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Transitivity of generic semigroups of area-preserving surface diffeomorphisms

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Abstract We prove that the action of the semigroup generated by a C^r generic pair of area-preserving diffeomorphisms of a compact orientable surface is transitive.

1 Introduction

In this article we consider a compact orientable surface S with a smooth volume form ω, and we denote by $\mathrm{Diff}^r_\omega(S)$ the space of C^r diffeomorphisms from S to itself that leave ω invariant (i.e. such that $f^*(\omega) = \omega$), endowed with the C^r topology. Our main result is the following

Theorem 1 *There is a residual set* $\mathcal{R} \subset \mathrm{Diff}^r_\omega(S) \times \mathrm{Diff}^r_\omega(S)$ $(r \in \mathbb{N} \cup \{\infty\})$ *with the product* C^r *topology such that if* $(f, g) \in \mathcal{R}$ *then the iterated function system* $\mathrm{IFS}(f, g)$ *is transitive.*

Saying that $\mathrm{IFS}(f_0, f_1)$ is transitive means that there is a point x and a sequence $\{i_k\}$ of 1's and 0's such that

$$\{f_{i_k} f_{i_{k-1}} \cdots f_{i_1}(x) : k \in \mathbb{N}\}$$

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is dense in S, and it is equivalent to say that the action of the semigroup generated by $\{f_0, f_1\}$ is transitive.

Theorem 1 generalizes a result of Moeckel [13], where he proves drift in the annulus for IFS(f, g), where f is a monotone twist map and g is a generic diffeomorphism. This is also related to the work of Le Calvez [9]. The most important difference in our case is that we do not require the twist condition, and combining the drift in annular invariant regions with generic arguments, we are able to obtain transitivity in the whole surface.

The dynamics of groups or semigroups of diffeomorphisms, besides its own importance, is a useful tool for understanding the dynamics of certain (single) diffeomorphisms. In fact, the topic of this paper is motivated by several open problems in conservative and Hamiltonian dynamics. One of them is the instability problem (or the so-called Arnold diffusion) in Hamiltonian dynamics. It has been conjectured [2] that for C^r generic perturbations of a high-dimensional integrable Hamiltonian system (or symplectomorphism), many orbits drift between KAM invariant tori. This problem remains open in its full generality, although many partial (but deep) results have been obtained in the last decades (c.f. [5,6,10,12,20] and references there).

Another important problem is the topological version of the Pugh–Shub conjecture [17], which says that C^r generic partially hyperbolic symplectic (or conservative) diffeomorphisms of a compact manifold are robustly transitive.

An approach to deal with these problems is using additional structure (e.g. partial hyperbolicity of certain sets, skew-product structure, invariant foliations) to obtain an associated iterated function system which reflects certain properties of the original system (c.f. [13,15,16]). In that setting, results like Theorem 1 can be helpful to obtain transitivity or drift in the original system.

For instance, for symplectic diffeomorphisms which are the product of an Anosov diffeomorphism and a conservative surface diffeomorphism close to an integrable system it is possible to prove (using Theorem 1) that C^r-generic (symplectic) perturbations are transitive. Such applications of Theorem 1 on transitivity of certain partially hyperbolic sets or diffeomorphisms will be treated elsewhere.

We remark that these problems have different nature in the C^1 topology. Bonatti and Crovisier [4], using a C^1 closing lemma, proved that C^1 generic conservative diffeomorphisms are transitive (see also [1]). However, if we consider surface diffeomorphisms, this result is clearly false if we use the C^r topology with r sufficiently large, due to the KAM phenomenon. In particular, the number of diffeomorphisms in Theorem 1 is optimal if r is large (for instance, $r \geq 16$ [7]), in the sense that just one generic diffeomorphism is generally not transitive.

The proof of Theorem 1 relies on the following result, which seems interesting by itself: for a C^r-generic f ($r \geq 16$), the invariant *frontiers* of f are pairwise disjoint (see Sect. 5 for a precise definition of frontier and Lemma 19 for a precise statement). The proof of this result is strictly two-dimensional, and that restricts our main theorem to dimension two. However, we conjecture that the conclusion of Theorem 1 also holds in higher dimensions.

The following relevant questions were motivated by the present work.

Question 2 *For a C^r-generic pair of area-preserving diffeomorphisms, does any of the following properties hold?*

(1) IFS(f, g) *is robustly transitive (i.e. if for small perturbations of f and g the iterated function system is still transitive);*
(2) IFS(f, g) *is ergodic (i.e. $\{f, g\}$-invariant sets have measure* 0 *or* 1*);*

(3) IFS(f, g) *is minimal (i.e. there are no compact* $\{f, g\}$*-invariant sets other than the whole surface and the empty set).*

Let us say a few words about the proof of Theorem 1. We divide it in several sections. In Sect. 2 we provide several definitions of transitivity of group actions and iterated function systems, and we prove that they are all equivalent in the area-preserving setting. This simplifies the proof of Theorem 1. In Sect. 3 we state a result from [8] (c.f. Theorem 6) which gives a good description of aperiodic invariant continua in the area-preserving setting , and plays a crucial role in the proof of our main theorem. When the surface is $\mathbb{T}^2$ or S^2, it is possible to prove the theorem without these results, using additional arguments, but [8] allows to unify these arguments to prove the theorem for arbitrary surfaces.

