

The unit ball of the complex $\mathcal{P}({}^3H)$

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Abstract Let H be a two-dimensional complex Hilbert space and $\mathcal{P}({}^3H)$ the space of 3-homogeneous polynomials on H . We give a characterization of the extreme points of its unit ball, $\mathbf{B}_{\mathcal{P}({}^3H)}$, from which we deduce that the unit sphere of $\mathcal{P}({}^3H)$ is the disjoint union of the sets of its extreme and smooth points. We also show that an extreme point of $\mathbf{B}_{\mathcal{P}({}^3H)}$ remains extreme as considered as an element of $\mathbf{B}_{\mathcal{L}({}^3H)}$. Finally we make a few remarks about the geometry of the unit ball of the predual of $\mathcal{P}({}^3H)$ and give a characterization of its smooth points.

1 Notation, terminology and preliminary results

Given n (real or complex) Banach spaces X_1, \dots, X_n we denote by $X_1 \otimes \dots \otimes X_n$ their tensor product and by π the projective norm. If $X_1 \hat{\otimes}_{\pi} \dots \hat{\otimes}_{\pi} X_n$ is the completion of $X_1 \otimes \dots \otimes X_n$ under the projective norm, then we have $(X_1 \hat{\otimes}_{\pi} \dots \hat{\otimes}_{\pi} X_n)^* = \mathcal{L}({}^n X_1, \dots, X_n)$, the space of continuous n -linear forms on $X_1 \times \dots \times X_n$ endowed with the supremum norm. However, as remarked in [6], most of the times the multilinear theory is far from being just a simple

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translation of the linear one, neither when it comes to algebraic nor analytical or geometrical properties [1, 2, 9, 13].

Let X be a Banach space over the scalar field $\mathbb{K} = \mathbb{R}$ or \mathbb{C} . We denote by $\mathcal{P}(^n X)$ the space of all scalar valued continuous n -homogeneous polynomials on X endowed with the natural supremum norm. For every n -homogeneous polynomial P there exists a unique symmetric n -linear form A , which we call the polar of P , such that $P(x) = A(x, \dots, x)$. The general polarization formula gives $\|P\| \leq \|A\| \leq \frac{n^n}{n!} \|P\|$. There are cases in which the constant $n^n/n!$ can be improved. For instance, for Hilbert spaces we have $\|P\| = \|A\|$ (see Propositions 1.8, page 10 and 1.44, page 52 in [7]).

Let $\otimes_{n,s} X$ be the n -fold symmetric tensor product of X , that is the subspace of $X \otimes \dots \otimes X$ generated by the diagonal tensors $x \otimes \dots \otimes x$. We endow it with the topology inherited from $X \otimes_{\pi} \dots \otimes_{\pi} X$ and denote its completion by $\hat{\otimes}_{n,s,\pi} X$. Note that the dual of $\hat{\otimes}_{n,s,\pi} X$ is isometrically the space of symmetric n -linear forms on X , endowed with the supremum norm, $\mathcal{L}^s(^n X)$. Ryan showed in [17] that the space $\otimes_{n,s,\pi} X$ can be renormed such that $\mathcal{P}(^n X)$ becomes its isometric dual. Indeed, every element u of $\otimes_{n,s} X$ can be expressed as a finite sum $\sum_{j=1}^k \lambda_j x_j \otimes \dots \otimes x_j$. Define the symmetric projective norm of u by

$$\|u\|_{\pi_s} = \inf \left\{ \sum_{j=1}^k |\lambda_j| \|x_j\|^n : u = \sum_{j=1}^k \lambda_j x_j \otimes \dots \otimes x_j \right\}.$$

We endow $\otimes_{n,s} X$ with this norm and we denote its completion by $\hat{\otimes}_{n,s,\pi_s} X$. Then, if we put $(u, P) = \sum_{j=1}^k \lambda_j P(x_j)$, where (\cdot, \cdot) denotes duality, then we have $(\hat{\otimes}_{n,s,\pi_s} X)^* = \mathcal{P}(^n X)$.

A unit vector x in a normed space X is an extreme point of its unit ball B_X if x is not the midpoint of a nontrivial segment lying in B_X . We denote by $\text{Ext}(B_X)$ the set of all extreme points of B_X . Also, a unit vector x is a smooth point of B_X if there exists exactly one linear functional ϕ in B_{X^*} such that $\phi(x) = 1$.

