Math. Z. (2009) 263:583–606
DOI 10.1007/s00209-008-0431-5

Mathematatische Zeitschrift

Uniformly continuous maps between ends of $\mathbb{R}$-trees

Álvaro Martínez-Pérez · Manuel A. Morón

Received: 4 July 2007 / Accepted: 10 September 2008 / Published online: 9 October 2008

Abstract There is a well-known correspondence between infinite trees and ultrametric spaces which can be interpreted as an equivalence of categories and comes from considering the end space of the tree. In this equivalence, uniformly continuous maps between the end spaces are translated to some classes of coarse maps (or even classes of metrically proper Lipschitz maps) between the trees.

Keywords Tree · Ultrametric · End space · Coarse map · Uniformly continuous · Non-expansive map

Mathematics Subject Classification (2000) Primary 54E35 · 53C23; Secondary 54C05 · 51K05

1 Introduction

This paper is mainly inspired by a recent, interesting and beautiful one due to Hughes [4] but it is also motivated by [8] where a complete ultrametric was defined on the sets of shape morphisms between compacta.

In [8] it was proved that every shape morphism induces a uniformly continuous map between the corresponding ultrametric spaces of shape morphisms which are, in particular, complete and bounded as metric spaces. Moreover Hughes established some categorical equivalences for some classes of ultrametric spaces and local similarity equivalences to certain categories of geodesically complete rooted $\mathbb{R}$-trees and certain equivalence classes of isometries at infinity.

M. A. Morón was partially supported by MTM 2006-00825.

Á. Martínez-Pérez · M. A. Morón (✉)
Departamento de Geometría y Topología, Universidad Complutense de Madrid,
28040 Madrid, Spain
e-mail: mamoron@mat.ucm.es

In view of that, it is natural for us to ask for a description of uniform types (the classification by means of uniform homeomorphism) of end spaces of geodesically complete rooted $\mathbb{R}$-trees in terms of some geometrical properties of the trees.

To answer these questions is the aim of this paper and we find herein that the bounded coarse geometry, see [10, 11], of $\mathbb{R}$-trees is an adequate framework to do that.

Also, we would like to point out some important differences between this paper and Hughes's. First of all we treat different, although related, categories:

The morphisms in every category of ultrametric spaces used in [4] are isomorphisms for the topological category of ultrametric spaces, i.e. they are always homeomorphisms, while in this paper we get results for the whole category of complete bounded ultrametric spaces and uniformly continuous maps between them (not only for uniformly continuous homeomorphism).

But above all, we get an explicit formula to construct a non-expansive map between two trees that induces a given uniformly continuous function between the corresponding end spaces. To obtain this formula we use a procedure described by Borsuk [3], to find a suitable modulus of continuity associated to a uniformly continuous function. This is the way in which we pass from the total disconnectedness of ultrametric spaces to the strong connectivity of any ray in the tree.

Our main results in this paper can be summarized as follows:

The category of complete ultrametric spaces with diameter bounded above by 1 *and uniformly continuous maps between them is isomorphic to any of the following categories*:

(1) *Geodesically complete rooted $\mathbb{R}$-trees and rooted metrically proper homotopy classes of metrically proper continuous maps between them.*
(2) *Geodesically complete rooted $\mathbb{R}$-trees and rooted coarse homotopy classes of coarse continuous maps between them.*
(3) *Geodesically complete rooted $\mathbb{R}$-trees and rooted metrically proper non-expansive homotopy classes of metrically proper non-expansive continuous maps between them.*

We finish this paper recovering, as a consequence of our constructions, the classical relation between the proper homotopy type of a locally finite simplicial tree and the topological type of its Freudenthal end space, see [1].

Although our main source of information on $\mathbb{R}$-trees is Hughes's paper [4], it must be also recommended the classical book [12] of Serre and the survey [2] of Bestvina for more information and to go further, let us say that in [7], Morgan treats a generalization of $\mathbb{R}$-trees called Λ-trees. Moreover, in [5], Hughes and Ranicki treat applications of ends, not only ends of trees, to topology.

2 Trees

We are going to recall some basic properties on trees mainly extracted from [4].

Definition 2.1 A *real tree* or $\mathbb{R}$-*tree* is a metric space (T, d) that is uniquely arcwise connected and $\forall\, x, y \in T$ the unique arc from x to y, denoted $[x, y]$, is isometric to the subinterval $[0, d(x, y)]$ of $\mathbb{R}$.

Lemma 2.2 *If T is an $\mathbb{R}$-tree and $v, w, z \in T$ then there exists $x \in T$ such that* $[v, w] \cap [v, z] = [v, x]$.

Definition 2.3 A *rooted* $\mathbb{R}$-*tree*, (T, v), is an $\mathbb{R}$-tree (T, d) and a point $v \in T$ called *the root*.

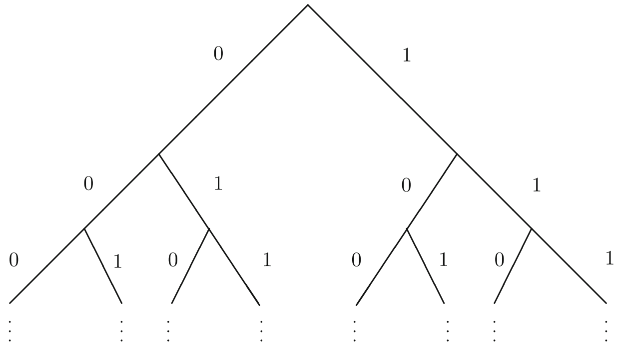
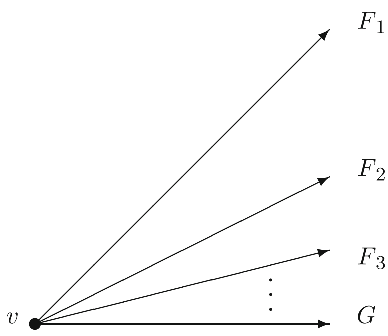


Fig. 1 The Cantor tree

Fig. 2 Non-locally finite tree



Definition 2.4 A rooted $\mathbb{R}$-tree is *geodesically complete* if every isometric embedding $f : [0, t] \to T,\ t > 0$ with $f(0) = v$ extends to an isometric embedding $F : [0, \infty) \to T$. In that case we say that $[v, f(t)]$ can be extended to a *geodesic ray*.

Remark 2.5 The single point v is a trivial rooted geodesically complete $\mathbb{R}$-tree.

Notation If (T, v) is a rooted $\mathbb{R}$-tree and $x \in T$, let $\|x\| = d(v, x)$,

$$B(v, r) = \{x \in T \mid \|x\| < r\}$$
$$\bar{B}(v, r) = \{x \in T \mid \|x\| \leq r\}$$
$$\partial B(v, r) = \{x \in T \mid \|x\| = r\}$$

Notation For any pair of metric spaces, $X \approx Y$ will denote that X, Y are isometric.

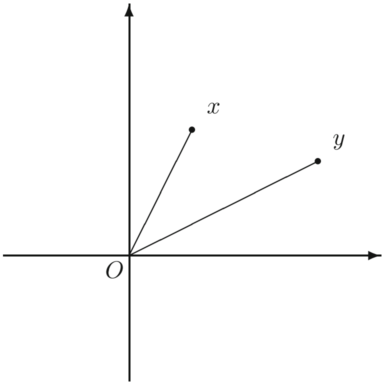
Example 2.6 Cantor tree. Assume that each edge of the tree has length 1 (Fig. 1).

Example 2.7 $\{(a, b) \in \mathbb{R}^2 \mid a \geq 0 \text{ and } b = 0, b = a \text{ or } b = \frac{a}{2^n} \text{ with } n \in \mathbb{N}\}$

For any two points collinear with the origin assume the euclidian metric, d_e, and if the points x,y are not collinear with the origin define $d(x, y) = d_e(x, v) + d_e(v, y)$ (Fig. 2).

Example 2.8 Consider $(\mathbb{R}^2, O)$, for any two points collinear with the origin assume the euclidian metric and for any two points x, y non-collinear with the origin define $d(x, y) = d_e(x, O) + d_e(O, y)$ (Fig. 3).

Fig. 3 $\mathbb{R}$-tree with a branching point of uncountable order



Definition 2.9 If c is any point of the rooted $\mathbb{R}$-tree (T, v), the *subtree of* (T, v) *determined by* c is:

$$T_c = \{x \in T \mid c \in [v, x]\}.$$

Also, let

$$T_c^i = T_c \backslash \{c\} = \{x \in T \mid c \in [v, x] \text{ and } x \neq c\}.$$

Lemma 2.10 *If (T, v) is a geodesically complete rooted $\mathbb{R}$-tree, T_c the subtree induced by any point c and $x \in (T, v)$ such that $x \notin T_c$ then $\forall\, y \in T_c \quad d(x, y) = d(x, c) + d(c, y)$.*

Proof It suffices to show that $c \in [x, y]$. Lemma 2.2 implies that there exists $z \in (T, v)$ such that $[v, x] \cap [v, y] = [v, z]$ and we start with $x \notin T_c$, that is, $c \notin [v, x]$; in particular, $c \notin [v, z]$ and $c \in [z, y]$. It is clear that $[x, y] = [x, z] \cup [z, y]$ and thus $c \in [x, y]$. □

Lemma 2.11 *Let (T, v) a geodesically complete rooted $\mathbb{R}$-tree, T_c the subtree induced by c and $x \in (T, v)$ such that $x \notin T_c$ then $d(x, T_c) = d(x, c)$.*

Proof It follows immediately from 2.10. □

Lemma 2.12 *If c is any point of a geodesically complete rooted $\mathbb{R}$-tree (T, v) then T_c is closed.*

Proof Let $x \notin T_c$ and $\varepsilon = d(x, T_c) = d(x, c) > 0$. By 2.10, $B(x, \varepsilon) \cap T_c = \emptyset$. Hence $T \backslash T_c$ is open. □

Lemma 2.13 *If c is any point of a geodesically complete rooted $\mathbb{R}$-tree (T, v) then T_c^i is open.*

Proof Let $x \in T_c^i$ ($c \in [v, x]$ and $x \neq c$) and $\varepsilon = d(x, c) > 0$. Then $B(x, \varepsilon) \subset T_c^i$ and hence T_c^i is open. □

Lemma 2.14 *If $F : [0, \infty) \to (T, v)$ is an isometric embedding such that $F(0) = v$, then $\forall\, t_0 \in [0, \infty) \quad F[t_0, \infty) \subset T_{F(t_0)}$.*

Proof Clearly $\forall\, t > t_0$, $F(t_0)$ must be in $[v, F(t)]$. Hence $F(t) \in T_{F(t_0)}$. □

Lemma 2.15 *Let $c \in (T, v)$ a geodesically complete rooted $\mathbb{R}$-tree, then T_c is also a geodesically complete rooted $\mathbb{R}$-tree.*

Proof T_c is a metric space since it is a subset of a metric space. It is clear that any point in T_c is connected with c by an arc; therefore any two points in T_c are connected by an arc which is obviously unique, since T_c is a subset of (T, v) which is uniquely arcwise connected.

We take c as the root of T_c.

Let $f : [0, t_0] \to T_c$ any isometric embedding such that $f(0) = c$. Then, consider the isometric embedding $f' : [0, t_0 + \|c\|] \to T$ such that $f'(0) = v$, $f'(\|c\|) = c$ and $f'(t + \|c\|) = f(t)$. The map f' extends f and, by definition of geodesically complete, there exists an isometric embedding $F'[0, \infty) \to T$ such that F' extends f'. $F'(\|c\|) = c$ and by Lemma 2.14 $F'[\|c\|, \infty) \subset T_c$. If we define $F(t) := F'(t + \|c\|)$ it is readily seen that $F : [0, \infty) \to T_c$ is an isometric embedding and extends f in T_c. □

Definition 2.16 A *cut set* for a geodesically complete rooted $\mathbb{R}$-tree (T, v) is a subset C of (T, v) such that $v \notin C$ and for every isometric embedding $F : [0, \infty) \to T$ with $F(0) = v$ there exists a unique $t_0 > 0$ such that $F(t_0) \in C$.