In Sect. 4 we introduce generic conditions and we state a result of Mather which relates open invariant sets with aperiodic invariant continua. The main idea in the proof of Theorem 1 is to reduce the problem to proving that for generic (f, g), there are no "nice annular continua" periodic by both f and g. We call these nice continua *frontiers*; in Sect. 5 we define them and prove some elementary facts about these sets. In Sect. 6 we prove that IFS(f, g) is transitive if there are no frontiers which are periodic for both f and g simultaneously. Using Theorem 6, we prove that all periodic frontiers of f are pairwise disjoint if f satisfies certain generic properties. Taking both f and g generic we are left with the problem of separating, by means of a perturbation of g, the family of periodic frontiers of f from the corresponding family of g. To do this, we extend to our setting the arguments used by Moeckel in [13] where he proves a similar result for twist maps. This is done in Sect. 7. Finally, in Sect. 8 we complete the proof of Theorem 1.

2 Transitivity of group actions and iterated function systems

Throughout this section we assume that M is an arbitrary manifold, not necessarily compact, and $\mathcal{F}$ is a family of homeomorphisms from M to itself. We denote by $\langle\mathcal{F}\rangle^+$ and $\langle\mathcal{F}\rangle$ the semigroup and the group generated by $\mathcal{F}$, respectively. For $x \in M$, we write the orbits of the actions of this semigroup and group as

$$\langle\mathcal{F}\rangle^+(x) = \{f(x) : f \in \langle\mathcal{F}\rangle^+\} \quad \text{and} \quad \langle\mathcal{F}\rangle(x) = \{f(x) : f \in \langle\mathcal{F}\rangle\}.$$

We say that a set $E \subset M$ is $\mathcal{F}$-invariant if $f(E) = E$ for each $f \in \mathcal{F}$. In this case, it is clear that $f(E) = E$ for each $f \in \langle\mathcal{F}\rangle$. If there is a point $x \in M$ such that $\langle\mathcal{F}\rangle(x)$ is dense in M, we say that $\langle\mathcal{F}\rangle$ is transitive. Similarly, if $\langle\mathcal{F}\rangle^+(x)$ is dense for some x we say that $\langle\mathcal{F}\rangle^+$ is transitive.

A sequence $\{x_n : n \in \mathbb{N}$ (resp. $n \in \mathbb{Z})\}$ is called a branch (resp. full branch) of an orbit of IFS$(\mathcal{F})$ if for each $n \in \mathbb{N}$ (resp. $n \in \mathbb{Z}$) there is $f_n \in \mathcal{F}$ such that $x_{n+1} = f_n(x_n)$. Here, IFS$(\mathcal{F})$ stands for the iterated function system associated to $\mathcal{F}$. We say that IFS$(\mathcal{F})$ is transitive if there is a branch of an orbit which is dense in M.

The following lemmas show that if the maps in question preserve a finite measure with total support, then all the different notions of transitivity are equivalent.

Lemma 3 *Let μ be a finite Borel measure on M with total support, and let $\mathcal{F}$ be a family of homeomorphisms from M, all of which leave μ invariant. Let U and V be open subsets of M such that $f(U) \cap V \neq \emptyset$ for some $f \in \langle\mathcal{F}\rangle$. Then there is $\hat{f} \in \langle\mathcal{F}\rangle^+$ such that $\hat{f}(U) \cap V \neq \emptyset$.*

Proof Any $f \in \langle\mathcal{F}\rangle$ has the form $f = f_n f_{n-1} \cdots f_1$ for some f_i such that one of f_i or f_i^{-1} belong to $\mathcal{F}$. We prove the result by induction on n. If $n = 1$, then either $f \in \mathcal{F} \subset \langle\mathcal{F}\rangle^+$

(and there is nothing to do) or $f^{-1} \in \mathcal{F}$. In this case, since f^{-1} preserves μ and the open set $f(U) \cap V$ has positive measure, there is $k \geq 2$ such that

$$f^{-k}(f(U) \cap V) \cap f(U) \cap V \neq \emptyset.$$

In particular, letting $\hat{f} = f^{-k+1}$, we have that $\hat{f} \in \langle \mathcal{F} \rangle^+$ and $\hat{f}(U) \cap V \neq \emptyset$ as required.