In the last 20 years many authors explored the geometry of various spaces of homogeneous polynomials or multilinear forms. Most of the results that are known are for real spaces, although some description of extreme and smooth points in complex spaces also exist [4, 5, 8, 10]. The geometry of other spaces of polynomials (not necessarily homogeneous) has also been studied by several authors [3, 14–16].

In this paper, we describe the geometry of the unit ball of the space of 3-homogeneous polynomials on a two-dimensional complex Hilbert space H . We give a characterization of the extreme points of the unit ball of $\mathcal{P}(^3 H)$ from which we deduce that the unit sphere of $\mathcal{P}(^3 H)$ is the disjoint union of the sets of its extreme and smooth points. Since the polarization constant for Hilbert spaces is 1, we can consider $\mathcal{P}(^3 H)$ as a closed subspace of $\mathcal{L}(^3 H)$. It is a well known fact that, in general, if Y is a subspace of X then the extreme points of B_Y are not necessarily extreme in B_X , but it is true that $\text{Ext}(B_X) \cap Y \subset \text{Ext}(B_Y)$. However, we show that an extreme point of $B_{\mathcal{P}(^3 H)}$ remains extreme as considered as an element of $B_{\mathcal{L}(^3 H)}$, that is $\text{Ext}(B_{\mathcal{P}(^3 H)}) = \text{Ext}(B_{\mathcal{L}(^3 H)}) \cap \mathcal{P}(^3 H)$. Actually this seems to be part of a pattern.

Proposition 1 *Let K be a (real or complex) Hilbert space. We have that*

$$\text{Ext}(B_{\mathcal{P}(^2 K)}) = \text{Ext}(B_{\mathcal{L}(^2 K)}) \cap \mathcal{P}(^2 K).$$

Proof Let us take $P \in \text{Ext}(B_{\mathcal{P}(^2 K)})$. Through the identification between $\mathcal{P}(^2 K)$ and $\mathcal{L}^s(^2 K)$, we have to show that the polar of P , which we call A , is an extreme point in $B_{\mathcal{L}(^2 K)}$.

By Theorems 1.2 and 2.4 in [10], there exists an orthonormal basis $\{e_i\}_i$ for K such that for all $x = \sum_i x_i e_i$ we have $P(x) = \sum_i P(e_i)x_i^2$ with $|P(e_i)| = 1$. There exists a mapping $T : K \rightarrow K$ such that $A(x, y) = \langle x, Ty \rangle$, with $\langle \cdot, \cdot \rangle$ the inner product in K . Let us note that T is linear when K is real and conjugate linear when K is complex. However, in both cases we have $\|T\| = \sup_{\|x\| \leq 1} \|Tx\| = \|A\| = \|P\| = 1$. Since $|P(e_i)| = 1$, it follows that $Te_i = \overline{P(e_i)}e_i$ and so $A(e_i, e_j) = 0$ if $i \neq j$.

Now suppose that $A = \frac{A_1+A_2}{2}$ with A_1 and A_2 (possibly nonsymmetric) bilinear forms of norm one. It is clear that $A_1(e_i, e_i) = A_2(e_i, e_i) = P(e_i)$ and it follows as above that $A_1(e_i, e_j) = A_2(e_i, e_j) = 0$ if $i \neq j$, which show that $A_1 = A_2 = A$.

The same result remains true for 3-homogeneous polynomials on a two dimensional *real* Hilbert space. To prove it we are going to use the characterizations of extreme 3-homogeneous polynomials and extreme trilinear forms which appear in [11, Proposition 7] and [12, Proposition 2], respectively.

Proposition 2 *Let K be a real two-dimensional Hilbert space. We have that*

$$\text{Ext}(\mathcal{B}_{\mathcal{P}(\mathbb{C}^3K)}) = \text{Ext}(\mathcal{B}_{\mathcal{L}(\mathbb{C}^3K)}) \cap \mathcal{P}(\mathbb{C}^3K).$$

Proof Let us take $P \in \text{Ext}(\mathcal{B}_{\mathcal{P}(\mathbb{C}^3K)})$. By [11] there exists an orthonormal basis $\{e_1, e_2\}$ of K relative to which P can be written $P(x) = x_1^3 + 3bx_1x_2^2 + cx_2^3$ with b and c having one of the following two forms:

- i) $b = \frac{1}{2}$ and $c = 0$,
- ii) $b = \frac{\cos \beta}{1+\cos \beta}$ and $c = \frac{\sin^2 \beta - 2\cos^2 \beta + 2\cos^3 \beta}{\sin^3 \beta}$, with $\beta \in [-2\pi/3, 2\pi/3] \setminus \{0\}$.

