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Nilpotent extensions of blocks

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1 Introduction

1.1

The *nilpotent blocks* over an algebraically closed field of characteristic $p > 0$ were introduced in [2] as a translation for blocks of the well-known Frobenius Criterion on p-nilpotency for finite groups. They correspond to the simplest situation with respect to the so-called *fusion* inside a defect group, and the structure of the source algebras of the nilpotent blocks determined in [9] confirms that these blocks represent indeed the easiest possible situation.

1.2

However, when the field of coefficients is not algebraically closed, together with Fan Yun we have seen in [3] that, in the general situation, the structure of the source algebra of a block which, after a suitable scalar extension, decomposes in a sum of nilpotent blocks—a structure that we determine in [3]—need not be so simple.

1.3

At that time, we already knew some examples of a similar fact in group extensions, namely that a *non-nilpotent* block of a normal subgroup H of a finite group G may decompose in a sum of nilpotent blocks of G. In this case, we also have been able to describe the source algebra structure, which is quite similar to (but easier than) the structure described in [3]. With a big delay, we explain this result here.

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1.4

Actually, this phenomenon is perhaps better described by saying that *a normal sub-block of a nilpotent block need not be nilpotent*. However, the normal sub-blocks of nilpotent blocks are quite special: they are *basically Morita equivalent* [15, §7] to the corresponding block of their *inertial subgroup*. Then, as a matter of fact, a normal sub-block of such a block still fulfills the same condition.

1.5

Thus, let us call *inertial block* any block of a finite group that is *basically Morita equivalent* [15, §7] to the corresponding block of its *inertial subgroup*; as a matter of fact, in [12, Corollaire 3.6], we already exhibit a large family of inertial blocks; see also [14, Appendix]. The main purpose of this paper is to prove that *a normal sub-block of an inertial block is again an inertial block*. Since a nilpotent block is *basically Morita equivalent* to its defect group [9, Theorem 1.6 and (1.8.1)], and the corresponding block of its *inertial subgroup* is also nilpotent, a nilpotent block is, in particular, an inertial block and thus, our main result applies.

2 Quoted results and inertial blocks

2.1

Throughout this paper p is a fixed prime number, k an algebraically closed field of characteristic p and $\mathcal{O}$ a complete discrete valuation ring of characteristic zero having the *residue field* k. Let G be a finite group; following Green [5], a G-*algebra* is a torsion-free $\mathcal{O}$-algebra A of finite $\mathcal{O}$-rank endowed with a G-action; we say that A is *primitive* if the unity element is primitive in A^G. A G-algebra homomorphism from A to another G-algebra A' is a *not necessarily unitary* algebra homomorphism $f: A \to A'$ compatible with the G-actions. We say that f is an *embedding* whenever

$$\mathrm{Ker}(f) = \{0\} \text{ and } \mathrm{Im}\,(f) = f(1_A)A'f(1_A), \tag{2.1.1}$$

and that f is a *strict semicovering* if f is *unitary*, the *radical* $J(A)$ of A contains $\mathrm{Ker}(f)$ and, for any p-subgroup P of G, $J(A'^P)$ contains $f(J(A^P))$ and $f(i)$ is primitive in A'^P for any primitive idempotent i of A^P [6, §3].

2.2

Recall that, for any subgroup H of G, a *point* α of H on A is an $(A^H)^*$-conjugacy class of primitive idempotents of A^H and the pair H_α is a *pointed group* on A [7, 1.1]; if $H = \{1\}$, we simply say that α is a *point* of A. For any $i \in \alpha$, iAi has an evident structure of H-algebra and we denote by A_α one of these mutually $(A^H)^*$-conjugate H-algebras and by $A(H_\alpha)$ the *simple quotient* of A^H determined by α; we call *multiplicity* of α the *square root* of the dimension of $A(H_\alpha)$. If $f: A \to A'$ is a G-algebra homomorphism and α' a point of H on A', we call *multiplicity* $\mathrm{m}(f)_\alpha^{\alpha'}$ of f at (α, α') the dimension of the image of $f(i)A'^H i'$ in $A'(H_{\alpha'})$ for $i \in \alpha$ and $i' \in \alpha'$; we still consider the H-algebra $A'_\alpha = f(i)A'f(i)$ together with the unitary H-algebra homomorphism induced by f and the embedding of H-algebras

$$A_\alpha \longrightarrow A'_\alpha \longleftarrow A'_{\alpha'}. \tag{2.2.1}$$

A second pointed group K_β on A is *contained* in H_α if $K \subset H$ and, for any $i \in \alpha$, there is $j \in \beta$ such that [7, 1.1]

$$ij = j = ji; \tag{2.2.2}$$

then, it is clear that the $(A^K)^*$-conjugation induces K-algebra embeddings

$$f_\beta^\alpha : A_\beta \longrightarrow \mathrm{Res}_K^H(A_\alpha). \tag{2.2.3}$$

2.3

Following Broué, for any p-subgroup P of G we consider the *Brauer quotient* and the *Brauer homomorphism* [1, 1.2]

$$\mathrm{Br}_P^A : A^P \longrightarrow A(P) = A^P \Big/ \sum_Q A_Q^P\ , \tag{2.3.1}$$

where Q runs over the set of proper subgroups of P, and call *local* any point γ of P on A not contained in $\mathrm{Ker}(\mathrm{Br}_P^A)$ [7, 1.1]. Recall that *a local pointed group P_γ contained in H_α is maximal if and only if* $\mathrm{Br}_P(\alpha) \subset A(P_\gamma)_P^{N_H(P_\gamma)}$ [7, Proposition 1.3] and then *the P-algebra A_γ*—called a *source algebra* of A_α—*is Morita equivalent to* A_α [17, 6.10]; moreover, *the maximal local pointed groups P_γ contained in H_α*—called the *defect pointed groups* of H_α—*are mutually H-conjugate* [7, Theorem 1.2].

2.4

Let us say that A is a *p-permutation G-algebra* if a Sylow p-subgroup of G stabilizes a basis of A [1, 1.1]. In this case, recall that if P is a p-subgroup of G and Q a normal subgroup of P then the corresponding Brauer homomorphisms induce a k-algebra isomorphism [1, Proposition 1.5]

$$(A(Q))\,(P/Q) \cong A(P); \tag{2.4.1}$$

moreover, choosing a point α of G on A, we call *Brauer* (α, G)*-pair* any pair (P, e_A) formed by a p-subgroup P of G such that $\mathrm{Br}_P^A(\alpha) \neq \{0\}$ and by a primitive idempotent e_A of the center $Z(A(P))$ of $A(P)$ such that

$$e_A \cdot \mathrm{Br}_P^A(\alpha) \neq \{0\}; \tag{2.4.2}$$

note that any local pointed group Q_δ on A *contained* in G_α determines a Brauer (α, G)-pair (Q, f_A) fulfilling $f_A \cdot \mathrm{Br}_Q^A(\delta) \neq \{0\}$.

2.5

Then, it follows from Theorem 1.8 in [1] that *the inclusion between the local pointed groups on A induces an inclusion between the Brauer* (α, G)*-pairs*; explicitly, if (P, e_A) and (Q, f_A) are two Brauer (α, G)-pairs then we have

$$(Q, f_A) \subset (P, e_A) \tag{2.5.1}$$

whenever there are local pointed groups P_γ and Q_δ on A fulfilling

$$Q_\delta \subset P_\gamma \subset G_\alpha, \quad f_A \cdot \mathrm{Br}_Q^A(\delta) \neq \{0\} \quad \text{and} \quad e_A \cdot \mathrm{Br}_P^A(\gamma) \neq \{0\}. \tag{2.5.2}$$

Actually, according to the same result, for any p-subgroup P of G, any primitive idempotent e_A of $Z(A(P))$ fulfilling $e_A \cdot \mathrm{Br}_P^A(\alpha) \neq \{0\}$ and any subgroup Q of P, there is a unique primitive idempotent f_A of $Z(A(Q))$ fulfilling

$$e_A \cdot \mathrm{Br}_P^A(\alpha) \neq \{0\} \quad \text{and} \quad (Q, f_A) \subset (P, e_A). \tag{2.5.3}$$

Once again, *the maximal Brauer* (α, G)*-pairs are pairwise* G*-conjugate* [1, Theorem 1.14].

2.6

Here, we are specially interested in the G-algebras A endowed with a group homomorphism $\rho\colon G \to A^*$ inducing the action of G on A, called G*-interior algebras*; in this case, for any pointed group H_α on A, $A_\alpha = iAi$ has a structure of H*-interior algebra* mapping $y \in H$ on $\rho(y)i = i\rho(y)$; moreover, setting $x \cdot a \cdot y = \rho(x)a\rho(y)$ for any $a \in A$ and any $x, y \in G$, a G-interior algebra homomorphism from A to another G-interior algebra A' is a G-algebra homomorphism $f\colon A \to A'$ fulfilling

$$f(x \cdot a \cdot y) = x \cdot f(a) \cdot y. \tag{2.6.1}$$

2.7

In particular, if H_α and K_β are two pointed groups on A, we say that an injective group homomorphism $\varphi\colon K \to H$ is an A-*fusion from* K_β *to* H_α whenever there is a K-interior algebra *embedding*

$$f_\varphi : A_\beta \longrightarrow \mathrm{Res}_K^H(A_\alpha) \tag{2.7.1}$$

such that the inclusion $A_\beta \subset A$ and the composition of $f_{\hat\varphi}$ with the inclusion $A_\alpha \subset A$ are A^*-conjugate; we denote by $F_A(K_\beta, H_\alpha)$ the set of H-conjugacy classes of A-fusions from K_β to H_α and, as usual, we write $F_A(H_\alpha)$ instead of $F_A(H_\alpha, H_\alpha)$. If $A_\alpha = iAi$ for $i \in \alpha$, it follows from [8, Corollary 2.13] that we have a group homomorphism

$$F_A(H_\alpha) \longrightarrow N_{A_\alpha^*}(H \cdot i) \Big/ H \cdot (A_\alpha^H)^* \tag{2.7.2}$$

and then we consider the k^*-group $\hat F_A(H_\alpha)$ defined by the *pull-back*

$$\begin{array}{ccc} F_A(H_\alpha) & \longrightarrow & N_{A_\alpha^*}(H \cdot i) \Big/ H \cdot (A_\alpha^H)^* \\ \uparrow & & \uparrow \\ \hat F_A(H_\alpha) & \longrightarrow & N_{A_\alpha^*}(H \cdot i) \big/ H \cdot \Big(i + J(A_\alpha^H)\Big). \end{array} \tag{2.7.3}$$

2.8

Recall that, for any subgroup H of G and any H-interior algebra B, the *induced* G*-interior algebra* is the induced bimodule

$$\mathrm{Ind}_H^G(B) = k_*G \otimes_{k_*H} B \otimes_{k_*H} k_*G, \tag{2.8.1}$$

endowed with the distributive product defined by the *formula*

$$(x \otimes b \otimes y)(x' \otimes b' \otimes y') = \begin{cases} x \otimes b.yx'.b' \otimes y' & \text{if} \quad yx' \in H \\ 0 & \text{otherwise} \end{cases} \tag{2.8.2}$$

where $x, y, x', y' \in G$ and $b, b' \in B$, and with the structural homomorphism

$$G \longrightarrow \mathrm{Ind}_H^G(B) \tag{2.8.3}$$

mapping $x \in G$ on the element

$$\sum_y xy \otimes 1_B \otimes y^{-1} = \sum_y y \otimes 1_B \otimes y^{-1}x \tag{2.8.4}$$

where $y \in G$ runs over a set of representatives for G/H.

