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Classification of solutions to the higher order Liouville's equation on $\mathbb{R}^{2m}$

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Abstract We classify the solutions to the equation $(-\Delta)^m u = (2m-1)!e^{2mu}$ on $\mathbb{R}^{2m}$ giving rise to a metric $g = e^{2u} g_{\mathbb{R}^{2m}}$ with finite total Q-curvature in terms of analytic and geometric properties. The analytic conditions involve the growth rate of u and the asymptotic behaviour of Δu at infinity. As a consequence we give a geometric characterization in terms of the scalar curvature of the metric $e^{2u} g_{\mathbb{R}^{2m}}$ at infinity, and we observe that the pull-back of this metric to S^{2m} via the stereographic projection can be extended to a smooth Riemannian metric if and only if it is round.

1 Introduction and statement of the main theorems

The study of the Paneitz operators has moved into the center of conformal geometry in the last decades, in part with regard to the problem of prescribing the Q-curvature. Given a four-dimensional Riemannian manifold (M, g), the Q-curvature Q_g^4 and the Paneitz operator P_g^4 have been introduced by Branson and Oersted [4] and Paneitz [18]:

$$Q_g^4 := -\frac{1}{6}\left(\Delta_g R_g - R_g^2 + 3|\operatorname{Ric}_g|^2\right)$$
$$P_g^4(f) := \Delta_g^2 f + \operatorname{div}\left(\frac{2}{3}R_g g - 2\operatorname{Ric}_g\right) df, \quad \forall f \in C^\infty(M),$$

where R_g and Ric_g denote the scalar and Ricci curvatures of g. Higher order Q-curvatures Q^n and Paneitz operators P^n have been introduced in [2,14]. Their interest lies in their covariant nature: considering in dimension $2m$ the conformal metric $g_u := e^{2u} g$, we have

$$P_{g_u}^{2m} = e^{-2mu} P_g^{2m}, \quad P_g^{2m} u + Q_g^{2m} = Q_{g_u}^{2m} e^{2mu}, \tag{1}$$

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see for instance [6, Chap. 4]. The last identity is a generalized version of Gauß's identity: in dimension 2

$$-\Delta_g u + K_g = K_{g_u} e^{2u},$$

where K_g is the Gaussian curvature, and Δ_g is the Laplace–Beltrami operator with the analysts' sign. Indeed, in dimension 2 we have $P_g^2 = -\Delta_g$ and $Q_g^2 = K_g$. Moreover $\Delta_{g_u} = e^{-2u}\Delta_g$. Another interesting fact is that the total Q-curvature is a global conformal invariant: if M is closed and $2m$-dimensional,

$$\int_M Q_{g_u}^{2m} \mathrm{dvol}_{g_u} = \int_M Q_g^{2m} \mathrm{dvol}_g.$$

Further evidence of the geometric relevance of the Q-curvatures is given by the Gauss–Bonnet–Chern's theorem [10]: on a locally conformally flat closed manifold of dimension $2m$, since Q_g^{2m} is a multiple of the Pfaffian plus a divergence term (see [3]), we have

$$\int_M Q_g^{2m} \mathrm{dvol}_g = (2m-1)!\,\mathrm{vol}(S^{2m}) \frac{\chi(M)}{2},$$

where $\chi(M)$ is the Euler–Poincaré characteristic of M.

Here we are interested in the special case when M is $\mathbb{R}^{2m}$ with the Euclidean metric $g_{\mathbb{R}^{2m}}$. In this case, we simply have $P^{2m}_{g_{\mathbb{R}^{2m}}} = (-\Delta)^m$ and $Q^{2m}_{g_{\mathbb{R}^{2m}}} \equiv 0$. We consider solutions to the equation

$$(-\Delta)^m u = (2m-1)! e^{2mu} \quad \text{on } \mathbb{R}^{2m}, \tag{2}$$

satisfying $\int_{\mathbb{R}^{2m}} e^{2mu} dx < \infty$. From the above remarks and (1) in particular, it follows that (2) has the following geometric meaning: if u solves (2), then the conformal metric $g := e^{2u} g_{\mathbb{R}^{2m}}$ has Q-curvature $Q_g^{2m} \equiv (2m-1)!$. As we shall see, every solution to (2) with $e^{2mu} \in L^1_{\mathrm{loc}}(\mathbb{R}^{2m})$ is smooth (Corollary 8).

Given such a solution u, define the auxiliary function

$$v(x) := \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|}\right) e^{2mu(y)} dy, \tag{3}$$

where γ_m is defined by the following property: $(-\Delta)^m \left(\frac{1}{\gamma_m} \log \frac{1}{|x|}\right) = \delta_0$ in $\mathbb{R}^{2m}$, see Proposition 22 below. Then $(-\Delta)^m v = (2m-1)! e^{2mu}$. We prove

Theorem 1 *Let u be a solution of (2) with*

$$\alpha := \frac{1}{|S^{2m}|} \int_{\mathbb{R}^{2m}} e^{2mu(x)} dx < +\infty. \tag{4}$$

Then

$$u(x) = v(x) + p(x), \tag{5}$$

where p is a polynomial of even degree at most $2m-2$, v is as in (3) and

$$\sup_{x \in \mathbb{R}^{2m}} p(x) < +\infty,$$

$$\lim_{|x|\to\infty} \Delta^j v(x) = 0, \quad j = 1, \ldots, m-1,$$

$$v(x) = -2\alpha \log|x| + o(\log|x|), \ \textit{as } |x| \to +\infty.$$

It is well known that the function

$$u(x) := \log \frac{2\lambda}{1 + \lambda^2|x - x_0|^2} \tag{6}$$

solves (2) and (4) with $\alpha = 1$ for any $\lambda > 0$, $x_0 \in \mathbb{R}^{2m}$. We call the functions of the form (6) *standard solutions*. They all arise as pull-back under the stereographic projection of metrics on S^{2m} which are round, i.e. conformally diffeomorphic to the standard metric. Chang and Yang [8] proved that the round metrics are the only metrics on S^{2m} having Q-curvature identically equal to $(2m - 1)!$.