In this basis the polar of P is

$$A(x, y, z) = x_1y_1z_1 + b(x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_1) + cx_2y_2z_2.$$

It is straightforward to check that we always have $3b^2 + c^2 + 2b^3 = 1$, which is exactly the condition required in [12, Proposition 2] so that A be an extreme point in the unit ball of $\mathcal{L}(\mathbb{C}^3K)$.

2 The characterization of $\text{Ext}(\mathcal{B}_{\mathcal{P}(\mathbb{C}^3H)})$

Let P be a 3-homogeneous polynomial of unit norm on the two-dimensional complex Hilbert space H , whose inner product we denote by $\langle \cdot, \cdot \rangle$. Let A be the polar of P , i.e. the unique symmetric trilinear form on H such that $A(x, x, x) = P(x)$ for every $x \in H$. By the Riesz representation theorem there exists a mapping $B : H \times H \rightarrow H$ such that $A(x, y, z) = \langle x, B(y, z) \rangle$. Furthermore

$$\|B\| = \sup_{\|y\|, \|z\| \leq 1} \|B(y, z)\| = \|A\| = \|P\|.$$

Thus if $P(x) = 1$ then from $\langle x, B(x, x) \rangle = 1$ it follows that $B(x, x) = x$. On the other hand, by continuity P attains its norm on the unit sphere of H and therefore there exists a unit vector w_0 such that $P(w_0) = 1$. Consider an orthonormal basis $\{e_1, e_2\}$ of H such that $e_1 = w_0$. Since $P(e_1) = 1$ we have $B(e_1, e_1) = e_1$ so $A(x, e_1, e_1) = \langle x, B(e_1, e_1) \rangle = \langle x, e_1 \rangle$ for all x in H . Thus $A(e_1, e_1, e_2) = 0$ and the expression of the polynomial P in the basis $\{e_1, e_2\}$ is

$$P(x) = x_1^3 + 3A(e_1, e_2, e_2)x_1x_2^2 + P(e_2)x_2^3.$$

For the sake of simplicity let $b = A(e_1, e_2, e_2)$ and $c = P(e_2)$. Without loss of generality, we can assume that $b \geq 0$, by substituting e_2 by $e^{-\frac{i \arg(b)}{2}} \cdot e_2$.

Of course $|b| \leq 1$ and $|c| \leq 1$ but the conditions $\|P\| \leq 1$ and the fact of P being extreme will impose further restrictions on b and c . For instance, it will follow that $b \leq \frac{1}{2}$. Indeed, since $\|P\| = 1$, for $x = (x_1, \pm x_2)$ with real coordinates we have

$$|x_1^3 + 3bx_1x_2^2 \pm cx_2^3| \leq \|x\|^3$$

and so

$$|x_1^3 + 3bx_1x_2^2| \leq \|x\|^3.$$

This shows that the 3-homogeneous polynomial $Q : \mathbb{R}^2 \rightarrow \mathbb{R}$ defined by $Q(x_1, x_2) = x_1^3 + 3bx_1x_2^2$ is of unit norm. Then, by the remarks preceding [11, Proposition 5], we necessarily have $b \leq \frac{1}{2}$.

We have that, if $x = (x_1, x_2)$, $y = (y_1, y_2)$ and $z = (z_1, z_2)$, in the basis $\{e_1, e_2\}$, then A can be expressed as

$$\begin{aligned} A(x, y, z) &= x_1y_1z_1 + b(x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_1) + cx_2y_2z_2 \\ &= x_1 \cdot \overline{B_1(y, z)} + x_2 \cdot \overline{B_2(y, z)}, \end{aligned}$$

where

$$\begin{aligned} B_1(y, z) &= \overline{y_1z_1 + by_2z_2}, \\ B_2(y, z) &= \overline{b(y_1z_2 + y_2z_1) + cy_2z_2}. \end{aligned}$$

Using the previous considerations, we are now in a position to state and prove the following characterization of the extreme points of the unit ball of $\mathcal{P}(\mathbb{C}^3H)$.