2.9

Obviously, the *group algebra* $\mathcal{O}G$ is a p-permutation G-interior algebra and, for any primitive idempotent b of $Z(\mathcal{O}G)$—called an $\mathcal{O}$*-block* of G—the conjugacy class $\alpha = \{b\}$ is a *point* of G on $\mathcal{O}G$. Moreover, for any p-subgroup P of G, the Brauer homomorphism $\mathrm{Br}_P = \mathrm{Br}_P^{kG}$ induces a k-algebra isomorphism [10, 2.8.4]

$$kC_G(P) \cong (\mathcal{O}G)(P); \tag{2.9.1}$$

thus, up to identification throughout this isomorphism, in a Brauer $(\{b\}, G)$-pair (P, e) as defined above—called *Brauer* (b, G)*-pair* from now on—e is nothing but a k-block of $C_G(P)$ such that $e\mathrm{Br}_P(b) \neq 0$. Setting

$$\bar{C}_G(P) = C_G(P)/Z(P), \tag{2.9.2}$$

recall that the image $\bar{e}$ of e in $k\bar{C}_G(P)$ is a k-block of $\bar{C}_G(P)$ and that the *Brauer First Main Theorem* affirms that (P, e) *is maximal if and only if the* k*-algebra* $k\bar{C}_G(P)\bar{e}$ *is simple and the inertial quotient*

$$E = N_G(P, e)/P \cdot C_G(P) \tag{2.9.3}$$

is a p'*-group* [17, Theorem 10.14].

2.10

For any p-subgroup P of G and any subgroup H of $N_G(P)$ containing $P \cdot C_G(P)$, we have

$$\mathrm{Br}_P\left((\mathcal{O}G)^H\right) = (\mathcal{O}G)(P)^H \tag{2.10.1}$$

and therefore *any* k*-block* e *of* $C_G(P)$ *determines a unique point* β *of* H *on* $\mathcal{O}G$ (cf. 2.2) *such that* H_β *contains* P_γ *for a local point* γ *of* P *on* $\mathcal{O}G$ *fulfilling* [9, Lemma 3.9]

$$e \cdot \mathrm{Br}_P(\gamma) \neq \{0\}. \tag{2.10.2}$$

Recall that, if Q is a subgroup of P such that $C_G(Q) \subset H$ then the k-blocks of $C_G(Q) = C_H(Q)$ determined by (P, e) from G and from H coincide [1, Theorem 1.8]. Note that if P is normal in G then the kernel of the obvious k-algebra homomorphism $kG \to k(G/P)$ is contained in the *radical* $J(kG)$ and contains $\mathrm{Ker}(\mathrm{Br}_P)$; thus, in this case, isomorphism 2.9.1 implies that *any point of* P *on* kG *is local.*

2.11

Moreover, for any local pointed group P_γ on $\mathcal{O}G$, the action of $N_G(P_\gamma)$ on the simple algebra $(\mathcal{O}G)(P_\gamma)$ (cf. 2.2) determines a central k^*-extension or, equivalently, a *k^*-group* $\hat{N}_G(P_\gamma)$ [10, §5] and it is clear that the Brauer homomorphism Br_P determines a $N_G(P_\gamma)$-stable injective group homomorphism from $C_G(P)$ to $\hat{N}_G(P_\gamma)$. Then, up to a suitable identification, we set

$$E_G(P_\gamma) = N_G(P_\gamma)/P \cdot C_G(P) \quad \text{and} \quad \hat{E}_G(P_\gamma) = \hat{N}_G(P_\gamma)/P \cdot C_G(P); \qquad (2.11.1)$$

recall that from [8, Theorem 3.1] and [10, Proposition 6.12] we obtain a *canonical* k^*-group isomorphism (cf. 2.7.3)

$$\hat{E}_G(P_\gamma)^\circ \cong \hat{F}_{\mathcal{O}G}(P_\gamma). \qquad (2.11.2)$$

2.12

In particular, a maximal local pointed group P_γ on $\mathcal{O}Gb$ determines a k-block e of $C_G(P)$, which is still a k-block of the group

$$N = N_G(P_\gamma) = N_G(P, e), \qquad (2.12.1)$$

called the *inertial subgroup* of b, and also determines a unique point ν of N on $\mathcal{O}Gb$ such that $P_\gamma \subset N_\nu$ (cf. 2.10); obviously, we have $E = E_G(P_\gamma)$ (cf. 2.9.3), P_γ is still a *defect pointed group* of N_ν and (P, e) is a maximal Brauer $(\hat{e}, N)$-pair, where $\hat{e}$ denotes the $\mathcal{O}$-block of N lifting e. As above, N acts on the simple k-algebra (cf. 2.9)

$$k\bar{C}_G(P)\bar{e} \cong (\mathcal{O}G)(P_\gamma) \qquad (2.12.2)$$

and therefore we get *k^*-groups* $\hat{N}$ and $\hat{E}^\circ = \hat{E}_G(P_\gamma)$.

2.13

Moreover, since E is a p'-group, it follows from [17, Lemma 14.10] that the short exact sequence

$$1 \longrightarrow P/Z(P) \longrightarrow N/C_G(P) \longrightarrow E \longrightarrow 1 \qquad (2.13.1)$$

splits and that all the splitings are conjugate to each other; thus, any spliting determines an action of E on P and it is easily checked that the semidirect products

$$L = P \rtimes E \quad \text{and} \quad \hat{L} = P \rtimes \hat{E} \qquad (2.13.2)$$

do not depend on our choice. At this point, it follows from [10, Proposition 14.6] that the source algebra of the block $\hat{e}$ of N is isomorphic to the P-interior algebra $\mathcal{O}_*\hat{L}$, and therefore it follows from [3, Proposition 4.10] that the multiplication in $\mathcal{O}Gb$ by a suitable idempotent $\ell \in \nu$ determines an injective unitary P-interior algebra homomorphism

$$\mathcal{O}_*\hat{L} \longrightarrow (\mathcal{O}G)_\gamma. \qquad (2.13.3)$$

2.14

On the other hand, a *Dade P-algebra* over $\mathcal{O}$ is a p-permutation P-algebra S which is a *full matrix algebra over $\mathcal{O}$* and fulfills $S(P) \neq \{0\}$ [11, 1.3]. For any subgroup Q of P, setting $\bar{N}_P(Q) = N_P(Q)/Q$ we have (cf. 2.4.1)

$$(S(Q))\left(\bar{N}_P(Q)\right) \cong S\left(N_P(Q)\right) \tag{2.14.1}$$

and therefore $\mathrm{Res}_Q^P(S)$ is a Dade Q-algebra; moreover, it follows from [11, 1.8] that the *Brauer quotient* $S(Q)$ is a Dade $\bar{N}_P(Q)$-algebra; thus, Q has a unique *local point* on S. In particular, if S is *primitive* (cf. 2.1) then $S(P) \cong k$ and therefore we have

$$\dim(S) \equiv 1 \pmod{p}, \tag{2.14.2}$$

so that the action of P on S can be lifted to a unique group homomorphism from P to the kernel of the determinant $\det_S$ over S; at this point, it follows from [11, 3.13] that the action of P on S always can be lifted to a well-determined P-interior algebra structure for S.

2.15

Recall that a block b of G is called *nilpotent* whenever the quotients $N_G(Q, f)/C_G(Q)$ are p-groups for all the Brauer (b, G)-pairs (Q, f) [2, Definition 1.1]; by the main result in [9], *the block b is nilpotent if and only if, for a maximal local pointed group P_γ on $\mathcal{O}Gb$, P stabilizes a unitary primitive Dade P-subalgebra S of $(\mathcal{O}Gb)_\gamma$ fulfilling*

$$(\mathcal{O}Gb)_\gamma = SP \cong S \otimes_{\mathcal{O}} \mathcal{O}P \tag{2.15.1}$$

where we denote by SP the obvious $\mathcal{O}$-algebra $\bigoplus_{u\in P} Su$ and, for the right-hand isomorphism, we consider the well-determined P-interior algebra structure for S.

2.16

Now, with the notation in 2.12 above, we say that the block b of G is *inertial* if it is *basically Morita equivalent* [15, 7.3] to the corresponding block $\hat{e}$ of the *inertial subgroup N* of b or, equivalently, if there is a primitive Dade P-algebra S such that we have a P-interior algebra embedding [15, Theorem 6.9 and Corollary 7.4]

$$(\mathcal{O}G)_\gamma \longrightarrow S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L}. \tag{2.16.1}$$

Note that, in this case, in fact *we have a P-interior algebra isomorphism*

$$(\mathcal{O}G)_\gamma \cong S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L} \tag{2.16.2}$$

and the Dade P-algebra S is uniquely determined; indeed, the uniqueness of S follows from [19, Lemma 4.5] and it is easily checked that

$$(S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L})(P) \cong S(P) \otimes_k (\mathcal{O}_*\hat{L})(P) \cong kZ(P) \tag{2.16.3}$$

and that the kernel of the Brauer homomorphism $\mathrm{Br}_P^{S\otimes_{\mathcal{O}}\mathcal{O}_*\hat{L}}$ is contained in the radical of $S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L}$, so that this P-interior algebra is also primitive.