In the next theorem, we give conditions under which an entire solution of Liouville's equation satisfying (4) is necessarily a standard solution.

Theorem 2 *Let u be a solution of* (2) *satisfying* (4)*. Then the following are equivalent*:

(i) *u is a standard solution,*
(ii) $\lim_{|x|\to\infty} \Delta u(x) = 0$
(ii') $\lim_{|x|\to\infty} \Delta^j u(x) = 0$ *for* $j = 1, \dots, m-1$,
(iii) $u(x) = o(|x|^2)$ *as* $|x| \to \infty$,
(iv) $\deg p = 0$, *where* p *is the polynomial in* (5).
(v) $\liminf_{|x|\to+\infty} R_{g_u} > -\infty$, *where* $g_u = e^{2u} g_{\mathbb{R}^{2m}}$.
(vi) $\pi^* g_u$ *can be extended to a Riemannian metric on* S^{2m}, *where* $\pi : S^{2m} \to \mathbb{R}^{2m}$ *is the stereographic projection.*

Moreover, if u is not a standard solution, there exist $1 \le j \le m-1$ *and a constant* $a < 0$ *such that*

$$\Delta^j u(x) \to a \quad \text{as } |x| \to +\infty. \tag{7}$$

The two-dimensional case ($m = 1$) of Theorem 2 was treated by Chen and Li [9], who proved that *every* solution with finite total Gaussian curvature is a standard one. The four-dimensional case was treated by Lin [15], with a classification of u in terms of its growth, or of the behaviour of Δu at ∞. The classification of C-S. Lin in terms of Δu was used by Robert and Struwe [20] to study the blow-up behaviour of sequences of solutions u_k to

$$\begin{cases} \Delta^2 u_k = \lambda u_k e^{32\pi^2 u_k^2} & \text{in } \Omega \subset \mathbb{R}^4 \\ u_k = \frac{\partial u_k}{\partial n} = 0 & \text{on } \partial\Omega, \end{cases}$$

and by Malchiodi [16] to show a compactness criterion for sequences of solutions u_k to the equation

$$P_g^4 u_k + Q_k^4 = h_k e^{4u_k}, \quad h_k \text{ constant}$$

on a closed 4-manifold. The same criterion could be used in higher dimension in the proof of an analogous compactness result. This was observed by Ndiaye [17], who then used a different technique to show compactness. We will discuss this in a forthcoming paper.

In higher dimension ($m > 2$), Wei and Xu [23] (see also [25]) treated a special case of Theorem 2: if $u(x) = o(|x|^2)$ at infinity, then u is always a standard solution. This result is not sufficient to prove compactness. Moreover, the proof appears to be overly simplified. For instance, in their Lemma 2.2 the argument for showing that $u \le C$ is not conclusive, and in the crucial Lemma 2.4 they simply refer to [15] for details. This latter lemma corresponds to Lemma 13 here and it is the main regularity result, as it implies that $u \le C$, hence that the right-hand side of (2) belongs to $L^\infty(\mathbb{R}^{2m})$. Its generalization is a major issue, because Lin's analysis is focused on the function Δu, and it makes use of the Harnack's inequality and of the fact that $\Delta(u - v) \equiv C$. In the general case, Harnack's inequality does not work and there are no uniform bounds for $\Delta^{(m-2)}(u - v)$ (while it is still true that $\Delta^{(m-1)}(u - v) \equiv C$). To overcome this difficulties, we spend a few pages in the following section to study polyharmonic functions. As a reward we obtain a Liouville-type theorem for polyharmonic functions (Theorem 6) which allows us to make the proof of [15] more direct and transparent.

The characterization in terms of the scalar curvature at infinity is new and quite interesting, as it shows that non-standard solutions have a geometry essentially different from standard solutions, and it also shows that the Q-curvature and the scalar curvature are independent of each other in dimension 4 and higher. On the other hand, since in dimension 2 we have $2Q_g = R_g$, (v) is consistent with the result of [9].

The characterization in (vi) implies the result of Chang and Yang [8] described above, which here follows from the general case.

The paper is organized as follows. In Sect. 2 we collect some relevant results about polyharmonic functions which will be needed later. Section 3 contains the proof of Theorems 1 and 2; at the end of the paper we give examples to show that the hypothesis of Theorem 2 are sharp in terms of the growth at infinity and of the degree of p. Recently Wei and Ye [24] proved that already in dimension 4 there is a great abundance of non-radially symmetric solutions.

In the following, the letter C denotes a generic constant, which may change from line to line and even within the same line.

2 A few remarks on polyharmonic functions

We briefly recall some properties of polyharmonic functions, which will be used in the sequel. For the standard elliptic estimates for the Laplace operator, we refer to [11] or [12]. The next lemma can be considered a generalized mean value inequality. We give the short proof for the convenience of the reader, and because identity (12) will be used in the next section.