Example 2.17 $\partial B(v, r)$ with $r > 0$ is a *cut set* for (T, v).

Proposition 2.18 *Given a cut set C for (T, v), the connected components of $T(C) := \{x \in T \mid [v, x] \cap C \neq \emptyset\}$ (that is, the part of (T, v) not between the root and the cut set) are exactly the subtrees $\{T_c\}_{c \in C}$.*

Proof $T(C) = \cup_{c \in C} T_c$ and we know that T_c is always connected (as it is in fact arcwise connected). Let's see that for any $c_0 \in C$, the connected component of c_0 in $T(C)$ is T_{c_0}.

If we remove from the tree any point $x \in (T, v)$ we disconnect the tree in two subsets: T_x^i and $T \backslash T_x$ which are open sets in (T, v), as we saw in Lemmas 2.12 and 2.13, and it is easy to verify that T_x^i is closed in $T \backslash \{x\}$. (Note that T_x^i need not be connected but we may remark that T_x^i is a union of connected components of an open set and these are open since (T, v) is locally connected).

Let $c' \in C$ such that $c' \neq c_0$ and $w \in (T, v)$ such that $[v, c_0] \cap [v, c'] = [v, w]$. Consider $x \in [w, c_0]$ such that $x \neq c_0$ and by definition of cut set it is clear that $x \notin T(C)$ and $T_x^i \cap T(C)$ is a clopen set in $T(C)$ that contains T_{c_0} and $T_x^i \cap T_{c'} = \emptyset$. The intersection of all the clopen sets that contain T_{c_0} (we already know that T_{c_0} is connected) is the quasi-component of T_{c_0}, which contains the connected component and does not intersect any other subtree $T_{c'}$ induced by any other point of the cut set. Hence, the connected component of c_0 is exactly T_{c_0}. □

Remark 2.19 If we consider in (T, v) the cut set $C := \partial B(v, r)$ with $r > 0$, then $T(C)$ is exactly $T \backslash B(v, r)$.

3 Metrically proper maps between trees

Main concepts in this section are taken from [10, 11]. Note that herein it is used the convention that a map need not be continuous. When continuity is required we will write it explicitly.

Definition 3.1 A map f between two metric spaces X, X' is *metrically proper* if for any bounded set A in X', $f^{-1}(A)$ is bounded in X.

Definition 3.2 A map between two rooted $\mathbb{R}$-trees, $f : (T, v) \to (T', w)$, is said to be *rooted* if $f(v) = w$.

To avoid repeating the expression: rooted, continuous and metrically proper map we define *metrically proper between trees* as follows.

Definition 3.3 A map f between two rooted R-trees is *metrically proper between trees* if it is rooted, metrically proper and continuous.

Remark 3.4 If $f : (T, v) \to (T', w)$ is a metrically proper map between trees, then:

$$\forall M > 0 \quad \exists N_{M,f} > 0 \quad \text{such that} \quad f^{-1}(B(w, M)) \subset B(v, N_{M,f}).$$

This is equivalent to saying that $f(T \backslash B(v, N_{M,f})) \subset T' \backslash B(w, M)$.

Proposition 3.5 *If $f : (T, v) \to (T', w)$ is a metrically proper map between trees, and $M > 0$ and $N > 0$ are such that $f^{-1}(B(w, M)) \subset B(v, N)$, then for any $c \in \partial B(v, N)$ there exists a unique $c' \in \partial B(w, M)$ such that $f(T_c) \subset T'_{c'}$.*

Proof If $f^{-1}(B(w, M)) \subset B(v, N)$ then $f(T \backslash B(v, N)) \subset T' \backslash B(w, M)$. The map f sends connected components of $T \backslash B(v, N)$ into connected components of $T' \backslash B(w, M)$; in particular, $\forall c \in \partial B(v, N) \quad f(T_c) \subset T' \backslash B(w, M)$. As it is a continuous image of a connected set is clearly contained in one of the connected components of $T' \backslash B(w, M)$ and those are, as we saw in Propositions 2.17 and 2.18, the subtrees determined by points of the cut set $\partial B(w, M)$. □

Equivalence relation on metrically proper maps between trees. In this paragraph we introduce an equivalence relation on the maps defined between trees. The resulting equivalence classes will be the morphisms of the category $\mathcal{T}$ whose objets are geodesically complete rooted $\mathbb{R}$-trees. The interest of this relation is that two maps will be in the same class if and only if they induce the same map between the end spaces (that will be uniformly continuous as we shall see). This allows us to establish the one-to-one correspondence of the morphisms in the equivalence of categories.

We define this equivalence relation in two steps: first we put it in terms of the behavior of the maps on the complement of closed balls centered at the root. Later, we prove that these classes are related to some natural concept of homotopy between metrically proper maps.

Let $M > 0, \; N > 0$ be such that $f(T \backslash B(v, N)) \subset T' \backslash B(w, M)$. For any $c \in \partial B(v, N)$ let T_c be the subtree determined by c. By Proposition 3.5, there is a unique $c' \in \partial B(w, M)$ such that $f(T_c) \subset T'_{c'}$. This allows us to consider the families $\mathcal{T}_N := \{T_c \mid c \in \partial B(v, N)\}$ and $\mathcal{T}'_M := \{T'_{c'} \mid c' \in \partial B(w, M)\}$ and a map $f_{\mathcal{T}_N} : \mathcal{T}_N \longrightarrow \mathcal{T}'_M$ given by $f_{\mathcal{T}_N}(T_c) = T'_{c'}$ if and only if $f(T_c) \subset T'_{c'}$.

Remark 3.6 This map can be defined for all $N > N_{M,f}$ since $\forall d \in \partial B(v, N)$ there exists a unique $c \in \partial B(v, N_{M,f})$ such that $T_d \subset T_c$, and obviously

$$f(T_d) \subset f(T_c) \subset T'_{c'} \Rightarrow f_{\mathcal{T}_{N'}}(T_d) = T'_{c'}.$$

Given $f, f' : (T, v) \to (T', w)$ two metrically proper maps between trees, then by $f \sim f'$ we understand the following

$$\forall M > 0, \; \exists N_{M,f,f'} > 0 \quad \text{such that} \quad \forall N > N_{M,f,f'} \quad f_{\mathcal{T}_N} = f'_{\mathcal{T}_N}. \tag{1}$$

Proposition 3.7 *$\sim$ defines an equivalence relation.*

Proof It is obviously *reflexive* and *symmetric*.
Transitive: If $f \sim f'$ and $f' \sim f''$ then there exist constants $N_{M,f,f'}$ and $N_{M,f',f''}$ such that $\forall N > N_{M,f,f'} \quad f_{\mathcal{T}_N} = f'_{\mathcal{T}_N}$ and $\forall N > N_{M,f',f''} \quad f'_{\mathcal{T}_N} = f''_{\mathcal{T}_N}$. Hence, for every $N > \max\{N_{M,f,f'}, N_{M,f',f''}\} \quad f_{\mathcal{T}_N} = f''_{\mathcal{T}_N}$ and $f \sim f''$. □

Definition 3.8 If $f, g : X \to T$ are two continuous maps from any topological space X to a tree T then the *shortest path homotopy of* f *to* g is a homotopy $H : X \times I \to T$ of f to g such that if $j_x : [0, d(f(x), g(x))] \to [f(x), g(x)]$ is the isometric embedding of the subinterval $[0, d(f(x), g(x))] \subset \mathbb{R}$ into T whose image is the shortest path between $f(x)$ and $g(x)$ then $H(x,t) = j_x(t \cdot d(f(x), g(x))) \; \forall t \in I \; \forall x \in X$.

Lemma 3.9 *If* $f, g : X \to T$ *are two continuous maps from any topological space* $(X, \mathcal{T})$ *to an* $\mathbb{R}$*-tree* T*, then there is a shortest path homotopy* $H : X \times I \to T$ *of* f *to* g*.*

Proof It suffices to prove that H with the definition above is continuous. Consider $(x_0, t_0) \in X \times I$. The continuity of f and g implies that $\forall \varepsilon > 0$ there exists $U \in \mathcal{T}$ with $x_0 \in U$ such that $f(U) \subset B(f(x_0), \frac{\varepsilon}{2})$ and $g(U) \subset B(g(x_0), \frac{\varepsilon}{2})$ where $B(., \gamma)$ is the corresponding ball in the $\mathbb{R}$-tree T. It is immediate to check that this implies that $H(U, t_0) \subset B(H(x_0, t_0), \frac{\varepsilon}{2})$. Let K be such that $d(f(x), g(x)) < K \quad \forall x \in U$. Then, $H(U, B(t_0, \frac{\varepsilon}{2K})) \subset B(H(x_0, t_0), \varepsilon)$ and H is continuous. Clearly, $H_0 \equiv f$ and $H_1 \equiv g$. □

Definition 3.10 Given $f, f' : (T, v) \to (T', w)$ two metrically proper maps between trees, we say that $H : T \times I \to T'$ is a *rooted metrically proper homotopy of* f *to* f' if H is continuous, $H_0 \equiv f$, $H_1 \equiv f'$, $H(v, t) = w \; \forall t \in I$ and $\forall M > 0$, $\exists N > 0$ such that $H^{-1}(B(v, M)) \subset B(v, N) \times I$. In this case we say that f, f' are *rooted metrically properly homotopic*, denoted by $f \simeq_{Mp} f'$.

Definition 3.11 Two trees $(T, v), (T', w)$ are said to be *rooted metrically properly homotopy equivalent*, $(T, v) \simeq_{Mp} (T', w)$, if there exist two metrically proper maps between trees $f : T \to T'$ and $f' : T' \to T$ such that $f \circ f' \simeq_{Mp} id_{T'}$ and $f' \circ f \simeq_{Mp} id_T$.

Proposition 3.12 $f \sim f'$ *if and only if* $f \simeq_{Mp} f'$*.*

Proof Suppose $f \sim f'$. $\forall n \in \mathbb{N}$ let $t_n > 0$ such that $f(T \backslash B(v, t_n)) \subset T' \backslash B(w, n)$ and $f'(T \backslash B(v, t_n)) \subset T' \backslash B(w, n)$. Without loss of generality suppose $t_{n+1} > t_n + 1$. If $f \sim f'$, by Proposition 3.5, $\forall c \in \partial B(v, t_n)$ there exists a unique point c' in $\partial B(w, n)$ such that the image under either f or f' of T_c is contained in $T'_{c'}$.

By 3.9, if we consider the shortest path homotopy of f to f' it remains to check that this homotopy is metrically proper. It suffices to show that $\forall t_n$ and $\forall t \in [0, 1] \quad H_t(T \backslash B(v, t_n)) \subset T' \backslash B(w, n)$. Given $x \in T \backslash B(v, t_n)$ we know that $f(x) \in T' \backslash B(w, n)$ and $f'(x) \in T' \backslash B(w, n)$ and also, by the meaning of the relation defined, there exists a unique $c' \in \partial B(w, n)$ such that $f(x) \in T'_{c'}$ and $f'(x) \in T'_{c'}$. As we saw in Remark 2.15, $T'_{c'}$ is an $\mathbb{R}$-tree, so there exists an arc in that tree from $f(x)$ to $f'(x)$ and, since T is uniquely arcwise connected, this arc must be contained in $T'_{c'}$. Hence the homotopy restricted to $T \backslash B(v, t_n)$ is contained in $T' \backslash B(w, n)$.