3 Normal sub-blocks of inertial blocks

3.1

Let G be a finite group, b an $\mathcal{O}$*-block* of G and (P, e) a maximal Brauer (b, G)-pair (cf. 2.9). Let us say that an $\mathcal{O}$*-block* c of a normal subgroup H of G is a *normal sub-block* of b if we have $cb \neq 0$; we are interested in the relationship between the source algebras of b and c, specially in the case where b is *inertial*.

3.2

Note that we have $b\mathrm{Tr}_{G_c}^G(c) = b$ where G_c denotes the stabilizer of c in G; since we know that $e\mathrm{Br}_P(b) \neq 0$ (cf. 2.9), up to modifying our choice of (P, e) we may assume that P stabilizes c; then, considering the G-stable semisimple k-subalgebra $\sum_x \mathcal{O} \cdot bc^x$ of $\mathcal{O}G$, where $x \in G$ runs over a set of representatives for G/G_c, it follows from [19, Proposition 3.5] that bc is an $\mathcal{O}$-block of G_c and that P remains a defect p-subgroup of this block, and then from [19, Proposition 3.2] that we have

$$\mathcal{O}Gb \cong \mathrm{Ind}_{G_c}^G(\mathcal{O}G_c\, bc), \tag{3.2.1}$$

so that the source algebras of the $\mathcal{O}$-block b of G and of the block bc of G_c are isomorphic.

3.3

Thus, from now on we assume that G fixes c, so that we have $bc = b$. Then, note that $\alpha = \{c\}$ is a point of G on $\mathcal{O}H$ (cf. 2.2), so that, choosing a block e^H of $C_H(P)$ such that $e^H e \neq 0$, (P, e^H) is a *Brauer* (α, G)*-pair* (cf. 2.4 and 2.9.1) and it follows from the proof of [18, Proposition 15.9] that we may choose a maximal Brauer (c, H)-pair (Q, f^H) fulfilling

$$(Q, f^H) \subset (P, e^H), \quad Q = H \cap P \quad \text{and} \quad e\mathrm{Br}_P(f^H) \neq 0. \tag{3.3.1}$$

Now, denote by γ^G and δ the respective local points of P and Q on $\mathcal{O}G$ and $\mathcal{O}H$ determined by e and f^H; as above, let us denote by F the *inertial quotient* of c; that is to say, we set (cf. 2.9 and 2.11)

$$F = E_H(Q_\delta) = F_{\mathcal{O}H}(Q_\delta) \quad \text{and} \quad \hat{F} = \hat{E}_H(Q_\delta)^\circ \cong \hat{F}_{\mathcal{O}H}(Q_\delta). \tag{3.3.2}$$

3.4

Since we have $e\mathrm{Br}_P(f^H) \neq 0$ and f^H is P-stable, from the obvious commutative diagram

$$\begin{array}{ccc} (\mathcal{O}H)(Q) & \longrightarrow & (\mathcal{O}G)(Q) \\ \cup & & \cup \\ (\mathcal{O}H)(Q)^P & \longrightarrow & (\mathcal{O}G)(Q)^P \\ \downarrow & & \downarrow \\ (\mathcal{O}H)(P) & \longrightarrow & (\mathcal{O}G)(P) \end{array} \tag{3.4.1}$$

we get a local point δ^G of Q on $\mathcal{O}G$ such that the multiplicity $\mathrm{m}_\delta^{\delta^G}$ of the inclusion $(\mathcal{O}H)^Q \subset (\mathcal{O}G)^Q$ at (δ, δ^G) (cf. 2.2) is not zero and Q_{δ^G} is contained in P_{γ^G}; similarly, we get a local point γ of P on $\mathcal{O}H$ fulfilling

$$\mathrm{m}_\gamma^{\gamma^G} \neq 0 \quad \text{and} \quad Q_\delta \subset P_\gamma. \tag{3.4.2}$$

At this point, the following commutative diagram (cf. 2.2.1)

$$\begin{array}{ccccc} & & \mathrm{Res}_Q^P(\mathcal{O}H)_\gamma & \longrightarrow & \mathrm{Res}_Q^P(\mathcal{O}G)_\gamma \\ & \nearrow & & \nearrow & \uparrow \\ (\mathcal{O}H)_\delta & \longrightarrow & (\mathcal{O}G)_\delta & & \mathrm{Res}_Q^P(\mathcal{O}G)_{\gamma^G} \\ & & \uparrow & \nearrow & \\ & & (\mathcal{O}G)_{\delta^G} & & \end{array}, \tag{3.4.3}$$

where all the Q-interior algebra homomorphisms but the horizontal ones are embeddings, already provides some relationship between the source algebras of b and c (cf. 2.2).

3.5

If R_ε is a local pointed group on $\mathcal{O}H$, we set

$$C_G(R_\varepsilon) = C_G(R) \cap N_G(R_\varepsilon) \quad \text{and} \quad E_G(R_\varepsilon) = N_G(R_\varepsilon)/R \cdot C_G(R_\varepsilon) \tag{3.5.1}$$

and denote by $b(\varepsilon)$ the block of $C_H(R)$ determined by ε, and by $\bar{b}(\varepsilon)$ the image of $b(\varepsilon)$ in $k\bar{C}_H(R) = k(C_H(R)/Z(R))$; recall that we have a canonical $\bar{C}_G(R)$-interior algebra isomorphism [19, Proposition 3.2]

$$k\bar{C}_G(R)\mathrm{Tr}_{\bar{C}_G(R_\varepsilon)}^{\bar{C}_G(R)}\left(\bar{b}(\varepsilon)\right) \cong \mathrm{Ind}_{\bar{C}_G(R_\varepsilon)}^{\bar{C}_G(R)}\left(k\bar{C}_G(R_\varepsilon)\bar{b}(\varepsilon)\right). \tag{3.5.2}$$

Moreover, note that if ε^G is a local point of R on $\mathcal{O}G$ such that $\mathrm{m}_\varepsilon^{\varepsilon^G} \neq 0$ then we have

$$E_G(R_{\varepsilon^G}) \subset E_G(R_\varepsilon); \tag{3.5.3}$$

indeed, the restriction to $C_H(R)$ of a simple $kC_G(R)$-module determined by ε^G is semisimple (cf. 2.9.1) and therefore $C_G(R)$ acts transitively on the set of local points ε' of R on $\mathcal{O}H$ such that $\mathrm{m}_{\varepsilon'}^{\varepsilon^G} \neq 0$, so that we have

$$N_G(R_{\varepsilon^G}) \subset C_G(R) \cdot N_G(R_\varepsilon). \tag{3.5.4}$$

Then, we also consider $E_H(R_{\varepsilon^G}) = E_H(R_\varepsilon) \cap E_G(R_{\varepsilon^G})$.

3.6

Since (Q, f^H) is a maximal Brauer (c, H)-pair, we have (cf. 2.12.2)

$$k\bar{C}_H(Q)\bar{f}^H \cong (\mathcal{O}H)(Q_\delta) \tag{3.6.1}$$

and, according to the very definition of the k^*-group $\hat{N}_G(Q_\delta)$, we also have a k^*-group homomorphism

$$\hat{N}_G(Q_\delta) \longrightarrow \left(k\bar{C}_H(Q)\bar{f}^H\right)^*; \tag{3.6.2}$$

then, denoting by $\hat{C}_G(Q_\delta)$ the corresponding k^*-subgroup of $\hat{N}_G(Q_\delta)$ and setting

$$Z = C_G(Q_\delta)/C_H(Q) \quad \text{and} \quad \hat{Z} = \hat{C}_G(Q_\delta)/C_H(Q), \tag{3.6.3}$$

it follows from [19, Theorem 3.7] that we have a canonical $\bar{C}_G(Q_\delta)$-interior algebra isomorphism

$$k\bar{C}_G(Q_\delta)\bar{f}^H \cong k\bar{C}_H(Q)\bar{f}^H \otimes_k (k_*\hat{Z})^\circ. \tag{3.6.4}$$

Now, this isomorphism and the corresponding isomorphism 3.5.2 determine a k-algebra isomorphism

$$Z\left(k\bar{C}_G(Q)\right)\mathrm{Tr}_{\bar{C}_G(Q_\delta)}^{\bar{C}_G(Q)}\left(\bar{f}^H\right) \cong Z(k_*\hat{Z}), \tag{3.6.5}$$

and induce a bijection between the set of local points δ^G of Q on $\mathcal{O}Gb$ such that $\mathrm{m}_\delta^{\delta^G} \neq 0$ and the set of points of the k-algebra $(k_*\hat{Z})^\circ\hat{b}_\delta$ where we denote by $\bar{\mathrm{Br}}_Q(b)$ the image of

$\mathrm{Br}_Q(b)$ in $k\bar{C}_G(Q)$ and by $\hat{b}_\delta$ the image of $\bar{\mathrm{Br}}_Q(b)\mathrm{Tr}_{\bar{C}_G(Q_\delta)}^{\bar{C}_G(Q)}(\bar{f}^H)$ in the right-hand member of isomorphism 3.6.5.

Proposition 3.7 *With the the notation above, the idempotent* $\hat{b}_\delta$ *is primitive in* $Z(k_*\hat{Z})^{E_G(Q_\delta)}$. *In particular, if* $E_G(Q_\delta)$ *acts trivially on* $\hat{Z}$ *then* P_{γ^G} *contains* Q_{δ^G} *for any local point* δ^G *of* Q *on* $\mathcal{O}Gb$ *such that* $\mathrm{m}_\delta^{\delta^G} \neq 0$.

