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Hilbert schemes and maximal Betti numbers over veronese rings

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Abstract Macaulay's Theorem (Macaulay in Proc. Lond Math Soc 26:531–555, 1927) characterizes the Hilbert functions of graded ideals in a polynomial ring over a field. We characterize the Hilbert functions of graded ideals in a Veronese ring R (the coordinate ring of a Veronese embedding of $\mathbf{P}^{r-1}$). We also prove that the Hilbert scheme, which parametrizes all graded ideals in R with a fixed Hilbert function, is connected; this is an analogue of Hartshorne's Theorem (Hartshorne in Math. IHES 29:5–48, 1966) that Hilbert schemes over a polynomial ring are connected. Furthermore, we prove that each lex ideal in R has the greatest Betti numbers among all graded ideals with the same Hilbert function.

1 Introduction

In this paper, S stands for the polynomial ring $k[x_1, \ldots, x_n]$ over a field k of characteristic zero. The ring S is graded by $\deg(x_i) = 1$ for each i. Let $q, r \geq 2$ be integers. Throughout, we consider the toric Veronese ring $R = S/I$, where I is the defining ideal of the q'th Veronese embedding of $\mathbf{P}^{r-1}$ (see Sect. 3 for a precise definition). Note that Veronese rings include the projective coordinate rings of all rational normal curves.

Let X be a graded ideal in S, and $D = S/X$. If J is a graded ideal in D, then J_i stands for the vector space of all polynomials in J of degree i. The Hilbert function

$$\begin{aligned} h:\ \mathbf{N} &\longrightarrow \mathbf{N} \\ i &\mapsto \dim_k J_i \end{aligned}$$

is an important numerical invariant, which measures the size of the ideal. The following question is very natural:

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Question 1.1 What are the possible Hilbert functions of graded ideals in the ring D?

A classical theorem by Macaulay [23] provides the answer for $D = S$. The key idea introduced by Macaulay is to consider lex ideals, which are special monomial ideals defined in a simple combinatorial way.

Macaulay's Theorem 1.2 [23] *For every graded ideal J in S there exists a unique lex ideal L_J with the same Hilbert function.*

Theorem 1.2 leads to a numerical characterization of Hilbert functions; see Theorem 4.3.

Hilbert functions of graded ideals in an exterior algebra (or in the quotient ring $D = S/(x_1^2, \ldots, x_n^2)$) are of interest on their own and also are of interest in Combinatorics because they correspond to f-vectors which count faces of simplicial complexes. It is proved by Kruskal–Katona [21,22] that Macaulay's theorem holds over every exterior algebra. Counting faces of simplicial complexes naturally generalizes to counting in multicomplexes; this leads to considering Clements–Lindström rings of the form $D = S/(x_1^{a_1}, \ldots, x_n^{a_n})$ where $2 \le a_1 \le \cdots \le a_n$. Clements–Lindström's Theorem [7] shows that Macaulay's theorem holds over every Clements–Lindström ring.

As outlined below, if Macaulay's Theorem holds over a ring D then lex ideals provide a powerful tool in the study of Hilbert schemes and syzygies over D. This motivates the open problem to find interesting classes of graded quotient rings over which Macaulay's Theorem holds. Unfortunately, the theorem fails over some very simple rings; for example, there exists no lex ideal with the same Hilbert function as the ideal (ab) in the quotient ring $k[a, b]/(a^2b, ab^2)$, cf. [25, Example 2.13]. It seems that it is rare and remarkable when Macaulay's Theorem holds over a quotient ring.

In this paper, we consider toric Veronese rings; such rings are of interest in Commutative Algebra (due to their nice homological properties; for example, R is well known to be a Koszul algebra) and in Algebraic Geometry (for example, the N_p-conjecture for R in [26] is still open). A classical result of Green [15] shows that the defining ideal I (recall that $R = S/I$) is generated by quadrics. We prove

Theorem 1.3 *For every graded ideal in the Veronese ring R there exists a lex ideal with the same Hilbert function.* (*We use a specific lex order, defined in Sect.* 3.)

Macaulay's Theorem and Kruskal–Katona's Theorem yield a numerical characterization of the Hilbert functions over a polynomial ring and over an exterior algebra, respectively. Similarly, Theorem 1.3 leads to a numerical characterization of the Hilbert functions over a Veronese ring; we present this in Sect. 4.

If Macaulay's Theorem holds over a graded quotient ring then a key role in the study of Hilbert functions and syzygies is played by lex ideals. We list the three most important applications in 1.4, 1.6, and 1.7. They concern the following problem, which has received broad attention.

Problem What can be said about the properties of ideals with a fixed Hilbert function?

1.4 Betti numbers

Evans raised the problem to study the possible Betti numbers of all graded ideals in S with a fixed Hilbert function. Since the problem is very complex in general, people focused on maximal and on minimal Betti numbers. It turned out that there exist examples of a Hilbert function for which there exists no ideal with minimal Betti numbers. In contrast, Bigatti, Hulett, and Pardue proved that every lex ideal in S attains maximal Betti numbers among all graded ideals with the same Hilbert function, cf. [6, Theorem 1.1]. This property also holds over exterior algebras by a result of Aramova–Herzog–Hibi [2, Theorem 4.4] and over Clements–Lindström rings by a result of Murai–Peeva [24, Theorem 1.5]. In Sect. 2 we prove the following theorem, and in Sect. 5 we prove a more precise (but more technical) version of it.

Theorem 1.5 *Every lex ideal in the Veronese ring R attains maximal Betti numbers among all graded ideals with the same Hilbert function.*

1.6 Gotzmann's theorems

Gotzmann's Regularity Theorem and Gotzmann's Persistence Theorem [13] are two important results on Hilbert functions in the polynomial ring S. Given a graded ideal J these results provide upper bounds for the Catelnuovo–Mumford regularity of J and for the degrees of the minimal monomial generators of the lex ideal L_J (see the notation in Theorem 1.2), respectively, in terms of the Hilbert function of J. Analogues are proved over exterior algebras in [2, Theorems 4.5 and 4.6] and over Clements–Lindström rings in [12]. In Sect. 4, we prove that Gotzmann's Persistence Theorem and Gotzmann's Regularity Theorem hold over a Veronese ring. Another important result on Hilbert functions is Green's Theorem [14] which shows how the Hilbert function changes when we take a generic hyperplane section; in 3.8 we observe that it holds over R.

1.7 Hilbert schemes

Grothendieck [16] introduced the classical Hilbert scheme, which parametrizes subschemes of $\mathbf{P}^{n-1}$ with a fixed Hilbert polynomial. When we work over a quotient ring it is natural to consider a slight variation of Grothendieck's construction: we consider the Hilbert scheme $\mathcal{H}_h^D$ that parametrizes all graded ideals in the ring D with a Hilbert function h. The structure of the Hilbert scheme is usually very complicated. The main known structural result is Hartshorne's famous Theorem 1.8.

Theorem 1.8 [17] *The Hilbert scheme, which parametrizes all graded ideals in S with a fixed Hilbert function, is connected. Every graded ideal in S with Hilbert function h is connected by a sequence of deformations to the lex ideal with Hilbert function h.*

Analogues of this result are proved over exterior algebras by Peeva–Stillman [29, Theorem 1.3] and over Clements–Lindström rings by Murai–Peeva [24, Theorem 1.3]. We prove

Theorem 1.9 *The Hilbert scheme, which parametrizes all graded ideals in the Veronese ring R with a fixed Hilbert function, is connected. Every graded ideal in R with Hilbert function h is connected by a sequence of deformations to the lex ideal with the same Hilbert function h.*

We raise the open problem whether the lex point on the Hilbert scheme is a smooth point. Reeves and Stillman [32] proved that the lex point is smooth on the classical Hilbert scheme, introduced by Grothendieck [16], which parametrizes subschemes of $\mathbf{P}^r$ with fixed Hilbert polynomial. It is an open problem whether the lexicographic point is smooth on the Hilbert scheme $\mathcal{H}_V^h$, that parametrizes all graded ideals with a fixed Hilbert function h in the ring V, where V is either a polynomial ring S, or an exterior algebra E, or a Clements–Lindstöm ring C, or a Veronese ring R. It is possible that the answer is different over different rings. The proof in [32] does not work because the tangent space is different, see [30, Proposition 2.1].

2 Lex ideals

In this section, we construct special monomial orders and examples that illustrate the subtleness of the lex definition in a toric ring.