Now suppose the proposition holds for some fixed $n \geq 1$, and suppose $f(U) \cap V \neq \emptyset$ where $f = f_{n+1} f_n \cdots f_1$ and $f_i \in \mathcal{F}$ or $f_i^{-1} \in \mathcal{F}$ for each i. Then $f_n \cdots f_1(U) \cap f_{n+1}^{-1}(V) \neq \emptyset$. By induction hypothesis (with $f_{n+1}^{-1}(V)$ instead of V) we find that there is $g \in \langle \mathcal{F} \rangle^+$ such that $g(U) \cap f_{n+1}^{-1}(V) \neq \emptyset$. Thus, $f_{n+1}(g(U)) \cap V \neq \emptyset$. Applying the case $n = 1$ with $g(U)$ instead of U, we see that there is $h \in \langle \mathcal{F} \rangle^+$ such that $h(g(U)) \cap V \neq \emptyset$. Hence, $\hat{f} = hg \in \langle \mathcal{F} \rangle^+$ satisfies the required condition. □

We will only use (4) $\implies$ (8) from the next lemma, but we state the other equivalences for the sake of completeness.

Lemma 4 *If $\mathcal{F}$ is a countable family of homeomorphisms from M to itself preserving a finite Borel measure with compact support, then the following are equivalent:*

(1) *$\langle \mathcal{F} \rangle$ is transitive;*
(2) *For any pair of nonempty open sets U and V of M, there is $f \in \langle \mathcal{F} \rangle$ such that $f(U) \cap V \neq \emptyset$;*
(3) *There is a residual set $\mathcal{R} \subset M$ such that* $\operatorname{cl} \langle \mathcal{F} \rangle (x) = M$ *for each $x \in \mathcal{R}$;*
(4) *$\langle \mathcal{F} \rangle^+$ is transitive;*
(5) *For any pair of nonempty open sets U and V of M, there is $f \in \langle \mathcal{F} \rangle^+$ such that $f(U) \cap V \neq \emptyset$;*
(6) *There is a residual set $\mathcal{R}^+ \subset M$ such that* $\operatorname{cl} \langle \mathcal{F} \rangle^+ (x) = M$ *for each $x \in \mathcal{R}^+$;*
(7) *There is a full branch of an orbit of* $\mathrm{IFS}(\mathcal{F})$ *which is dense in M;*
(8) $\mathrm{IFS}(\mathcal{F})$ *is transitive;*
(9) *There is a residual subset $\mathcal{R}' \subset M$ such that for each $x \in \mathcal{R}'$ there is a branch of an orbit of* $\mathrm{IFS}(\mathcal{F})$ *starting at x which is dense in M.*

Proof Note that (2) $\implies$ (5) because of the previous lemma. To prove that (5) $\implies$ (9), consider a countable basis of open sets $\{V_n : n \in \mathbb{N}\}$ of M, and let R_n be the set of all $x \in M$ such that $f(x) \in V_n$ for some $f \in \langle \mathcal{F}^+ \rangle$. Clearly, R_n is open, and it is dense by (5). Thus $\mathcal{R} = \cap_n R_n$ is a residual set such that $\langle \mathcal{F} \rangle^+ (x)$ is dense in M for each $x \in \mathcal{R}$. Now $\mathcal{R}' = \bigcap_{f \in \langle \mathcal{F} \rangle} f(\mathcal{R})$ is a residual $\mathcal{F}$-invariant set, and for every $x \in \mathcal{R}'$ we can find a sequence $f_n \in \langle \mathcal{F} \rangle^+$ such that $f_n \cdots f_1(x) \in V_n$, so that $\{f_n \cdots f_1(x) : n \in \mathbb{N}\}$ is a dense branch of an orbit starting at x. It is clear that (9) implies all the other conditions, and all the other conditions imply (2), so we are done. □

3 Invariant continua

By a continuum, we mean a compact connected set.

Definition 5 As in [8], we say that a continuum K is annular if it has an annular neighborhood A such that $A \setminus K$ has exactly two connected components, both homeomorphic to annuli. We call any such A an *annular neighborhood* of K.

This definition is equivalent to saying that K is the intersection of a sequence $\{A_i\}$ of closed topological annuli such that A_{i+1} is an essential subset of A_i (i.e. it separates the two boundary components of A_i), for each $i \in \mathbb{N}$.

Recall that $\Omega(f)$ denotes the nonwandering set of f, that is, the set of points $x \in S$ such that for each neighborhood U of x there is $n > 0$ such that $f^n(U) \cap U \neq \emptyset$. We will need the following result:

Theorem 6 **([8])** *Let $f : S \to S$ be a homeomorphism of a compact orientable surface such that $\Omega(f) = S$. If K is an f-invariant continuum, then one of the following holds:*

(1) *f has a periodic point in K;*
(2) *K is annular;*
(3) *$K = S = \mathbb{T}^2$ (a torus);*

Remark 7 If f is area-preserving, then $\Omega(f) = S$, so the first hypothesis of the above theorem always holds.

Remark 8 Theorem 6 implies that if f is a C^r-generic area-preserving diffeomorphism and K is an aperiodic invariant continuum (i.e. one which contains no periodic points of f), then K is annular. This follows from the well known fact that a C^r-generic surface diffeomorphism has periodic points (see for example [3, Corollary 2] and [21, Sect. 8]).