Theorem 1 *Let H be a two-dimensional complex Hilbert space. A polynomial P is extreme in $\mathbb{B}_{\mathcal{P}(\mathbb{C}^3H)}$ if and only if $\|P\| = 1$ and P attains its norm at two or more linearly independent points.*

Proof Let $P \in \mathcal{P}(\mathbb{C}^3H)$ of unit norm, and take $\{e_1, e_2\}$ the basis of H described in the comments preceding this theorem. From now on all coordinates in this proof shall be referred to this basis. Let us suppose that there exists $w = (w_1, w_2)$ with $w_2 \neq 0$ such that $P(w) = 1$. Then $(w, B(w, w)) = 1$ and hence, by the Cauchy–Schwartz inequality, we have that $B(w, w) = w$. From the latter it follows that

$$\begin{aligned} B_1(w, w) &= \overline{w_1^2 + bw_2^2} = w_1, \\ B_2(w, w) &= \overline{2bw_1w_2 + cw_2^2} = w_2, \end{aligned}$$

from which

$$\begin{cases} w_1^2 + bw_2^2 = \overline{w_1}, \\ 2bw_1w_2 + cw_2^2 = \overline{w_2}. \end{cases} \tag{1}$$

Now, since $w_2 \neq 0$, we obtain $b = \frac{\overline{w_1} - w_1^2}{w_2^2}$ and $c = \frac{\overline{w_2} - 2bw_1w_2}{w_2^2}$, i.e. the system (1) determines b and c uniquely. This shows that if $\|P\| = 1$, and P attains its norm at two or more linearly independent points, then P is extreme. Indeed, if $P = \frac{P_1 + P_2}{2}$ with $\|P_1\| = \|P_2\| \leq 1$, then

$$1 = P(e_1) = \frac{P_1(e_1) + P_2(e_1)}{2},$$

and therefore $P(e_1) = P_1(e_1) = P_2(e_1) = 1$. This lets us write P_1 and P_2 as follows

$$P_1(x) = x_1^3 + 3b_1x_1x_2^2 + c_1x_2^3,$$

$$P_2(x) = x_1^3 + 3b_2x_1x_2^2 + c_2x_2^3,$$

where $x = (x_1, x_2)$. Next, since

$$1 = P(w) = \frac{P_1(w) + P_2(w)}{2},$$

we also have that $P_1(w) = P_2(w) = 1$, which means that (b_1, c_1) and (b_2, c_2) satisfy (1), whose solution is unique. This shows that $b_1 = b_2 = b$ and $c_1 = c_2 = c$, i.e. $P = P_1 = P_2$ and hence P is extreme.

For the converse, let us now work with an extreme polynomial P . We shall analyze separately two different situations: $0 \leq b < \frac{1}{2}$ and $b = \frac{1}{2}$.

(1) If $0 \leq b < \frac{1}{2}$ then for $x = (x_1, x_2)$ in the unit sphere of H let us put $|x_1| = \cos \alpha$ and $|x_2| = \sin \alpha$. Define $f : [0, \pi/2] \rightarrow \mathbb{R}$ in the following way:

$$f(\alpha) = \cos^3 \alpha + 3b \cos \alpha \sin^2 \alpha + |c| \sin^3 \alpha,$$

with derivative

$$f'(\alpha) = 3 \sin \alpha (2b \cos^2 \alpha - \cos^2 \alpha - b \sin^2 \alpha + |c| \sin \alpha \cos \alpha) = 3 \sin \alpha g(\alpha).$$

Since $\lim_{\alpha \rightarrow 0} g(\alpha) = 2b - 1 < 0$ and g is continuous, there is $\delta > 0$ such that $g(\alpha) < b - \frac{1}{2} < 0$ on $[0, \delta)$. Now, let

$$K_\delta = \{(x_1, x_2) \in \mathbb{C}^2 : \|(x_1, x_2)\| = 1, |x_1| \leq \cos \delta\}.$$

Next, if we assume that P attains its norm only at e_1 , we have that $m = \sup_{K_\delta} |P(x_1, x_2)| < 1$. Choose $\varepsilon > 0$ such that

$$b - \frac{1}{2} + \varepsilon < 0,$$

$$m + \varepsilon < 1$$

and let

$$P_\pm(x_1, x_2) = P(x_1, x_2) \pm \varepsilon x_2^3.$$

In K_δ , we have that

$$|P_\pm(x_1, x_2)| \leq |P(x_1, x_2)| + \varepsilon \leq m + \varepsilon < 1.$$

In a similar manner as above, we associate with P_\pm the functions $f_\pm : [0, \pi/2] \rightarrow \mathbb{R}$ in the following way:

$$f_\pm(\alpha) = \cos^3 \alpha + 3b \cos \alpha \sin^2 \alpha + |c| \sin^3 \alpha \pm \varepsilon \sin^3 \alpha,$$

with derivative

$$f'_\pm(\alpha) = 3 \sin \alpha [(2b - 1) \cos^2 \alpha - b \sin^2 \alpha + (|c| \pm \varepsilon) \sin \alpha \cos \alpha].$$