Let $\mathcal{A} = \{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$ be a subset of $\mathbf{N}^r \setminus \{\mathbf{0}\}$, A be the matrix with columns $\mathbf{a}_i$, and suppose that $\operatorname{rank}(A) = r$. For $1 \le i \le n$, denote $\mathbf{t}^{\mathbf{a}_i} = t_1^{a_{i1}} \ldots t_r^{a_{ir}}$, where $\mathbf{a}_i = (a_{i1}, \ldots, a_{ir})$. We denote by $I_{\mathcal{A}}$ the kernel of the homomorphism

$$\varphi_{\mathcal{A}} : k[x_1, \ldots, x_n] \to k[t_1, \ldots, t_r]$$
$$x_i \mapsto \mathbf{t}^{\mathbf{a}_i}.$$

It is a prime ideal, and is called the *toric ideal* associated to $\mathcal{A}$. The *toric ring* associated to $\mathcal{A}$ is $R_{\mathcal{A}} = S/I_{\mathcal{A}}$. We have the isomorphism $Q_{\mathcal{A}} := k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}] \cong R_{\mathcal{A}}$. The toric ideal $I_{\mathcal{A}}$ is *projective* (or $S/I_{\mathcal{A}}$ is a *projective* toric ring) if $I_{\mathcal{A}}$ is homogeneous with respect to the standard grading of S with $\deg(x_i) = 1$ for $1 \le i \le n$. Then $R_{\mathcal{A}}$ inherits the grading from S. Let $p \ge 0$ be an integer; a *p-monomial space* W is a vector subspace of $(R_{\mathcal{A}})_p$ spanned by monomials of degree p. In the rest, we consider only projective toric rings.

Now, we focus on the definition of a lex ideal in a projective toric ring. For simplicity we will assume that the monomials $\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}$ have the same degree.

Construction 2.1 Order the variables in $T = k[t_1, \ldots, t_r]$ by $t_1 > t_2 > \cdots > t_r$ and consider the following monomial order $>$ in $Q_{\mathcal{A}}$: if v and v' are monomials in $Q_{\mathcal{A}}$, then $v > v'$ if v is degree-lex-greater than v'. This is a total order on the monomials in $Q_{\mathcal{A}}$ (and also in T). We denote it by $\mathbf{dlex}_T$.

Order the monomials in S so that $m >_T u$ in S if $\varphi_{\mathcal{A}}(m) >_{\mathbf{dlex}_T} \varphi_{\mathcal{A}}(u)$ in T. This is a partial order on the monomials in S. Two monomials m and m' in S are incomparable by $>_T$ if and only if $\varphi_{\mathcal{A}}(m) = \varphi_{\mathcal{A}}(m')$, which holds if and only if $m - m' \in I_{\mathcal{A}}$.

Construction 2.2 We define a partial monomial order $<_{toric}$ on S using the weight orders with respect to the rows in the matrix A. For $1 \le i \le r$, denote by $\mathbf{w}_i$ the weight order of the monomials in S with respect to the vector $((\mathbf{a}_1)_i, \ldots, (\mathbf{a}_n)_i)$. Let $\mathbf{w}_0 = \sum_{i=1}^{r} \mathbf{w}_i$. Let m and u be two monomials in S. We define that $m >_{toric} u$ if there exists a $0 \le j \le r$ such that

$$\mathbf{w}_j(m) > \mathbf{w}_j(u) \text{ and } \mathbf{w}_i(m) = \mathbf{w}_i(u) \quad \text{for } 0 \le i < j.$$

This is a partial order on the monomials in S.

Two monomials m and m' are incomparable by $<_{toric}$ if and only if $\mathbf{w}_i(m) = \mathbf{w}_i(m')$ for all $1 \le i \le r$; this happens if and only if $m - m' \in I_{\mathcal{A}}$. Hence, the following two properties hold:

(a) $\operatorname{in}_{<_{toric}}(I_{\mathcal{A}}) = I_{\mathcal{A}}$.
(b) if m and m' are incomparable monomials, then $m - m' \in I_{\mathcal{A}}$.

It is easy to check that the following lemma holds.

Lemma 2.3 *The two partial orders on S constructed in Constructions 2.1 and 2.2 coincide.*

Note that $<_T = <_{toric}$ induces a well-defined total monomial order $<_T$ in the ring $R_{\mathcal{A}}$ since $m - m' \in I_{\mathcal{A}}$ for two monomials m and m' in S if and only if the monomials are equal with respect to $<_T$.

Let M be a monomial ideal in $R_{\mathcal{A}}$. Denote by $\varphi_{\mathcal{A}}(M)$ its image ideal in $Q_{\mathcal{A}} = k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$. It looks natural to define that M is lex if $\varphi(M_q)$ is lex in $Q_{\mathcal{A}}$ for every $q \ge 0$. Example 2.4 shows that this is not a satisfactory definition since a lex monomial space (in a fixed degree) may not generate a lex monomial space in the next degree.

Example 2.4 Consider the defining ideal $I_{\mathcal{A}}$ of $\mathbf{P}^1 \times \mathbf{P}^1$. It is the kernel of the homomorphism

$$\begin{aligned} \varphi_{\mathcal{A}} : \ S = k[x_1, x_2, x_3, x_4] &\to T = k[t_1, t_2, t_3, t_4] \\ x_1 &\mapsto t_1 t_3 \\ x_2 &\mapsto t_1 t_4 \\ x_3 &\mapsto t_2 t_3 \\ x_4 &\mapsto t_2 t_4. \end{aligned}$$

Here the set $\mathcal{A}$ is

$$\mathcal{A} = \{(1, 0, 1, 0),\ (1, 0, 0, 1),\ (0, 1, 1, 0),\ (0, 1, 0, 1)\}.$$

Denote by $\mathbf{dlex}_T$ the degree-lex monomial order in the polynomial ring T so that the variables are ordered by $t_1 > t_2 > t_3 > t_4$. Furthermore, we order the variables in the polynomial ring S so that $x_i > x_p$ if $\varphi_{\mathcal{A}}(x_i) >_{\mathbf{lex}_T} \varphi_{\mathcal{A}}(x_p)$. Therefore, $x_1 > x_2 > x_3 > x_4$. Consider the degree-lex monomial order $\mathbf{dlex}_S$ on S determined by the equivalent Constructions 2.1 and 2.2.

Consider the ideal $M = (x_1)$ in the Segre toric ring $S/I_{\mathcal{A}}$. On the one hand, $\varphi_{\mathcal{A}}(M_1)$ is spanned by $t_1 t_3$, so it is a lex monomial space in $\varphi_{\mathcal{A}}(S)$.

On the other hand, $\varphi_{\mathcal{A}}(M_2)$ contains the monomial $\varphi_{\mathcal{A}}(x_1 x_3) = t_1 t_2 t_3^2$, but does not contain the $\mathbf{dlex}_T$-bigger monomial $t_1^2 t_4^2 = \varphi_{\mathcal{A}}(x_2^2)$. Therefore, $\varphi_{\mathcal{A}}(M_2)$ is not a lex monomial space in $\varphi_{\mathcal{A}}(S)$.

This example explains why a more intricate definition of lex ideals is introduced in [11].

Definition 2.5 [11, Definition 3.7] We fix the order of the variables in S to be $x_1 > \cdots > x_n$, and consider the induced dlex order $\mathbf{dlex}_S$ on S. Recall that for $\mathbf{c} \in \mathbf{N}^r$, the set $\{m \text{ is a monomial in } S \mid \varphi(m) = \mathbf{t^c}\}$ is called the *fiber* of $\mathbf{c}$ (or the fiber of $\mathbf{t^c}$). The lex-greatest monomial in a fiber will be called the *top-representative* of the fiber.

We say that a p-monomial space W in $R_{\mathcal{A}}$ is a *lex space* if the following property is satisfied: if $m \in W$ is a monomial, $v \in S$ is the top-representative of the fiber of m, and $u \in S_p$ is a monomial such that $u >_{\mathbf{dlex}_S} v$, then $u \in W$ (by abuse of notation $u \in W$ means that the image of u in $R_{\mathcal{A}}$ is a monomial in W).

It is proved in [11, Theorem 3.4] that a lex p-monomial space in $(R_{\mathcal{A}})_p$ generates a lex monomial space in $(R_{\mathcal{A}})_{p+1}$. A monomial ideal L in $R_{\mathcal{A}}$ is called *lex* if for every $i \geq 0$, we have that L_i is spanned by a lex monomial space in $(R_{\mathcal{A}})_i$.

In Example 2.4, it is easy to check that M_2 is lex monomial space according to the definition above. The fact that $\varphi_{\mathcal{A}}(x_2^2) = t_1^2 t_4^2 \notin \varphi_{\mathcal{A}}(M_2)$ is not a problem anymore since x_2^2 is not $\mathbf{dlex}_S$-bigger than any of the top-representatives of the fibers of the monomials in M_2.