Remark 9 We will use the following observation several times: If $\Omega(f) = S$ (in particular, if f is area-preserving) and $\{U_i\}_{i\in\mathbb{N}}$ is a family of pairwise disjoint open sets which are permuted by f (e.g. the connected components of the complement of a compact periodic set) then each U_i is periodic for f.

4 Generic conditions

From now on, S is a compact orientable surface and ω is a smooth area element on S.

The main generic properties that we will require are comprised in the following

Definition 10 We say that $f \in \mathrm{Diff}^r_\omega(S)$ is *Moser generic* if the following conditions are met:

(1) All periodic points of f are either elliptic or hyperbolic (saddles);
(2) There are no saddle connections between hyperbolic periodic points (i.e. the stable and unstable manifold of any pair of hyperbolic periodic points are transverse);
(3) The elliptic periodic points are Moser stable. This means that if $f^n(p) = p$ and p is elliptic then
   - there are arbitrarily small periodic curves surrounding p;
   - the restriction of f^n to each of these curves is an irrational rotation, and all the rotation numbers are different.

Remark 11 Conditions (1) and (2) are C^r-generic for any $r \geq 1$, due to Robinson [18]. These are usually referred to as the Kupka-Smale property for area-preserving diffeomorphisms. Condition (3) is C^r-generic due to KAM theory if r is large enough (for instance, $r \geq 16$ [7]). Thus, Moser generic diffeomorphisms are C^r generic for $r \geq 16$. Also note that if f is Moser generic, then f^n is Moser generic for any $n \neq 0$.

In [11] Mather studies the prime ends compactification of open sets invariant by an area-preserving diffeomorphism. We will use the following result, which is a direct consequence of the corollary to Theorem 5.1 and Lemma 2.3 from that article. We say that an open set $U \subset S$ is a *residual domain* if it is a connected component of $S \setminus K$ for some continuum K, where K is not a single point.

Theorem 12 **([11])** *Let $f \in \mathrm{Diff}^r_\omega(S)$ be a Moser generic diffeomorphism and U an f-invariant residual domain. Then U has finitely many boundary components, and if K is one such component, then there are no periodic points in K.*

5 Annular continua and frontiers

The following definitions and observations will be important to obtain disjoint families of invariant continua in the next sections.

Definition 13 Let K be an annular continuum with annular neighborhood A, and denote by ∂A^- and ∂A^+ the two boundary components of A. Then $A \setminus K$ has exactly two (annular) components, which we denote by K^- and K^+, where K^- is the one whose closure contains $\partial^- A$. We denote by $\partial^- K$ and $\partial^+ K$ the boundaries of K^- and K^+ in A, respectively. If $K = \partial^- K = \partial^+ K$, we say that K is a *frontier*.

Lemma 14 *If K is an annular continuum with empty interior, then K contains a unique frontier.*

Proof If $U_+ = \operatorname{int} \overline{K^+}$ and $U_- = \operatorname{int} \overline{K^-}$, then $U_+ \cup U_-$ is open and dense in A. Since U_+ and U_- are disjoint and since $\operatorname{int} \overline{U^+} = U^+$ (and similarly for U^-), it follows that $\partial U_+ = \partial U_-$. Let $\alpha = \partial U_+ = \partial U_-$. It is clear that α is a frontier and $\alpha \subset K$.

Now suppose that $\beta \subset K$ is a frontier. Since $K^- \subset \beta^-$ and K has empty interior, $\beta^- \setminus K^-$ has empty interior as well. Thus, $U_- = \operatorname{int} \overline{K^-} = \operatorname{int} \overline{\beta^-} = \beta^-$. Therefore, $\alpha = \partial U_- = \partial \beta^- = \beta$. This completes the proof. □

Corollary 15 *Let $f : S \to S$ be a homeomorphism and $K \subset S$ an f-invariant annular continuum with empty interior. Then the unique frontier $\hat{K} \subset K$ is f-invariant.*

Proof It is clear that $f(\hat{K}) \subset K$ is a frontier, so $f(\hat{K}) = \hat{K}$ by uniqueness. □

Lemma 16 *Let $f : S \to S$ be a homeomorphism such that $\Omega(f) = S$, and K_1, K_2 two periodic frontiers containing no periodic point of f. Then either $K_1 \cap K_2 = \emptyset$, or $K_1 = K_2$.*

Proof By considering an appropriate power of f instead of f, we may assume that K_1 and K_2 are f-invariant. If $K_1 \cap K_2 \neq \emptyset$, then $K = K_1 \cup K_2$ is connected, has empty interior, and contains no periodic point of f, so by Theorem 6 it is an annular continuum. By Lemma 14 there is a unique frontier $\hat{K}$ contained in K; hence $K_1 = K_2 = \hat{K}$. □

6 Periodic frontiers from invariant open non-dense sets

Lemma 17 *Let $f, g \in \mathrm{Diff}^r_\omega(S)$ be Moser generic. Suppose that there is an open nonempty $\{f, g\}$-invariant set $U \subset S$ which is not dense in S. Then there is a continuum $K \subset S$ with empty interior and $n > 0$ such that $f^n(K) = g^n(K) = K$ and there are no periodic points of f or g in K.*