Conversely, let $f, f' : (T, v) \to (T', w)$ be metrically proper maps between trees and $H : T \times I \to T'$ a rooted metrically proper homotopy of f to f'. Let $M > 0$, $N > 0$ such that $H_t(T \backslash B(v, N)) \subset T' \backslash B(w, M) \; \forall t \in I$. For any $c \in \partial B(v, N)$ and $c' \in \partial B(w, M)$ such that $f(T_c) \subset T'_{c'}$ it is clear that $H(T_c \times I) \subset T'_{c'}$ (as it is the continuous image of a connected set into $T' \backslash B(w, M)$); in particular (if $t = 1$) $f'(T_c) \subset T'_{c'}$ and hence, $f \sim f'$. □

Remark 3.13 This implies that $\simeq_{Mp}$ defines an equivalence relation.

Notation For any metrically proper map between trees f, let $[f]_{Mp}$ be the class of all metrically proper maps between trees metrically properly homotopic to f. Some places in section 7 where the context is clear this class will be simply denoted by $[f]$.

4 The end space of a tree

In this section we define the functor ξ on the objects of the categories, from trees to ultrametric spaces, following step by step [4].

Definition 4.1 If (X, d) is a metric space and $d(x, y) \leq \max\{d(x, z), d(z, y)\}$ for all $x, y, z \in X$, then d is an *ultrametric* and (X, d) is an *ultrametric space*.

Elementary properties of ultrametric spaces can be found in [9].

Definition 4.2 The *end space* of a rooted $\mathbb{R}$-tree (T, v) is given by:
$end(T, v) = \{F : [0, \infty) \to T \mid F(0) = v$ and F is an isometric embedding$\}$.

For $F, G \in end(T, v)$, define:

$$d_u(F, G) = \begin{cases} 0 & \text{if } F = G, \\ e^{-t_0} & \text{if } F \neq G \text{ and } t_0 = sup\{t \geq 0 \mid F(t) = G(t)\}. \end{cases}$$

Note that since T is uniquely arcwise connected:

$$\{t \geq 0 \mid F(t) = G(t)\} = \begin{cases} [0, \infty) & \text{if } F = G, \\ [0, t_0] & \text{if } F \neq G. \end{cases}$$

Proposition 4.3 *If* (T, v) *is a rooted* $\mathbb{R}$*-tree, then* $(end(T, v), d_u)$ *is a complete ultrametric space of diameter* ≤ 1.

Remark 4.4 Abusing of the notation, we sometimes identify the element of the end space with its image on the tree. This will be usually called *branch*. Also, for non-geodesically complete $\mathbb{R}$-trees, we also use *branch* to call any rooted non-extendable isometric embedding, making distinction between finite and infinite branches.

Proposition 4.5 *For any* $x \in (T, v)$*, a geodesically complete rooted* $\mathbb{R}$*-tree, there exists* $F \in end(T, v)$ *and* $t \in [0, \infty)$ *such that* $F(t) = x$ *(in fact,* $t = \|x\|$*).*

Proof The unique arc $[v, x]$ is isometric to the interval $[0, d(v, x)] = [0, \|x\|]$. If $f : [0, \|x\|] \to [v, x]$ is an isometry with $f(0) = v$, by 2.4, it extends to an isometric immersion F (an element of the end space of the tree) and clearly $F(\|x\|) = x$. □

5 The tree of an ultrametric space

We follow again [4] to define the functor η from ultrametric spaces to trees.

Definition 5.1 Let U a complete ultrametric space with diameter ≤ 1 and define:

$$T_U := \frac{U \times [0, \infty)}{\sim}$$

with $(\alpha, t) \sim (\beta, t')$ if and only if $t = t'$ and $\alpha, \beta \in U$ such that $d(\alpha, \beta) \leq e^{-t}$.

Given two points in T_U represented by equivalence classes $[x, t], [y, s]$ with $(x, t), (y, s) \in U \times [0, \infty)$ define a metric on T_U by:

$$D([x, t], [y, s]) = \begin{cases} |t - s| & \text{if } x = y, \\ t + s - 2\min\{-ln(d(x, y)), t, s\} & \text{if } x \neq y. \end{cases}$$

Remark 5.2 Instead of defining the tree as in [4] for any ultrametric space of finite diameter we restrict ourselves to ultrametric spaces of diameter ≤ 1. We define the root to be $[(x, 0)]$ and thus the ultrametric space is isometric to the end space of the tree.

The next two propositions are in [4].

Proposition 5.3 *D is a metric on T_U.*

Proposition 5.4 *(T_U, D) is a geodesically complete rooted $\mathbb{R}$-tree.*

Proposition 5.5 *$U \approx end(T_U)$.*

Proof Consider the map $\gamma : U \to end(T_U)$ which sends each $\alpha \in U$ to the isometric embedding $f_\alpha : [0, \infty) \to T_U$ such that $f_\alpha(t) = (\alpha, t)$ (clearly $f_\alpha \in end(T_U)$).

Given $\alpha, \beta \in U$ let $d_0 := d(\alpha, \beta)$. Then $(\alpha, t) = (\beta, t)$ on $[0, -ln(d_0)]$ and, in the end space, $d(f_\alpha, f_\beta) = e^{ln(d_0)} = d_0$ and hence γ is an isometry. It is immediate to see that it is surjective by the completeness of U. □

6 Constructing the functors

6.1 Maps between trees induced by a uniformly continuous map between the end spaces

The purpose in this section is to show how to construct a map between trees from a uniformly continuous map between their end spaces.

For this purpose, we are going to use the following.

Definition 6.1.1 A function $\gamma : [0, \infty) \longrightarrow [0, \infty)$ is called a *modulus of continuity* if γ is non-decreasing, continuous at 0 and $\gamma(0) = 0$.

If $f : (X_1, d_1) \to (X_2, d_2)$ is a map between two metric spaces and (X_2, d_2) is bounded, let

$$\gamma_f(\delta) := \sup_{x,y \in X_1 \mid d_1(x,y) \leq \delta} \{d_2(f(x), f(y))\}. \tag{2}$$

Lemma 6.1.2 *Consider (X_1, d_1), (X_2, d_2) two metric spaces with X_2 bounded and let $f : X_1 \to X_2$ be a uniformly continuous map. Then $\gamma_f : [0, \infty) \to [0, \infty)$ is a modulus of continuity such that $\forall\, x, y \in X_1 \;\; d_2(f(x), f(y)) \leq \gamma_f(d_1(x, y))$.*

Proof γ_f is well-defined since X_2 is bounded, and it is immediate to see that it is non-decreasing and $\gamma_f(0) = 0$. It remains to check the continuity at 0. Since f is uniformly continuous, $\forall\, \varepsilon > 0 \;\; \exists\, \delta > 0$ such that if $d(x, y) < \delta$ then $d(f(x), f(y)) < \varepsilon$. Therefore, $\forall\, \delta' < \delta \; \gamma_f(\delta') \leq \varepsilon$, and hence,

$$\lim_{\delta \to 0} \gamma_f(\delta) = 0.$$

□

To define the map between the trees we need the modulus of continuity to be continuous in order to pass from the total disconnectedness of ultrametric spaces to the strong connectivity of the tree. In fact, to define the functor and to prove the equivalence of categories, 7.6, it would suffice to show that given a uniformly continuous map between the end spaces, there exists a continuous modulus of continuity holding the condition of Lemma 6.1.2.

Nevertheless, considering the function γ_f shown in (2) in certain examples, the construction of the function (4) may give us an analytic expression of this map between the trees as we shall see later on in 6.1.15.

First, since the spaces we are considering are of diameter bounded by 1, we restrict the map γ_f to a map from the unit interval to itself. Then, following the construction of Borsuk [3], we take something similar to a convex hull of the image to make it continuous and concave.

Let us recall the definition of concave function.

Definition 6.1.3 A real-valued function f defined on a convex set C is called *concave* if for any pair of points x,y in C and any $t \in [0, 1]$, we have

$$f(tx + (1-t)y) \geq tf(x) + (1-t)f(y).$$

If $f : X_1 \to X_2$ is a uniformly continuous map and X_1, X_2 have diameter ≤ 1 let $\varrho_f : [0, 1] \to [0, 1]$ be a function such that $\varrho_f(1) = 1$ and $\varrho_f(t) = \gamma_f(t)\ \forall t \in [0, 1)$.

Remark 6.1.4 Similarly to 6.1.2, ϱ_f is non-decreasing, continuous at 0, $\varrho_f(0) = 0$ and $\forall x, y \in X_1 \quad d_2(f(x), f(y)) \leq \varrho_f(d_1(x, y))$.

For every $x \in [0, 1]$ let $\Gamma(x)$ denote the set of ordered pairs (x_1, x_2) such that $x_1, x_2 \in [0, 1]$, $x_1 < x_2$ and $x \in [x_1, x_2]$. For any $x \in [x_1, x_2]$ there exists a unique $t \in [0, 1]$ such that $x = tx_1 + (1-t)x_2$. Then, let

$$\varrho_{f,x_1,x_2}(x) := t\varrho_f(x_1) + (1-t)\varrho_f(x_2). \tag{3}$$

Finally, this allows us to construct the following function which is the key to construct the map between the trees:

$$\lambda_f(x) := \sup_{x_1,x_2 \in \Gamma(x)} \varrho_{f,x_1,x_2}(x). \tag{4}$$

Remark 6.1.5 Note that by definition $\lambda_f(x) \geq \varrho_f(x) \quad \forall x \in [0, \infty)$.

Proposition 6.1.6 *$\lambda_f(0) = 0$, $\lambda_f(1) = 1$ and $\lambda_f(x)$ is increasing, concave and uniformly continuous.*

Proof Clearly $\lambda_f(0) = \varrho_f(0) = 0$ and $\lambda_f(1) = \varrho_f(1) = 1$. It is immediate to see that it is increasing since ϱ_f is, and concave obviously by construction. The proof that it is continuous at 0 is similar to the analogous result in [3]. □

Hence, from a uniformly continuous map f between two metric spaces U_1, U_2 with diameter$(U_i) \leq 1$, we get a map $\lambda_f : [0, 1] \to [0, 1]$ uniformly continuous, concave and non-decreasing with $\lambda_f(0) = 0$, $\lambda_f(1) = 1$ and by Remark 6.1.5:

$$\forall x,\ y \in U_1 \quad d(f(x), f(y)) \leq \lambda_f(d(x, y)). \tag{5}$$

Using this map we are now in position to induce, from a uniformly continuous map f between two complete ultrametric spaces of diameter ≤ 1, a map between the corresponding trees. As we saw in 5.5, we can identify these ultrametric spaces with the end spaces of their trees. Given a uniformly continuous map $f : U_1 \to U_2$, by abuse of notation, consider $f : end(T_{U_1}, v) \to end(T_{U_2}, w)$ in the canonical way.

Let (T, v), (T', w) be two geodesically complete rooted $\mathbb{R}$-trees and $f : end(T, v) \to end(T', w)$ a uniformly continuous map. Our purpose is to prove that the formula

$$\hat{f}(x) = f(F)\left(-ln(\lambda_f(e^{-t}))\right). \tag{6}$$

where $x = F(t)$ with $F \in end(T, v)$, $t \in [0, \infty)$ and ln represents the natural logarithm, defines a non-expansive metrically proper map between the trees.

The following fact is obvious:

Lemma 6.1.7 *If $f : end(T_{U_1}, v) \to end(T_{U_2}, w)$ is uniformly continuous and λ_f is defined as in (4), then $-ln(\lambda_f(e^{-t}))$ is non-decreasing.*

Remark 6.1.8 Moreover, if $d_0 = \min\{d > 0 | \lambda_f(d) = 1\}$, then λ_f is strictly increasing on $[0, d_0]$ since it is concave, and hence, it is immediate to check that $-ln(\lambda_f(e^{-t}))$ is strictly increasing for t on $[-ln(d_o), \infty)$.