Thus, $f'_\pm(0) = 0$ and, since in $[0, \delta)$ we have

$$g(\alpha) \pm \varepsilon \sin \alpha \cos \alpha < b - \frac{1}{2} + \varepsilon < 0,$$

it follows that f_{\pm} are decreasing functions on the interval $[0, \delta)$, therefore on this interval $f_{\pm}(\alpha) \leq f_{\pm}(0) = 1$. Now, if $(x_1, x_2) \notin K_{\delta}$ then $|x_1| = \cos \alpha > \cos \delta$ which means that $\alpha \in [0, \delta)$ and so $|P_{\pm}(x_1, x_2)| \leq f_{\pm}(\alpha) \leq 1$.

All of this shows that $\|P_{\pm}\| = 1$ and, since

$$P = \frac{P_+ + P_-}{2},$$

it follows that P is not extreme, a contradiction.

(2) If $b = \frac{1}{2}$ then for all unitary elements x with real coordinates x_1, x_2 , one has that

$$1 \geq |P(x_1, x_2)| \geq \left| x_1^3 + \frac{3}{2}x_1x_2^2 + \Re(c)x_2^3 \right|$$

and, by the remarks preceding [11, Proposition 5] it follows that $\Re(c) = 0$. If we set $c = i\lambda$, $\lambda \in \mathbb{R}$, we claim that $\|P\| = 1$ if and only if $|\lambda| \leq \frac{1}{\sqrt{2}}$.

Indeed, let us take $x_1 = \pm\frac{1}{3}$ and $x_2 = \pm\frac{2\sqrt{2}}{3}i$. Since $|P(x_1, x_2)| \leq 1$, we obtain two inequalities from which it follows that $|\lambda| \leq \frac{1}{\sqrt{2}}$.

On the other hand, we show that for all $|\lambda| \leq \frac{1}{\sqrt{2}}$, the polynomial $P(x_1, x_2) = x_1^3 + \frac{3}{2}x_1x_2^2 + i\lambda x_2^3$ has norm one. Replacing x_2 by ix_2 and keeping in mind that $P(1, 0) = 1$, it follows that $\|P\|$ is the same as the norm of the real 3-homogeneous polynomial $(x_1, x_2) \mapsto x_1^3 - \frac{3}{2}x_1x_2^2 + \lambda x_2^3$. Using either Lagrange multipliers or the method described in [11] it can be shown that the critical points of this polynomial over the Euclidean unit sphere of \mathbb{R}^2 are

$$\pm \left(\pm \frac{\lambda \pm \sqrt{\lambda^2 + 4}}{\sqrt{16 + (\lambda \pm \sqrt{\lambda^2 + 4})^2}}, \frac{4}{\sqrt{16 + (\lambda \pm \sqrt{\lambda^2 + 4})^2}} \right).$$

Evaluating at the above points, we obtain that the norm of P is the maximum of the absolute values of the following two expressions over the interval $\left[-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right]$:

$$\frac{\sqrt{2} \left(\mp 5 + \sqrt{\lambda^2 + 4} + \lambda(13 + \lambda(\lambda \pm \sqrt{\lambda^2 + 4})) \right)}{\left(10 + \lambda(\lambda \pm \sqrt{\lambda^2 + 4}) \right)^{\frac{3}{2}}}$$

and

$$\frac{\sqrt{2} \left(\mp 5 + \sqrt{\lambda^2 + 4} + \lambda(19 + \lambda(\lambda \pm \sqrt{\lambda^2 + 4})) \right)}{\left(10 + \lambda(\lambda \pm \sqrt{\lambda^2 + 4}) \right)^{\frac{3}{2}}},$$

which are monotonous, therefore they attain their extreme values at the endpoints of the above interval. Since all of these extreme values are between -1 and 1 , we are done.