We close this section by an example which shows that the $\mathbf{dlex}_T$ order in $Q_{\mathcal{A}}$ does not agree with the $\mathbf{dlex}_S$ order on the set of the top-representatives in S. This illustrates the subtleness of the definition of a lex ideal in a toric ring.

Example 2.6 We continue Example 2.4. On the one hand, we have the inequality

$$x_2^2 <_{\mathbf{dlex}_S} x_1 x_3.$$

On the other hand, we have that

$$\varphi_{\mathcal{A}}(x_2^2) = t_1^2 t_4^2 >_{\mathbf{dlex}_T} t_1 t_2 t_3^2 = \varphi_{\mathcal{A}}(x_1 x_3).$$

Thus, the following property fails: $m >_{\mathbf{dlex}_S} u$ for two top-representative monomials in S if and only if $\varphi_{\mathcal{A}}(m) >_{\mathbf{dlex}_T} \varphi_{\mathcal{A}}(u)$. In this example, the $\mathbf{dlex}_T$ order in $Q_{\mathcal{A}}$ does not agree with the $\mathbf{dlex}_S$ order on the set of the top-representatives in S.

3 Veronese rings

After notation is introduced (say in some lemma or construction), it will be used in the rest of the paper. We will prove Theorems 1.3 and 1.9 in a series of constructions.

3.1 Veronese rings

Fix integer numbers $r, q \geq 1$. Set $n = \binom{r+q-1}{r-1}$.

For an integer column vector $\mathbf{v} = \begin{pmatrix} v_1 \\ \vdots \\ v_n \end{pmatrix}$, set $\mathbf{x}^{\mathbf{v}} = x_1^{v_1}, \ldots, x_n^{v_n}$.

T stands for the polynomial ring $k[t_1, \ldots, t_r]$, graded by $\deg(t_j) = 1$ for $1 \leq j \leq r$. Let R be the q'th Veronese ring in r variables which defines the qth Veronese embedding of $\mathbf{P}^{r-1}$. Thus,

$$R \cong \oplus_{j=0}^{\infty} T_{jq} = k[\text{all monomials of degree } q \text{ in } T].$$

The set of points defining the toric ideal can be taken to be

$$\mathcal{A} = \left\{ (i_1, \ldots, i_r) \in \mathbf{N}^r \backslash \mathbf{0} \mid \sum_{j=1}^{r} i_j = q \right\}.$$

We order the vectors in $\mathcal{A}$ lexicographically so that the first vector is the greatest, and then denote them by $\{\mathbf{a}_1, \ldots, \mathbf{a}_n\}$. Let A be the matrix with columns $\mathbf{a}_i$. We denote $k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}]$ by Q.

We denote by I the kernel of the homomorphism

$$\begin{aligned} \varphi :\ k[x_1, \ldots, x_n] &\to k[t_1, \ldots, t_r] \\ x_i &\mapsto \mathbf{t}^{\mathbf{a}_i}. \end{aligned}$$

It is a prime ideal, and is called the toric *Veronese ideal* associated to $\mathcal{A}$. The toric *Veronese ring* associated to $\mathcal{A}$ is

$$R := S/I \cong k[\mathbf{t}^{\mathbf{a}_1}, \ldots, \mathbf{t}^{\mathbf{a}_n}] = Q \subseteq T.$$

Furthermore, φ_S stands for the map $S \to R$, and φ_R stands for the isomorphism $R \cong Q$ induced by φ. Thus, $\varphi = \varphi_R \varphi_S$.

The Veronese toric ideal I is projective, that is, I is homogeneous with respect to the standard grading of S with $\deg(x_i) = 1$ for $1 \leq i \leq n$. The ring R inherits the grading from S.

3.2 The lex property in Veronese rings

In contrast to Example 2.6, it is easy to see that we have the following very useful lemma over Veronese rings. This lemma is the foundation for our arguments.

Lemma 3.2.1 *In the notation of Sect. 2, over a Veronese ring we have that the* $\mathbf{dlex}_T$ *order in* Q *agrees with the* $\mathbf{dlex}_S$ *order on the set of the top-representatives in* S*, that is,* $m >_{\mathbf{dlex}_S} u$ *for two top-representative monomials in* S *if and only if* $\varphi(m) >_{\mathbf{dlex}_T} \varphi(u)$.

For simplicity we call this order lex throughout.

Proof Let $\mathbf{a} \in \mathbf{N}^r$. We will describe the top-representative of the fiber of $\mathbf{t^a}$.

Consider the lex order on $\mathbf{N}^r$ such that $\mathbf{e_1} > \cdots > \mathbf{e_r}$, where $\mathbf{e_1}, \ldots, \mathbf{e_r}$ are the standard vectors. We can write

$$\mathbf{a} = \mathbf{b_1} + \cdots + \mathbf{b_p},$$

where $\mathbf{b_i}$ is the lex-greatest vector in $\mathcal{A}$ such that $\mathbf{a} - \mathbf{b_1} + \cdots + \mathbf{b_{i-1}}$ is coordinatewise bigger or equal to $\mathbf{b_i}$ (we assume $\mathbf{b}_0 = \mathbf{0}$ here). Note that for each i, we have that $\varphi^{-1}(\mathbf{t^{b_1}})$ is a variable in S. Denote by $\sigma(\mathbf{t^a})$ the monomial $\varphi^{-1}(\mathbf{t^{b_1}}) \ldots \varphi^{-1}(\mathbf{t^{b_p}}) \in S$. Clearly, $\varphi(\sigma(\mathbf{t^a})) = \mathbf{t^a}$. We will show that $\sigma(\mathbf{t^a})$ is the top-representative in the fiber of $\mathbf{t^a}$.

Let $m = x_{j_1}, \ldots, x_{j_p}$ be a monomial in the fiber of $\mathbf{t^a}$, and assume that $j_1 \leq \cdots \leq j_p$. We will show that $m <_{\mathbf{dlex}_S} \sigma(\mathbf{t^a})$. Recall that $\varphi(x_{j_i}) = \mathbf{t^{a_{j_1}}}$.

If $\mathbf{a_{j_1}} = \mathbf{b_1}$, then $\dfrac{m}{x_{j_1}} <_{\mathbf{dlex}_S} \dfrac{\sigma(\mathbf{t^a})}{x_{\mathbf{b_1}}}$ by induction hypothesis on the degree, so we are done. Suppose that $\mathbf{a_{j_1}} \neq \mathbf{b_1}$. Let i be the smallest number such that the i'th coordinates of $\mathbf{a_{j_1}}$ and $\mathbf{b_1}$ are different. It follows that $(\mathbf{a_{j_1}})_i < (\mathbf{b_1})_i$ by the construction of $\mathbf{b_1}$. Hence $x_{j_1} <_{\mathbf{dlex}_S} x_{\mathbf{b_1}}$. Therefore, $\sigma(\mathbf{t^a}) >_{\mathbf{dlex}_S} m$.

Denote by $\mathcal{M}_S$ the set of all monomials in S that are top-representatives. Denote by $\mathcal{M}_Q$ the set of all monomials in Q. By Construction 2.1, if $m >_{\mathbf{dlex}_S} u$ for two top-representative monomials in S then we have that $\varphi(m) >_{\mathbf{dlex}_T} \varphi(u)$. If $\mathbf{a}, \mathbf{c} \in \mathbf{N}^r$ and $\mathbf{t^a} >_{\mathbf{dlex}_T} \mathbf{t^c}$, then $\sigma(\mathbf{t^a}) >_{\mathbf{dlex}_S} \sigma(\mathbf{t^c})$ by the construction of σ. We have the dlex-order preserving bijection

$$\begin{aligned} \varphi &: \mathcal{M}_S \longrightarrow \mathcal{M}_Q \\ \tau &: \mathcal{M}_Q \longrightarrow \mathcal{M}_S. \end{aligned}$$

Thus, the following property holds: $m >_{\mathbf{dlex}_S} u$ for two top-representative monomials in S if and only if $\varphi(m) >_{\mathbf{dlex}_T} \varphi(u)$. □

An immediate consequence is the following result.

Theorem 3.2.2 *A monomial ideal M in the Veronese ring R is lex if and only if its image $\varphi(M)$ generates a lex ideal in the polynomial ring T.*

Proof Let M be a lex ideal in R. Let $p \in \mathbf{N}$ and $d_p = \dim(M_p)$. By Lemma 3.2.1, it follows that $\varphi(M_p)$ is the vector space spanned by the d_p lex-greatest monomials of degree pq in T starting with t_1^{pq}. Therefore, $\varphi(M)$ generates a lex ideal in the polynomial ring T.