Remark 6.1.9 Note that

$$\lim_{\delta \to 0} \lambda_f(\delta) = 0 \Rightarrow \lim_{t \to \infty} \left(-ln(\lambda_f(e^{-t}))\right) = \infty.$$

Now we are going to verify that the map $\tilde{f}$ is well-defined and then we shall study its properties.

Proposition 6.1.10 *Let (T, v), $(T'w)$ be two geodesically complete rooted ℝ-trees and suppose that $f : end(T, v) \to end(T', w)$ is a uniformly continuous map. Then, the formula in (6) defines a map between the trees.*

Proof We need to prove that each point in (T, v) has a unique image.

Let $x \in (T, v)$, $F, G \in end(T, v)$ and $t_0, t_1 \in [0, \infty)$ such that $F(t_0) = x = G(t_1)$. As we saw in 5.1, necessarily $t_0 = t_1$ and $F(t) = G(t)\ \forall t \in [0, t_0]$. Then, $d(F, G) \le e^{-t_0}$ and by (5), $d(f(F), f(G)) \le \lambda_f(e^{-t_0})$.

Since $d(f(F), f(G)) = e^{-sup\{s \ge 0 / f(F)(s) = f(G)(s)\}} \le \lambda_f(e^{-t_0}) \Leftrightarrow sup\{s \ge 0 \,|\, f(F)(s) = f(G)(s)\} \ge -ln(\lambda_f(e^{-t_0}))$ it is clear that, in particular, $f(F)\left(-ln(\lambda_f(e^{-t_0}))\right) = f(G)\left(-ln(\lambda_f(e^{-t_0}))\right)$. Thus, the image by $\hat{f}$ does not depend on the representative and it is well-defined. □

Definition 6.1.11 A map between two metric spaces $f : X \to Y$ is *Lipschitz of constant C* if for any pair of points $x, x' \in X$, $d(f(x), f(x')) \le C \cdot d(x, x')$. If $C = 1$ the map is called *non-expansive*.

Proposition 6.1.12 *If f is a uniformly continuous map between the end spaces, then $\hat{f}$ is non-expansive.*

Proof Given $x, x' \in (T, v)$, we are going to prove that $d(\hat{f}(x), \hat{f}(x')) \le d(x, x')$:

Case I. If the points are in the same branch of the tree.

Then, there exists $F \in end(T, v)$ such that $x = F(t_0)$ and $x' = F(t_1)$, suppose $t_1 > t_0$, and hence $d(x, x') = t_1 - t_0$.

The images are $f(F)\left(-ln(\lambda_f(e^{-t_0}))\right)$ and $f(F)\left(-ln(\lambda_f(e^{-t_1}))\right)$ and it is clear that

$$d(\hat{f}(x), \hat{f}(x')) = \Big| - ln\left(\lambda_f(e^{-t_0})\right) - \left(-ln\left(\lambda_f(e^{-t_1})\right)\right)\Big|.$$

We can avoid the absolute value, since $\lambda_f : [0, 1] \to [0, 1]$ is non-decreasing:

$$t_1 > t_0 \Rightarrow e^{-t_1} < e^{-t_0} \Rightarrow \lambda_f(e^{-t_1}) \le \lambda_f(e^{-t_0}) \Rightarrow ln(\lambda_f(e^{-t_1})) \le ln(\lambda_f(e^{-t_0})).$$

Hence,

$$d(\hat{f}(x), \hat{f}(x')) = ln(\lambda_f(e^{-t_1})) - ln(\lambda_f(e^{-t_0})). \tag{7}$$

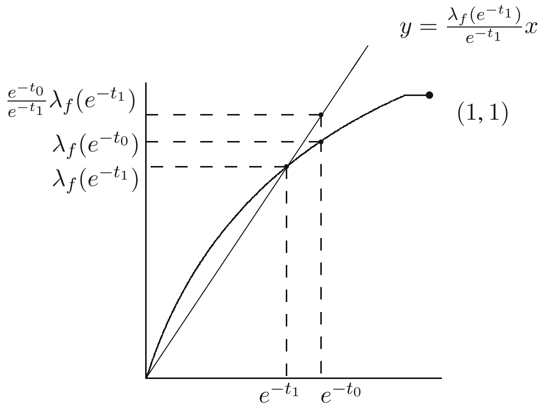


Fig. 4 The function λ_f is concave

The concavity of λ_f will allow us to relate this distance with $t_1 - t_0$. The idea is that if we have two points on the line $y = Kx$, $y_1 = Kx_1$, $y_2 = Kx_2$, the difference between the logarithms only depends on the proportion between x_1 and x_2 since $ln(Kx_1) - ln(Kx_2) = ln(\frac{Kx_1}{Kx_2}) = ln(\frac{x_1}{x_2})$ and in our case, this proportion between two points in the image of λ_f may be bounded using the line which joins the (0, 0) with the first point since λ_f is concave (see Fig. 4).

Since $\lambda_f : [0, 1] \to [0, 1]$ is concave and $e^{-t_1} < e^{-t_0}$, we have that $\lambda_f(e^{-t_0}) \leq \frac{e^{-t_0}}{e^{-t_1}}\lambda_f(e^{-t_1})$. Then, since the natural logarithm is an increasing function, substituting in (7) we finally obtain that

$$d(\hat{f}(x), \hat{f}(x')) \leq ln\left(\frac{e^{-t_0}}{e^{-t_1}}\lambda_f(e^{-t_1})\right) - ln(\lambda_f(e^{-t_1})) = ln(e^{t_1-t_0}) = t_1 - t_0 = d(x, x').$$

Case II. Suppose that x, x' are not in the same branch. Then there exist $F, G \in end(T, v)$ and $t_0, t_1 \in R$ such that $x = F(t_0)$, $x' = G(t_1)$; also let $t_2 = sup\{s| F(s) = G(s)\}$. Then $t_2 \leq t_0, t_1$ (otherwise x and x' would be in the same branch) and $d(x, x') = t_0 - t_2 + t_1 - t_2 = d(x, y) + d(y, x')$ with $y = F(t_2) = G(t_2)$.

Thus, $\hat{f}(F(t_2)) = \hat{f}(y) = \hat{f}(G(t_2))$ and by case I, we can see that $d(\hat{f}(x), \hat{f}(x')) \leq d(\hat{f}(x), \hat{f}(y)) + d(\hat{f}(y), \hat{f}(x')) \leq d(x, y) + d(y, x') = d(x, x')$. □

Remark 6.1.13 Being Lipschitz, the induced map $\hat{f}$ is uniformly continuous.

Now we are going to prove:

Proposition 6.1.14 *If f is a uniformly continuous map between the end spaces, then $\hat{f}$ is metrically proper between trees.*

Proof We have already proved the continuity.
Rooted. We assumed $\lambda_f(1) = 1$ and the image of the root will be the image of $F(0)$ for any $F \in end(T, v)$. Thus

$$\hat{f}(v) = \hat{f}(F(0)) = f(F)\left(-ln(\lambda_f(e^0))\right) = f(F)(0) = w.$$

Metrically proper. We need to show that $\forall\, M > 0\ \ \exists\, N_{M,f} > 0$ such that $\hat{f}^{-1}(B(w, M)) \subset B(v, N_{M,f})$. (This is equivalent to saying that the inverse image of bounded sets is bounded).

$\hat{f}^{-1}(B(w, M)) = \{x \in T|\ -ln(\lambda_f(e^{-\|x\|})) < M\}$. By Remark 6.1.7, we know that $-ln(\lambda_f(e^{-t}))$ is non-decreasing, and by Remark 6.1.9 it is clear that $\exists\, N_{M,f} > 0$ such that $\forall\, t \geq N_{M,f}\ \ -ln(\lambda_f(e^{-t})) > M$, and hence, $\hat{f}^{-1}(B(w, M)) \subset B(v, N_{M,f})$. □

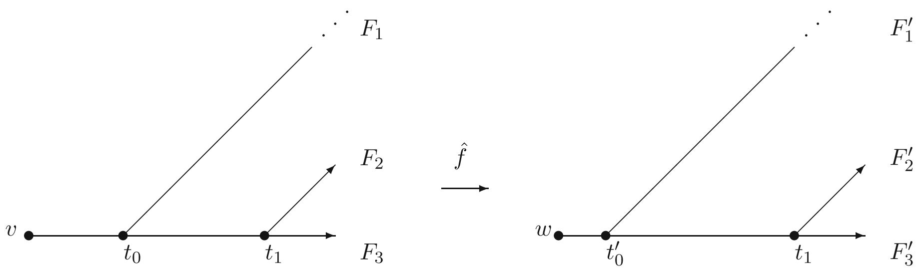
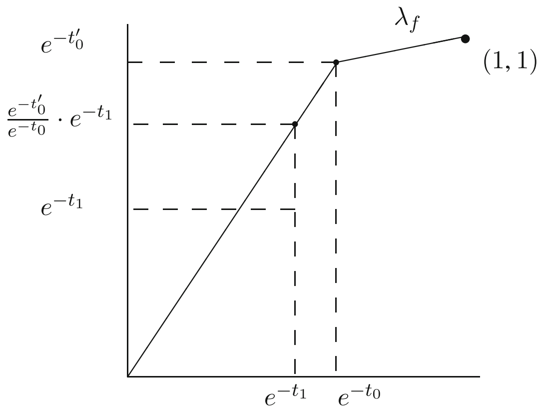


Fig. 5 $\hat{f} : (T, v) \to (T', w)$ induced by a map f between the ends

Fig. 6 The resulting λ_f



Now we present some examples to illustrate the above construction and we will show how coarse category naturally appears.

Example 6.1.15 Let $f : end(T, v) \to end(T', w)$ for the trees in Fig. 5 with $t'_0 < t_0$ and such that $f(F_i) = F'_i$ for $i = 1, 2$ or 3.

A modulus of continuity can be defined as in Lemma 6.1.2

$$\varrho_f(\delta) := \begin{cases} 0 & \text{if } \delta < e^{-t_1}, \\ e^{-t_1} & \text{if } e^{-t_1} \leq \delta < e^{-t_0}, \\ e^{-t'_0} & \text{if } e^{-t_0} \leq \delta < 1, \\ 1 & \text{if } 1 \leq \delta. \end{cases}$$

Now, if we construct λ_f as in (4), we have (Fig. 6)

$$\lambda_f(\delta) := \begin{cases} \frac{e^{-t'_0}}{e^{-t_0}} \cdot \delta & \text{if } \delta < e^{-t_0}, \\ \varrho_{e^{-t_0},1}(\delta) & \text{if } e^{-t_0} \leq \delta < 1, \\ 1 & \text{if } 1 \leq \delta. \end{cases}$$

It can be readily seen that $\hat{f}$ is Lipschitz of constant < 1 from $F_i[0, t_0]$ to $F'_i[0, t'_0]$ and an isometry between $F_i[t_0, \infty)$ and $F_i[t'_0, \infty)$ for $i = 1, 2$ or 3 with $\hat{f}(F_i(t_0)) = F'_i(t'_0)$ and $\hat{f}(F_j(t_1)) = F'_j(t_1 - t_0 + t'_0) \in F'_j(t'_0, t_1)$ for $j = 2, 3$. Thus, f is a non-expansive map.

Definition 6.1.16 A map $f : X_1 \to X_2$ between two metric spaces is *bornologous* if for every $R > 0$ there is $S > 0$ such that for any two points $x, x' \in X_1$ with $d(x, x') < R$, $d(f(x), f(x')) < S$.