Proof Since $Q = H \cap P$, for any $a \in (\mathcal{O}G)^P$ it is easily checked that

$$\mathrm{Br}_Q\left(\mathrm{Tr}_P^G(a)\right) = \mathrm{Tr}_P^{N_G(Q)}\left(\mathrm{Br}_Q(a)\right) \tag{3.7.1}$$

and, in particular, we have $\mathrm{Br}_Q\left((\mathcal{O}G)_P^G\right) \cong kC_G(Q)_P^{N_G(Q)}$ (cf. 2.9.1); consequently, since the idempotent $b \in (\mathcal{O}G)_P^G$ is primitive in $Z(\mathcal{O}G)$, setting $E_G(Q) = N_G(Q)/\big(Q \cdot C_G(Q)\big)$, $\mathrm{Br}_Q(b)$ is still primitive in [17, Proposition 3.23]

$$kC_G(Q)^{N_G(Q)} = Z\left(kC_G(Q)\right)^{E_G(Q)}, \tag{3.7.2}$$

which amounts to saying that $N_G(Q)$ acts transitively over the set of k-blocks of $C_G(Q)$ involved in $\mathrm{Br}_Q(b)$; hence, since any k-block of $C_G(Q)$ maps on a k-block of $\bar{C}_G(Q)$ (cf. 2.9), $\bar{\mathrm{Br}}_Q(b)$ is also primitive in $Z(k\bar{C}_G(Q))^{E_G(Q)}$ and then, it suffices to apply isomorphism 3.6.5.

On the other hand, identifying $(\mathcal{O}G)(Q)$ with $kC_G(Q)$ (cf. 2.9.1), it is easily checked that $\mathrm{Br}_Q((\mathcal{O}G)^P) = kC_G(Q)^P$ and therefore, for any $i \in \gamma^G$, the idempotent $\mathrm{Br}_Q(i)$ is primitive in $kC_G(Q)^P$ [17, Proposition 3.23]; thus, since the canonical P-algebra homomorphism $kC_G(Q) \to k\bar{C}_G(Q)$ is a *strict semicovering* [16, Theorem 2.9], it follows from [6, Proposition 3.15] that the image $\bar{\mathrm{Br}}_Q(i)$ of $\mathrm{Br}_Q(i)$ in $k\bar{C}_G(Q)^P$ remains a primitive idempotent and that, denoting by $\bar{\gamma}^G$ the point of P on $k\bar{C}_G(Q)$ determined by $\bar{\mathrm{Br}}_Q(i)$, $P_{\bar{\gamma}^G}$ remains a maximal local pointed group on $k\bar{C}_G(Q)$.

Moreover, since P fixes $\bar{f}^H$ (cf. 3.3), we may choose $i \in \gamma^G$ fulfilling $\mathrm{Br}_Q(i) = \mathrm{Br}_Q(i)\bar{f}^H$; in this case, it follows from isomorphism 3.5.2 and from [19, Proposition 3.5] that $\bar{\mathrm{Br}}_Q(i)$ is a primitive idempotent of $(k\bar{C}_G(Q_\delta)\bar{f}^H)^P$ and that $P_{\bar{\gamma}^G}$ is also a maximal local pointed group on $k\bar{C}_G(Q_\delta)\bar{f}^H$.

But, it follows from isomorphism 3.6.4 that we have

$$\left(k\bar{C}_G(Q_\delta)\bar{f}^H\right)(P) \cong \left(k\bar{C}_H(Q)\bar{f}^H\right)(P) \otimes_k (k_*\hat{Z})^\circ(P) \tag{3.7.3}$$

and therefore, since evidently $ib = i$, $P_{\bar{\gamma}^G}$ determines a maximal local pointed group $P_{\hat{\bar{\gamma}}^G}$ on $(k_*\hat{Z})^\circ\hat{b}_\delta$ [9, Theorem 5.3]; moreover, if $E_G(Q_\delta)$ acts trivially on $\hat{Z}$ then $\hat{b}_\delta$ is a block of $\hat{Z}$ and therefore all the maximal local pointed groups on $(k_*\hat{Z})^\circ\hat{b}_\delta$ are mutually conjugate (cf. 2.5). Then, any idempotent $\hat{\bar{\imath}} \in \hat{\bar{\gamma}}^G$ has a nontrivial image in all the simple quotient of $(k_*\hat{Z})^\circ$ (cf. 2.2.2); now, the last statement follows from 3.6. □

Proposition 3.8 *Let* δ^G *be a local point of* Q *on* $\mathcal{O}G$ *such that* $\mathrm{m}_\delta^{\delta^G} \neq 0$. *The commutator in* $\hat{N}_G(Q_\delta)/Q \cdot C_H(Q)$ *induces a group homomorphism*

$$\varpi : F \longrightarrow \mathrm{Hom}(Z, k^*) \tag{3.8.1}$$

and $\mathrm{Ker}(\varpi)$ *is contained in* $E_H(Q_{\delta^G})$. *In particular,* $E_H(Q_{\delta^G})$ *is normal in* F, $F/E_H(Q_{\delta^G})$ *is an Abelian* p'*-group and, denoting by* $\hat{K}^\delta$ *and* $\hat{K}^{\delta^G}$ *the respective converse images in*

$\hat{C}_G(Q_\delta)$ of the fixed points of F and $E_H(Q_{\delta^G})$ over $\hat{Z}$, we have the exact sequence

$$1 \longrightarrow \hat{K}^\delta \longrightarrow \hat{K}^{\delta^G} \longrightarrow \mathrm{Hom}\left(F/E_H(Q_{\delta^G}), k^*\right) \longrightarrow 1. \tag{3.8.2}$$

Proof It is quite clear that F and Z are normal subgroups of the quotient $N_G(Q_\delta)/Q \cdot C_H(Q)$ and therefore their converse images $\hat{F}$ and $\hat{Z}$ in the quotient $\hat{N}_G(Q_\delta)/Q \cdot C_H(Q)$ still normalizes each other; but, since we have

$$N_H(Q_\delta) \cap C_G(Q_\delta) = C_H(Q), \tag{3.8.3}$$

their commutator is contained in k^*; hence, indentifying $\mathrm{Hom}(Z, k^*)$ with the group of the automorphisms of the k^*-group $\hat{Z}$ which act trivially on Z, we easily get homomorphism 3.8.1.

In particular, $\mathrm{Ker}(\varpi)$ acts trivially on the k^*-group $\hat{Z}$ and therefore, since its action is compatible with the bijection in 3.6 above, it is contained in $E_H(Q_{\delta^G})$; hence, since the p'-group $\mathrm{Hom}(Z, k^*)$ is Abelian, $E_H(Q_{\delta^G})$ is normal in $E_H(Q_\delta)$ (cf. 3.5.3) and $F/E_H(Q_{\delta^G})$ is Abelian.

Symmetrically, the commutator in $\hat{N}_G(Q_\delta)/Q \cdot C_H(Q)$ also induces surjective group homomorphisms

$$\begin{aligned} \hat{C}_G(Q_\delta) &\longrightarrow \mathrm{Hom}\left(F/\mathrm{Ker}(\varpi), k^*\right) \\ \hat{C}_G(Q_\delta) &\longrightarrow \mathrm{Hom}\left(E_H(Q_{\delta^G})/\mathrm{Ker}(\varpi), k^*\right) \end{aligned} \tag{3.8.4}$$

and it is quite clear that the kernels, respectively, coincide with $\hat{K}^\delta$ and $\hat{K}^{\delta^G}$; consequently, the kernel of the surjective group homomorphism

$$\hat{C}_G(Q_\delta)/\hat{K}^\delta \longrightarrow \hat{C}_G(Q_\delta)/\hat{K}^{\delta^G} \tag{3.8.5}$$

is canonically isomorphic to $\mathrm{Hom}(F/E_H(Q_{\delta^G}), k^*)$. We are done.

3.9

Assume that b is an inertial block of G or, equivalently, that there is a primitive Dade P-algebra S such that, with the notation in 2.13 above, we have a P-interior algebra isomorphism

$$(\mathcal{O}G)_{\gamma^G} \cong S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L} \tag{3.9.1}$$

where we consider S endowed with the unique P-interior algebra structure fulfilling $\det_S(P) = \{1\}$ (cf. 2.14). In this case, it follows from [6, Lemma 1.17] and [8, Proposition 2.14 and Theorem 3.1] that

$$E = F_{\mathcal{O}G}(P_{\gamma^G}) = F_S(P_{\{1_S\}}) \cap F_{\mathcal{O}_*\hat{L}}(P_{\{1_{\hat{L}}\}}) \tag{3.9.2}$$

and, in particular, that S is E-*stable* [8, Proposition 2.18]. Moreover, since we have a P-interior algebra embedding (cf. 2.14)

$$\mathcal{O} \longrightarrow \mathrm{End}_{\mathcal{O}}(S) \cong S^\circ \otimes_{\mathcal{O}} S, \tag{3.9.3}$$

we still have a P-interior algebra embedding

$$\mathcal{O}_*\hat{L} \longrightarrow S^\circ \otimes_{\mathcal{O}} (\mathcal{O}G)_{\gamma^G}. \tag{3.9.4}$$

3.10

Conversely, always with the notation in 2.13, assume that S is an E-stable Dade P-algebra or, equivalently, that E is contained in $F_S(P_\pi)$ where π denotes the unique local point of P on S (cf. 2.14); since we have [9, Proposition 5.9]

$$F_S(P_\pi) \cap F_{\mathcal{O}G}(P_{\gamma^G}) \subset F_{S^\circ \otimes_{\mathcal{O}} \mathcal{O}G}(P_{\pi \times \gamma^G}) \tag{3.10.1}$$

where $\pi \times \gamma^G$ denotes the local point of P on $S^\circ \otimes_{\mathcal{O}} \mathcal{O}G$ determined by π and γ^G [9, Proposition 5.6], and we still have [18, Theorem 9.21]

$$\hat{F}_S(P_\pi) \cong k^* \times F_S(P_\pi), \tag{3.10.2}$$

it follows from [9, proposition 5.11] that the k^*-group $\hat{E}$ is isomorphic to a k^*-subgroup of $\hat{F}_{S^\circ \otimes_{\mathcal{O}} \mathcal{O}G}(P_{\pi \times \gamma^G})$; then, since E is a p'-group, it follows from [10, Proposition 7.4] that there is an injective unitary P-interior algebra homomorphism

$$\mathcal{O}_* \hat{L} \longrightarrow (S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi \times \gamma^G} \tag{3.10.3}$$

and, in particular, we have

$$|P||E| \le \mathrm{rank}_{\mathcal{O}}(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi \times \gamma^G}. \tag{3.10.4}$$

□

Proposition 3.11 *With the notation above, the block b is inertial if and only if there is an E-stable Dade P-algebra S such that*

$$\mathrm{rank}_{\mathcal{O}} \left(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G\right)_{\pi \times \gamma^G} = |P||E| \tag{3.11.1}$$

Proof If b is inertial then the equality 3.11.1 follows from the existence of embedding 3.9.4. □

Conversely, we claim that if equality 3.11.1 holds then the corresponding homomorphism 3.10.3 is an isomorphism; indeed, since this homomorphism is injective and we have $\mathrm{rank}_{\mathcal{O}}(\mathcal{O}_* \hat{L}) = |P||E|$, it suffices to prove that the reduction to k of homomorphism 3.10.3 remains injective; but, it also follows from [10, Proposition 7.4] that, setting ${}^k S = k \otimes_{\mathcal{O}} S$, there is an injective unitary P-interior algebra homomorphism

$$k_* \hat{L} \longrightarrow \left({}^k S^\circ \otimes_k kG\right)_{\bar{\pi} \times \bar{\gamma}^G}, \tag{3.11.2}$$

where $\bar{\pi}$ and $\bar{\gamma}^G$ denote the respective images of π and γ^G in ${}^k S^\circ$ and kG, which is a conjugate of the reduction to k of homomorphism 3.10.3.