Let M be a monomial ideal in R such that $\varphi(M)$ generates a lex ideal in the polynomial ring T. Let $p \in \mathbf{N}$ and $d_p = \dim \varphi(M_p)$. We have that $\varphi(M_p)$ is the vector space spanned by the d_p lex-greatest monomials of degree pq in T starting with t_1^{pq}. By Lemma 3.2.1, it follows that M_p is a lex p-monomial space in R. Hence, M is lex in R. □

Construction 3.2.3 If G is an ideal in the polynomial ring T, then we set $G_Q = Q \cap G$ and $G_R = \varphi_R^{-1}(G_Q)$, and denote by G_S the preimage of G_R in S. Thus, G_R, and G_S are ideals in R and S, respectively. We also set $G_T = G$.

Construction 3.2.4 If N is an ideal in the Veronese ring R, then we denote by ${}_QN$ the isomorphic ideal $\varphi_R(N)$ in Q. Furthermore, denote by ${}_TN$ the ideal in T generated by ${}_QN$, and denote by ${}_SN$ the preimage of N in S. Thus, ${}_TN$ and ${}_SN$ are ideals in T, and S, respectively. We also set ${}_RN = N$.

An ideal in S is called *lex+I* if it is the preimage of a lex ideal in R.

Proposition 3.2.5 (1) *If G is a lex ideal in T, then G_R is lex in R, and G_S is lex+I in S.* (2) *If N is a lex ideal in R, then ${}_TN$ is lex in T, and ${}_SN$ is lex+I in S.*

We will consider three types of deformations, which we call Type-A deformations, Type-B deformations, and Type-C deformations.

3.3 Type-A deformations using change of coordinates

The general linear group $\mathcal{G} = \mathrm{GL}(r, k)$ of invertible $(r \times r)$-matrices over k, acts as a group of algebra automorphisms on the polynomial ring T by acting on the variables as follows: if $E \in \mathcal{G}$ has entries e_{ij}, then $E(t_j) = \sum_{i=1}^{r} e_{ij} t_i$, and furthermore $E(t_1^{p_1}, \dots, t_r^{p_r}) = E(t_1)^{p_1}, \dots, E(t_r)^{p_r}$. This is called a *change of coordinates* in T. Let ψ_T be such a change of coordinates. Note that $\psi_T(Q) = Q$. We denote by ψ_Q its restriction on Q. We denote by ψ_S the change of variables (coordinates) in the polynomial ring S that is induced by ψ_Q. Therefore,

$$\varphi\psi_S = \psi_T\varphi.$$

If $f \in I = \mathrm{Ker}(\varphi)$, then $\psi_S(f) \in I$ since $\varphi\psi_S(f) = \psi_T\varphi(f) = 0$. Hence, $\psi_S(I) = I$.

Let G be an ideal in the polynomial ring T. If $H = \psi_T(G)$, then we have that

$$H_Q = \psi_Q(G_Q),$$
$$H_R = \psi_R(G_R),$$
$$H_S = \psi_S(G_S),$$

where the last equality follows from the fact that $\psi_S(I) = I$.

We say that H, H_Q, H_R, and H_S are *Type-A deformations* of G, G_Q, G_R, and G_S, respectively. Such a deformation preserves the Betti numbers.

3.4 Type-B deformations using initial ideals

Let H be a graded ideal in the polynomial ring T, and let F be an initial ideal of H. By [3], we can choose a vector $\mathbf{z} = (z_1, \dots, z_n)$ with strictly positive integer coordinates, such that F is the initial ideal of H with respect to the weight order induced by the weight vector $\mathbf{z}$, cf. [8, Theorem 15.16]. Let $\tilde{H}$ be the homogenization of H in the polynomial ring $\tilde{T} = T[t]$; here $\tilde{T}$ is graded by $\deg(t_i) = z_i$ for $1 \le i \le r$ and $\deg(t) = 1$. Then t and $t-1$ are regular elements on $\tilde{T}/\tilde{H}$, cf. [8, Theorem 15.17]. Set $\tilde{Q} = Q[t]$. Furthermore, set $\tilde{H}_{\tilde{Q}} = \tilde{H} \cap \tilde{Q}$. Clearly, $\tilde{H}_{\tilde{Q}} = \widetilde{H_Q}$.

We will show that t and $t-1$ are regular elements on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$. Suppose that $t\tilde{h} \in \tilde{H}_{\tilde{Q}}$ for some $\tilde{h} \in \tilde{Q}$. It follows that $t\tilde{h} \in \tilde{H}$. Since t is a regular element on $\tilde{T}/\tilde{H}$, we conclude that $\tilde{h} \in \tilde{H}$. Hence, $\tilde{h} \in \tilde{H}_{\tilde{Q}} = \tilde{H}_{\tilde{Q}} \cap \tilde{H}$. Therefore, t is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$. The argument for $t-1$ is similar. Suppose that $(t-1)\tilde{h} \in \tilde{H}_{\tilde{Q}}$ for some $\tilde{h} \in \tilde{Q}$. It follows that $(t-1)\tilde{h} \in \tilde{H}$. Since $t-1$ is a regular element on $\tilde{T}/\tilde{H}$, we conclude that $\tilde{h} \in \tilde{H}$. Hence, $\tilde{h} \in \tilde{H}_{\tilde{Q}} = \tilde{H}_{\tilde{Q}} \cap \tilde{H}$. Therefore, $t-1$ is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$.

Furthermore, set $\tilde{R} = R[t]$. Let $\tilde{H}_{\tilde{R}} = \tilde{\varphi}_{\tilde{R}}^{-1}(\tilde{H}_{\tilde{Q}})$, where $\tilde{\varphi}_{\tilde{R}}$ is the isomorphism $\tilde{R} \cong \tilde{Q}$ induced by φ_R. We conclude that t and $t-1$ are regular elements on $\tilde{R}/\tilde{H}_{\tilde{R}}$.

Set $\tilde{S} = S[t]$, and let $\tilde{\varphi} : \tilde{S} \to \tilde{R}$ be the map induced by φ. Let $\tilde{H}_{\tilde{S}}$ be the preimage of $\tilde{H}_{\tilde{R}}$ in $\tilde{S}$. We will show that t and $t-1$ are regular elements on $\tilde{S}/\tilde{H}_{\tilde{S}}$. Suppose that $th \in \tilde{H}_{\tilde{S}}$ for some $h \in \tilde{S}$. It follows that $t\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Since t is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$, it follows that $\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Hence, $h \in \varphi^{-1}(\tilde{H}_{\tilde{Q}}) = \tilde{H}_{\tilde{S}}$. Hence, t is a non-zerodivisor on $\tilde{S}/\tilde{H}_{\tilde{S}}$. The argument for $t-1$ is similar. Suppose that $(t-1)h \in \tilde{H}_{\tilde{S}}$ for some $h \in \tilde{S}$. It follows that $(t-1)\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Since $t-1$ is a non-zerodivisor on $\tilde{Q}/\tilde{H}_{\tilde{Q}}$, it follows that $\varphi(h) \in \tilde{H}_{\tilde{Q}}$. Hence, $h \in \varphi^{-1}(\tilde{H}_{\tilde{Q}}) = \tilde{H}_{\tilde{S}}$. Hence, $t-1$ is a non-zerodivisor on $\tilde{S}/\tilde{H}_{\tilde{S}}$.

By construction we have that

$$T/F = \tilde{T}/\tilde{H} \otimes \tilde{T}/t \quad \text{and} \quad T/H = \tilde{T}/\tilde{H} \otimes \tilde{T}/(t-1).$$

It follows that

$$Q/F_Q = \tilde{Q}/\tilde{H}_{\tilde{Q}} \otimes \tilde{Q}/t \quad \text{and} \quad Q/H_Q = \tilde{Q}/\tilde{H}_{\tilde{Q}} \otimes \tilde{Q}/(t-1).$$

As $R \cong Q$,we get

$$R/F_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/t \quad \text{and} \quad R/H_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/(t-1).$$

Consider the grading of S with $\deg(x_i) = \deg(\varphi(x_i)) = \mathbf{a_i} \cdot \mathbf{z}$. Homogenize the ideal H_S using the variable t with respect to that grading. We denote this homogenization by $\widetilde{H_S}$. Recall that $\tilde{H}_{\tilde{S}}$ is the preimage of the ideal $\tilde{H}$ in $\tilde{S}$. Then we have $\widetilde{H_S} = \tilde{H}_{\tilde{S}}$. Therefore,

$$S/F_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/t \quad \text{and} \quad S/H_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/(t-1).$$