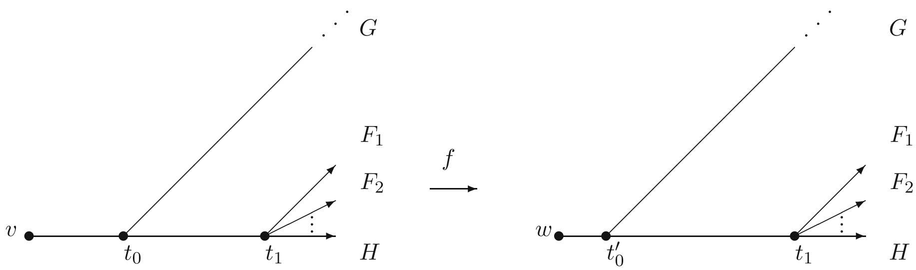


Fig. 7 A metrically proper map between the trees which is not proper

Definition 6.1.17 A map is *coarse* if it is metrically proper and bornologous.

Lemma 6.1.18 *If f is a uniformly continuous map between end spaces, then the induced map between the trees, $\hat{f}$, is coarse.*

Proof We have already seen that it is metrically proper and since $\hat{f}$ is Lipschitz of constant 1 it suffices to consider $R = S$. □

Definition 6.1.19 A map is *proper* if the inverse image of any compact set is compact.

The following example shows that $\hat{f}$ need not be proper.

Counterexample. Let U a ultrametric space consisting of a countable family, non-finite, of points $\{x_n\}_{n\in\mathbb{N}}$ with $d(x_i, x_j) = d_1\ \forall\, i \neq j$ and another point, $\{y\}$ with $d(y, x_i) = d_0\ \forall\, i$, suppose $d_0 > d_1$, and let U' a similar family of points $\{x'_n\}_{n\in\mathbb{N}}$ with distance d_1 among them and another point, $\{y'\}$ with $d(y', x'_i) = d'_0$ and $d'_0 > d_0$. Both spaces are uniformly discrete and the map f which sends y to y', and x_i to x'_i is obviously uniformly continuous. Now we can find a compact set K in $T_{U'}$ such that its inverse image under $\hat{f} : T_U \to T_{U'}$ is not compact.

Consider $t_0 = -ln(d_0),\ t'_0 = -ln(d'_0)$ and $t_1 = -ln(d_1)$. The induced trees are as shown in Fig. 7.

Let $K = \bar{B}(w, t_1)$ which is obviously compact, we can see that $\hat{f}^{-1}(K)$ is not compact.

The image by $\hat{f}$ of the arc $[v, x_i(t_0)] \approx [0, t_0]$ will be $[w, x'_i(t'_0)] \approx [0, t'_0]$ (with $t'_0 < t_0$). By concavity of λ_f, $\forall\, t > t_0\quad e^{-t} < e^{-t_0} \Rightarrow \lambda_f(e^{-t}) \geq \frac{e^{-t}}{e^{-t_0}}\lambda_f(e^{-t_0}) \Rightarrow -ln(\lambda_f(e^{-t})) \leq -ln(e^{t_0-t}\cdot\lambda_f(e^{-t_0})) = t - t_0 + t'_0$. If $\varepsilon = t_0 - t'_0 > 0$ then $\hat{f}(B(v, t)) \subset B(w, t - \varepsilon)$; in particular $B(v, t_1 + \varepsilon) \subset \hat{f}^{-1}(B(w, t_1))$, and so the inverse image by $\hat{f}$ of K is a closed ball about v of radius greater than t_1, and since T_U is not locally compact at t_1, this set is not compact.

6.2 Uniformly continuous map between end spaces induced by a metrically proper map between trees

In this subsection we treat some kind of opposite correspondence, to that of the previous one.

Proposition 6.2.1 *If (T, v) and (T', w) are two geodesically complete rooted $\mathbb{R}$-trees and $f : (T, v) \to (T', w)$ is a metrically proper map between trees, then for each $F \in end(T, v)$ there exists a unique $G \in end(T', w)$ such that $G[0, \infty) \subset im\left(\hat{f}(F[0, \infty))\right)$. Thus, f induces a map between the end spaces of the trees.*

Fig. 8 Uniqueness

Proof *Existence*. Let $F \in end(T, v)$. By 3.4, $\forall n \in \mathbb{N}$, $\exists t_n > 0$ such that $\hat{f}^{-1}(B(w, n)) \subset B(v, t_n)$. By Proposition 3.5, there exists a unique $c'_n \in \partial B(w, n)$ such that $f\left(T_{F(t_n)}\right) \subset T'_{c'_n}$.

Define $G : [0, \infty) \to T$ such that $G|_{[0,n]} \equiv [w, c'_n]$ $\forall n \in \mathbb{N}$. It is clear that G is well-defined, $G \in end(T, v)$ and $G[0, \infty) \subset im\left(\hat{f}(F[0, \infty))\right)$ proving the existence.

Uniqueness: Let $H \in end(T', w)$ with $H \neq G$ and let $d(H, G) = d_0 > 0$. We are going to show that $H[0, \infty)$ cannot be contained in the image of $F[0, \infty)$ by f (Fig. 8).

Let $M > -ln(d_0)$. As we know, $\exists N_{M,f} > 0$ such that $\hat{f}^{-1}(B(w, M)) \subset B(v, N)$. By Proposition 3.5 there exists a unique $c'_M \in \partial B(w, M)$ such that $f\left(T_{F(N)}\right) \subset T'_{c'_M}$ and it is clear that $c'_M = G(M)$ but since $M > -ln(d_0) = sup\{s/G(s) = H(s)\}$ then $H(M) \neq c'_M$ and $\hat{f}(F[N, \infty)) \cap H[0, \infty) = \emptyset$.

Moreover (T, v), (T', w) are metric spaces and $\hat{f}$ is continuous. Hence $\hat{f}(F[0, N])$ is the continuous image of a compact set and so it is compact in a metric space and it is bounded. Therefore $H[0, \infty) \not\subset \hat{f}(F[0, N])$.

Thus, $H[0, \infty) \not\subset \hat{f}(F[0, \infty))$ and G is unique. □

If $f : (T, v) \to (T', w)$ is a metrically proper map between trees, let us denote by $\tilde{f} : end(T, v) \to end(T', w)$ the map constructed in the previous proposition. Thus $\tilde{f}(F) = G \in end(T', w)$ where $G[0, \infty) \subset f(F[0, \infty))$.

Proposition 6.2.2 *$\tilde{f}$ is uniformly continuous.*

Proof Let $\varepsilon > 0$. Then there exists $\delta > 0$ such that $f^{-1}(B(w, -ln\varepsilon)) \subset B(v, -ln\delta) \Rightarrow f(T \backslash B(v, -ln\delta)) \subset T' \backslash B(w, -ln\varepsilon)$. Once again, the idea of 3.5.

Consider two branches $F, G \in end(T, v)$ with $d(F, G) \leq \delta$, this is, $F(t) = G(t)$ on $[0, -ln\delta]$. If $c = F(-ln\delta) = G(-ln\delta)$ then $f(c) \in T' \backslash B(w, -ln\varepsilon)$ and $\tilde{f}(F) = \tilde{f}(G)$ at least on $[0, -ln\varepsilon]$. Therefore, $d(\hat{f}(F), \hat{f}(G)) \leq \varepsilon$ and $\tilde{f}$ is uniformly continuous. □

Proposition 6.2.3 *If $f, f' : (T, v) \to (T', w')$ are two metrically proper maps between trees, then $f \sim f'$ if and only if $\tilde{f} = \tilde{f}'$. Where $\sim$ represents the relation defined in (1).*

Proof Suppose $f \sim f'$ and $\tilde{f} \neq \tilde{f}'$ (i.e. f and f' do not induce the same map between the end spaces). Then, there exists some $F \in end(T, v)$ such that $\tilde{f}(F) = G \neq H = \tilde{f}'(F)$. If $M > -ln(d(G, H)) > 0$, since f, f' are metrically proper, there exists some $N_{M,f,f'} > 0$ such that $f^{-1}(B(w, M)) \subset B(v, N_{M,f,f'})$ and $f'^{-1}(B(w, M)) \subset B(v, N_{M,f,f'})$. Then, $\forall N > N_{M,f,f'}$, let $c = F(N) \in \partial B(v, N)$ and by 6.2.1 $f_{\mathcal{T}_N}(T_c) = T'_{G(M)} \neq T'_{H(M)} = f'_{\mathcal{T}_N}(T_c)$ which are different because $M > -ln(d(G, H))$. This is a contradiction with $f \sim f'$.

Conversely, suppose that f and f' induce the same map between the end spaces. Since they are metrically proper, see 3.4, $\forall M > 0 \, \exists N_{M,f} > 0$ such that $f(T \backslash B(v, N_{M,f})) \subset T' \backslash B(w, M)$ and $\exists N_{M,f'} > 0$ such that $f(T \backslash B(v, N_{M,f'})) \subset T' \backslash B(w, M)$. If $N_{M,f,f'} := \max\{(N_{M,f}, N_{M,f'})\}$ then $\forall N > N_{M,f,f'}$, we have two maps as we saw in 3.6.

$$f_{\mathcal{T}_N}, f'_{\mathcal{T}_N} : \mathcal{T}_N \longrightarrow \mathcal{T}'_M.$$

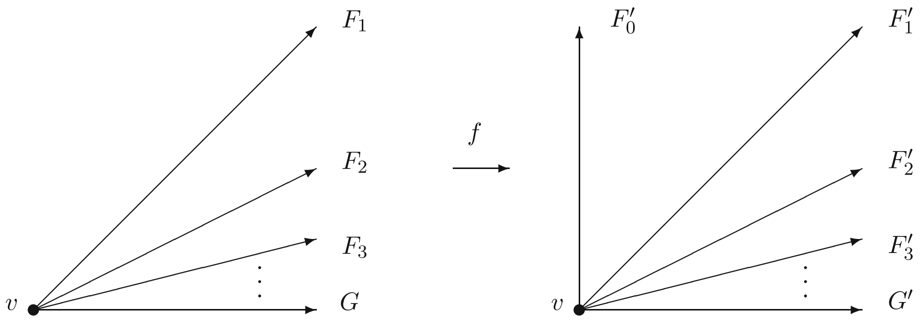


Fig. 9 A surjective metrically proper map between the trees which induces a non-surjective map between the ends

Since the induced map between the end spaces is the same, $\forall\, F \in end(T, v)$ there exists a unique $G \in end(T', w)$ such that $\tilde{f}(F) = G = \tilde{f}'(F)$. Consider $T_{F(N)}$ the subtree of $T \backslash B(v, N)$. It is clear that the image of $F[N, \infty)$ whether by f or f' must be contained in $T'_{G(M)}$ since $G[0, \infty)$ is contained in the image of $F[0, \infty)$. Thus $f_{\mathcal{T}_N}(T_{F(N)}) = T'_{G(M)} = f'_{\mathcal{T}_N}(T_{F(N)})$ and $f \sim f'$. □

This, together with 3.12 gives the following.

Corollary 6.2.4 *If $f, f' : (T, v) \to (T', w')$ are two metrically proper maps between trees, then $f \simeq_{Mp} f'$ if and only if $\tilde{f} = \tilde{f}'$.*

Corollary 6.2.5 *In any class of metrically proper maps between trees there is a representative which is non-expansive.*

Given $f : (T, v) \to (T', w')$ a surjective metrically proper map between trees, the question arises whether the induced map between the end spaces is also surjective. It need not be as the following example shows.

Counterexample. Consider the trees shown in Fig. 9.