Lemma 3 (Pizzetti [19]) *Let $\Delta^m h = 0$ in $B_R(x_0) \subset \mathbb{R}^n$, for some m, n positive integers. Then*

$$\fint_{B_R(x_0)} h(z)dz = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h(x_0), \tag{8}$$

where

$$c_0 = 1, \quad c_i = \frac{n}{n+2i} \frac{(n-2)!!}{(2i)!!(2i+n-2)!!}, \quad i \geq 1. \tag{9}$$

Proof We can translate and assume that $x_0 = 0$. We first prove by induction on m that there are constants $b_0^{(m)}, \dots, b_{m-1}^{(m)}$ such that

$$\fint_{\partial B_r} h(z)dS = \sum_{i=0}^{m-1} b_i^{(m)} r^{2i} \Delta^i h(0), \quad 0 < r < R, \ B_r := B_r(0). \tag{10}$$

For $m = 1$ this reduces to the mean value theorem for harmonic functions. Assume now that the assertion has been proved up to $m - 1$, and that $\Delta^m h = 0$. Let G_r be the Green function of Δ^m in B_r:

$$\Delta^m G_r = \delta_0 \quad \text{in } B_r, \quad G_r = \Delta G_r = \dots = \Delta^{m-1} G_r = 0 \quad \text{on } \partial B_r. \tag{11}$$

For simplicity, let us only consider the case $n = 2m$. Then $G_r(x) = G_1\left(\frac{x}{r}\right)$,

$$G_1(x) = \alpha_0 \log|x| + \alpha_1|x|^2 + \cdots + \alpha_{m-1}|x|^{2m-2},$$

where the constants can be computed inductively starting with α_0 up to α_{m-1} in order to satisfy (11). Notice that G_1 is radial. Integrating by parts

$$\begin{aligned} 0 &= \int_{B_r} G_r \Delta^m h dx \\ &= h(0) - \sum_{i=0}^{m-1} \int_{\partial B_r} \frac{\partial \Delta^{m-1-i} G_r}{\partial n} \Delta^i h dS \\ &= h(0) - \sum_{i=0}^{m-1} \fint_{\partial B_r} a_i r^{2i} \Delta^i h dS, \end{aligned} \tag{12}$$

where each a_i depends only on n and m. For each term on the right-hand side with $i \geq 1$, we can use the inductive hypothesis

$$r^{2i} \fint_{\partial B_r} \Delta^i h dS = r^{2i} \sum_{j=0}^{m-i-1} b_j^{(m-1)} r^{2j} \Delta^{j+i} h(0), \quad 0 \leq i \leq m-1,$$

and substituting we obtain (10). To conclude the induction it is enough to multiply (10) by r^{n-1}, integrate with respect to r from 0 to R and divide by $\frac{R^n}{n}$.

To compute the c_i's, we test with the functions $h(x) = r^{2i} := |x|^{2i}$, $i \geq 1$ (for the case $i = 0$ use the function $h(x) \equiv 1$). Since $\Delta r^{2i} = 2i(2i + n - 2) r^{2i-2}$, we have that $\Delta^k h(0) = 0$ for $k \neq i$ and $\Delta^i h(0) = \frac{(2i)!!(2i+n-2)!!}{(n-2)!!}$. Hence Pizzetti's formula reduces to

$$c_i R^{2i} \frac{(2i)!!(2i + n - 2)!!}{(n - 2)!!} = \fint_{B_R} r^{2i} dx = \frac{n}{n + 2i} R^{2i},$$

whence (9). □

Remark From (12), moreover, for an arbitrary C^{2m}-function u it follows that

$$\fint_{B_R(x_0)} u(z)dz = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i u(x_0) + c_m R^{2m} \Delta^m u(\xi), \tag{13}$$

for some $\xi \in B_R(x_0)$.

Proposition 4 *Let $\Delta^m h = 0$ in $B_4 \subset \mathbb{R}^n$. For every $0 \leq \alpha < 1$, $p \in [1, \infty)$ and $k \geq 0$ there are constants $C(k, p)$, $C(k, \alpha)$ independent of h such that*

$$\begin{aligned} \|h\|_{W^{k,p}(B_1)} &\leq C(k, p)\|h\|_{L^1(B_4)} \\ \|h\|_{C^{k,\alpha}(B_1)} &\leq C(k, \alpha)\|h\|_{L^1(B_4)}. \end{aligned}$$

The proof of Proposition 4 is given in the appendix. As a consequence of Proposition 4 and Pizzetti's formula we have the following Liouville-type theorem, compare [1].

Theorem 5 *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h(x) \leq C(1 + |x|^\ell)$, for some $\ell \geq 2m - 2$. Then $h(x)$ is a polynomial of degree at most ℓ.*

Proof Thanks to Proposition 4, we have for any $x \in \mathbb{R}^n$

$$|D^{\ell+1}h(x)| \le \frac{C}{R^{\ell+1}} \fint_{B_R(x)} |h(y)|dy = -\frac{C}{R^{\ell+1}} \fint_{B_R(x)} h(y)dy + O(R^{-1}), \quad \text{as } R \to \infty. \tag{14}$$