Denote by $\tilde{\mathbf{F}}_{\tilde{R}}$ a graded minimal free resolution of $\tilde{R}/\tilde{H}_R$ over $\tilde{R}$. Note that this resolution is infinite. Then $\tilde{\mathbf{F}}_{\tilde{R}} \otimes \tilde{R}/t$ is a minimal free resolution of $R/F_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/t$. Thus, the graded Betti numbers of R/F_R and $\tilde{R}/\tilde{H}_{\tilde{R}}$ coincide. On the other hand, $\tilde{\mathbf{F}}_R \otimes \tilde{R}/(t-1)$ is a non-minimal graded free resolution of $R/H_R = \tilde{R}/\tilde{H}_{\tilde{R}} \otimes \tilde{R}/(t-1)$. Therefore,

$$\tilde{\mathbf{F}}_R \otimes \tilde{R}/(t-1) \cong \mathbf{H}_R \oplus \mathbf{U},$$

where $\mathbf{H}_R$ is a minimal graded free resolution of R/H_R and $\mathbf{U}$ is a trivial complex, cf. [8, Theorem 20.2]. The triviality of the complex $\mathbf{U}$ implies that the graded Betti numbers of R/H_R are obtained from those of $\tilde{R}/\tilde{H}_{\tilde{R}}$ by consecutive cancellations. Therefore, the graded Betti numbers of R/H_R are smaller or equal to those of R/F_R and are obtained by consecutive cancellations.

The same argument can be applied over S as follows. Denote by $\tilde{\mathbf{G}}_{\tilde{S}}$ a graded minimal free resolution of $\tilde{S}/\tilde{H}_{\tilde{S}}$ over $\tilde{S}$. Note that this resolution is finite. Then $\tilde{\mathbf{G}}_{\tilde{S}} \otimes \tilde{S}/t$ is a minimal free resolution of $S/F_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/t$. Thus, the graded Betti numbers of S/F_S and $\tilde{S}/\tilde{H}_S$ coincide. On the other hand, $\tilde{\mathbf{G}}_{\tilde{S}} \otimes \tilde{S}/(t-1)$ is a non-minimal graded free resolution of $S/H_S = \tilde{S}/\tilde{H}_{\tilde{S}} \otimes \tilde{S}/(t-1)$. Therefore,

$$\tilde{\mathbf{G}}_S \otimes \tilde{S}/(t-1) \cong \mathbf{E}_S \oplus \mathbf{V},$$

where $\mathbf{E}_S$ is a minimal graded free resolution of S/H_S and $\mathbf{V}$ is a trivial complex, cf. [8, Theorem 20.2]. The triviality of the complex $\mathbf{V}$ implies that the graded Betti numbers of S/H_S are obtained from those of $\tilde{S}/\tilde{H}_{\tilde{S}}$ by consecutive cancellations. Therefore, the graded Betti numbers of S/H_S are smaller or equal to those of S/F_S and are obtained by consecutive cancellations.

We say that F, F_Q, F_R and F_S are *Type-B deformations* of H, H_Q, H_R and H_S, respectively. A Type-B deformation increases the graded Betti numbers and the smaller Betti numbers can be obtained from the new greater ones by consecutive cancellations.

3.5 Type-C deformations using polarization

Let X be a monomial ideal in T. Fix an $1 \le i \le r$. We recall the definition of the partial polarization of X using the variable t_i. Let $\bar{T} = T[y]$. If m is a minimal monomial generator of X, then we set

$$\bar{m} = \begin{cases} \frac{m}{t_i} y & \text{if } t_i \text{ divides } m \\ m & \text{otherwise.} \end{cases}$$

Let $m_1, \ldots, m_l$ be the minimal monomial generators of X. Set

$$X_{pol} = (\bar{m}_1, \ldots, \bar{m}_l).$$

Then

$$\bar{T} / \big(X_{pol} + (t_i - y)\big) = T/X.$$

The ideal X_{pol} in $\bar{T}$ is called the *partial polarization with respect to* t_i of X. Let $\alpha_1 t_1 + \cdots + \alpha_r t_r$ be a generic linear form in T (here $\alpha_1, \ldots, \alpha_r \in k$). Furthermore, let X' be the ideal in T such that

$$\bar{T} / \big(X_{pol} + (\alpha_1 t_1 + \cdots + \alpha_r t_r - y)\big) = T/X'.$$

Note that X' is usually not a monomial ideal. The elements $t_i - y$ and $\alpha_1 t_1 + \cdots + \alpha_r t_r - y$ are non-zerodivisors on $\bar{T}/X_{pol}$, since we use the usual construction of partial polarization in the polynomial ring T. Therefore, the ideals X and X' have the same Hilbert function. Let $\mathbf{dlex}_i$ be a degree-lex order in the polynomial ring T such that t_i is greatest. By construction, it follows that $\mathrm{in}_{\mathbf{dlex}_i}(X') \supseteq X$. Since the two ideals have the same Hilbert function, we conclude that $\mathrm{in}_{\mathbf{dlex}_i}(X') = X$. Therefore, the ideals X and X' are connected by a deformation over T, cf. [8, Chap. 15].

By 3.4, it follows that the ideals X, X_Q, X_R, and X_S are deformations of the ideals X', X'_Q, X'_R, and X'_S, respectively. We call these deformations *Type-C deformations*.

3.6 A proof of Theorems 1.3 and 1.9

We need the following observation from [11, Lemma 2.3].

Lemma 3.6.1 *Let P and U be homogeneous ideals in R. Let $\tilde{P}$ and $\tilde{U}$ be the preimages of these ideals in S, respectively. The ideals P and U have the same Hilbert function over R, if and only if, the ideals $\tilde{P}$ and $\tilde{U}$ have the same Hilbert function over S.*

We are ready to prove the theorem.

Proof of Theorems 1.3 and 1.9 Let J be a graded ideal in R. Denote by ${}_S J$ its preimage in S, and by ${}_Q J$ its isomorphic image in Q. Denote by ${}_T J$ the ideal in T generated by ${}_Q J$. Also, set ${}_R J = J$.

By Macaulay's Theorem 1.1 there exists a lex ideal L_T in the polynomial ring T with the same Hilbert function as ${}_T J$. By [17,27], there exists a sequence of Type-A, Type-B, and Type-C deformations that connects ${}_T J$ to L_T. The argument above implies that the ideal J is connected to L_R by a a sequence of Type-A, Type-B, and Type-C deformations, and also that the ideal J_S is connected to L_S by a a sequence of Type-A, Type-B, and Type-C deformations.

The ideal L_T is lex in the polynomial ring T. By Proposition 3.2.5 it follows that the ideal L_R is lex in the Veronese ring R. Thus, we have proved (1) and (2). □

3.7 A proof of Theorem 1.5

Let J be a graded ideal in R. We denote by $\beta^R_{i,j}(J)$ and $\beta^R_i(J)$ the graded Betti numbers and the total Betti numbers of J over R, respectively.

Definition 3.7.1 A monomial ideal M in the polynomial ring T is called *Borel* if $mt_j \in M$ implies $mt_i \in M$ for all $1 \le i < j$. We say that a monomial ideal B in the Veronese ring R is *Borel* if ${}_TB$ is Borel.

Note that lex ideals are Borel. Borel ideals play an important role in Commutative Algebra since every generic initial ideal (assuming that char$(k) = 0$) is Borel. The minimal free resolution of a Borel ideal over the polynomial ring S is the well known Eliahou–Kervaire resolution [9, Sect. 2]. It is proved using iterated mapping cones, cf. [31]. It yields a formula for the Betti numbers and shows that if a Borel ideal is generated in one degree then its resolution is linear. In this section, we prove analogues results for the Betti numbers over R in Theorem 3.7.4.

First, recall that a graded ideal J in R is said to have a *j-linear resolution* if $\beta^R_{i,i+s}(J) = 0$ for all i, s with $s \neq j$. In particular, J is generated in degree j in this case.

For a monomial m in the polynomial ring T, let $\max(m)$ [respectively, $\min(m)$] be the maximal (respectively, minimal) integer j such that t_j divides m. Set $F(1) = \{0\}$ and

$$F(i) = \{x \in S : x \text{ is a variable such that } \min(\varphi(x)) < i\} \quad \text{for } i = 2, 3, \ldots, r.$$

Let $J_{F(i)}$ be the ideal in R generated by the image of $F(i)$ in R. The ideal $J_{F(i)}$ has a 1-linear resolution; indeed the following fact is known (cf. [18]).