Proof Let U_0 be a connected component of U. Since f and g are area-preserving, U_0 must be periodic for each of them. Let V be a connected component of $S \setminus \overline{U}_0$. Then V is also periodic for both f and g, and it is a residual domain, so Theorem 12 applied to some power of f and some power of g implies that there are finitely many boundary components of V, none of which contains a periodic point of f or g. If K is one such component, since there are finitely many components, it must be periodic for both f and g, so there is n such that $f^n(K) = g^n(K) = K$ as required. □

The characterization of transitivity of IFS(f, g) used to prove Theorem 1 is the following:

Lemma 18 *If $f, g \in \mathrm{Diff}^r_\omega(S)$ are Moser generic, then* IFS(f, g) *is transitive if there is no frontier $K \subset S$ which is periodic for both f and g.*

Proof Suppose that IFS(f, g) is not transitive. Then by Lemma 4, there is an open nonempty set $U \subset S$ such that

$$\langle f, g\rangle (U) = \bigcup_{h\in\langle f,g\rangle} h(U)$$

is not dense in S. The above set is open, nonempty and $\langle f, g\rangle$-invariant (hence, f-invariant and g-invariant). Thus by the previous lemma there is a continuum K which has no periodic points such that $f^n(K) = g^n(K) = K$. By Theorem 6, K must be annular. By Lemma 14 and its corollary, K contains a unique frontier which is both f^n-invariant and g^n-invariant. This completes the proof. □

We will use the following consequence of Mather's Theorem and Theorem 6:

Lemma 19 *If $f \in \mathrm{Diff}^r_\omega(S)$ is Moser generic, then the invariant frontiers of f are aperiodic and pairwise disjoint.*

Proof If K is an invariant frontier, then K has at most two residual domains, and K is the union of their boundaries. Since obviously any boundary component of these residual domains consists of more than one point, it follows from Theorem 12 that K is aperiodic. Now our claim follows from Lemma 16. □

7 Separating two families of disjoint continua

Let $Q = [0, 1]^2$ denote the unit square. We say that a continuum $K \subset Q$ *crosses the square* horizontally (resp. vertically) if the two horizontal (resp. vertical) sides of Q are contained in different connected components of $Q \setminus K$.

The main result of this section (Lemma 22) is very similar to a result of Moeckel [13] for families of Lipschitz graphs, which is in turn inspired by the embedding theory in [19]. It says that for any pair of disjoint families of continua horizontally crossing the unit square, we can find a smooth, area-preserving map C^∞-close to the identity, coinciding with the identity in a neighborhood of ∂Q, such that no element of the first family is mapped to an element of the second one.

First we state a simple lemma, which is proved in [13].

Lemma 20 *Let A and A' be two subsets of $\mathbb{R}^3$ with upper box dimensions at most* 1. *Then for almost every $(x, y, z) \in \mathbb{R}^3$, the sets $A + (x, y, z)$ and A' are disjoint.*

We will also need the following result which is proved, although not explicitly stated, in [13]:

Lemma 21 *Let $S \subset \mathbb{R}^3$ be a set that is linearly ordered by the partial ordering defined as $(x_1, x_2, x_3) \preceq (y_1, y_2, y_3)$ if $x_i \leq y_i$ for each i. Then the upper box dimension of S is at most* 1.

Proof Consider the map $\phi\colon \mathbb{R}^3 \to \mathbb{R}$ defined by $\phi(x_1, x_2, x_3) = x_1 + x_2 + x_3$. Note that ϕ is order-preserving in S; in fact, if $x, y \in S$ and $x \prec y$ (i.e. $x \preceq y$ and $x \neq y$), we have that all three coordinates of $y - x$ are nonnegative, and at least one is positive, so that $0 < \phi(y - x) = \phi(y) - \phi(x)$. In particular, $\phi|_S$ is a bijection onto its image $E = \phi(S)$. Denoting by $\psi\colon E \to S$ its inverse, we have that if $x = (x_1, x_2, x_3) \prec (y_1, y_2, y_3) = y$, then denoting by $\|\cdot\|_1$ the l^1 norm,

$$\begin{aligned}\|\psi(\phi(y)) - \psi(\phi(x))\|_1 &= \|y - x\|_1 = |y_1 - x_1| + |y_2 - x_2| + |y_3 - x_3| \\ &= |\phi(y) - \phi(x)|,\end{aligned}$$

because $y_i - x_i \geq 0$ for all $i \in \{1, 2, 3\}$. This means that ψ is an isometry if we use the l^1 norm in $\mathbb{R}^3$. Since the upper box dimension is independent of the norm used to compute it (and $E \subset \mathbb{R}$ implies that E has upper box dimension at most 1), the claim follows. □