On the other hand, Pizzetti's formula implies that

$$\fint_{B_R(x)} h(y)dy = \sum_{i=0}^{m-1} c_i R^{2i} \Delta^i h(x) = O(R^{2m-2}),$$

and letting $R \to \infty$, we obtain $D^{\ell+1}h = 0$. □

A variant of the above theorem, which will be used later is the following

Theorem 6 *Consider $h : \mathbb{R}^n \to \mathbb{R}$ with $\Delta^m h = 0$ and $h(x) \le u - v$, where $e^{pu} \in L^1(\mathbb{R}^n)$ for some $p > 0$, $v \in L^1_{\mathrm{loc}}(\mathbb{R}^n)$ and $-v(x) \le C(\log(1+|x|)+1)$. Then h is a polynomial of degree at most $2m-2$.*

Proof The only thing to change in the proof of Theorem 5, is the estimate of the term $\frac{2C}{R^{2m-1}} \fint_{B_R(x)} h^+ dy$, corresponding to the $O(R^{-1})$ in (14). We have

$$\begin{aligned} \fint_{B_R(x)} h^+ dy &\le \fint_{B_R(x)} u^+ dy + C \fint_{B_R(x)} \log(1+|y|)dy + C \\ &\le \frac{1}{p} \fint_{B_R(x)} e^{pu} dy + C\log R + C, \end{aligned}$$

and all terms go to 0 when divided by R^{2m-1} and for $R \to \infty$. □

The following estimate has been obtained by Brézis and Merle [5] in dimension 2 and by Lin [15] and Wei [22] in dimension 4. Notice that the constant γ_m, defined by the relation

$$(-\Delta)^m \left(\frac{1}{\gamma_m} \log \frac{1}{|x|} \right) = \delta_0, \quad \text{in } \mathbb{R}^{2m}$$

(see Proposition 22 in the appendix), plays an important role.

Theorem 7 *Let $f \in L^1(B_R(x_0))$ and let v solve*

$$\begin{cases} (-\Delta)^m v = f & \text{in } B_R(x_0) \subset \mathbb{R}^{2m}, \\ v = \Delta v = \cdots = \Delta^{m-1} v = 0 & \text{on } \partial B_R(x_0). \end{cases}$$

Then, for any $p \in \left(0, \frac{\gamma_m}{\|f\|_{L^1(B_R(x_0))}}\right)$, we have $e^{2mp|v|} \in L^1(B_R(x_0))$ and

$$\int_{B_R(x_0)} e^{2mp|v|} dx \le C(p) R^{2m},$$

where γ_m is given by (48).

Proof We can assume $x_0 = 0$ and, up to rescaling, that $\|f\|_{L^1(B_R)} = 1$. Define

$$w(x) := \frac{1}{\gamma_m}\int_{B_R} \log\frac{2R}{|x-y|}|f(y)|dy, \quad x \in \mathbb{R}^{2m}.$$

Extend f to be zero outside $B_R(x_0)$; then

$$(-\Delta)^m w = |f| \quad \text{in } \mathbb{R}^{2m}.$$

We claim that $w \geq |v|$ in B_R. Indeed by (49) and from $|x-y| \leq 2R$ for $x, y \in B_R$, we immediately see that

$$(-\Delta)^j w \geq 0, \quad j = 0, 1, 2, \dots$$

In particular the function $z := w - v$ satisfies

$$\begin{cases} (-\Delta)^m z \geq 0 & \text{in } B_R \\ (-\Delta)^j z \geq 0 & \text{on } \partial B_R \text{ for } 0 \leq j \leq m-1. \end{cases}$$

By Proposition 21, $(-\Delta)^j z \geq 0$ in B_R, $0 \leq j \leq m-1$ and the case $j = 0$ corresponds $w \geq v$. Working also with $-v$ we complete the proof of our claim.

Now it suffices to show that for $p \in (0, \gamma_m)$ we have $\|e^{2mpw}\|_{L^1(B_R)} \leq C(p)R^{2m}$. By Jensen's inequality we have

$$\begin{aligned}\int_{B_R} e^{2mpw}dx &= \int_{B_R} e^{\frac{2mp}{\gamma_m}\int_{B_R}\log\frac{2R}{|x-y|}|f(y)|dy}dx \\ &\leq \int_{B_R}\int_{B_R} |f(y)|e^{\frac{2mp}{\gamma_m}\log\frac{2R}{|x-y|}}dydx \\ &= \int_{B_R}|f(y)|\left(\int_{B_R}\left(\frac{2R}{|x-y|}\right)^{\frac{2mp}{\gamma_m}}dx\right)dy\end{aligned}$$

On the other hand

$$\begin{aligned}\int_{B_R}\left(\frac{2R}{|x-y|}\right)^{\frac{2mp}{\gamma_m}}dx &\leq \int_{B_R}\left(\frac{2R}{|x|}\right)^{\frac{2mp}{\gamma_m}}dx \\ &= \omega_{2m}\int_0^R r^{2m-1-\frac{2mp}{\gamma_m}}(2R)^{\frac{2mp}{\gamma_m}}dr \\ &= \omega_{2m}\frac{\gamma_m}{2m\gamma_m - 2mp}R^{2m}2^{\frac{2mp}{\gamma_m}}.\end{aligned}$$

We then conclude

$$\int_{B_R} e^{2mpw}dx \leq \frac{C(m)}{\gamma_m - p}R^{2m}.$$

□

Corollary 8 *Every solution u to (2) with $e^{2mu} \in L^1_{\text{loc}}(\mathbb{R}^{2m})$ is smooth.*