Let

$$f(F_n(t)) = \begin{cases} F'_0(t) & \text{if } t \in \left[0, \frac{1}{4}\right], \\ F'_0\left(\frac{1}{4} + (4n-1)\left(t - \frac{1}{4}\right)\right) & \text{if } t \in \left[\frac{1}{4}, \frac{1}{2}\right], \\ F'_0(2n(1-t)) & \text{if } t \in \left(\frac{1}{2}, 1\right], \\ F'_n(t-1) & \text{if } t \in (1, \infty), \end{cases}$$

and

$$f(G(t)) = G'(t).$$

The map f is clearly rooted, continuous, surjective and metrically proper but if we consider the induced map between the end spaces we find that F'_0 is not contained in the image of any branch of T.

7 Equivalence of categories

Consider the categories,

$\mathcal{T}$: Geodesically complete rooted $\mathbb{R}$-trees and rooted metrically proper homotopy classes of metrically proper maps between trees.

$\mathcal{U}$: Complete ultrametric spaces of diameter ≤ 1 and uniformly continuous maps.

Define the functors,

$\xi : \mathcal{T} \longrightarrow \mathcal{U}$ such that $\xi(T, v) = end(T, v)$ for any geodesically complete rooted $\mathbb{R}$-tree and $\xi([f]_{Mp}) = \tilde{f}$ for any rooted metrically proper homotopy class of a metrically proper map between trees.

$\eta : \mathcal{U} \longrightarrow \mathcal{T}$ such that $\eta(U) = T_U$ for any complete ultrametric space of diameter ≤ 1 and $\eta(f) = [\hat{f}]_{Mp}$ for any uniformly continuous map.

Proposition 7.1 $\xi : \mathcal{T} \longrightarrow \mathcal{U}$ *is a functor.*

Proof $\xi(id_{(T,v)}) = id_{end(T,v)}$ is obvious.

If $[f] : (T, v) \to (S, w)$, $[g] : (S, w) \to (R, z)$ represent classes of metrically proper maps between trees then

$$\xi([g] \circ [f]) = \xi([g]) \circ \xi([f]).$$

By 6.2.1, the induced maps between the end spaces are clearly the same. □

Proposition 7.2 $\eta : \mathcal{U} \longrightarrow \mathcal{T}$ *is a functor.*

Proof $\eta(id_U) = \eta(id_{end(T_U)}) = id_{T_U}$ is obvious.

If $f : U_1 \to U_2$, and $g : U_2 \to U_3$ are uniformly continuous maps then

$$\eta(g \circ f) = \eta(g) \circ \eta(f).$$

This follows immediately from 6.2.3 since the maps between the end spaces are the same. □

To prove the equivalence of the categories we use the following lemma from [6].

Definition 7.3 A functor $S : A \to C$ between two categories is *full* if for every pair of objects a, a' in A and every morphism $g : S(a) \to S(a')$ in C, there exists a morphism $f : a \to a$ with $g = S(f)$.

Definition 7.4 A functor $S : A \to C$ between two categories is *faithful* if for every pair of objects a, a' in A and every pair of morphisms $f_1, f_2 : a \to a'$ in A, the equality $S(f_1) = S(f_2) : S(a) \to S(a')$ implies that $f_1 = f_2$.

Lemma 7.5 *Let $S : A \to C$ be a functor between two categories. S is an equivalence of categories if and only if S is $full$, $faithful$ and each object $c \in C$ is isomorphic to $S(a)$ for some object $a \in A$.*

Theorem 7.6 (Main theorem) $\xi : \mathcal{T} \longrightarrow \mathcal{U}$ *is an equivalence of categories.*

Proof ξ *is full*. This is immediate since for every morphism $f \in \mathcal{U}$, $f = \xi(\hat{f}))$.

ξ *is faithful*. This follows immediately from Proposition 6.2.3.

It remains to check that $\forall U \in \mathcal{U}$ $\exists T \in \mathcal{T}$ such that $\xi(T) \approx U$. As we saw in 5.3, $\xi(T_U) \approx U$, finishing the proof. □

Corollary 7.7 *Two geodesically complete rooted $\mathbb{R}$-trees are rooted metrically proper homotopy equivalent if and only if their end spaces are uniformly homeomorphic.*

The following example illustrates the importance of asking the map between the trees to be metrically proper.

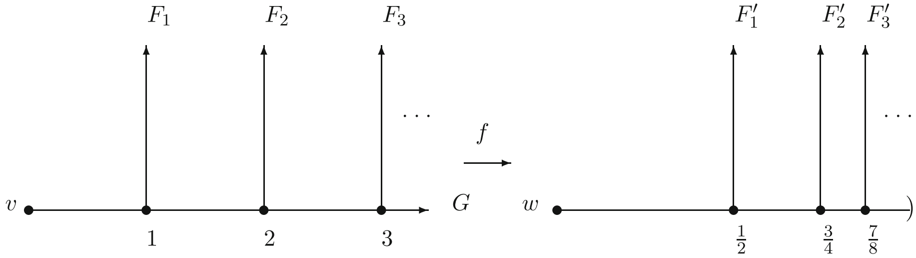


Fig. 10 A homeomorphism between the trees which does not induce a map between the ends

Example 7.8 A homeomorphism between the trees need not induce a map between the ends.

Consider (T, v), (T', w) the geodesically complete rooted $\mathbb{R}$-trees shown in Fig. 10.

We can easily define a homeomorphism between these trees. Let f be such that $f(G[n-1, n]) = F'_n([1 - \frac{1}{2^{n-1}}, 1 - \frac{1}{2^n}])$ linearly $\forall n \in \mathbb{N}$, and an isometry on the rest (the vertical lines) with $f(F_n) = F'_n \ \forall n \in \mathbb{N}$. Then it is obviously a homeomorphism but clearly not uniform since f^{-1} is not uniformly continuous.

Since f is a non-expansive map, f^{-1} is metrically proper and hence, it induces a map $\widetilde{f^{-1}}$ from $end(T', w)$ to $end(T, v)$ which is uniformly continuous but f is not metrically proper (for example $f^{-1}(B(w, 1))$ is not bounded) and it does not induce any map from $end(T, v)$ to $end(T', w)$ since $f(G)$ is not geodesically complete.

f is bornologous but it is not coarse (fails to be metrically proper) and f^{-1} is not bornologous.

Example 7.9 We can define also a homeomorphism f between two rooted geodesically complete $\mathbb{R}$-trees such that $\tilde{f}$ is a non-uniform homeomorphism between the end spaces.

Consider the trees (T, v) and (T', w) in Fig. 11. (T, v) has branches $\{F_i\}_{i=1}^{\infty}$ such that $F_i \cap F_j = \{v\}$, and $\forall i$ there are branches $\{F_{i,k}\}_{k=1}^{\infty}$ such that $F_{i,k} = F_i$ on $[0, k]$. (T', w) is quite similar but $\forall i$ the branches $\{F'_{i,k}\}_{k=1}^{\infty}$ are such that $F'_{i,k} = F'_i$ on $[0, \frac{k}{i}] \ \ \forall k \leq i$ and $F'_{i,k} = F'_i$ on $[0, k-i] \ \ \forall k > i$.

Define $f : (T, v) \to (T', w)$ such that $f(F_i(t)) = F'_i(\frac{t}{i}) \ \forall t \in [0, i]$ and $f(F_i(t)) = F'_i(t - i + 1) \ \forall t \in [i, \infty) \ \forall i \in \mathbb{N}$, and also $f(F_{i,k}(t) = F'_{i,k}(t - i + \frac{k}{i}) \ \forall t \in [i, \infty), \forall k \leq i$ and $f(F_{i,k}(t) = F'_{i,k}(t - i) \ \forall t \in [i, \infty), \forall k > i$. Hence, the induced map between the end spaces $\tilde{f} : end(T, v) \to end(T', w)$ is $\tilde{f}(F_i) = F'_i$ and $\tilde{f}(F_{i,k}) = F'_{i,k} \ \ \forall i, k \in \mathbb{N}$. It is easy to verify that $\tilde{f}$ is a homeomorphism but this homeomorphism is not uniform. Let $\varepsilon < e^{-1}$, and $\forall \delta > 0$ let $N_\delta > 0$ such that $e^{-i} < \delta \ \forall i \geq N$. Then, $\forall i > N_\delta \ d(F_i, F_{i,i}) = e^{-i} < \delta$ and $d(\tilde{f}(F_i), \tilde{f}(F_{i,i})) = d(F'_i, F'_{i,i}) = e^{-1} > \varepsilon$.

Define $g := \widetilde{f^{-1}}$. Then it easy to check that g is uniformly continuous and the induced map $\hat{g}$ is such that $\hat{g}|_{F'[0,\infty)} \to F[0, \infty)$ is an isometric embedding $\forall F' \in end(T', w)$.

Nevertheless, the end spaces of these trees are in fact uniformly homeomorphic, and hence, as it has been proved, there are $f : (T, v) \to (T', w)$, and $f' : (T', w) \to (T, v)$ metrically proper maps between trees such that $f \circ f' \simeq_{Mp} id_{T'}$ and $f' \circ f \simeq_{Mp} id_T$. Let us define $f : (T, v) \to (T', w)$ such that $f(F[0, 1]) = w \ \forall F \in end(T, v)$, $f(F_i(t)) = F'_i(t-1) \ \forall t \in [1, \infty), \forall i \in \mathbb{N}$, $f(F_{i,k}(t)) = F_{i,k+i-1}(t-1) \ \forall t \in [1, \infty), \forall k \geq 2$ and finally $f(F_{\frac{i \cdot (i-1)}{2}+k,1}) = F'_{i,k} \ \forall k \leq i, \forall i \in \mathbb{N}$. The induced map $\tilde{f}$ between the end spaces is a uniform homeomorphism and therefore, if $h = \tilde{f}^{-1}$, it suffices to define $f' := \hat{h}$.

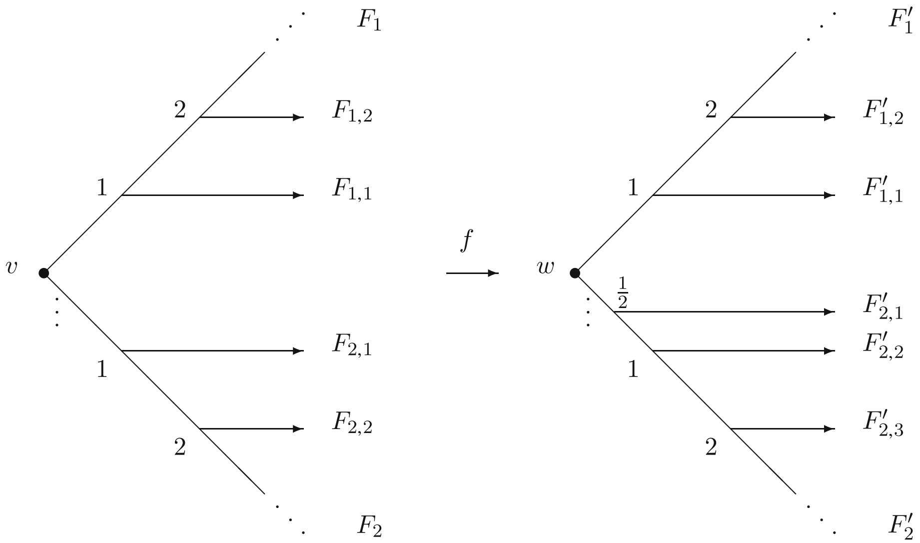


Fig. 11 A homeomorphism between the trees which induces a non-uniform homeomorphism between the ends

8 Lipschitz maps and coarse maps between trees

Lemma 8.1 *If* $x_1, x_2, y_1, y_2 \in \mathbb{R}$*, then for any* $t \in [0, 1]$*,*

$$d(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) \leq \max\{d(x_1, y_1), d(x_2, y_2)\}.$$

Proof $d(tx_1 + (1-t)x_2, ty_1 + (1-t)y_2) = |tx_1 + (1-t)x_2 - [ty_1 + (1-t)y_2]| = |t(x_1 - y_1) + (1-t)(x_2 - y_2)| \leq t \cdot |x_1 - y_1| + (1-t) \cdot |x_2 - y_2| \leq \max\{d(x_1, y_1), d(x_2, y_2)\}$. □

Lemma 8.2 *If* $f, g : T \to T'$ *are two metrically proper maps between trees and* $H : T \times I \to T'$ *is the shortest path homotopy of* f *to* g*, then for any two points* $x, y \in T$*,*

$$d(H_t(x), H_t(y)) \leq \max\{d(f(x), f(y)), d(g(x), g(y))\}.$$

Proof Suppose $d(f(x), f(y)) < d(g(x), g(y))$. If for some $t \in I\, d(H_t(x), H_t(y)) > d(g(x), g(y))$ then there must be some $t_0 > t \in I$ such that $d(H_{t_0}(x), H_{t_0}(y)) = d(g(x), g(y))$. So let us assume $d(f(x), f(y)) = d(g(x), g(y)) = d_0$ and it suffices to show that in this case the condition is satisfied.

