \eta)$.*

Proof a) We take a fixed small enough $\alpha > 0$; this will imply that $\vartheta(\alpha)$ is also small enough such that Theorem 3 works. Hence f is hyperbolic as an endomorphism on Λ. Also we notice that it is conformal on stable manifolds as these have real dimension 1, and also that f does not have any critical points in Λ. We proved in [8] that in this case, $\delta^s(z) \le t_s^-$, where t_s^- is the unique zero of the inverse pressure function $t \to P_f^-(t\Phi^s)$.

From Theorem 3, an arbitrary fiber Λ_x does not contain intervals. Hence since local stable manifolds are contained in the vertical fibers in our case, it follows that no local stable manifold is contained in Λ. Thus Λ is not a local repellor, in the sense of [5]. Translating to our case the result of [5] we obtain then, that $t_s^- < 1$.

Therefore we obtain $\delta^s(z) \le t_s^- < 1, z \in \Lambda$.

b) The estimate $\delta^s(z) \ge t_\eta$ follows immediately from [7], since $d(\cdot) \le \eta(\cdot)$ and since f is hyperbolic on Λ, conformal on stable manifolds and does not have critical points in Λ.

□

Remark (1) Theorem 5 permits us to get **better and better lower estimates** for $\delta^s(z)$, if we take continuous functions η which approximate better and better the preimage counting function $d(\cdot)$. Indeed in the notation of Theorem 3, let an open subset $W_1 \subset V_1$ so that $\bar{W}_1 \subset V_1$. Then let $\eta(\cdot)$ a continuous real function on Λ such that $\eta(z) \equiv 1, z \in W_1, \eta(z) \equiv 2, z \in V_2$, and $1 \le \eta \le 2$ in rest. Then from Theorem 5 it follows that $\delta^s(z) \ge t_\eta$. When the set W_1 is increased inside V_1, we will obtain smaller and smaller maps η, and thus larger and larger zeros t_η. Still, from part a) of Theorem 5, we know that always $t_\eta \le \delta^s(z) \le t_s^- < 1$ for these functions η.

(2) Theorem 5 shows that from the point of view of stable dimension, when the fiber contraction factors s_1, s_2, s_3, s_4 are all equal to $\frac{1}{2}$, then $f|_\Lambda$ is far from being a homeomorphism for **all choices of parameters** $\psi_j, j = 1, 2, 3$ satisfying condition (16). Indeed if $\delta^s(z)$ were the zero t_s of the pressure $t \to P(t\Phi^s)$, then since $\Phi^s \equiv -\log 2$ on Λ, we would obtain $t^s = 1$. But we saw in Theorem 5 that $\delta^s(z) < 1, z \in \Lambda$.

(3) Also we notice that in the setting of Theorem 5, $f|_\Lambda$ **is not 2-to-1** either. Indeed assume that the parameters $s_i, \psi_j, i = 1, \dots, 4, j = 1, 2$ satisfy (16) and α is small enough and let the function f given by these parameters in (15) (we do not record now the dependence of f on α, but are keeping it in mind). Then from Theorem 3, we know that f is hyperbolic as an endomorphism on Λ. In that case, from [8] it follows that for all $z \in \Lambda$, $\delta^s(z)$ would be equal to the unique zero t_2^s of $t \to P(t\Phi^s - \log 2)$. But we saw above that $h_{top}(f|_\Lambda) = \log 2$; hence it would follow that $t_2^s = 0$. On the other hand, we know from the proof of Theorem 3 that there are points $z \in \Lambda_x$ with $\delta^s(z) = HD(\Lambda_z) \ge \frac{\log 2}{\log(2+\frac{1}{\sqrt{\Delta(\alpha)}})} > 0$.

We have thus obtained a contradiction; hence $f|_\Lambda$ is not 2-to-1. □

Let us assume now that α is fixed and $s_i = \frac{1}{2}, i = 1, \dots, 4$ in the definition (15) of f. Now if there were only at most m prehistories in the natural extension $\hat{\Lambda}$ (associated to this f) for each point of Λ, then the stable dimension would still be equal to the zero t_s of the pressure $t \to P(t\Phi^s)$; this follows from [8]. Therefore for any integer $m \ge 1$, there must exist a point $z_m \in \Lambda$ with more than m prehistories in $\hat{\Lambda}$. One may assume that the points z_m converge to some point $z \in \Lambda$. We will prove that z has infinitely many prehistories in $\hat{\Lambda}$. First we know that each z_m has at least m prehistories in $\hat{\Lambda}$ for all $m \ge 1$. Denote the set of all these prehistories by $\mathcal{P}$; clearly $\mathcal{P}$ is infinite. Hence there is some level k_1 so that there exists an infinite set $\mathcal{P}(k_1, 1) \subset \mathcal{P}$ of prehistories whose k_1 entries are close to some k_1-preimage $z(k_1, 1)$ of z, and another disjoint infinite set of prehistories $\mathcal{P}(k_1, 2) \subset \mathcal{P}$ whose elements have their k_1 entries close to another k_1-preimage of z called $z(k_1, 2)$, with $d(z(k_1, 2), z(k_1, 1)) > \varepsilon_0(k_1) > 0$; this estimate follows from the fact that f does not have critical points in Λ. Next we take the family $\mathcal{P}(k_1, 2)$ and show as above that there exists some $k_2 > k_1$ and two disjoint subcollections of $\mathcal{P}(k_1, 2)$, denoted by $\mathcal{P}(k_2, 1)$ and $\mathcal{P}(k_2, 2)$, s. t the distance between the k_2 entry of any prehistory from $\mathcal{P}(k_2, 1)$ and the k_2 entry of any prehistory from $\mathcal{P}(k_2, 2)$, is larger than a fixed $\varepsilon_0(k_2) > 0$. This is done similarly as in the previous step and we use again that f does not have critical points in Λ. We obtain thus by induction a sequence of disjointed infinite families of prehistories $\mathcal{P}(k_n, 1)$, $n \geq 1$. Also recall that $\hat{\Lambda}$ is compact so any infinite family has accumulation points in $\hat{\Lambda}$. Hence from the above procedure we see that we can separate at some level any accumulation point of $\mathcal{P}(k_n, 1)$ from any accumulation point of $\mathcal{P}(k_p, 1)$ whenever $n \neq p$; thus any two such accumulation points must be different. But on the other hand we assumed the points z_m converge to z, so any accumulation point of an infinite family $\mathcal{P}(k_p, 1)$ must be some prehistory of z in $\hat{\Lambda}$, for any $p \geq 1$. We thus obtain an infinite collection of different prehistories of z in $\hat{\Lambda}$.