Now, embedding 3.9.3 and the structural embedding

$$(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi \times \gamma^G} \longrightarrow S^\circ \otimes_{\mathcal{O}} (\mathcal{O}G)_{\gamma^G} \tag{3.11.3}$$

determine P-interior algebra embeddings

$$\begin{array}{ccc} S \otimes_{\mathcal{O}} (S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi \times \gamma^G} & \longrightarrow & S \otimes_{\mathcal{O}} S^\circ \otimes_{\mathcal{O}} (\mathcal{O}G)_{\gamma^G} \\ \wr\| & & \uparrow \\ S \otimes_{\mathcal{O}} \mathcal{O}_* \hat{L} & & (\mathcal{O}G)_{\gamma^G} \end{array} ; \tag{3.11.4}$$

thus, since P has a unique local point on $S \otimes S^\circ \otimes_{\mathcal{O}} (\mathcal{O}G)_{\gamma^G}$ [9, Theorem 5.3], we get a P-interior algebra embedding

$$(\mathcal{O}G)_{\gamma^G} \longrightarrow S \otimes_{\mathcal{O}} \mathcal{O}_* \hat{L} \tag{3.11.5}$$

which proves that b is inertial. We are done.

3.12

With the notation above, assume that the block b is inertial; then, denoting by χ the unique local point of Q on S (cf. 2.14) and by δ^G a local point of Q on $\mathcal{O}Gb$ such that $\mathrm{m}_\delta^{\delta^G} \neq 0$, there is a unique local point $\hat{\delta}^L$ of Q on $\mathcal{O}_*\hat{L}$ such that isomorphism 3.9.1 induces a Q-interior algebra embedding [9, Proposition 5.6]

$$(\mathcal{O}G)_{\delta^G} \longrightarrow S_\chi \otimes_{\mathcal{O}} (\mathcal{O}_*\hat{L})_{\hat{\delta}^L}\,; \tag{3.12.1}$$

but, the image of Q in $(S_\chi)^*$ need not be contained in the kernel of the corresponding *determinant map*. Note that, as above, it follows from this embedding and from [6, Lemma 1.17] and [8, Proposition 2.14 and Theorem 3.1] that

$$E_G(Q_{\delta^G}) = F_{\mathcal{O}G}(Q_{\delta^G}) = F_S(Q_\chi) \cap F_{\mathcal{O}_*\hat{L}}(Q_{\hat{\delta}^L}), \tag{3.12.2}$$

so that *the Dade Q-algebra S_χ is $E_G(Q_{\delta^G})$-stable*; as in 2.13 above, let us consider the corresponding semidirect products

$$M = Q \rtimes F \quad \text{and} \quad \hat{M} = Q \rtimes \hat{F}. \tag{3.12.3}$$

We are ready to state our main result.

Theorem 3.13 *With the notation above, assume that the block b of G is inertial. Then, there is a Q-interior algebra isomorphism*

$$(\mathcal{O}H)_\delta \cong S_\chi \otimes_{\mathcal{O}} \mathcal{O}_*\hat{M} \tag{3.13.1}$$

and, in particular, the block c of H is inertial too.

Proof We argue by induction on $|G/H|$; in particular, if H' is a proper normal subgroup of G which properly contains H, it suffices to choose a block c' of H' fulfilling $c'b \neq 0$ to get $c'c \neq 0$ and the induction hypothesis successively proves that the block c' of H' is inertial and then that the block c is inertial too; moreover, setting $Q' = H' \cap P$, the corresponding Dade Q'-algebra comes from S and therefore the final Dade Q-algebra also comes from S. Consequently, since G fixes c, it follows from the *Frattini argument* that we have (cf. 2.3)

$$G = H \cdot N_G(Q_\delta) \tag{3.13.2}$$

and therefore we may assume that either $C_G(Q_\delta) \subset H$ or $G = H \cdot C_G(Q_\delta)$.

Firstly assume that $C_G(Q_\delta) \subset H$; in this case, it follows from [18, Proposition 15.10] that $b = c$; moreover, since $C_G(Q_\delta) = C_H(Q)$, it follows from 3.6 above that Q has a unique local point δ^G on $\mathcal{O}Gb$ such that $\mathrm{m}_\delta^{\delta^G} \neq 0$, and from isomorphism 3.6.4 that we have

$$(\mathcal{O}H)(Q_\delta) \cong k\bar{C}_H(Q)\bar{f}^H \cong k\bar{C}_G(Q_\delta)\bar{f}^H\,; \tag{3.13.3}$$

in particular, $N_G(Q_\delta)$ normalizes Q_{δ^G} and therefore the inclusion 3.5.3 becomes an equality

$$E_G(Q_{\delta^G}) = E_G(Q_\delta); \tag{3.13.4}$$

thus, since F is obviously contained in $E_G(Q_\delta)$, S_χ is F-stable too. Consequently, according to Proposition 3.11, it suffices to prove that

$$\mathrm{rank}_{\mathcal{O}}(S_\chi^\circ \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta} = |Q||F|. \tag{3.13.5}$$

As in 3.12 above, the P-interior algebra embedding 3.9.4 induces a Q-interior algebra embedding [9, Theorem 5.3]

$$(\mathcal{O}_*\hat{L})_{\hat{\delta}^L} \longrightarrow S_\chi^\circ \otimes_{\mathcal{O}} (\mathcal{O}G)_{\delta G} \tag{3.13.6}$$

and it suffices to apply again [6, Lemma 1.17] and [8, Proposition 2.14 and Theorem 3.1] to get

$$E_L(Q_{\hat{\delta}^L}) = F_{\mathcal{O}_*\hat{L}}(Q_{\hat{\delta}^L}) = F_S(Q_\chi) \cap F_{\mathcal{O}G}(Q_{\delta^G}), \tag{3.13.7}$$

so that we obtain

$$E_L(Q_{\hat{\delta}^L}) = E_G(Q_{\delta^G}) \subset F_S(Q_\chi). \tag{3.13.8}$$

In particular, it follows from [8, Proposition 2.12] that for any $x \in N_G(Q_\delta)$ there is $s_x \in (S_\chi)^*$ fulfilling

$$s_x \cdot u = u^x \cdot s_x \tag{3.13.9}$$

for any $u \in Q$, and therefore, choosing a set of representatives $X \subset N_G(Q_\delta)$ for G/H (cf. 3.13.2), we get an $\mathcal{O}Q$-bimodule direct sum decomposition

$$S_\chi^\circ \otimes_{\mathcal{O}} \mathcal{O}G = \bigoplus_{x\in X} (s_x \otimes x)(S_\chi^\circ \otimes_{\mathcal{O}} \mathcal{O}H). \tag{3.13.10}$$

But, for any $x \in N_G(Q_\delta)$, the element $s_x \otimes x$ normalizes the image of Q in $S_\chi^\circ \otimes_{\mathcal{O}} \mathcal{O}H$ and it is clear that it also normalizes the local point $\chi \times \delta$ of Q on this Q-interior algebra; more precisely, if $S_\chi = \ell S \ell$ for $\ell \in \chi$ and $(\mathcal{O}H)_\delta = j(\mathcal{O}H)j$ for $j \in \delta$, there is $j' \in \chi \times \delta$ such that [9, Proposition 5.6]

$$j'(\ell \otimes j) = j' = (\ell \otimes j)j'; \tag{3.13.11}$$

thus, for any $x \in N_G(Q_\delta)$ the idempotent $j'^{s_x \otimes x}$ still belongs to $\chi \times \delta$ and therefore there is an inversible element a_x in $(S_\chi^\circ \otimes_{\mathcal{O}} \mathcal{O}H)^Q$ fulfilling

$$j'^{s_x \otimes x} = j'^{a_x}, \tag{3.13.12}$$

so that we get the new $\mathcal{O}Q$-bimodule direct sum decomposition

$$j'(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)j' = \bigoplus_{x\in X} (s_x \otimes x)(a_x)^{-1} j'(S^\circ \otimes_{\mathcal{O}} \mathcal{O}H)j'. \tag{3.13.13}$$

Moreover, the equality in 3.13.8 forces the group $E_G(Q_\delta) = E_G(Q_{\delta^G})$ to have a normal Sylow p-subgroup and therefore, since we are assuming that $C_G(Q_\delta) \subset H$, it follows from equality 3.13.2 that the quotient G/H also has a normal Sylow p-subgroup. At this point, arguing by induction, we may assume that G/H is either a p-group or a p'−group.

Firstly assume that G/H is a p-group or, equivalently, that $G = H \cdot P$ [9, Lemma 3.10]; in this case, it follows from [6, Proposition 6.2] that the inclusion homomorphism $\mathcal{O}H \to \mathcal{O}G$ is a *strict semicovering* of Q-interior algebras (cf. 2.1) and, in particular, we have $\delta \subset \delta^G$ since $\mathrm{m}_\delta^{\delta^G} \neq 0$; similarly, since for any subgroup R of Q we have [9, Proposition 5.6]

$$\begin{aligned} (S^\circ \otimes_{\mathcal{O}} \mathcal{O}H)(R) &\cong S(R)^\circ \otimes_k (\mathcal{O}H)(R) \\ (S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)(R) &\cong S(R)^\circ \otimes_k (\mathcal{O}G)(R), \end{aligned} \tag{3.13.14}$$

it follows from [6, Theorem 3.16] that the corresponding Q-interior algebra homomorphism $S^\circ \otimes_{\mathcal{O}} \mathcal{O}H \to S^\circ \otimes_{\mathcal{O}} \mathcal{O}G$ is also a *strict semicovering* and, in particular, we have $\chi \times \delta \subset \chi \times \delta^G$, so that j' belongs to $\chi \times \delta^G$.