Lemma 3.7.2 *Any ideal in R generated by variables has a 1-linear resolution.*

Let B be a Borel ideal in the Veronese ring R, and $\text{mingens}({}_TB) = \{v_1, \ldots, v_p\}$ denote the set of the minimal monomial generators of ${}_TB$ (note, that here each v_i is a monomial in T). We may assume that $\deg v_1 \le \cdots \le \deg v_p$ and that the ideals $(v_1, \ldots, v_j)$ are Borel for $j = 1, \ldots, p$. Fix monomial generators $u_1, \ldots, u_p$ of B such that $\varphi(u_j) = v_j$ for $j = 1, 2, \ldots, p$.

Lemma 3.7.3 *For $j = 1, 2, \ldots, p$, one has that*

$$((u_1, \ldots, u_{j-1}) :_R u_j) = J_{F(\max(v_j))}.$$

Proof It is enough to prove the case $j = p$. Let B' be the ideal in R generated by $u_1, \ldots, u_{p-1}$. Then ${}_TB'$ is the Borel ideal in T generated by $v_1, \ldots, v_{p-1}$. We will show that

$$({}_TB' :_T v_p) = \left(\left\{t_i : i < \max(v_p)\right\}\right).$$

Indeed, $({}_TB' :_T v_p) \supset \{t_i : i < \max(v_p)\}$ is obvious by the definition of Borel ideals. Let $m \in ({}_TB' :_T v_p)$ be a monomial. Then $mv_p \in {}_TB'$ and it follows from [9, Lemma 1.1] that there exists the unique generator v_k of ${}_TB'$ which divides mv_p and satisfies $\max(v_k) \le \min(\frac{mv_p}{v_k})$. Since $v_k \neq v_p$, it follows from [9, Lemma 1.2] that $\min(m) < \max(v_p)$. Thus $m \in (\{t_i : i < \max(v_p)\})$ and $({}_TB' :_T v_p) \subset (\{t_i : i < \max(v_p)\})$ as desired.

Since

$$({}_QB' :_Q v_p) = ({}_TB' :_T v_p) \cap Q = \left(\left\{\varphi(x) : x \in F\left(\max(v_p)\right)\right\}\right),$$

it follows that $(B' :_R u_p) = \varphi_R^{-1}({}_QB' :_Q v_p) = J_{F(\max(v_p))}$. □

Recall that the *Castelnuovo–Mumford regularity* (or simply regularity) of a graded ideal J in R is

$$\operatorname{reg}_R(J) = \max\{j \mid \beta^R_{i,i+j}(J) \neq 0 \quad \text{for some } i\}.$$

Theorem 3.7.4 *Let B be a Borel ideal in R. With the same notation as above, one has that*

$$\beta^R_{i,i+j}(B) = \sum_{v \in \operatorname{mingens}({}_T B),\ \deg v = qj} \beta^R_i\big(R/J_{F(\max(v))}\big) \quad \textit{for all } i, j.$$

We have that $\operatorname{reg}_R(B)$ is equal to the maximal degree of a minimal monomial generator of B. In particular, if B is generated in degree j, then its minimal free resolution over R is j-linear.

Proof Note that $\deg v_j = q \deg u_j$ for all j. We use induction on p. Since $\max(v_1) = 1$ and since R is a domain, the statement is obvious if $p = 1$.

Suppose $p > 1$. Let $B' = (u_1, \ldots, u_{p-1})$. Consider the short exact sequence

$$0 \longrightarrow R/(B' :_R u_p)(-\deg u_p) \longrightarrow R/B' \longrightarrow R/B \longrightarrow 0,$$

where the first map is a multiplication by u_p. By Lemma 3.7.3 we have that $(B' :_R u_p) = J_{F(\max(v_p))}$. Let $\mathbf{G}$ be the graded minimal free resolution of $R/J_{F(\max(v_p))}(-\deg u_p)$ over R, and let $\mathbf{F}$ be that of R/B'. We will consider the mapping cone of the complex homomorphism $\mathbf{G} \to \mathbf{F}$ that is a lifting of the map $F/(B' :_R u_p)(-\deg u_p) \longrightarrow R/B'$. In the mapping cone construction, we have that the free module in homological degree i is $\mathbf{F}_i \oplus \mathbf{G}_{i-1}$. Since B' is Borel and $\deg u_p \geq \cdots \geq \deg u_1$, by induction hypothesis we have that the degree of any basis element of $\mathbf{F}_i$ is less than or equal to $\deg u_p + i - 1$. Also, since $J_{F(\max(v_p))}$ has a 1-linear resolution, the degree of each basis element of $\mathbf{G}_{i-1}$ is equal to $\deg u_p + i - 1$. These facts show that the mapping cone is a minimal free resolution of R/B. Then the theorem follows from the induction hypothesis. □

For a graded ideal J in R, we write $J_{\langle j\rangle}$ for the ideal in R generated by all elements of degree j in J. Let $\mathbf{m} = (x_1, \ldots, x_n) \subset R$ be the graded maximal ideal in R.

Corollary 3.7.5 *Let B be a Borel ideal in R. For all integers i, j, one has that*

$$\beta^R_{i,i+j}(B) = \beta^R_i(B_{\langle j\rangle}) + \beta^R_{i+1}(B_{\langle j-1\rangle}) - (\dim_k B_{j-1})\beta^R_{i+1}(k).$$

Proof Note that $B_{\langle j\rangle}$ and $\mathbf{m}B_{\langle j-1\rangle}$ are Borel ideals generated in degree j. By Theorem 3.7.4 they have j-linear minimal free resolutions. Since

$$\big\{v \in \operatorname{mingens}({}_T B) : \deg(v) = qj\big\} = \operatorname{mingens}({}_T B_{\langle j\rangle}) \backslash \operatorname{mingens}\big({}_T(\mathbf{m}B_{\langle j-1\rangle})\big),$$

Proposition 3.7.4 implies that

$$\beta^R_{i,i+j}(B) = \beta^R_i(B_{\langle j\rangle}) - \beta^R_i(\mathbf{m}B_{\langle j-1\rangle}).$$

Consider the short exact sequence

$$0 \longrightarrow \mathbf{m}B_{\langle j-1\rangle} \longrightarrow B_{\langle j-1\rangle} \longrightarrow B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle} \longrightarrow 0.$$

For each s, the short exact sequence leads to the long exact sequence

$$\begin{aligned} \cdots \to\ & \operatorname{Tor}^R_{i+1}(B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle}, k)_s \\ \to\ & \operatorname{Tor}^R_i(\mathbf{m}B_{\langle j-1\rangle}, k)_s \to \operatorname{Tor}^R_i(B_{\langle j-1\rangle}, k)_s \to \operatorname{Tor}^R_i(B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle}, k)_s \\ \to\ & \operatorname{Tor}^R_{i-1}(\mathbf{m}B_{\langle j-1\rangle}, k)_s \to \cdots \end{aligned}$$

Since the Veronese ring R is Koszul, the residue field k has a linear resolution over R. Thus both $B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle} \cong \bigoplus_{k=1}^{\dim_k B_{j-1}} k$ and $B_{\langle j-1\rangle}$ have $(j-1)$-linear minimal free resolutions, and $\mathbf{m}B_{\langle j-1\rangle}$ has a j-linear minimal free resolution. Therefore, for each i, the long exact sequences on Tor yield the short exact sequence

$$0 \to \operatorname{Tor}_i^R(B_{\langle j-1\rangle}, k)_{i+j-1} \to \operatorname{Tor}_i^R(B_{\langle j-1\rangle}/\mathbf{m}B_{\langle j-1\rangle}, k)_{i+j-1} \\ \to \operatorname{Tor}_{i-1}^R(\mathbf{m}B_{\langle j-1\rangle}, k)_{i-1+j} \to 0.$$

Hence we get

$$\beta_i^R(\mathbf{m}B_{\langle j-1\rangle}) = (\dim_k B_{j-1})\beta_{i+1}^R(k) - \beta_{i+1}^R(B_{\langle j-1\rangle}).$$

Then it follows that

$$\beta_{i,i+j}^R(B) = \beta_i^R(B_{\langle j\rangle}) - \beta_i^R(\mathbf{m}B_{\langle j-1\rangle}) = \beta_i^R(B_{\langle j\rangle}) + \beta_{i+1}^R(B_{\langle j-1\rangle}) - (\dim_k B_{j-1})\beta_{i+1}^R(k)$$

as desired. □

In the rest of this section we prove Theorem 1.5.

We have proved that Type-A deformations preserve the graded Betti numbers, and also that Type-B deformations increase the graded Betti numbers so that the smaller Betti numbers can be obtained from the new greater ones by consecutive cancellations. Since every generic initial ideal is Borel when $\operatorname{char}(k) = 0$, for any graded ideal J in R there exists a sequence of Type-A and Type-B deformations that connects ${}_TJ$ to some Borel ideal in T (cf. [8, Chap. 15.9]). Thus, to prove Theorem 1.5, it is enough to prove the following claim.