Then by employing also Theorem 2, we have proved the following:

Corollary 1 *Let a small $\alpha > 0$ and a function f as in (15), and assume that $s_i = \frac{1}{2}, i = 1, \dots, 4$ and that the parameters $\psi_j, j = 1, \dots, 3$ satisfy (16). Then there exists a point $z \in \Lambda$ having infinitely many different prehistories in $\hat{\Lambda}$, and thus infinitely many different unstable directions of type $E^u_{\hat{z}}$.*

We show now that our examples are both far from having a homeomorphism-type behaviour, and also far from the constant-to-1 maps of [8]:

Corollary 2 *Let a small $\alpha > 0$ and a function f defined as in (15), s. t the parameters $s_i, \psi_j, i = 1, \dots, 4, j = 1, \dots, 3$ of f satisfy (16). Write Λ as the union $V_1 \cup V_2$, where V_1 is defined as the set of points having only one f-preimage inside Λ and V_2 is the set of points having exactly two f-preimages in Λ.*

a) *Then $\delta^s(z) \in (\frac{\log 2}{\log(2+\frac{1}{\Delta(\alpha)})}, 1)$, $z \in \Lambda$. So if α tends to 0, then the stable dimension at an arbitrary point of Λ may be made as close as we want to 1, but always strictly smaller than 1.*
b) *V_1 is an open uncountable set in Λ, and V_2 is a closed set in Λ.*
c) *Assume moreover that in the definition (15) of f, the contraction factors $s_i, i = 1, \dots, 4$ are all equal to $\frac{1}{2}$. Then V_2 is uncountable as well.*

Proof a) We apply a result of Palis and Takens [11] giving an estimate for the Hausdorff dimension of a Cantor set K in the line, as follows:

$$HD(K) \geq \frac{\log 2}{\log(2 + \frac{1}{\tau(K)})}$$

We showed in Theorem 3 that we have $\tau(\Lambda_x) \geq \Delta(\alpha)$; and as in the proof of Theorem 1, $\Delta(\alpha) \underset{\alpha \to 0}{\to} \infty$. Hence combining also with Theorem 5, we obtain the estimates.

b) We recall that $d(z)$ was defined as the number of f-preimages of z belonging to Λ.

We can partition now Λ into two subsets, $V_1 := \{z \in \Lambda, d(z) = 1\}$ and $V_2 := \{z \in \Lambda, d(z) = 2\}$. It can be seen easily that V_1 is open and V_2 is closed, since $d(\cdot)$ is upper semi-continuous. We remark that V_1, V_2 are not necessarily f-invariant.

If V_1 would be countable, then we can approximate any point from Λ by points from V_2; given an arbitrary $n > 1$, we can even approximate any point $z \in \Lambda$ with points w such that $w \in V_2$ and all its preimages of order less than n are also in V_2; i.e. if $w_{-i} \in \Lambda \cap f^{-i} w$, then $w_{-i} \in V_2, i < n$. This makes the proof of Theorem 3.1 of [8] to work (holomorphicity in that Theorem is not essential); all that is important is that f be conformal on stable manifolds, and this is satisfied in our case as the stable manifolds are 1-dimensional.

Hence it would follow that $\delta^s(z) = t_2^s$, where t_2^s is the unique zero of the pressure $t \to P(t\Phi^s - \log 2)$. Now $h_{top}(f|_\Lambda) = \log 2$, since the expansion takes place mainly in the horizontal direction and since $g|_{J_*}$ is topologically conjugate to σ_2 on the one-sided Bernoulli space Σ_2^+. Thus it follows that $t_2^s = 0$.

However we saw in part a) that $HD(\Lambda_x) \geq \frac{\log 2}{\log(2+\frac{1}{\Delta(\alpha)})} > 0$. So we cannot have $HD(\Lambda_x) = t_2^s$, and we obtain a contradiction. Therefore V_1 is **uncountable**.

c) Now assume that all the contraction factors of f on fibers are equal to $\frac{1}{2}$. Let us suppose also that V_2 is countable. Then as above, for any $n > 1$ we can approximate any point of Λ with points having only one preimage of order n and the proof of Theorem 3.1 from [8] gives that $\delta^s(z) = t^s$, $z \in \Lambda$, where t^s is the unique zero of the pressure function $t \to P(t\Phi^s)$. But since in our case $\Phi^s \equiv -\log 2$ on Λ, we obtain $t^s = 1$. But from Theorem 5, $t^s < 1$, thus a contradiction. In conclusion V_2 is **uncountable** as well. □

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