But, since $Q_{\delta^G} \subset P_{\gamma^G}$ (cf. 3.4), it is easily checked that $Q_{\chi\times\delta^G} \subset P_{\pi\times\gamma^G}$, where as above π is the unique local point of P on S, and therefore we get the Q-interior algebra embedding (cf. embeddings 2.2.3 and 3.9.4)

$$(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\chi\times\delta^G} \longrightarrow \mathrm{Res}^P_Q(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi\times\gamma^G} \cong \mathrm{Res}^P_Q(\mathcal{O}_*\hat{L}); \tag{3.13.15}$$

in particular, it follows from equality 3.13.13 that we have

$$|X|\,\mathrm{rank}_{\mathcal{O}}\,(S^\circ_\chi \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta} \le |L|. \tag{3.13.16}$$

Moreover, we have $|X| = |G/H| = |P/Q|$ and, since $C_P(Q) \subset Q$, it follows from [4, Ch. 5, Theorem 3.4] that $E \subset L$ acts faithfully on $Q = H \cap P$; in particular, $\hat{\delta}^L$ is the unique local point of Q on $\mathcal{O}_*\hat{L}$ (actually, we have $\hat{\delta}^L = \{1_{\mathcal{O}_*\hat{L}}\}$) and therefore, since (cf. 3.13.4 and 3.13.8)

$$E_L(Q_{\hat{\delta}^L}) = E_G(Q_{\delta^G}) = E_G(Q_\delta) \supset F \tag{3.13.17}$$

and $E_G(Q_\delta)/F$ is a p-group, the p'-group E is actually isomorphic to F.

Consequently, it follows from the inequalities 3.10.4 and 3.13.16 that

$$|F||Q| \le \mathrm{rank}_{\mathcal{O}}(S^\circ_\chi \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta} \le |L|/|X| = |F||Q| \tag{3.13.18}$$

which forces equality in 3.13.6.

Secondly assume that G/H is a p'-group; in this case, we have $Q = P$, $\delta = \gamma$ and $\delta^G = \gamma^G$; in particular, since we are assuming that

$$C_G(Q_\delta) \subset H \quad \text{and} \quad E_G(Q_{\delta^G}) = E_G(Q_\delta), \tag{3.13.19}$$

we actually get

$$|X| = |G/H| = |E_G(P_{\gamma^G})|/|E_H(Q_\delta)| = |E|/|F|. \tag{3.13.20}$$

Moreover, we claim that, as above, the idempotent j' remains primitive in $(S \otimes_{\mathcal{O}} \mathcal{O}G)^P$[1], so that it belongs to $\pi \times \gamma^G$; indeed, setting

$$A' = j'(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)j' \quad \text{and} \quad B' = j'(S^\circ \otimes_{\mathcal{O}} \mathcal{O}H)j', \tag{3.13.21}$$

let i' be a primitive idempotent of A'^P such that $\mathrm{Br}_P(i') \ne 0$; in particular, i' belongs to $\pi \times \gamma^G$ and we may assume that

$$i'A'i' = (S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi\times\gamma^G} \cong \mathcal{O}_*\hat{L}. \tag{3.13.22}$$

It is clear that the multiplication by B' on the left and the action of P by conjugation endows A' with a $B'P$-module structure and, since the idempotent j' is primitive in B'^P, equality 3.13.13 provides a direct sum decomposition of A' in indecomposable $B'P$-modules. More explicitly, note that B' is an indecomposable $B'P$-module since we have $\mathrm{End}_{B'P}(B') = B'^P$; but, for any $x \in X$, the inversible element

$$a'_x = (s_x \otimes x)(a_x)^{-1} j' \tag{3.13.23}$$

[1] The corresponding argument has been forgotten in [18] at the end of the proof of Proposition 15.19!

of A' together with the action of x on P determine an automorphism g_x of $B'P$; thus, equality 3.13.13 provides the following direct sum decomposition on indecomposable $B'P$-modules

$$A' \cong \bigoplus_{x\in X} \mathrm{Res}_{g_x}(B'). \tag{3.13.24}$$

Moreover, we claim that the $B'P$-modules $\mathrm{Res}_{g_x}(B')$ and $\mathrm{Res}_{g_{x'}}(B')$ for $x, x' \in X$ are isomorphic if and only if $x = x'$; indeed, a $B'P$-module isomorphism

$$\mathrm{Res}_{g_x}(B') \cong \mathrm{Res}_{g_{x'}}(B') \tag{3.13.25}$$

is necessarily determined by the multiplication on the right by an inversible element b' of B' fulfilling

$$(xux^{-1}) \cdot b' = b' \cdot (x'ux'^{-1}) \tag{3.13.26}$$

or, equivalently, $(u \cdot j')^{b'} = u^{xx'^{-1}} \cdot j'$ for any $u \in P$, which amounts to saying that the automorphism of P determined by the conjugation by $x'x^{-1}$ is a B'-*fusion* from P_γ to P_γ [8, Proposition 2.12]; but, once again from [6, Lemma 1.17] and [8, Proposition 2.14 and Theorem 3.1] we get

$$F_{A'}(P_{\gamma^G}) = E_G(P_{\gamma^G}) = E \quad \text{and} \quad F_{B'}(P_\gamma) = E_H(P_\gamma); \tag{3.13.27}$$

hence our claim now follows from equalities 3.13.20.

On the other hand, it is clear that $A'i'$ is a direct summand of the $B'P$-module A' and therefore there is $x \in X$ such that $\mathrm{Res}_{g_x}(B')$ is a direct summand of the $B'P$-module $A'i'$; but, it follows from [8, Proposition 2.14] that we have

$$F_{i'A'i'}(P_{\gamma^G}) = F_{A'}(P_{\gamma^G}) = E \tag{3.13.28}$$

and therefore, once again applying [8, Proposition 2.12], for any $y \in N_G(P_{\gamma^G})$ there is an inversible element c'_y in A' fulfilling

$$c'_y(u \cdot i')(c'_y)^{-1} = yuy^{-1} \cdot i' \tag{3.13.29}$$

for any $u \in P$; then, for any $x' \in X$, it is clear that $A'i' = A'i'c'_{x^{-1}x'}$ has a direct summand isomorphic to $\mathrm{Res}_{g_{x'}}(B')$, which forces the equality of the $\mathcal{O}$-ranks of $A'i'$ and A', so that $A'i' = A'$ and $i' = j'$, which proves our claim. Consequently, it follows from the equalities 3.13.13 and 3.13.20 that

$$\mathrm{rank}_{\mathcal{O}}\,(S^\circ_\chi \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta} = |L|/|X| = |F||Q|, \tag{3.13.30}$$

so that equality holds in 3.13.6.

From now on, we assume that $H \cdot C_G(Q_\delta) = G$; in particular, $C_G(Q)$ stabilizes δ, we have $E_G(Q_\delta) = E_H(Q_\delta) = F$ and we can choose the set of representatives X for G/H contained in $C_G(Q)$ so that this time we get the $\mathcal{O}Q$-bimodule direct sum decomposition

$$S^\circ_\chi \otimes_{\mathcal{O}} \mathcal{O}G = \bigoplus_{x\in X}(1_S \otimes x)(S^\circ_\chi \otimes_{\mathcal{O}} \mathcal{O}H). \tag{3.13.31}$$

Since any $z \in C_G(Q)$ stabilizes δ choosing again $\ell \in \chi$, $j \in \delta$ and $j' \in \chi \times \delta$ such that [9, Proposition 5.6]

$$j'(\ell \otimes j) = j' = (\ell \otimes j)j', \tag{3.13.32}$$

there is an inversible element a_z in $(\mathcal{O}H)^Q$ fulfilling $j^z = j^{a_z}$; consequently, with the notation above, from these choices and equality 3.13.31 we have

$$A' = \bigoplus_{x\in X}(1_S \otimes x(a_x)^{-1})B'. \tag{3.13.33}$$

As in Proposition 3.8, denote by $\hat{K}^\delta$ the converse image in $\hat{C}_G(Q)$ of the fixed points of F in $\hat{Z}$ and by K^δ the k^**-quotient* $\hat{K}^\delta/k^*$ of $\hat{K}^\delta$; since $\hat{K}^\delta$ is a normal k^*-subgroup of $\hat{C}_G(Q)$, $H \cdot K^\delta$ is a normal subgroup of G and therefore, arguing by induction, we may assume that it coincides with H or with G.