Now, if $|\lambda| < \frac{1}{\sqrt{2}}$ then P is not extreme. To prove this choose $\varepsilon > 0$ such that $|\lambda \pm \varepsilon| < \frac{1}{\sqrt{2}}$. Therefore the two polynomials

$$P_{\pm}(x) = x_1^3 + \frac{3}{2}x_1x_2^2 + (\lambda \pm \varepsilon)x_2^3,$$

have norm one, and since $P = \frac{P_+ + P_-}{2}$, we conclude that P is not extreme. There remains to study the case where $\lambda = \pm \frac{1}{\sqrt{2}}$. As we have seen above, when $\lambda = -\frac{1}{\sqrt{2}}$, we have that

$P(\frac{1}{3}, \frac{2\sqrt{2}}{3}i) = -1$ and when $\lambda = \frac{1}{\sqrt{2}}$, we have that $P(-\frac{1}{3}, \frac{2\sqrt{2}}{3}i) = 1$. Thus in both cases P attains its norm at two linearly independent points.

This shows that for $b = \frac{1}{2}$, the only extreme polynomials are of the form $x_1^3 + \frac{3}{2}x_1x_2^2 + i\lambda x_3^3$ with $\lambda = \pm \frac{1}{\sqrt{2}}$ and they have at least two points of norm attainment.

Remark 1 When $b = \frac{1}{2}$, the fact that the polynomial $P(x) = x_1^3 + \frac{3}{2}x_1x_2^2 + i\lambda x_3^3$ has norm one if and only if $|\lambda| \leq \frac{1}{\sqrt{2}}$ can be obtained too using [5, Theorem 1]. Indeed, if A is the polar of P then we have that $\|P\| = \|A\| = 1$ if and only if

$$3\left(\frac{1}{2}\right)^2 + \frac{|\lambda|^2}{2} + \left|2\left(\frac{1}{2}\right)^3 - \frac{\lambda^2}{2}\right| \leq 1,$$

which easily gives $|\lambda|^2 \leq \frac{1}{2}$.

Corollary 1 *Let H be a two-dimensional complex Hilbert space. The unit sphere of $\mathcal{P}(\mathcal{H}^3)$ is the disjoint union of the set of its extreme points and the set of its smooth points.*

Proof A 3-homogeneous polynomial of unit norm P must attain its norm on \mathbf{B}_H . If it does so at only one point, then P is a smooth point of $\mathbf{B}_{\mathcal{P}(\mathcal{H}^3)}$ [11, Corollary 3]. If P has at least two points of norm attainment, then P is extreme. Since there are no extreme polynomials that attain the norm at only one point, there do not exist any polynomials that are extreme and smooth at the same time.

3 The connection between $\text{Ext}(\mathbf{B}_{\mathcal{P}(\mathcal{H}^3)})$ and $\text{Ext}(\mathbf{B}_{\mathcal{L}(\mathcal{H}^3)})$

The following theorem will be useful in order to prove our next result. Its proof can be found in [5].

Theorem 2 [5] *A trilinear form $A : H^3 \rightarrow \mathbb{C}$ on the Hilbert space $H = \mathbb{C}^2$ is an extreme point in $\mathbf{B}_{\mathcal{L}(\mathcal{H}^3)}$ if and only if there are three orthonormal bases $\{e_1, e_2\}$, $\{f_1, f_2\}$, and $\{g_1, g_2\}$ of H and complex numbers b_1, b_2, b_3, c such that, if $A_{jkl} = A(e_j, f_k, g_l)$, then $A_{111} = 1$, $A_{112} = A_{121} = A_{211} = 0$, $A_{122} = b_1$, $a_{212} = b_2$, $A_{221} = b_3$, $A_{222} = c$ and we have that*

$$|b_1|^2 + |b_2|^2 + |b_3|^2 + \frac{|c|^2}{2} + \left|2b_1b_2b_3 + \frac{c^2}{2}\right| = 1, \tag{2}$$

and in addition either

$$2b_1b_2b_3 + \frac{c^2}{2} = 0, \tag{3}$$

or the strict inequality

$$|2b_1b_2b_3| < \left|2b_1b_2b_3 + \frac{c^2}{2}\right| + \frac{|c|^2}{2} \tag{4}$$

holds.

This characterization will allow us to prove for the two-dimensional complex Hilbert space H a similar result to Propositions 1 and 2.