Lemma 22 *Let $\mathcal{K}$ and $\mathcal{K}'$ be two families of pairwise disjoint continua horizontally crossing the unit square $Q = [0, 1]^2$ which are disjoint from $[0, 1] \times [0, \delta]$ and $[0, 1] \times [1 - \delta, 1]$ for some small $\delta > 0$. Then there exists an area-preserving C^∞ diffeomorphism $h\colon Q \to Q$ arbitrarily close to the identity in the C^∞ topology, such that h coincides with the identity in a neighborhood of ∂Q and $h(\mathcal{K}) = \{h(K) : K \in \mathcal{K}\}$ is disjoint from $\mathcal{K}'$.*

Proof Denote by $\mathrm{pr}_2\colon Q \to [0, 1]$ the projection onto the second coordinate. Given $K_1, K_2 \in \mathcal{K}$, and $x \in [0, 1]$, we write $K_1 \prec_x K_2$ if

$$\max \mathrm{pr}_2(K_1 \cap \ell) < \max \mathrm{pr}_2(K_2 \cap \ell),$$

where $\ell = \{x\} \times [0, 1]$. It is easily verified that $\prec_x$ is a total ordering of $\mathcal{K}$. Moreover, this ordering is independent of x. In fact, $\ell \setminus K_1$ has exactly one connected component containing $(x, 1)$, which is included in $\ell \cap K_1^+$, where K_1^+ is the connected component of $Q \setminus K_1$ containing $[0, 1] \times \{1\}$. If $K_1 \prec_x K_2$ then $K_2 \cap K_1^+ \neq \emptyset$, and since K_2 is connected and K_1 is disjoint from K_2 it follows that $K_2 \subset K_1^+$. This in turn implies that $K_1 \prec_{x'} K_2$ for any $x' \in [0, 1]$. Thus we may unambiguously write $K_1 \prec K_2$ if $K_1 \prec_x K_2$ for some x.

Fix three numbers $x_i \in (0, 1)$ such that $x_1 < x_2 < x_3$, and define $\ell_i = \{x_i\} \times [\delta, 1-\delta]$, $i \in \{1, 2, 3\}$. Note that for each $K \in \mathcal{K}$, the line ℓ_i intersects K. Thus, for each i we may define a map

$$y_i\colon \mathcal{K} \to \mathbb{R}, \quad y_i(K) = \max \mathrm{pr}_2(K \cap \ell_i).$$

From the previous observations, $y_i(K_1) < y_i(K_2)$ for some i if and only if $y_i(K_1) < y_i(K_2)$ for all $i \in \{1, 2, 3\}$ (i.e. iff $K_1 \prec K_2$). Thus the hypotheses of Lemma 21 hold for the set

$$A = \{(y_1(K), y_2(K), y_3(K)) : K \in \mathcal{K}\} \subset [0, 1]^3,$$

so we know that the upper box dimension of A is at most one.

We can order the elements of $\mathcal{K}'$ in a similar way and define maps $y_i'\colon \mathcal{K}' \to \mathbb{R}$ in an analogous way, obtaining a set

$$A' = \left\{(y_1'(K), y_2'(K), y_3'(K)) : K \in \mathcal{K}'\right\}$$

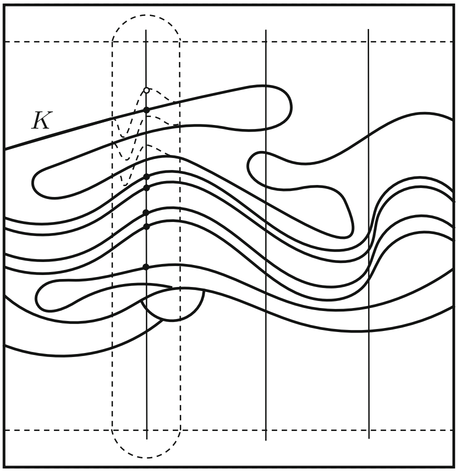

Fig. 1 Construction of h

which also has upper box dimension at most one. It follows from Lemma 20 that we can choose $(t_1, t_2, t_3) \subset [0, 1]^3$ arbitrarily close to the origin such that $A + (t_1, t_2, t_3)$ is disjoint from A'.

For each i, we take small disjoint rectangular neighborhoods R_i of each ℓ_i, and we can define C^∞ functions $\alpha_i : Q \to \mathbb{R}$ such that $\alpha_i(x, y) = x$ for (x, y) in a small neighborhood of ℓ_i, $\alpha_i(x, y) = 0$ for $(x, y) \in Q \setminus R_i$, and $|\alpha_i(x, y)| \leq 1$ for all $(x, y) \in Q$. It is easy to see that the Hamiltonian flow ϕ_i^t induced by the vector field $(-\partial_y \alpha_i, \partial_x \alpha_i)$ has the property that $\phi_i^t(Q) = Q$ for $t \in \mathbb{R}$, $\phi_i^t(x, y) = (x, y)$ if $(x, y) \in Q \setminus R_i$, and there is $\epsilon > 0$ such that $\phi_i^t(x, y) = (x, y) + (0, t)$ if $|t| < \epsilon$ and $(x, y) \in \ell_i$ (Fig. 1).