Firstly assume that $H \cdot K^\delta = G$; in this case, since we have $K^\delta = C_G(Q)$, F acts trivially on $\hat{Z}$ and we have $F = E_H(Q_{\delta^G})$ for any local point δ^G of Q on $\mathcal{O}Gb$ such that $\mathrm{m}_\delta^{\delta^G} \neq 0$, so that S_χ is F-stable (cf. 3.12.2); consequently, according to Proposition 3.11, once again it suffices to prove that

$$\mathrm{rank}_\mathcal{O}(S_\chi^\circ \otimes_\mathcal{O} \mathcal{O}H)_{\chi\times\delta} = |Q||F|. \tag{3.13.34}$$

For any $z \in C_G(Q)$, the element $z(a_z)^{-1}$ stabilizes $j(\mathcal{O}H)j = (\mathcal{O}H)_\delta$ and actually it induces a Q-interior algebra automorphism g_z of the source algebra $(\mathcal{O}H)_\delta$; but, symmetrically, $C_G(Q)$ acts trivially on [8, Proposition 2.14 and Theorem 3.1]

$$\hat{F} = \hat{E}_H(Q_\delta)^\circ \cong \hat{F}_{(\mathcal{O}H)_\delta}(Q_\delta); \tag{3.13.35}$$

hence, it follows from [10, Proposition 14.9] that g_z is an *inner automorphism* and therefore, up to modifying our choice of a_z, we may assume that $z(a_z)^{-1}$ centralizes $(\mathcal{O}H)_\delta$; then, for any $x \in X$ the element $1_S \otimes x(a_x)^{-1}$ centralizes

$$B' = j'(S^\circ \otimes_\mathcal{O} \mathcal{O}H)j' \tag{3.13.36}$$

and therefore, denoting by C the centralizer of B' in A', it follows from equality 3.13.33 that we have

$$A' = C \otimes_{Z(B')} B'; \tag{3.13.37}$$

in particular, we get $A'^Q = C \otimes_{Z(B')} B'^Q$ which induces a k-algebra isomorphism [10, 14.5.1]

$$A'(Q) \cong C \otimes_{Z(B')} kZ(Q) \tag{3.13.38}$$

and then it follows from isomorphism 3.6.4 that

$$k \otimes_{Z(B')} C \cong (k_*\hat{Z})^\circ. \tag{3.13.39}$$

At this point, for any local point δ^G of Q on $\mathcal{O}Gb$ such that $\mathrm{m}_\delta^{\delta^G} \neq 0$, it follows from Proposition 3.7 that $Q_{\delta^G} \subset P_{\gamma^G}$, so that $Q_{\chi\times\delta^G} \subset P_{\pi\times\gamma^G}$ [9, Proposition 5.6] and therefore $\chi \times \delta^G$ is also a local point of Q on the P-interior algebra (cf. embedding 3.9.4)

$$(S^\circ \otimes_\mathcal{O} \mathcal{O}G)_{\pi\times\gamma^G} \cong \mathcal{O}_*\hat{L}; \tag{3.13.40}$$

actually, since $N_G(P)$ normalizes $Q = H \cap P$, Q is normal in L and therefore all the points of Q on $\mathcal{O}_*\hat{L}$ are local (cf. 2.10). In conclusion, since $\{1_L\}$ is the unique point of P on $\mathcal{O}_*\hat{L}$, isomorphism 3.13.40 induces a bijective correspondence between the sets of local points of Q on

$$j'(S^\circ \otimes_\mathcal{O} \mathcal{O}Gb)j' = A'(1 \otimes b) \tag{3.13.41}$$

and on $\mathcal{O}_*\hat{L}$; moreover, note that if two local points $\chi \times \delta^G$ and $\chi \times \varepsilon^G$ of Q on the left-hand member of 3.13.40 correspond to two local points $\hat{\delta}^G$ and $\hat{\varepsilon}^G$ of Q on $\mathcal{O}_*\hat{L}$, choosing suitable $j^G \in \delta^G$, $k^G \in \varepsilon^G$, $\hat{j}^G \in \hat{\delta}^G$ and $\hat{k}^G \in \hat{\varepsilon}^G$, from isomorphism 3.13.40 we still get an $\mathcal{O}Q$-bimodule isomorphism

$$j^G A' k^G \cong \hat{j}^G (\mathcal{O}_*\hat{L})\hat{k}^G . \tag{3.13.42}$$

Consequently, since we have $A'^Q = C \otimes_{Z(B')} B'^Q$ and C is a free $Z(B')$-module, for suitable primitive idempotents $\bar{j}^G$ and $\bar{k}^G$ of C we have (cf. 3.13.37 and 3.13.38)

$$\begin{aligned} \dim \left(k \otimes_{Z(B')} (\bar{j}^G C \bar{k}^G)\right) \operatorname{rank}_{\mathcal{O}}(B') &= \operatorname{rank}_{\mathcal{O}} \left(\hat{j}^G (\mathcal{O}_*\hat{L})\hat{k}^G\right) \\ \dim \left(k \otimes_{Z(B')} (\bar{j}^G C \bar{k}^G)\right) &= \operatorname{rank}_{kZ(Q)} \left(\hat{j}^G (\mathcal{O}_*\hat{L})\hat{k}^G\right)(Q); \end{aligned} \tag{3.13.43}$$

thus, since the respective *multiplicities* (cf. 2.2) of points $\hat{\delta}^G$ and $\mathrm{Br}_Q^{\mathcal{O}_*\hat{L}}(\hat{\delta}^G)$ of Q on $\mathcal{O}_*\hat{L}$ and on $(\mathcal{O}_*\hat{L})(Q) \cong k_* C_{\hat{L}}(Q)$ coincide with each other, we finally get

$$|L| = \operatorname{rank}_{\mathcal{O}}(\mathcal{O}_*\hat{L}) = |\bar{C}_L(Q)| \operatorname{rank}_{\mathcal{O}}(B'). \tag{3.13.44}$$

But, according to 3.5.4, $N_G(P_{\gamma^G})$ normalizes γ which determines f^H (cf. 3.3.1) and therefore γ determines the unique local point δ of Q on $\mathcal{O}H$ associated with f^H; thus, $N_G(P_{\gamma^G})$ is contained in $N_G(Q_\delta)$ which acts trivially on $\hat{Z}$, and therefore $N_G(P_{\gamma^G})$ stabilizes all the local points δ^G of Q on $\mathcal{O}Gb$ fulfilling $\mathrm{m}_\delta^{\delta^G} \neq 0$ (cf. 3.6); hence, it follows from isomorphism 3.13.40 above that, denoting by $\hat{\delta}^G$ the point of Q on $\mathcal{O}_*\hat{L}$ determined by δ^G, L normalizes $Q_{\hat{\delta}^G}$; in particular, we have

$$\begin{aligned} F = E_G(Q_\delta) = E_G(Q_{\delta^G}) &= F_{(\mathcal{O}G)_{\gamma^G}}(Q_{\delta^G}) \\ &= E_L(Q_{\hat{\delta}^G}) = L/Q \cdot C_L(Q) \end{aligned} \tag{3.13.45}$$

and therefore from equality 3.13.44 we get

$$|F||Q| = |L|/|\bar{C}_L(Q)| = \operatorname{rank}_{\mathcal{O}}(B'), \tag{3.13.46}$$

which proves that c is inertial.

Finally, assume that $K^\delta = C_H(Q)$; in this case, since the commutator in $\hat{N}_G(Q_\delta)/\big(Q \cdot C_H(Q)\big)$ induces a group isomorphism

$$\hat{C}_G(Q_\delta)/\hat{K}^\delta \cong \operatorname{Hom}\left(F/\operatorname{Ker}(\varpi), k^*\right), \tag{3.13.47}$$

the quotient G/H is an Abelian p'-group and, in particular, we have $P = Q$. But, since with our choices above we still have (cf. 3.13.33)

$$(\mathcal{O}G)_\delta = j(\mathcal{O}G)j = \bigoplus_{x \in X} x(a_x)^{-1}(\mathcal{O}H)_\delta \tag{3.13.48}$$

where the element $x(a_x)^{-1}$ determines a Q-interior algebra automorphism of $(\mathcal{O}H)_\delta$, it suffices to consider the k^*-group

$$\hat{U} = \bigcup_{x \in X} x(a_x)^{-1} \left((\mathcal{O}H)_\delta^Q\right)^* \tag{3.13.49}$$

to get the Q-interior algebra $(\mathcal{O}G)_\delta$ as the *crossed product* [3, 1.6]

$$(\mathcal{O}G)_\delta \cong (\mathcal{O}H)_\delta \otimes_{((\mathcal{O}H)_\delta^Q)^*} \hat{U}. \tag{3.13.50}$$

Then, since G/H is a p'-group, denoting by U the k^*-quotient of $\hat{U}$ it follows from [10, Proposition 4.6] that the exact sequence

$$1 \longrightarrow j + J\left((\mathcal{O}H)_\delta^Q\right) \longrightarrow U \longrightarrow G/H \longrightarrow 1 \tag{3.13.51}$$

is *split* and therefore, for a suitable central k^*-extension $\widehat{G/H}$ of G/H, we still get an evident Q-interior algebra isomorphism

$$(\mathcal{O}G)_\delta \cong (\mathcal{O}H)_\delta \otimes_{k^*} \widehat{G/H}; \tag{3.13.52}$$

at this point, it suffices to compute the *Brauer quotients* at Q of both members to get

$$k \otimes_{kZ(Q)} (\mathcal{O}G)_\delta(Q) \cong k_*\widehat{G/H} \tag{3.13.53}$$

and therefore, comparing this k-algebra isomorphism with isomorphism 3.6.4, we obtain a Q-interior algebra isomorphism

$$(\mathcal{O}G)_\delta \cong (\mathcal{O}H)_\delta \otimes_{k^*} \hat{Z}^\circ \tag{3.13.54}$$

for a suitable action of Z over $(\mathcal{O}H)_\delta$ defined, up to *inner automorphisms* of the Q-interior algebra $(\mathcal{O}H)_\delta$, by the group homomorphism

$$Z \longrightarrow \mathrm{Aut}_{k^*}\left(\hat{E}_H(Q_\delta)\right) \tag{3.13.55}$$

induced by the commutator in $\hat{N}_G(Q_\delta)\big/\big(Q \cdot C_H(Q)\big)$ [10, Proposition 14.9].