Proof Given $B_4(x_0) \subset \mathbb{R}^{2m}$, write $(2m-1)!e^{2mu}\big|_{B_4(x_0)} = f_1 + f_2$ with

$$\|f_1\|_{L^1(B_4(x_0))} < \gamma_m, \quad f_2 \in L^\infty(B_4(x_0)),$$

and $u = u_1 + u_2 + u_3$, with

$$\begin{cases} (-\Delta)^m u_i = f_i & \text{in } B_4(x_0) \\ u_i = \Delta u_i = \cdots = \Delta^{m-1} u_i = 0 & \text{on } \partial B_4(x_0) \end{cases}$$

for $i = 1, 2$, and $\Delta^m u_3 = 0$. Then, by Theorem 7, $e^{2mu_1} \in L^p(B_4(x_0))$ for some $p > 1$, while, by standard elliptic estimates $u_2 \in L^\infty(B_4(x_0))$ and u_3 is smooth, hence $u_3 \in L^\infty(B_3(x_0))$. Then $e^{2mu} \in L^p(B_3(x_0))$. Write now $u\big|_{B_3(x_0)} = v_1 + v_2$, where

$$\begin{cases} (-\Delta)^m v_1 = (2m-1)!e^{2mu} & \text{in } B_3(x_0) \\ v_1 = \Delta v_1 = \cdots = \Delta^{m-1} v_1 = 0 & \text{on } \partial B_3(x_0) \end{cases}$$

and $\Delta^m v_2 = 0$. Then, by L^p-estimates and Sobolev's embedding theorem, $v_1 \in W^{2m,p}(B_3(x_0)) \hookrightarrow C^{0,\alpha}(B_3(x_0))$ for some $0 < \alpha < 1$, while v_2 is smooth. Then $u \in C^{0,\alpha}(B_2(x_0))$ and with the same procedure of writing u as the sum of a polyharmonic (hence smooth) function plus a function with vanishing Navier boundary condition, we can bootstrap and use Schauder's estimate to prove that $u \in C^\infty(B_1(x_0))$. □

3 Proof of Theorems 1 and 2

The proof of Theorems 1 and 2 will be divided into several lemmas. It consists of a careful study of the functions v, defined in (3), and $u - v$. In what follows the generic constant C may depend also on u.

Remark In general $v \neq u$, even if u is a standard solution. To see that, rescale u by a factor $r > 0$ as follows:

$$\widetilde{u}(x) := u(rx) + \log r.$$

Then $\widetilde{u}$ is again a solution, with the same energy. On the other hand the corresponding $\widetilde{v}$ satisfies

$$\begin{aligned} \widetilde{v}(x) &= \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y|}{|x-y|}\right) e^{2mu(ry)} r^{2m} dy \\ &= \frac{(2m-1)!}{\gamma_m} \int_{\mathbb{R}^{2m}} \log\left(\frac{|y'|}{|rx-y'|}\right) e^{2mu(y')} dy' = v(rx). \end{aligned} \tag{15}$$

That shows that after rescaling, $u - v$ changes by a contant.

Lemma 9 *Let u be a solution of (2), (4). Then, for $|x| \geq 4$,*

$$v(x) \geq -2\alpha \log|x| + C. \tag{16}$$

Proof The proof is similar to that in dimension 4, compare [15]. Fix x with $|x| \geq 4$, and decompose $\mathbb{R}^{2m} = A_1 \cup A_2 \cup B_2$, where $B_2 = B_2(0)$ and

$$A_1 := B_{|x|/2}(x), \quad A_2 := \mathbb{R}^{2m} \backslash (A_1 \cup B_2).$$

For $y \in A_1$ we have

$$|y| \geq |x| - |x-y| \geq \frac{|x|}{2} \geq |x-y|, \quad \log \frac{|y|}{|x-y|} \geq 0,$$

hence

$$\int_{A_1} \log \frac{|y|}{|x-y|} e^{2mu(y)} dy \geq 0. \tag{17}$$

For $y \in A_2$, since $|x|, |y| \geq 2$, we have

$$|x-y| \leq |x| + |y| \leq |x||y|, \quad \log \frac{|y|}{|x-y|} \geq \log \frac{1}{|x|},$$

hence

$$\int_{A_2} \log \frac{|y|}{|x-y|} e^{2mu(y)} dy \geq -\log|x| \int_{A_2} e^{2mu(y)} dy. \tag{18}$$

For $y \in B_2$, $\log|x-y| \leq \log|x| + C$ and, since u is smooth, we find

$$\begin{aligned}\int_{B_2} \log \frac{|y|}{|x-y|} e^{2mu(y)} dy &\geq \int_{B_2} \log|y| e^{2mu(y)} dy - \log|x| \int_{B_2} e^{2mu} dy - C \int_{B_2} e^{2mu} dy \\ &\geq -\log|x| \int_{B_2} e^{2mu} dy + C.\end{aligned} \tag{19}$$

Putting together (17)–(19) and observing that $\log \frac{1}{|x|} < 0$, we conclude that

$$\begin{aligned}v(x) &\geq \frac{(2m-1)!}{\gamma_m} \int_{A_2 \cup B_2} \log\left(\frac{|y|}{|x-y|}\right) e^{2mu(y)} dy \\ &\geq -\frac{(2m-1)!}{\gamma_m} \log|x| \int_{A_2 \cup B_2} e^{2mu} dy + C \\ &\geq -\frac{(2m-1)!|S^{2m}|}{\gamma_m} \alpha \log|x| + C.\end{aligned}$$

Finally, observing that $(2m-2)!! = 2^{m-1}(m-1)!$, we infer

$$\frac{(2m-1)!|S^{2m}|}{\gamma_m} = \frac{(2m-1)!2(2\pi)^m(2m-2)!!}{(2m-1)!!2^{3m-2}[(m-1)!]^2\pi^m} = 2.$$

□

Lemma 10 *Let u be a solution of (2) and (4), with $m \geq 2$. Then $u = v + p$, where p is a polynomial of degree at most $2m-2$. Moreover*

$$\begin{aligned}\Delta^j u(x) &= \Delta^j v(x) + p_j \\ &= (-1)^j \frac{2^{2j}(j-1)!(m-1)!}{(m-j-1)!|S^{2m}|} \int_{\mathbb{R}^{2m}} \frac{e^{2mu(y)}}{|x-y|^{2j}} dy + p_j,\end{aligned}$$

where p_j is a polynomial of degree at most $2(m-1-j)$.