We choose $(t_1, t_2, t_3) \in \mathbb{R}^3$ with $|t_i| < \epsilon$ for each i, such that $A + (t_1, t_2, t_3)$ is disjoint from A'. Note that the maps $\phi_1^{t_1}$, $\phi_2^{t_2}$ and $\phi_3^{t_3}$ have disjoint supports. Letting $h = \phi_1^{t_1} \circ \phi_2^{t_2} \circ \phi_3^{t_3}$, we have that $h(A) = A + (t_1, t_2, t_3)$, which is disjoint from A'. Thus, if $K \in \mathcal{K}$ and $K' \in \mathcal{K}'$ then $h(K) \neq K'$, since $\max \mathrm{pr}_2\, h(K) \cap \ell_i \neq \max \mathrm{pr}_2\, K \cap \ell_i$ for some $i \in \{1, 2, 3\}$. If ϵ is chosen sufficiently small, we get h arbitrarily C^∞ close to the identity. This completes the proof. □

8 Proof of Theorem 1

First we state a finer result, and we show how it implies Theorem 1. Then, after introducing a useful definition, we complete the proof.

Theorem 23 *For $r \geq 16$, if $f \in \mathrm{Diff}^r_\omega(S)$ is Moser generic, there is a C^r-residual set $\mathcal{R}_f \subset \mathrm{Diff}^r_\omega(S)$ such that if $g \in \mathcal{R}_f$ then $\mathrm{IFS}(f, g)$ is transitive.*

Proof of Theorem 1 assuming Theorem 23. It is easy to see that the set of pairs (f, g) such that $\mathrm{IFS}(f, g)$ is transitive is a G_δ set in $\mathrm{Diff}^r_\omega(S) \times \mathrm{Diff}^r_\omega(S)$ with the product C^0 topology (and hence, also with the product C^r topology, for any $r \in \mathbb{N} \cup \{\infty\}$). From Theorem 23 and Remark 11, this set is dense if $r \geq 16$. To conclude the proof, it is enough to note that $\mathrm{Diff}^r_\omega(S)$ is dense in $\mathrm{Diff}^k_\omega(S)$ if $r \geq k$ (see [22]). □

Definition 24 We say that a continuum K is *essential* in an open annular set A if K is annular and A is an annular neighborhood of K (see Definition 5). If E is a closed annulus, we say that K is essential in E if K is essential in some annular neighborhood of E.

Proof of Theorem 23. Let $E \subset S$ be diffeomorphic to a closed annulus, $n > 0$, and $f \in \mathrm{Diff}^r_\omega(S)$ be Moser generic. Let $\mathcal{R}^n_{f,E}$ be the set of all $g \in \mathrm{Diff}^r_\omega(S)$ such that there is no frontier which is essential in E and invariant by both f^n and g^n. We will prove that $\mathcal{R}^n_{f,E}$ is open and dense.

Let A be an open annular neighborhood of E. Denote by $\mathcal{K}(f^n)$ the family of all f^n-invariant frontiers which are essential in E. To prove density, we will find $h \in \mathrm{Diff}^\infty_\omega(S)$, arbitrarily C^r-close to the identity, such that $\mathcal{K}(hg^nh^{-1}) \cap \mathcal{K}(f^n) = \emptyset$.

Denote by A_1 and A_2 the two (annular) components of $A \setminus E$. Let $\phi\colon Q = [0,1]^2 \to S$ be a C^∞ embedding such that $\phi([0,1]\times\{0\}) \subset A_1$, $\phi([0,1]\times\{1\}) \subset A_2$ and $\phi(Q) \subset A$ (this can easily be obtained using a small tubular neighborhood of a simple arc joining the boundary components of E). By a result of Moser [14], we may assume that ϕ maps the area element of Q to the restriction of ω to $\phi(Q)$. Let

$$\hat{\mathcal{K}}_0(f^n) = \left\{\phi^{-1}(K \cap \phi(Q)) : K \in \mathcal{K}(f^n)\right\},$$

and denote by $\hat{\mathcal{K}}(f^n)$ the family of all connected components of $\hat{\mathcal{K}}_0(f^n)$ that separate Q. Since elements of $\mathcal{K}(f^n)$ are pairwise disjoint (by Lemma 19), $\hat{\mathcal{K}}(f^n)$ is a family of pairwise disjoint continua. It is easy to see that each of these continua horizontally separates Q, and they satisfy the hypotheses of Lemma 22.