Theorem 3 *Let H be a two-dimensional complex Hilbert space. A polynomial P is extreme in the unit ball of $\mathcal{P}(\mathfrak{L}^3 H)$ if and only if its polar A is extreme in the unit ball of $\mathcal{L}(\mathfrak{L}^3 H)$. In other words,*

$$\text{Ext}(\mathbf{B}_{\mathcal{P}(\mathfrak{L}^3 H)}) = \text{Ext}(\mathbf{B}_{\mathcal{L}(\mathfrak{L}^3 H)}) \cap \mathcal{P}(\mathfrak{L}^3 H).$$

Proof Let $P \in \text{Ext}(\mathbf{B}_{\mathcal{P}(\mathfrak{L}^3 H)})$ and let us show that its polar A belongs to $\text{Ext}(\mathbf{B}_{\mathcal{L}(\mathfrak{L}^3 H)})$. As we did earlier, P can be expressed as follows

$$P(x_1, x_2) = x_1^3 + 3bx_1x_2^2 + cx_2^3,$$

in an orthonormal basis $\{e_1, e_2\}$ where $P(1, 0) = \|P\| = 1$ and $0 \leq b \leq \frac{1}{2}$. For the rest of this proof all coordinates shall be referred to $\{e_1, e_2\}$. We shall show that the polar of P , given by

$$A(x, y, z) = x_1y_1z_1 + b(x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_1) + cx_2y_2z_2$$

is an extreme point of $\mathbf{B}_{\mathcal{P}(\mathfrak{L}^3 H)}$.

We will analyze the following two cases.

(1) If $b = \frac{1}{2}$, then we have necessarily that $c = \pm \frac{i}{\sqrt{2}}$ (see the proof of Theorem 1). It is straightforward to check that the 3-linear forms

$$\begin{aligned} A_-(x, y, z) &= x_1y_1z_1 + \frac{1}{2}(x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_1) - \frac{i}{\sqrt{2}}x_2y_2z_2, \\ A_+(x, y, z) &= x_1y_1z_1 + \frac{1}{2}(x_1y_2z_2 + x_2y_1z_2 + x_2y_2z_1) + \frac{i}{\sqrt{2}}x_2y_2z_2, \end{aligned}$$

satisfy the conditions (2) and (3) from Theorem 2.

(2) If $0 \leq b < \frac{1}{2}$, since P attains its norm at a point (w_1, w_2) with $w_2 \neq 0$, we have from (1) that

$$\begin{aligned} b &= \frac{\overline{w_1} - w_1^2}{w_2^2}, \\ c &= \frac{1 - 3|w_1|^2 + 2w_1^3}{w_2^2}. \end{aligned}$$

Now, some calculations show that

$$X := 2b^3 + \frac{c^2}{2} = \frac{4\overline{w_1}^3 + 4w_1^3 - 3|w_1|^4 - 6|w_1|^2 + 1}{2w_2^6}. \tag{5}$$

We can show that the numerator in the right hand side of (5) is positive. Indeed, since $|b|^2 < \frac{1}{4}$,

$$\begin{aligned} |b|^2 &= \frac{\overline{w_1} - w_1^2}{w_2^2} \cdot \frac{w_1 - \overline{w_1}^2}{w_2^2} \\ &= \frac{|w_1|^2 - \overline{w_1}^3 - w_1^3 + |w_1|^4}{|w_2|^4} \\ &= \frac{|w_1|^2 - \overline{w_1}^3 - w_1^3 + |w_1|^4}{(1 - |w_1|^2)^2} < \frac{1}{4}, \end{aligned}$$

from which we obtain that

$$4|w_1|^2 - 4\overline{w_1}^3 - 4w_1^3 + 4|w_1|^4 < 1 - 2|w_1|^2 + |w_1|^4,$$

which proves our claim. Thus

$$|X| = \frac{4\bar{w}_1^3 + 4w_1^3 - 3|w_1|^4 - 6|w_1|^2 + 1}{2|w_2|^6}.$$

On the other hand

$$3|b|^2 + \frac{|c|^2}{2} + |X| = 3 \frac{|w_1|^2 - \bar{w}_1^3 - w_1^3 + |w_1|^4}{|w_2|^4} + \frac{c\bar{c}}{2} + |X|,$$

and since

$$\begin{aligned} |c|^2 &= \frac{1 - 3|w_1|^2 + 2w_1^3}{w_2^3} \cdot \frac{1 - 3|w_1|^2 + 2\bar{w}_1^3}{\bar{w}_2^3} \\ &= \frac{1 - 6|w_1|^2 + 2\bar{w}_1^3 + 9|w_1|^4 - 6|w_1|^2\bar{w}_1^3 + 2w_1^3 - 6|w_1|^2w_1^3 + 4|w_1|^6}{|w_2|^6}, \end{aligned}$$

after making all the calculations we arrive at

$$3|b|^2 + \frac{|c|^2}{2} + |X| = 1.$$

Thus A verifies (2). Let us now see that (4) holds. From the latter identity we have

$$|X| + \frac{|c|^2}{2} = 1 - 3b^2.$$

In other words, there remains to prove that $2b^3 < 1 - 3b^2$, which is straightforward since $0 \leq b < \frac{1}{2}$.