Proof Let $p := u - v$. Then $\Delta^m p = 0$. By Lemma 9 we have

$$p(x) \le u(x) + 2\alpha \log|x| + C,$$

and Theorem 6 implies that p is a polynomial of degree at most $2m-2$. To compute $\Delta^j v$, one can use (49) and the definition of γ_m. □

Lemma 11 *Let p be the polynomial of Lemma 10. Then*

$$\sup_{x\in\mathbb{R}^{2m}} p(x) < +\infty.$$

In particular deg p *is even.*

Proof Define

$$f(r) := \sup_{\partial B_r} p.$$

If $\sup_{\mathbb{R}^{2m}} p = +\infty$, there exists $s > 0$ such that

$$\lim_{r\to+\infty} \frac{f(r)}{r^s} = +\infty, \tag{20}$$

see [13, Theorem 3.1].[1] Moreover $|\nabla p(x)| \le C|x|^{2m-3}$ hence, also taking into account Lemma 9, there is $R > 0$ such that for every $r \ge R$, we can find x_r with $|x_r| = r$ such that

$$u(y) = v(y) + p(y) \ge r^s \quad \text{for } |y - x_r| \le \frac{1}{r^{2m-3}}.$$

Then, using Fubini's theorem,

$$\begin{aligned}\int_{\mathbb{R}^{2m}} e^{2mu}dx &\ge \int_R^{+\infty} \int_{\partial B_r(0)\cap B_{r^{3-2m}}(x_r)} e^{2mr^s} d\sigma dr\\ &\ge C\int_R^{+\infty} \frac{\exp(2mr^s)}{r^{(2m-3)(2m-1)}}dr = +\infty,\end{aligned}$$

contradicting the hypothesis $e^{2mu} \in L^1(\mathbb{R}^{2m})$. □

The following lemma will be used in the proof of Lemma 13.

Lemma 12 *Let $G = G(|x|)$ be the Green's function for Δ^m in $B_1 \subset \mathbb{R}^n$ for n, m given positive integers. Then there are constants c_i depending on m and n such that for $|x| = 1$, and* $0 \le i \le m-1$,

$$(-1)^i \frac{\partial \Delta^{m-1-i} G(x)}{\partial r} = c_i > 0.$$

Proof Since $G = G(|x|)$, we only need to show that $c_i > 0$. Fix i and let h solve

$$\begin{cases} \Delta^m h = 0 & \text{in } B_1\\ (-\Delta)^i h = -1 & \text{on } \partial B_1\\ (-\Delta)^j h = 0 & \text{on } \partial B_1 \text{ for } 0 \le j \le m-1,\ \ j \ne i.\end{cases}$$

[1] The statement of Theorem 3.1 in [13] is about $\mu(r) := \inf_{\partial B_r} |p|$, but the proof works in our case too.

By Proposition 21, $h(0) < 0$, hence (12) implies

$$0 < -h(0) = (-1)^i \int\limits_{\partial B_1} \frac{\partial \Delta^{m-1-i} G}{\partial r} dS = c_i \omega_n.$$

□

Lemma 13 *Let $v : \mathbb{R}^{2m} \to \mathbb{R}$ be defined as in (3). Then*

$$\lim_{|x|\to\infty} \Delta^{m-j} v(x) = 0, \quad j = 1, \ldots, m-1 \tag{21}$$

and for any $\varepsilon > 0$ there is $R > 0$ such that for $|x| > R$

$$v(x) \le (-2\alpha + \varepsilon) \log |x|. \tag{22}$$

Proof We proceed by steps.
Step 1. For any $\varepsilon > 0$ there is $R > 0$ such that for $|x| \ge R$

$$v(x) \le \left(-2\alpha + \frac{\varepsilon}{2}\right) \log |x| - \frac{(2m-1)!}{\gamma_m} \int\limits_{B_\tau(x)} \log |x-y| e^{2mu(y)} dy, \tag{23}$$

where $\tau \in (0, 1)$ will be fixed later. The simple proof of (23) is very similar to the proof of Lemma 9 (see [15, p. 213]), and it is omitted. Notice that the second term on the right-hand side may be very large. Together with Fubini's theorem, (23) implies

$$\begin{aligned} \int\limits_{\mathbb{R}^{2m}\setminus B_R(0)} v^+ dx &\le C \int\limits_{\mathbb{R}^{2m}} \int\limits_{\mathbb{R}^{2m}} \chi_{|x-y|\le\tau} \log \frac{1}{|x-y|} e^{2mu(y)} dy dx \\ &= C \int\limits_{\mathbb{R}^{2m}} e^{2mu(y)} \int\limits_{B_\tau(y)} \log \frac{1}{|x-y|} dx dy \\ &\le C \int\limits_{R^{2m}} e^{2mu(y)} dy \le C. \end{aligned} \tag{24}$$