If g is another Moser generic element of $\mathrm{Diff}^r_\omega(S)$, then we can define $\mathcal{K}(g^n)$ and $\hat{\mathcal{K}}(g^n)$ similarly. By Lemma 22, there is $h_0 \in \mathrm{Diff}^\infty_{\mathrm{Leb}}(Q)$ arbitrarily C^∞ close to the identity, which coincides with the identity in a neighborhood of ∂Q and such that $h_0(\hat{\mathcal{K}}(g^n))$ and $\hat{\mathcal{K}}(f^n)$ are disjoint. Defining $h \in \mathrm{Diff}^\infty_\omega(S)$ by $h(x) = \phi(h_0(\phi^{-1}(x)))$ if $x \in \phi(Q)$ and $h(x) = x$ otherwise, we have that $h(\mathcal{K}(g^n))$ and $\mathcal{K}(f^n)$ are disjoint, and

$$\mathcal{K}((hgh^{-1})^n) = \mathcal{K}(hg^nh^{-1}) = h(\mathcal{K}(g^n)).$$

Since we may assume that h is arbitrarily C^r close to the identity, we may also assume that $\tilde{g} = hgh^{-1}$ is C^r close to g, and by construction $\tilde{g}^n$ and f^n have no common invariant frontiers which are essential in E. This proves the C^r density of $\mathcal{R}^n_{f,E}$.

We now prove that the complement of $\mathcal{R}^n_{f,E}$ is C^0 (thus C^r) closed. Let $\{g_n\}$ be a sequence of diffeomorphisms not in $\mathcal{R}^n_{f,E}$, such that $g_n \to g \in \mathrm{Diff}^r_\omega(S)$ in the C^0 topology. Then, for each m there is $K_m \in \mathcal{K}(f^n)$ such that $g^n(K_m) = K_m$. By compactness, there is a subsequence $\{K_{m_i}\}$ which converges in the Hausdorff topology, and it is easy to see that its limit must be a continuum K which is essential in E, and $f^n(K) = K = g^n(K)$. Moreover, since K is a Hausdorff limit of a sequence of disjoint frontiers, it follows that K has empty interior. In fact, the sequence K_{m_i} can be chosen so that it is either increasing or decreasing with respect to the ordering defined (using the notation of Definition 13) by $K_0 < K_1$ if $K_0^- \subset K_1^-$ (and consequently $K_0^+ \supset K_1^+$) for $K_0, K_1 \in \mathcal{K}(f^n)$; if the sequence is increasing then it is easy to see that $K = \partial \cup_i K^-_{m_i}$, which has empty interior (and similarly if the sequence is decreasing). Since K is f^n-invariant and g^n-invariant and has empty interior, it contains a unique frontier $\hat{K} \in \mathcal{K}(f^n) \cap \mathcal{K}(g^n)$ (by Lemma 14 and its corollary). Thus, $g \notin \mathcal{R}^n_{f,E}$. This proves that $\mathrm{Diff}^r_\omega(S) \setminus \mathcal{R}^n_{f,E}$ is closed; hence $\mathcal{R}^n_{f,E}$ is open and dense.

Now consider a countable family $\mathcal{A}$ of closed annuli in S such that for any closed annulus $E \subset S$ there is $E' \in \mathcal{A}$ such that E is essential in the interior of E'. Such a family can be obtained as follows: Consider a sequence of (finite) triangulations $\{\mathcal{T}_i\}_{i\in\mathbb{N}}$ of S, such that the mesh of $\mathcal{T}_i$ tends to 0 when $i \to \infty$. For each i, consider the family $\mathcal{F}_i$ of simple closed curves formed by sides of elements of $\mathcal{T}_i$. Note that every simple closed curve can be C^0-approximated by elements of $\mathcal{F}_i$. Let $\mathcal{A}$ be the family of all annuli whose boundaries are elements of $\mathcal{F}_i$ for some i. That family is clearly countable. Furthermore, if $E \subset S$ is a closed annulus then we can consider an open annular neighborhood A of E so that $A \setminus E$ is a union of two open annuli A_1 and A_2. If i is large enough, A_1 and A_2 both contain some element of $\mathcal{F}_i$, one of which is essential in A_1 and the other in A_2. These curves bound an annulus $E' \in \mathcal{A}$, and E essential in the interior of E', so $\mathcal{A}$ has the required property.

We know that for each $E \in \mathcal{A}$ and $n > 0$, the set $\mathcal{R}^n_{f,E}$ is open and dense; thus

$$\mathcal{R}_f = \bigcap_{n\in\mathbb{N},\, E\in\mathcal{A}} \mathcal{R}^n_{f,E}$$

is a residual subset of $\mathrm{Diff}^r_\omega(S)$. Let $g \in \mathcal{R}_f$ and let $K \subset S$ be a frontier such that $g^n(K) = K$ for some $n > 0$. Note that, since K is contained in an annulus, $K \subset E$ for some $E \in \mathcal{A}$. If $f^m(K) = K$ for some $m > 0$, then K is f^{mn}-periodic and g^{mn}-periodic, which is not possible because $g \in \mathcal{R}^{mn}_{f,E}$. Thus no frontier in S is periodic for both f and g simultaneously. By Lemma 18, we conclude that $\mathrm{IFS}(f, g)$ is transitive for any $g \in \mathcal{R}_f$. This completes the proof. □

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