Proposition 3.7.6 *The graded Betti numbers of a Borel ideal B in R are smaller than or equal to those of the lex ideal L in R having the same Hilbert function as B.*

Proof By Corollary 3.7.5, it is enough to show that $\beta_i^R(B_{\langle j\rangle}) \le \beta_i^R(L_{\langle j\rangle})$ for all i, j. Let $V = \{v_1, \ldots, v_s\}$ and $V' = \{v'_1, \ldots, v'_s\}$ be the sets of monomials of degree jq in ${}_TB$ and in ${}_TL$, respectively. By Green's Theorem [14] we have that

$$|\{v \in V : \max(v) \le \ell\}| \ge |\{v \in V' : \max(v) \le \ell\}|$$

for $\ell = 1, 2, \ldots, r$. Thus we may assume that $\max(v_\ell) \le \max(v'_\ell)$ for $\ell = 1, 2, \ldots, s$. Note that both ideals $J_{F(\max(v_\ell))}$ and $J_{F(\max(v'_\ell))}$ are Borel. Since $\operatorname{mingens}({}_TJ_{F(s)}) \subset \operatorname{mingens}({}_TJ_{F(s')})$, Theorem 3.7.4 implies that

$$\beta_i^R(R/J_{F(\max(v_\ell))}) \le \beta_i^R(R/J_{F(\max(v'_\ell))})$$

for all i. By applying Theorem 3.7.4 to $B_{\langle j\rangle}$ and $L_{\langle j\rangle}$, we obtain

$$\beta_i^R(B_{\langle j\rangle}) = \sum_{\ell=1}^{s} \beta_i^R(R/J_{F(\max(v_\ell))}) \le \sum_{\ell=1}^{s} \beta_i^R(R/J_{F(\max(v'_\ell))}) = \beta_i^R(L_{\langle j\rangle})$$

for all i as desired. □

3.8 Green's Theorem

We close this section by observing that Green's Theorem [14] (cf. [14, Theorem 3.4], [5, Theorem 4.2.12]) holds over R. In [14, Theorem 3.4] this result is called the Hyperplane Restriction Theorem.

Green's Theorem 3.8.1 *Let J be a graded ideal in R, and L be the lex ideal with the same Hilbert function as J. Let h be a generic linear form. For every $s \geq 0$ we have*

$$\dim_k \ (R/(L,h))_s \geq \dim_k \ (R/(J,h))_s \,.$$

Proof Since h is a generic linear form in R, we have that $\varphi_R(h)$ is a generic form of degree q in the polynomial ring T. As in 3.7, let L_T be the lex ideal in T with the same Hilbert function as ${}_T J$. By [19, Proposition 1.4], t_r^q is a generic form for L_T, so

$$\dim_k \ (T/(L_T, \varphi_R(h)))_i = \dim_k \ \left(T/(L_T, t_r^q)\right)_i$$

for every $i \geq 0$. A result of Gasharov [10, Theorem 2.4] shows that

$$\dim_k \ (T/(L_T, \varphi_R(h)))_i \geq \dim_k \ (T/({}_T J, \varphi_R(h)))_i$$

for every $i \geq 0$.

Since $Q = T \cap Q$, $L_Q = L_T \cap Q$ and ${}_Q J = {}_T J \cap Q$, and since $\varphi_R(h) \in Q$, we conclude that

$$\dim_k \ \left(Q/(L_Q, \varphi_R(h))\right)_{sq} \geq \dim_k \ \left(Q/({}_Q J, \varphi_R(h))\right)_{sq}$$

for every $s \geq 0$. Then the isomorphism $R \cong Q$ implies

$$\begin{aligned} \dim_k \ (R/(L,h))_s &= \dim_k \ \left(Q/(L_Q, \varphi_R(h))\right)_{sq} \\ &\geq \dim_k \ \left(Q/({}_Q J, \varphi_R(h))\right)_{sq} = \dim_k \ (R/(J,h))_s \end{aligned}$$

for every $s \geq 0$. □

4 Numerical versions of Macaulay's theorem and Gotzmann's persistence theorem over Veronese rings

In this section, we derive some corollaries of the proof of Theorem 1.3: a numerical characterization of the possible Hilbert functions in the Veronese ring R, Gotzmann's Persistence Theorem over R, and Gotzmann's Regularity Theorem over R.

Macaulay's Theorem 1.2 leads to a numerical criterion on what sequences of positive numbers are Hilbert functions. We will describe this criterion. Let i be a positive integer. It is easy to prove (cf. [5, Lemma 4.2.6]) that for every $p \in \mathbf{N}$ there exist unique natural numbers $g_i > \cdots > g_1 \geq 0$ such that

$$p = \binom{g_i}{i} + \binom{g_{i-1}}{i-1} + \cdots + \binom{g_1}{1} \tag{4.1}$$

This is called the *ith Macaulay representation* of p. For example, the third Macaulay representation of 14 is $14 = \binom{5}{3} + \binom{3}{2} + \binom{1}{1}$. Set

$$\begin{aligned} p^{\langle i \rangle} &= \binom{g_i + 1}{i+1} + \binom{g_{i-1}+1}{i} + \cdots + \binom{g_1+1}{2} \\ 0^{\langle i \rangle} &= 0. \end{aligned}$$

For example, $14^{\langle 3 \rangle} = \binom{6}{4} + \binom{4}{3} + \binom{2}{2} = 20.$

Lex ideals are highly structured and it is easy to derive the inequalities characterizing their possible Hilbert functions. The following proposition is easy to prove, cf. [5, Corollary 4.2.9], [14, Proposition 3.7].

Proposition 4.2 *Let $\alpha = \{\alpha_0 = 1, \alpha_1 = n, \alpha_2, \dots\}$ be a sequence of non-negative integer numbers. If this sequence is the Hilbert function of S/L for a lex ideal L in the polynomial ring S, then $\alpha_{i+1} \geq \alpha_i^{\langle i \rangle}$ for each $i \geq 0$, and a strict inequality holds if and only if L has a minimal monomial generator in degree $i+1$.*

By Macaulay's Theorem 1.2 it follows that we have the following numerical characterization of Hilbert functions, cf. [14, Theorem 3.3], [5, Theorem 4.2.10].

Numerical version of Macaulay's Theorem 4.3 *Let $\alpha = \{\alpha_0 = 1, \alpha_1 = n, \alpha_2, \dots\}$ be a sequence of non-negative integer numbers. This sequence is the Hilbert function of S/J for a graded ideal J in the polynomial ring S if and only if $\alpha_{i+1} \geq \alpha_i^{\langle i \rangle}$ for each $i \geq 1$.*

Two other important results on Hilbert functions in a polynomial ring are proved in [13], also cf. [14, Theorems 3.8 and 3.11], [5, Theorems 4.3.2 and 4.3.3].

Theorem 4.4 *Let J be a graded ideal in the polynomial ring S. Set $\alpha_i = \dim_k (S/J)_i$ for $i \geq 0$. Let s be such that $\alpha_{s+1} = \alpha_s^{\langle s \rangle}$ and J is generated in degrees $\leq s$.*

(1) **Gotzmann's Persistence Theorem.** *$\alpha_{i+1} = \alpha_i^{\langle i \rangle}$ for each $i \geq s$. If L_J is the lex ideal in S with the same Hilbert function as J, then L_J is generated in degrees $\leq s$.*
(2) **Gotzmann's Regularity Theorem.** *$\operatorname{reg}_S(J) \leq s$.*

A similar Numerical version of Macaulay's Theorem holds over an exterior algebra. Gotzmann's Persistence Theorem and Gotzmann's Regularity Theorem also hold over an exterior algebra by [2, Theorems 4.5 and 4.6]. We will prove Theorems 4.5 and 4.7, which give analogues of Theorems 4.3 and 4.4 over the q'th Veronese ring R.

Consider the qi'th Macaulay representation (4.1) of a number $p \in \mathbf{N}$

$$p = \binom{g_{qi}}{qi} + \binom{g_{qi-1}}{qi-1} + \dots + \binom{g_1}{1}.$$

We define a new operation $\langle q, i \rangle$ as follows

$$p^{\langle q,i \rangle} = \binom{g_{qi}+q}{qi+q} + \binom{g_{qi-1}+q}{qi-1+q} + \dots + \binom{g_1+q}{1+q}.$$

For example, the 6th Macaulay representation of 14 is $14 = \binom{6}{1} + \binom{5}{1} + \binom{3}{1}$, and so for $q = 3, i = 2$ we get $14^{\langle 3,2 \rangle} = \binom{9}{4} + \binom{8}{4} + \binom{6}{4} = 211$. Note that $p^{\langle i \rangle} = p^{\langle 1,i \rangle}$.