Now if we show that in this conditions there is always some $\varepsilon > 0$ such that for any $0 < t < \varepsilon$ or $1 - \varepsilon < t < 1 \quad d(H_t(x), H_t(y)) \leq d_0$, then we have that this happens for any t in an open set of I and by continuity of the metric, this will be also a closed set of I and hence, $d(H_t(x), H_t(y)) \leq d_0\ \forall t \in I$.

Now to prove the lemma it suffices to distinguish the following cases.

Case 1. If there is an arc on the tree containing $f(x), f(y), g(x)$ and $g(y)$, then we can apply Lemma 8.1.

Case 2. If there is an arc containing three of the points (and no one containing all). Consider the subtree given by the points and the minimal paths between them. In this subtree there is a unique point, b, of order 3 and either $f(x), f(y)$ or $g(x), g(y)$ are border points. Let us suppose $f(x)$, $f(y)$ are in the border. If $\delta < \min\{d(f(x), b), d(f(y), b)\}$, defining $\varepsilon := \frac{\delta}{d_0}$ it is immediate to check that $d(H_t(x), H_t(y)) \leq d_0\ \forall\, t < \varepsilon$.

Case 3. If there is no arc containing three of the points. In this case, the subtree has $f(x), f(y), g(x), g(y)$ as border points and there are either one or two branching points different from $\{f(x), f(y), g(x), g(y)\}$: two points of order three or one point of order four. Let δ less than the distance from any of the border points to a branching point. Thus, defining $\varepsilon := \frac{\delta}{d_0}$ it is immediate that $d(H_t(x), H_t(y)) \leq d_0\ \forall\, t < \varepsilon$. □

Definition 8.3 Two maps are *rooted metrically proper non-expansive homotopic*, $f \simeq_L f'$, if there exists $H : T \times I \to T'$ a rooted metrically proper homotopy of f to f' such that H_t is non-expansive for every $t \in I$.

Definition 8.4 Two maps are *rooted coarse homotopic*, $f \simeq_C f'$ if there exists $H : T \times I \to T'$ a rooted (metrically proper) homotopy of f to f' such that H_t is coarse for every $t \in I$. Being metrically proper is already supposed by definition of coarse.

The next propositions follow immediately from the lemma and Proposition 6.2.3.

Proposition 8.5 *If $f, f' : T \to T'$ are two non-expansive metrically proper maps between trees, then $\tilde{f} = \tilde{f}'$ if and only if $f \simeq_L f'$.*

Corollary 8.6 *There is an equivalence of categories between $\mathcal{U}$ and the category of geodesically complete rooted $\mathbb{R}$-trees with rooted metrically proper non-expansive homotopy classes of non-expansive, metrically proper maps between trees.*

Corollary 8.7 *Two geodesically complete rooted $\mathbb{R}$-trees are rooted metrically proper non-expansive homotopy equivalent if and only if their end spaces are uniformly homeomorphic.*

Corollary 8.8 *There is an equivalence of categories between $\mathcal{U}$ and the category of geodesically complete rooted $\mathbb{R}$-trees with rooted coarse homotopy classes of rooted, coarse and continuous maps.*

Corollary 8.9 *Two geodesically complete rooted $\mathbb{R}$-trees are rooted coarse homotopy equivalent if and only if their end spaces are uniformly homeomorphic.*

Proposition 8.10 *If $f, f' : T \to T'$ are two coarse metrically proper maps between trees, then $\tilde{f} = \tilde{f}'$ if and only if $f \simeq_C f'$.*

Corollary 8.11 *There is an equivalence of categories between $\mathcal{U}$ and the category of geodesically complete rooted $\mathbb{R}$-trees with coarse, (metrically proper) homotopy classes of coarse, (metrically proper) maps between trees.*

Corollary 8.12 *If $f : T \to T'$ is a metrically proper map between trees, then there exists a rooted continuous metrically proper non-expansive map $f' : T \to T'$ such that $f \simeq_{Mp} f'$.*

Corollary 8.13 *If $f : T \to T'$ is a rooted continuous coarse map between trees, then there exists a rooted continuous metrically proper non-expansive map $f' : T \to T'$ such that $f \simeq_C f'$.*

9 Freudenthal ends and classical results

This work allows us to give some new proofs of already known results and to look at them from a new perspective. We also extend in this section the field of our study to include some considerations about non-rooted and non-geodesically complete trees and how can we use or adapt our tools with them.

Pruning the tree When we have a non-geodesically complete rooted $\mathbb{R}$-tree and we are only interested in the geodesically complete branches we can prune the rest as follows.

Theorem 9.1 *If (T, v) is a rooted $\mathbb{R}$-tree then, there exists $(T_\infty, v) \subset (T, v)$ a unique geodesically complete subtree that is maximal.*

Proof Using Zorn's lemma. Consider $(\mathcal{T}_{gc}, \leq)$ with $\mathcal{T}_{gc}$ geodesically complete subtrees of (T, v) containing the root, and $T_1 \leq T_2 \Leftrightarrow T_1 \subset T_2$. This is an ordered structure.

It is not empty since the root is a trivial geodesically complete subtree.

To prove that every chain of $(\mathcal{T}_{gc}, \leq)$ admits an upper bound T_M it suffices to show that the union of elements of the chain is also a geodesically complete subtree of (T, v). It is a subset of the tree where every point is arcwise connected to the root and hence it is obviously a subtree. Let $f : [0, t] \to T_M, t > 0$ any isometric embedding such that $f(0) = v$, then there exists an element T_0 in the chain such than $f(t) \in T_0 \Rightarrow f[0, t] \in T_0$ and f extends to an isometric embedding $\tilde{f} : [0, \infty) \to T_0 \subset T_M$, and hence, T_M is geodesically complete.

Then, by Zorn's lemma $(\mathcal{T}_{gc}, \leq)$ possesses a maximal element.

The union of two elements of $(\mathcal{T}_{gc}, \leq)$ is also a geodesically complete subtree and hence, the maximal element (T_∞, v) is unique. □

Lemma 9.2 *If the metric of (T_∞, v) is proper then it is a deformation retract of (T, v).*

Proof Since the metric is proper, for any $x \in T\backslash T_\infty$ there is a point $y \in T_\infty$ such that $d(x, T_\infty) = d(x, y)$ and it is unique since the tree is uniquely arcwise connected. Let $r : T \to T_\infty$ such that $r(x) = y \; \forall x \in T\backslash T_\infty$ and the identity on T_∞. Then r is a retraction and the shortest path homotopy makes the deformation retract. □

Proper homotopies and Freudenthal ends

Definition 9.3 Two proper maps $f, g : X \to Y$ are *properly homotopic* $f \simeq_p g$ in the usual sense if there exists a homotopy $H : X \times I \to Y$ of f to g such that H is proper.

Definition 9.4 X, Y are of the same proper homotopy type or *properly homotopy equivalent* in the usual sense if there exist two proper maps $f : X \to Y$ and $g : Y \to X$ such that $g \circ f \simeq_p Id_X$ and $f \circ g \simeq_p Id_Y$.

Notation $\simeq_{RP}$ means properly homotopic such that the proper maps and the homotopy are rooted, and $\simeq_p$ is the usual sense of proper homotopy equivalence.

Lemma 9.5 *Let S_1, S_2 two locally finite simplicial trees and consider any two points $x_1 \in S_1, x_2 \in S_2$. Then $(S_1, x_1) \simeq_{RP} (S_2, x_2)$ if and only if $S_1 \simeq_p S_2$.*

Proof The only if part is clear since it is a particular case.

The other part is rather technical. Consider $f : S_1 \to S_2$ and $g : S_2 \to S_1$ proper maps and proper homotopies H^1 of $g \circ f$ to Id_{S_1} and H^2 of $f \circ g$ to Id_{S_2}. First we construct two rooted proper maps 'near' f and g. Consider the unique arc in S_2 $[x_2, f(x_1)]$ and, in order to define the rooted proper map from S_1 to S_2, we are going to send this arc with a proper homotopy to the root x_2 and to pull somehow the rest of the tree after it.

Since $[x_2, f(x_1)]$ is compact and the tree is locally finite, there are finitely many vertices $v_1, \ldots, v_n$ in this arc. Let us denote also $f(x_1) = v_0$. The tree is locally compact, hence consider $\overline{B}(v_i, \varepsilon_i)$ compact neighborhoods of v_i with $i = 0, \ldots, n$ (we may assume that they are disjoint).

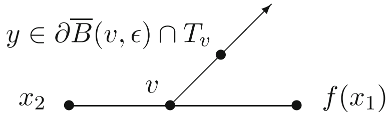


Fig. 12 The homotopy sends $[x_2, f(x_1)]$ to x_2 and $[v, y]$ to $[x_2, y]$

We define a homotopy H that sends $[x_2, f(x_1)]$ to x_2 (Fig. 12), that for each point $y \in T_{v_i} \cap \partial\overline{B}(v_i, \varepsilon_i)$, with $y \notin [x_2, f(x_1)]$, goes linearly from the arcs $[v_i, y]$ to $[x_2, y]$, and it is the identity for every t on the rest. If $x \in [x_2, f(x_1)]$ and $j_x : [0, d(x_2, x)] \to [x_2, x]$ is an isometry with $j_x(0) = x$ then let $H(x,t) = j_x(t \cdot d(x_2, x))$. If $x \in T_{v_i} \cap \overline{B}(v_i, \varepsilon_i)$ such that $x \notin [x_2, f(x_1)]$, then for $j_x : [0, d(x_2, x)] \to [x_2, x]$ an isometry such that $j_x(0) = x$ let $H(x,t) = j_x\left(t \cdot \left[\frac{d(v_i,x_2)+\varepsilon_i}{\varepsilon_i}(\varepsilon_i - d(x, v_i)) - (\varepsilon_i - d(x,v))\right]\right) = j_x\left(t \cdot \frac{d(v_i,x_2)}{\varepsilon_i}(\varepsilon_i - d(x,v))\right)$.

It is easy to check that $H(v_i \times I) = [x_2, v_i]$ with $H(v_i, 0) = v_i$ and $H(v_i, 1) = x_2$, and $\forall\, y \in \partial\overline{B}(v_i, \varepsilon_i) \cap T_{v_i}$ with $y \notin [x_2, f(x_1)]$ $H(y,t) = y\ \forall t$. $H(x,t) = x$ on the rest of the tree. This map is continuous. To see that it is proper first consider $K_0 := [x_2, f(x_1)] \cup (\bigcup_{i=1}^{n} \overline{B}(v_i, \varepsilon_i))$ which is a compact subset of the tree S_2, and hence $K_0 \times I$ is a compact subset of $S_2 \times I$. For any compact set $K \in S_2$, $H^{-1}(K)$ is a closed (since H is continuous) subset of the compact set $K_0 \cup K$. Thus, H is proper.