In the light of this result and of Propositions 1 and 2 we would like to finish this section with the following question: Is it true that $\text{Ext}(\mathbf{B}_{\mathcal{P}({}^nH)}) = \text{Ext}(\mathbf{B}_{\mathcal{L}({}^nH)}) \cap \mathcal{P}({}^nH)$ for any (real or complex) Hilbert space H and for any degree n ?

4 Some remarks regarding $\mathbf{B}_{\otimes_{s,\pi_s}^3 H}$

From [8], it follows that the extreme points of the unit ball of $\otimes_{s,\pi_s}^3 H$ are the vectors $x \otimes x \otimes x$ with x in the unit sphere of H . Thus, any element of the unit ball of $\otimes_{s,\pi_s}^3 H$ is a convex combination of a finite number of such points and we know, by Caratheodory’s Theorem, that, like in all finite dimensional spaces, this number is limited.

However, in certain cases we have more precise information. For instance, if K is a two dimensional real Hilbert space then any element u in the unit sphere of $\otimes_{s,\pi_s}^3 K$ is a convex combination of at most three points of the form $x \otimes x \otimes x$ with x in the unit sphere of K . This comes from the fact that for u there exists a polynomial $P \in \text{Ext}(\mathbf{B}_{\mathcal{P}({}^3K)})$ such that $(u, P) = 1$ (where (\cdot, \cdot) denotes duality) and P cannot attain its norm at more than three points in the unit sphere of K (see [11, Proposition 13]). Thus, on the four dimensional space $\otimes_{s,\pi_s}^3 K$, any tensor u of unit norm can be written as a convex combination of three elementary tensors (not five, as it would be the best given by Caratheodory’s Theorem).

In our case the situation is more complicated since it is possible that an extreme point of the unit sphere of $\mathcal{P}({}^3H)$ attains its norm at an infinity of points. Indeed, let us consider

the extreme polynomial $P(x_1, x_2) = x_1^3 + \frac{3}{2}x_1x_2^2 + \frac{i}{\sqrt{2}}x_2^3$. Its points of norm attainment are given by the equations

$$\begin{cases} x_1^2 + \frac{1}{2}x_2^2 = \bar{x}_1, \\ x_1x_2 + \frac{i}{\sqrt{2}}x_2^2 = \bar{x}_2, \end{cases}$$

which have an infinity of solutions (x_1, x_2) with $|x_1|^2 + |x_2|^2 = 1$. Indeed, for any ρ with $0 < |\rho| < \frac{4}{3\sqrt{2}}$ and θ such that $4 \sin 3\theta + 3\sqrt{2}\rho = 0$, taking

$$\begin{aligned} x_2 &= \rho(\cos \theta + i \sin \theta) \\ x_1 &= \frac{\bar{x}_2 - \frac{i}{\sqrt{2}}x_2^2}{x_2} \end{aligned}$$

we have that $|x_1|^2 + |x_2|^2 = 1$ and the equations above are verified. This shows that $P(x_1, x_2) = 1$ for all of these points. Therefore we cannot hope to obtain more precise information than what is given by Caratheodory's Theorem. However, as a consequence of Theorem 1 we can give a characterization of the smooth points of $\mathbf{B}_{\mathcal{P}(\mathfrak{H})}$.

Proposition 3 *Let H be a two-dimensional complex Hilbert space. A unit vector u is a smooth point of the unit ball of $\otimes_{s,\pi_s}^3 H$ if and only if u can be written as a convex combination of at least two points of the form $x \otimes x \otimes x$ with x in the unit sphere of H .*

Proof Suppose that u is a convex combination of two such points, $u = \lambda x \otimes x \otimes x + \mu y \otimes y \otimes y$ with x and y unit vectors, $0 \leq \lambda, \mu \leq 1$ and $\lambda + \mu = 1$. Let P be a 3-homogeneous polynomial that norms u . Then necessarily $P(x) = P(y) = 1$ and from the proof of Theorem 1 it follows that P is uniquely determined and so u is a smooth point.

For the converse, it is easy to see that, given x in the unit sphere of H , it is not possible that $x \otimes x \otimes x$ be a smooth point since there are many 3-homogeneous polynomials that attain their norm at x .

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