Theorem 4.5 (1) **Numerical version of Macaulay's Theorem over Veronese rings.** *Let $\alpha = \{\alpha_0 = 1,\ \alpha_1 = n,\ \alpha_2, \dots\}$ be a sequence of non-negative integer numbers. This sequence is the Hilbert function of R/J for a graded ideal J in the q'th Veronese ring R if and only if $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ for each $i \geq 0$.*
(2) **Gotzmann's Persistence Theorem over Veronese rings.** *Let J be a graded ideal in the q'th Veronese ring R. Set $\alpha_i = \dim_k (R/J)_i$ for $i \geq 0$. If s is a number such that $\alpha_{s+1} = \alpha_s^{\langle q,s \rangle}$ and J is generated in degrees $\leq s$, then $\alpha_{i+1} = \alpha_i^{\langle q,i \rangle}$ for each $i \geq s$. If L_J is the lex ideal in R with the same Hilbert function as J, then L_J is generated in degrees $\leq s$.*

Proof Define a new sequence γ as follows:

$$\gamma_0 = 1$$
$$\gamma_1 = \cdots = \gamma_{q-1} = 0$$

and for $j \geq q-1$ define

$$\gamma_{j+1} = \begin{cases} \frac{\alpha_j+1}{q} & \text{if } j+1 \text{ is divisible by } q \\ \gamma_j^{\langle j \rangle} & \text{otherwise.} \end{cases}$$

Clearly, the sequence γ satisfies the equalities $\gamma_{j+1} = \gamma_j^{\langle j \rangle}$ for each $j \geq 0$ such that $j+1$ is not divisible by q. If $j+1$ is divisible by q, then the inequality $\gamma_{j+1} \geq \gamma_j^{\langle j \rangle}$ holds if and only if the inequality $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ holds for $i = \frac{j+1}{q} - 1$.

First, we will prove (1). Suppose that $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ for each $i \geq 0$. Therefore, $\gamma_{j+1} \geq \gamma_j^{\langle j \rangle}$ holds for all $j \geq 0$. By Theorem 4.3, γ is the Hilbert function of a lex ideal U in T. Set $L = U_R = \varphi^{-1}(Q \cap U)$. It is a lex ideal in R by Proposition 3.2.5, and its Hilbert function is given by the sequence α.

On the other hand, let $\alpha = \{\alpha_0 = 1,\ \alpha_1 = n,\ \alpha_2, \dots\}$ be the Hilbert function of a graded ideal J in the Veronese ring R. By Theorem 1.3, α is the Hilbert function of a lex ideal L in R. Let U be the ideal generated by $\varphi(L)$ in the polynomial ring T. It is a lex ideal by Theorem 3.2.2. Since U is lex, it follows that the sequence γ is the Hilbert function of U. By Theorem 4.3, we have that $\gamma_{j+1} \geq \gamma_j^{\langle j \rangle}$ holds for all $j \geq 0$. Therefore, $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ for each $i \geq 0$. We have proved (1).

Now, we will prove (2). Denote by V the ideal in T generated by $\varphi(J)$. Let W be the lex ideal in T with the same Hilbert function as V. It follows that $U \subseteq W$, and $U_{iq} = W_{iq}$ for every $i \geq 0$. Denote by $\nu_i = \dim_k W_i$ for all i. Therefore, $\gamma_i \leq \nu_i$ for $i \geq 0$ and furthermore, $\gamma_{j+1} = \nu_{j+1}$ whenever q divides $j+1$.

Since we have $\alpha_{s+1} = \alpha_s^{\langle q,s \rangle}$ by assumption, it follows that $\gamma_{c+1} = \gamma_c^{\langle c \rangle}$ for $c = q(s+1) - 1$. Hence,

$$\nu_{c+1} = \gamma_{c+1} = \gamma_c^{\langle c \rangle} \leq \nu_c^{\langle c \rangle} \leq \nu_{c+1},$$

where the last inequality holds by Macaulay's Theorem 4.3. Therefore,

$$\nu_{c+1} = \nu_c^{\langle c \rangle}.$$

Since J is generated in degrees $\leq s$, it follows that V is generated in degrees $\leq qs \leq q(s+1) - 1 = c$. By Gotzmann's Persistence Theorem 4.4 in the polynomial ring T, we have that $\nu_{j+1} = \nu_j^{\langle j \rangle}$ holds for all $j \geq c$. Hence, $\alpha_{i+1} = \alpha_i^{\langle q,i \rangle}$ for each $i \geq s$. □

The following is an immediate corollary of Theorem 4.5(1) (which can also be proved using Proposition 4.2 and the argument in the proof above).

Proposition 4.6 *Let $\alpha = \{\alpha_0 = 1, \alpha_1 = n, \alpha_2, \dots\}$ be a sequence of non-negative integer numbers. If this sequence is the Hilbert function of R/L for a lex ideal L in the q'th Veronese ring R, then $\alpha_{i+1} \geq \alpha_i^{\langle q,i \rangle}$ for each $i \geq 0$, and a strict inequality holds if and only if L has a minimal monomial generator in degree $i+1$.*

Gotzmann's Regularity Theorem 4.7 *Let J be a graded ideal in the q'th Veronese ring R. Set $\alpha_i = \dim_k (R/J)_i$ for $i \geq 0$. If s is a number such that $\alpha_{s+1} = \alpha_s^{\langle q,s \rangle}$ and J is generated in degrees $\leq s$, then we have the following inequality for the Catelnuovo–Mumford regularity*

$$\operatorname{reg}_R(J) \leq s.$$

Proof Denote by L_J the lex ideal in R with the same Hilbert function as J. We have that L_J is generated in degrees $\leq s$ by Theorem 4.5(2). Every lex ideal is Borel. Hence, we can apply Theorem 3.7.4 and get $\operatorname{reg}_R(L_J) \leq s$. Finally, Theorem 1.5 yields $\operatorname{reg}_R(J) \leq \operatorname{reg}_R(L_J) \leq s$. □

5 Consecutive cancellations

Given a sequence of numbers $\{c_{i,j}\}$, we obtain a new sequence by a *cancellation* as follows: fix a j, and choose i and i' so that one of the numbers is odd and the other is even; then replace $c_{i,j}$ by $c_{i,j} - 1$, and replace $c_{i',j}$ by $c_{i',j} - 1$. We have a *consecutive cancellation* when $i' = i + 1$. If we need to be specific, we call it a consecutive i, j-cancellation. The term "consecutive" is justified by the fact that we consider cancellations in Betti numbers of consecutive homological degrees.

A more precise version of Theorem 1.5 says:

Theorem 5.1 [28, Theorem 1.1] *If J is a graded ideal in S and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta^S_{i,j}(S/J)$ can be obtained from the graded Betti numbers $\beta^S_{i,j}(S/L)$ by a sequence of consecutive cancellations.*

The analogue of this result holds over an exterior algebra.

Theorem 5.2 *Let E be an exterior algebra on variables $e_1, \ldots, e_n$ over k. The ring E is graded by* $\deg(e_i) = 1$ *for each i. If J is a graded ideal in E and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta^E_{i,j}(E/J)$ can be obtained from the graded Betti numbers $\beta^E_{i,j}(E/L)$ by a sequence of consecutive cancellations.*

The above theorem is not published, but it follows from Theorem 5.1 and a formula in [1, Proposition 2.1] relating a finite monomial resolution over S and an infinite monomial resolution over E.

We have the following more precise versions of Theorem 1.5.

Theorem 5.3 *Let $R = S/I$ be a Veronese toric ring. If J is a graded ideal in R and L is the lex ideal with the same Hilbert function, then the graded Betti numbers $\beta^R_{i,j}(R/J)$ can be obtained from the graded Betti numbers $\beta^R_{i,j}(R/L)$ by a sequence of consecutive cancellations.*

Proof By the proof of Theorem 1.5 it follows that we may assume that J is Borel. Set $d_{i,i+j} = \beta_{i+1}(L_{\langle j \rangle}) - \beta_{i+1}(J_{\langle j \rangle})$. We showed in the proof of Proposition 3.7.6 that $d_{i,j}$ are nonnegative integers. Also, by Corollary 3.7.5 $\beta_{i,j}(J) = \beta_{i,j}(L) - d_{i-1,j-1} - d_{i,j-1}$. This formula and the non-negativity of $d_{i,j}$ show that the graded Betti numbers of J is obtained from those of L by a sequence of consecutive cancellations (a consecutive i, j-cancellation occurs $d_{i,j}$ times).

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