Clearly $f(x) = H(x,0)$ and let $\tilde{f}(x) := H(x,1)$. The map $\tilde{f}$ is proper, $\tilde{f}(x_1) = x_2$ (it is rooted) and $f \simeq_p \tilde{f}$.

We do the same for $g : S_2 \to S_1$ and we get a rooted proper map $\tilde{g} : (S_2, x_2) \to (S_1, x_1)$ such that $g \simeq_p \tilde{g}$.

Hence we have a proper homotopy H^1 of $\tilde{g} \circ \tilde{f}$ to Id_{S_1} (conversely H^2 of $\tilde{f} \circ \tilde{g}$ to Id_{S_2}) such that $H^1(x_1, 0) = x_1 = H^1(x_1, 1)$.

To finish the proof we need this homotopy to be rooted.

Since γ is compact and S_1 is locally finite, and hence locally compact, let $\epsilon > 0$ such that $\bar{B}(x_1, \varepsilon)$ is compact. Then consider $\tilde{H} : S_1 \times I \to S_1$ such that $\tilde{H}(x,t) = H(x,t)\ \forall\,(x,t) \in (S_1 \backslash \overline{B}(x_1, \varepsilon)) \times I$ and $\forall\,(x,t) \in S_1 \times \{0,1\}$, and in the rest of the domain we change H to make it rooted. To do this, the points $H(x_1, t)$ must be sent to x_1 and pull the image of the points in $\overline{B}(x_1, \varepsilon)$ to keep continuity. Also, notice that H_0 and H_1 must not change. Therefore, let us define $d_t = d(H(x_1, t), x_1)$ to change the homotopy at each level proportionally.

Consider, for each $(x,t) \in \overline{B}(x_1, \varepsilon) \times I$,

$$d_{x,t} := \min\left\{\left(1 - \frac{d(x, x_1)}{\varepsilon}\right) \cdot d_t,\ d(x_1, H(x,t))\right\}$$

and for the isometric embedding $j_{x,t} : [0, d(x_1, H(x,t))] \to [x_1, H(x,t)]$ with $j_{x,t}(0) = H(x,t)$ let

$$\tilde{H}(x,t) = j_{x,t}(d_{x,t}).$$

It is immediate to check that this makes $\tilde{H}$ a rooted homotopy of $\tilde{g} \circ \tilde{f}$ to Id_{S_1}.

It remains to see that it is proper but for any compact set $K \in S_1$, $\tilde{H}^{-1}(K) \subset H^{-1}(K) \cup \bar{B}(x_1, \varepsilon)$ and then $\tilde{H}$ is also proper.

The same works with H^2 and we finally obtain that $(S_1, x_1) \simeq_{RP} (S_2, x_2)$. □

We can now give another proof of the following corollary in [1].

Proposition 9.6 *Two locally finite simplicial trees are properly homotopy equivalent (in the usual sense) if and only if their Freudenthal ends are homeomorphic.*

Proof Let S_1, S_2 two simplicial, locally finite trees. Let $v \in S_1$ and $w \in S_2$ any two points, hence (S_1, v) and (S_2, w) are two rooted trees, and by Lemma 9.5 $(S_1, v) \simeq_{RP} (S_2, w)$ if and only if $S_1 \simeq_p S_2$.

We can change the metric on the simplices and assume length 1 for each simplex. Then we have two homeomorphic copies of the simplicial rooted trees $(S_1', v) \cong (S_1, v)$ and $(S_2', w) \cong (S_2, w)$ (in particular $(S_1', v) \simeq_{RP} (S_1, v)$ and $(S_2', w) \simeq_{RP} (S_2, w)$), such that the non-compact branches are geodesically complete.

The metrics on (S_1', v) and (S_2', w) are proper. It suffices to check that any closed ball centered at the root is compact and this can be easily done by induction on the radius. Since the trees are locally finite and the distance between two vertices is at least 1, the closed ball $\overline{B}(v, 1)$ (similarly $\overline{B}(w, 1)$) is a finite union of compact sets (isometric to the subinterval $[0, 1]$ in $\mathbb{R}$). Let $\overline{B}(v, n)$ a finite union of compact sets, $\partial\overline{B}(v, n)$ is a finite number of vertices and, since the trees are locally finite and the distance between two vertices is at least 1, $\overline{B}(v, n+1)$ is also a finite union of compact sets. Thus every closed ball centered at the root is compact.

(S_1', v) and (S_2', w) are proper length spaces, and by the Hopf–Rinow theorem for metric spaces, see [10], (S_1', v) and (S_2', w) are complete and locally compact.

Now consider the maximal geodesically complete subtrees (T_1, v) and (T_2, w) of (S_1', v) and (S_2', w) (Note that these are empty sets if and only if (S_1, v) and (S_2, w) are compact). These trees are locally finite, complete, geodesically complete and their metrics are proper. We can now find a proper homotopy equivalence between the pruned tree T_i and S_i'. The retractions $r_i : (S_i', v) \to (T_i, v)$, $i = 1, 2$, such that $r_i(x) = y$ with $d(x, T_i) = d(x, y)$ defined in Lemma 9.2 are proper maps since after the change of metric the bounded branches are compact and the tree is supposed to be locally finite. Clearly this retraction and the inclusion give us rooted proper homotopy equivalences between the trees, $(S_1', v) \simeq_{RP} (T_1, v)$ and $(S_2', w) \simeq_{RP} (T_2, w)$. Thus

$$(S_1, v) \simeq_{RP} (T_1, v) \quad \text{and} \quad (S_2, w) \simeq_{RP} (T_2, w)$$

It is well known, see for example 9.20 in [1], that since S_i' are locally finite simplicial trees, $end(T_1, v) = Fr(S_1', v) = Fr(S_1)$ and $end(T_2, w) = Fr(S'2, w) = Fr(S_2)$ and, by 7.7, $end(T_1, v) \cong end(T_2, w) \Leftrightarrow (T_1, v) \simeq_{Mp} (T_2, w)$. If the metric is proper $(T_1, v) \simeq_{Mp} (T_2, w) \Leftrightarrow (T_1, v) \simeq_{RP} (T_2, w)$ and hence $Fr(S_1) \cong Fr(S_2) \Leftrightarrow (T_1, v) \simeq_{RP} (T_2, w)$.

Thus, $Fr(S_1) \cong Fr(S_2) \Leftrightarrow (S_1, v) \simeq_{RP} (S_2, w) \Leftrightarrow S_1 \simeq_p S_2$. □

There is also an immediate proof of the following corollary in [4].

Proposition 9.7 *Two geodesically complete rooted $\mathbb{R}$-trees, (T, v) and (S, w), are rooted isometric if and only if $end(T, v)$ and $end(S, w)$ are isometric.*

Proof If there is an isometry between the trees then the induced map between their end spaces is clearly an isometry.

Let $f : end(T, v) \to end(S, w)$ be an isometry between the end spaces. Then, to induce the map between the trees we can take the identity as modulus of continuity. If $\lambda_f = Id_{[0,1]}$ then $f(F)(-ln(\lambda_f(e^{-t}))) = f(F)(t) \forall F \in end(T, v) \forall t \in [0, \infty)$ and the map restricted to the branches is an isometry. Consider any two points in different branches $x = F(t)$, $y = G(t')$ with $-ln(d(F, G)) < t, t'$. Since the end spaces are isometric, the distance between two branches is the same between their images and hence $d(\hat{f}(x), \hat{f}(y)) = t + t' - 2(-ln(d(f(F), f(G)))) = t + t' - 2(-ln(d(F, G))) = d(x, y)$ and $\hat{f}$ is an isometry between the trees. □

Non-rooted maps between the trees If the map is not rooted we can extend the idea of the rooted case and define how a non-rooted metrically proper map induces a map between the end spaces.

Let $f : (T, v) \to (T', w)$ be any metrically proper (non-rooted) map between two geodesically complete rooted $\mathbb{R}$-trees. Then $\forall M > 0 \ \ \exists N > 0$ such that $f(B(v, N)) \subset B(f(v), M)$. If $d_0 := d(w, f(v))$, hence $f(B(v, N)) \subset B(w, M + d_0)$ and this is equivalent to $f^{-1}(T' \backslash B(w, M + d_0)) \subset T \backslash f(B(v, N))$. Now we can induce a uniformly continuous map between the end spaces almost like in 6.2.1, since for each branch $F \in (T, v)$ there is a unique branch $F' \in (T', w)$ such that $F'[d_0, \infty) \subset f(F)$ and so we define $\tilde{f} : end(T, v) \to (T', w)$ such that $\tilde{f}(F) = F'$.

The results then are not so strong, as an example of this we can give the following proposition.

Proposition 9.8 *An isometry (non-rooted) $f : (T, v) \to (S, w)$ between two geodesically complete rooted $\mathbb{R}$-trees, induces a bi-Lipschitz homeomorphism between $end(T, v)$ and $end(S, w)$.*

Proof Let $f : (T, v) \to (S, w)$ be a non-rooted isometry. Consider F, G any two branches in $end(T, v)$ and let $x \in T$ such that $F[0, \infty) \cap G[0, \infty) = [v, x] \approx [0, -ln(d(F, G))] \subset \mathbb{R}$. Then $f(F[0, \infty)) \cap f(G[0, \infty)) = [f(v), f(x)] \approx [0, -ln(d(F, G))] \subset \mathbb{R}$ since f is an isometry. If $d_0 = d(w, f(v))$, hence $\tilde{f}(F) =: F'$ and $\tilde{f}(G) =: G'$ coincide at least on $[0, -ln(d(F, G)) - d_0]$ and at most on $[0, -ln(d(F, G)) + d_0]$ and so, $e^{ln(d(F,G)) - d_0} \leq d(F', G') \leq e^{ln(d(F,G)) + d_0}$, this is, $e^{-d_0} \cdot d(F, G) \leq d(F', G') \leq e^{d_0} \cdot d(F, G)$ and therefore $\tilde{f}$ is bi-Lipschitz. □

Acknowledgments The authors would like to thank the referee for his careful reading and the helpful comments which significantly improved this manuscript.

References

1. Baues, H.J., Quintero, A.: Infinite Homotopy Theory. *K*-Monographs in Mathematics, no. 6. Kluwer, Boston (2001)
2. Bestvina, M.: $\mathbb{R}$-trees in Topology, Geometry and Group Theory. Handbook of geometric topology, pp. 55–91. North-Holland, Amsterdam (2002)
3. Borsuk, K.: On some metrization of the hyperspace of compact sets. Fund. Math. **41**, 168–202 (1954)
4. Hughes, B.: Trees and ultrametric spaces: a categorical equivalence. Adv. Math. **189**, 148–191 (2004)
5. Hughes, B., Ranicki: Ends of Complexes. Cambridge Tracts in Mathematics, vol. 123. Cambridge University Press, Cambridge (1996)
6. Mac Lane, S.: Categories for the Working Mathematician. Springer, New York (1971)
7. Morgan, J.W.: Λ-trees and their applications. Bull. Am. Math. Soc. **26**(1), 87–112 (1992)
8. Morón, M.A., Ruiz del Portal, F.R.: Shape as a Cantor completion process. Mathematische Zeitschrift **225**, 67–86 (1997)
9. Robert, A.M.: A Course in p-adic Analysis. Grad. Text Math., vol. 198. Springer, New York (2000)
10. Roe, J.: Lectures on Coarse Geometry. University Lecture Series, vol. 31. American Mathematical Society, Providence (2003)
11. Roe, J.: Coarse Cohomology and Index Theory on Complete Riemannian Manifolds. Memoirs of the American Mathematical Society, vol. 104, no. 497 (1993)
12. Serre, J.P.: Trees. Springer, New York (1980)