Step 2. From now on, x will be a point in $\mathbb{R}^{2m}$ with $|x| > R$, where R is as in Step 1. Fix $p > 1$ such that $p(2m-2) < 2m$, and $p' = \frac{p}{p-1}$. By Theorem 7, there is $\delta > 0$ such that if

$$\int\limits_{B_4(x)} e^{2mu} dy < \delta, \tag{25}$$

then

$$\int\limits_{B_4(x)} e^{2mp'|z|} dy \le C, \tag{26}$$

with C independent of x, where z solves

$$\begin{cases} (-\Delta)^m z = (2m-1)! e^{2mu} & \text{in } B_4(x) \\ \Delta^j z = 0 & \text{on } \partial B_4(x) \text{ for } 0 \le j \le m-1. \end{cases}$$

We now choose $R > 0$ such that (25) is satisfied whenever $|x| \ge R$, and claim that for such x,

$$\int\limits_{B_\tau(x)} e^{2mp'u} dy \le C \int\limits_{B_\tau(x)} e^{2mp'|z|} dy \le C\varepsilon. \tag{27}$$

We now observe that for any $\sigma > 0$,

$$\int_{\mathbb{R}^{2m}\setminus B_\sigma(x)} \frac{e^{2mu(y)}}{|x-y|^{2j}}dy \to 0 \quad \text{as } |x| \to \infty \tag{28}$$

by dominated convergence; by Hölder's inequality and (27), if σ is small enough,

$$\int_{B_\sigma(x)} \frac{e^{2mu}}{|x-y|^{2j}}dy \leq \left(\int_{B_\sigma(x)} e^{2mp'u}dy\right)^{\frac{1}{p'}} \left(\int_{B_\sigma(x)} \frac{1}{|x-y|^{2jp}}dy\right)^{\frac{1}{p}} \leq C\varepsilon^{\frac{1}{p'}}.$$

Therefore

$$(-\Delta)^j v(x) = C\int_{\mathbb{R}^{2m}} \frac{e^{2mu}}{|x-y|^{2j}}dy \to 0, \quad \text{as } |x| \to \infty.$$

Finally (22) follows from (23), (27) and Hölder's inequality.
Step 3. It remains to prove (27). Set $h := v - z$, so that

$$\begin{cases} \Delta^m h = 0 & \text{in } B_4(x) \\ \Delta^j h = \Delta^j v & \text{on } \partial B_4(x) \text{ for } 0 \leq j \leq m-1, \end{cases}$$

Integrating $(-\Delta)^m v = (2m-1)!e^{2mu}$ and then integrating by parts we get

$$(-1)^m \int_{\partial B_\rho(x)} \frac{\partial}{\partial r}(\Delta^{m-1}v)dS = (2m-1)! \int_{B_\rho(x)} e^{2mu}dy.$$

Dividing by $\omega_{2m}\rho^{2m-1}$, integrating on $[0, R]$ and using Fubini's, we find

$$\begin{aligned}\int_0^R \fint_{\partial B_\rho(x)} \frac{\partial}{\partial r}(\Delta^{m-1}v)d\sigma d\rho &= \int_0^R \fint_{\partial B_1(x)} \frac{\partial}{\partial r}(\Delta^{m-1}v(\rho,\theta))d\theta d\rho \\ &= \fint_{\partial B_1(x)} \int_0^R \frac{\partial}{\partial r}(\Delta^{m-1}v(\rho,\theta))d\rho d\theta \\ &= \fint_{\partial B_R(x)} \Delta^{m-1}v d\sigma - \Delta^{m-1}v(x).\end{aligned}$$

Similarly

$$\begin{aligned}\int_0^R \frac{1}{\rho^{2m-1}} \int_{B_\rho(x)} e^{2mu(y)}dyd\rho &= \int_0^R \frac{1}{\rho^{2m-1}} \int_{B_R(x)} e^{2mu(y)}\chi_{|x-y|\leq\rho}dyd\rho \\ &= \int_{B_R(x)} e^{2mu(y)} \int_{|x-y|}^R \frac{1}{\rho^{2m-1}}d\rho dy \\ &= \frac{1}{(2m-2)} \int_{B_R(x)} \left[\frac{1}{|x-y|^{2m-2}} - \frac{1}{R^{2m-2}}\right] e^{2mu(y)}dy.\end{aligned}$$

Hence, multiplying above by $\frac{(2m-1)!}{\omega_{2m}}$ and setting $C_{m-1} := \frac{(2m-1)!}{(2m-2)\omega_{2m}}$,

$$\fint_{\partial B_R} (-\Delta)^{m-1} v d\sigma = (-\Delta)^{m-1} v(x) - C_{m-1} \int_{B_R(x)} \left[\frac{1}{|x-y|^{2m-2}} - \frac{1}{R^{2m-2}} \right] e^{2mu(y)} dy$$
$$= C_{m-1} \left[\int_{|x-y| \geq R} \frac{e^{2mu(y)}}{|x-y|^{2m-2}} dy + \int_{B_R(x)} \frac{e^{2mu(y)}}{R^{2m-2}} dy \right]$$

which implies at once, setting $R = 4$,

$$\fint_{\partial B_4(x)} (-\Delta)^{m-1} v dS \leq C, \tag{29}$$

with C independent of x. Similarly, one can show that

$$\fint_{\partial B_4(x)} (-\Delta)^{i} v dS \leq C, \quad 1 \leq i \leq m-1. \tag{30}$$