Similarly, considering the trivial action of Z over S, we also obtain the Q-interior algebra isomorphism

$$S^\circ \otimes_{\mathcal{O}} (\mathcal{O}G)_\delta \cong \big(S^\circ \otimes_{\mathcal{O}} (\mathcal{O}H)_\delta\big) \otimes_{k^*} \hat{Z}^\circ; \tag{3.13.56}$$

since $\chi \times \delta$ is the unique local point of Q on $S^\circ \otimes_{\mathcal{O}} (\mathcal{O}H)_\delta$, we have $j'^{\bar{z}} = j'^{b_{\bar{z}}}$ for a suitable inversible element $b_{\bar{z}}$ in $(S^\circ \otimes_{\mathcal{O}} (\mathcal{O}H)_\delta)^Q$; hence, arguing as above, we finally obtain a Q-interior algebra isomorphism

$$(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\chi\times\delta} \cong (S^\circ \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta} \otimes_{k^*} \hat{Z}^\circ. \tag{3.13.57}$$

Moreover, since the k-algebra $k_*\hat{Z}$ is now semisimple, for any pair of primitive idempotents $\hat{\imath}$ and $\hat{\imath}'$ of $\mathcal{O}_*\hat{Z}$ we have $\hat{\imath}(\mathcal{O}_*\hat{Z})\hat{\imath}' = \mathcal{O}$ *or* $\{0\}$, and, since $\mathcal{O}_*\hat{Z}$ is contained in $(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\chi\times\delta} \subset S^\circ \otimes_{\mathcal{O}} \mathcal{O}G$, in the first case from isomorphism 3.13.56 we get

$$\mathrm{rank}_{\mathcal{O}}\left(\hat{\imath}(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)\hat{\imath}'\right) \le \mathrm{rank}_{\mathcal{O}}(S^\circ \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta}; \tag{3.13.58}$$

hence, since isomorphism 3.13.57 implies that

$$\mathrm{rank}_{\mathcal{O}}(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\chi\times\delta} = \mathrm{rank}_{\mathcal{O}}(S^\circ \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta}\,|Z|, \tag{3.13.59}$$

all the inequalities 3.13.58 are actually equalities and, in particular, we get (cf. embedding 3.9.4)

$$|L| = \mathrm{rank}_{\mathcal{O}}(S^\circ \otimes_{\mathcal{O}} \mathcal{O}G)_{\pi\times\gamma^G} = \mathrm{rank}_{\mathcal{O}}(S^\circ \otimes_{\mathcal{O}} \mathcal{O}H)_{\chi\times\delta} \tag{3.13.60}$$

since $P = Q$ and $\pi \times \gamma^{G} = \chi \times \delta^{G}$ (cf. 3.4). Consequently, according to Proposition 3.11, it suffices to prove that S is F-stable.

On the other hand, it follows from Proposition 3.7 that F acts transitively over the set of primitive idempotents of $Z(k_*\hat{Z})\,\hat{b}_\delta$; but, since $k_*\hat{Z}$ is semisimple, this set is canonically isomorphic to the set of points of this k-algebra (cf. 2.2), so that F acts transitively over the set of local points δ^{G} of Q on $\mathcal{O}Gb$ fulfilling $\mathrm{m}_\delta^{\delta^G} \neq 0$ (cf. 3.6). More precisely, choosing $\delta^{G} = \gamma^{G}$ and denoting by $\hat{K}^{\delta^G}$ the converse image in $\hat{C}_G(Q)$ of the fixed points of $E_H(Q_{\delta^G})$ in $\hat{Z}$ and by K^{δ^G} the k^*-*quotient* of $\hat{K}^{\delta^G}$, as above $H \cdot K^{\delta^G}$ is a normal subgroup of G and therefore, arguing by induction, we may assume that either $C_H(Q) = K^{\delta^G}$ or $G = H \cdot K^{\delta^G}$.

In the first case, it follows from Proposition 3.8 that

$$F = E_H(Q_{\delta^G}) \subset E_G(Q_{\delta^G}) = E \tag{3.13.61}$$

so that S is indeed F-stable (cf. 3.9). In the second case, since we have (cf. Proposition 3.8)

$$F/E_H(Q_{\delta^G}) \cong K^{\delta^G}/K^{\delta} \cong G/H \cong Z, \tag{3.13.62}$$

the number of points of $\mathcal{O}_*\hat{Z}$ coincides with its $\mathcal{O}$-rank which forces the k^*-group isomorphism $\hat{Z} \cong k^* \times Z$; in particular, isomorphism 3.13.54 becomes the Q-interior algebra isomorphism

$$(\mathcal{O}G)_\delta \cong (\mathcal{O}H)_\delta\, Z = \bigoplus_{z \in Z} (\mathcal{O}H)_\delta \cdot z \tag{3.13.63}$$

and therefore we have $(\mathcal{O}G)_\delta^Q \cong (\mathcal{O}H)_\delta^Q Z$.

Thus, since $Q = P$, we may assume that the image i of $\frac{1}{|Z|}\sum_{z \in Z} z$ in $(\mathcal{O}G)_\delta \subset \mathcal{O}G$ belongs to $\delta^{G} = \gamma^{G}$ and then we get (cf. 3.9.1)

$$S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L} \cong i(\mathcal{O}G)i \cong (\mathcal{O}H)_\delta^Z. \tag{3.13.64}$$

But, it follows from [10, Proposition 7.4] that there is a unique $j + J\left((\mathcal{O}H)_\delta^Q\right)$-conjugacy class of k^*-group homomorphisms

$$\hat{\alpha} : Q \rtimes \hat{F} \longrightarrow ((\mathcal{O}H)_\delta)^* \tag{3.13.65}$$

mapping $u \in Q$ on $u \cdot j$; then, since Z is a p'-group, it follows from [3, Lemma 3.3 and Proposition 3.5] that we can choose α in such a way that Z normalizes $\alpha(\hat{F})$ and then we have $[Z, \alpha(\hat{F})] \subset k^*$ In this case, $\alpha(\hat{F})$ stabilizes $(\mathcal{O}H)_\delta^Z$ and therefore, throughout isomorphism 3.13.64, F acts on $S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L}$ normalizing the structural image of Q; hence, F acts on

$$S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L} \Big/ J(S \otimes_{\mathcal{O}} \mathcal{O}_*\hat{L}) \cong S \otimes_{\mathcal{O}} k_*\hat{E} \tag{3.13.66}$$

stabilizing the simple k-subalgebra $S \otimes_{\mathcal{O}} k$ and the image of Q inside; finally, it follows from [11, 1.5.2] that S is also F-stable. We are done.

4 Normal sub-blocks of nilpotent blocks

4.1

With the notation of sect. 3, assume now that the block b of G is nilpotent; since we already know that $(\mathcal{O}G)_\gamma \cong S \otimes_O \mathcal{O}P$ for a suitable Dade P-algebra S [9, Main Theorem], the block b is also inertial and therefore we already have proved that the normal sub-block c of H is inertial too; let us show with the following example—as a matter of fact, the example which has motivated this note—that the block c need not be nilpotent.

Example 4.2 Let $\mathfrak{F}$ be a finite field of characteristic different from p, q the cardinal of $\mathfrak{F}$ and $\mathfrak{E}$ a field extension of $\mathfrak{F}$ of degree $n \neq 1$; denoting by Φ_n the n-th *cyclotomic polynomial*, assume that p divides $\Phi_n(q)$ but not $q-1$, that $\Phi_n(q)$ and $q-1$ have a nontrivial common divisor r—which has to be a prime number[2] —and that n is a power of r. For instance, the triple (p, q, n) could be $(3, 5, 2)$, $(5, 3, 4)$, $(7, 4, 3)\ldots$

Set $G = GL_{\mathfrak{F}}(\mathfrak{E})$ and $H = SL_{\mathfrak{F}}(\mathfrak{E})$, and, respectively, denote by T and by W the images in G of the multiplicative group of $\mathfrak{E}$ and of the Galois group of the extension $\mathfrak{E}/\mathfrak{F}$; since p does not divide $q-1$, $T \cap H$ contains the Sylow p-subgroup P of T and, since p divides $\Phi_n(q)$, we have

$$C_G(P) = T \quad \text{and} \quad N_G(P) = T \rtimes W; \tag{4.2.1}$$

consequently, since W acts regularly on the set of generators of a Sylow r-subgroup of T, a generator φ of the Sylow r-subgroup of $\mathrm{Hom}(T, \mathbb{C}^*)$ determines a local point γ of P on $\mathcal{O}G$ such that

$$N_G(P_\gamma) = T = C_G(P) \tag{4.2.2}$$

and, by the *Brauer First Main Theorem*, P_γ is a defect pointed group of a block b of G which, according to [13, Proposition 5.2], is *nilpotent* by equality 4.2.2.

On the other hand, since r divides $q-1$, the restriction ψ of φ to the intersection $T \cap H = C_H(P)$ has an order strictly smaller than φ and therefore, since we clearly have

$$N_H(P)/C_H(P) \cong W, \tag{4.2.3}$$

r divides $|N_H(P_\delta)/C_H(P)|$ where δ denotes the local point of P on $\mathcal{O}H$ determined by ψ; once again by the *Brauer First Main Theorem*, P_δ is a defect pointed group of a block c of H, which is clearly a normal sub-block of the block b of G and it is *not* nilpotent since r divides $|N_H(P_\delta)/C_H(P)|$.

Corollary 4.3 *A block c of a finite group H is a normal sub-block of a nilpotent block b of a finite group G only if it is inertial and has an Abelian inertial quotient.*

Proof We already have proved that c has to be inertial. For the second statement, we borrow the notation of Proposition 3.8; on the one hand, since the block b is nilpotent, we know that $E_G(Q_{\delta^G})$ is a p-group; on the other hand, it follows from this proposition that $E_H(Q_{\delta^G})$ is a normal subgroup of F and that $F/E_H(Q_{\delta^G})$ is Abelian; since the inertial quotient F is a p'-group, we have $E_H(Q_{\delta^G}) = \{1\}$ and F is Abelian. We are done. □

[2] We thank Marc Cabanes for this remark.

Remark 4.4 Conversely, if P is a finite p-group and E a finite Abelian p'-group acting faithfully on P, the unique block of $\hat{L} = P \rtimes \hat{E}$ for any central k^*-extension of E is a normal sub-block of a nilpotent block of a finite group obtained as follows. Setting

$$Z = \mathrm{Hom}(E, k^*), \tag{4.4.1}$$

it is clear that Z acts faithfully on $\hat{E}$ fixing the k^*-quotient E; thus, the semidirect product $\hat{E} \rtimes Z$ still acts on P and we finally consider the semidirect product

$$\hat{M} = P \rtimes (\hat{E} \rtimes Z) = \hat{L} \rtimes Z. \tag{4.4.2}$$

Then, we clearly have

$$(\mathcal{O}_* \hat{M})(P) \cong k\,(Z(P) \times Z) \tag{4.4.3}$$

and therefore any group homomorphism $\varepsilon : Z \to k^*$ determines a local point of P on $\mathcal{O}_* \hat{M}$—still noted ε; but E acts on kZ, regularly permuting the set of its points; hence, we get

$$N_{\hat{M}}(P_\varepsilon) = k^* \times P \times Z. \tag{4.4.4}$$

and therefore P_ε is a defect pointed group of the nilpotent block $\{1_{\mathcal{O}_* \hat{M}}\}$ of $\hat{M}$.

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