By Lemma 12 and by (12) rescaled and translated to $B_4(x)$ and with the function $-\Delta h$ instead of h, $m-1$ instead of m, we obtain

$$-\Delta h(x) = -\sum_{i=0}^{m-2} \int_{\partial B_4(x)} \frac{\partial \Delta^{m-2-i} G}{\partial n} \Delta^i (\Delta h) dS$$
$$= \sum_{i=1}^{m-1} \int_{\partial B_4(x)} c_{i-1} (-\Delta)^i h dS \leq C, \tag{31}$$

where G is the Green function for Δ^{m-1} on $B_4(x)$:

$$\Delta^{m-1} G = \delta_x, \quad \Delta^i G = 0, \quad \text{on } \partial B_4(x), \quad \text{for } 0 \leq i \leq m-2.$$

On the other hand, since the $c_i > 0$, there is some $\tau > 0$ such that the following holds: if $\xi \in B_{2\tau}(x)$ and G_ξ is the Green's function defined by

$$\Delta^{m-1} G_\xi = \delta_\xi, \quad \Delta^i G_\xi = 0, \quad \text{on } \partial B_4(x), \quad \text{for } 0 \leq i \leq m-2,$$

then also

$$0 \leq (-1)^i \frac{\partial \Delta^{m-2-i} G_\xi(\eta)}{\partial r} \leq C, \quad \text{for } \eta \in \partial B_4(x), \ r := \frac{\eta - x}{4}.$$

Therefore, as in (31), we infer

$$-\Delta h \leq C \quad \text{on } B_{2\tau}(x), \tag{32}$$

for some $\tau \in (0, 2)$.

On the other hand, thanks to (24) and (26),

$$\int_{B_4(x)} h^+ dy \leq \int_{B_4(x)} (v^+ + |z|) dy \leq C.$$

By elliptic estimates,

$$\sup_{B_\tau(x)} h \leq \fint_{B_4(x)} h^+ dy + C \sup_{B_{2\tau(x)}} (-\Delta h) \leq C,$$

C independent of x, as usual. Since the polynomial p is bounded from above, we infer

$$u \leq h + p + |z| \leq C + |z|,$$

and (27) follows at once. □

Corollary 14 *Any solution u of (2), (4) is bounded from above.*

Proof Indeed u is continuous, $u = v + p$, and

$$\lim_{|x|\to\infty} v(x) = -\infty, \quad \sup_{x\in\mathbb{R}^{2m}} p(x) < +\infty,$$

by Lemma 11. □

Lemma 15 *Assume that $|u(x)| = o(|x|^2)$ as $|x| \to \infty$. Then $u = v + C$. Furthermore, for any $\varepsilon > 0$ there exists $R > 0$ such that*

$$-2\alpha \log|x| - C \leq u(x) \leq (-2\alpha + \varepsilon) \log|x|, \tag{33}$$

for $|x| \geq R$.

Proof Since $v(x) = -2\alpha \log|x| + o(\log|x|)$ at ∞, if $\deg p \geq 2$, we have that $u(x) = v(x) + p(x)$ cannot be $o(|x|^2)$. Hence, knowing that $\deg p$ is even, we get $u = v + C$ for some constant C. Then (33) follows at once from Lemmas 9 and 13. □

Lemma 16 *Set $g_u = e^{2u} g_{\mathbb{R}^{2m}}$. If u is a standard solution, then*

$$R_{g_u} \equiv 2m(2m-1).$$

If u is not a standard solution, then

$$\liminf_{|x|\to+\infty} R_{g_u}(x) = -\infty. \tag{34}$$

Proof Assume that u is a standard solution and set

$$u_\lambda(x) := \log \frac{2\lambda}{1+\lambda^2|x|^2}, \quad g_\lambda := e^{2u_\lambda} g_{\mathbb{R}^{2m}}. \tag{35}$$

Then, up to translation, $u = u_\lambda$ for some $\lambda > 0$. Since $g_1 = (\pi^{-1})^* g_{S^{2m}}$, where π is the stereographic projection, we have $R_{g_1} \equiv 2m(2m-1)$. Then consider the diffeomorphism of $\mathbb{R}^{2m}$ defined by $\varphi_\lambda(x) := \lambda x$. Then $g_\lambda = \varphi_\lambda^* g_1$, hence $R_{g_\lambda} = R_{g_1} \circ \varphi_\lambda \equiv 2m(2m-1)$.

Assume now that $u = v + p$ is not a standard solution. Since $g_{\mathbb{R}^{2m}}$ is flat, the formula for the conformal change of scalar curvature, in the case $m > 1$, reduces to

$$R_{g_u} = -2(2m-1)e^{-2u}\left(\Delta u + (m-1)|\nabla u|^2\right), \tag{36}$$

see for instance [21, p. 184]. Then differentiating the expression (3) for v and using that $u \leq C$, we find that $|\nabla v(x)| \to 0$ as $|x| \to \infty$. We have already seen that $\Delta v(x) \to 0$ as $|x| \to \infty$; since $\deg p \geq 2$ implies

$$\deg \Delta p < \deg |\nabla p|^2,$$

we then have

$$\limsup_{|x|\to\infty}\left(\Delta u+(m-1)|\nabla u|^2\right)=\limsup_{|x|\to\infty}\left(\Delta p+(m-1)|\nabla p|^2\right)=+\infty.$$

Observing that $e^{-2u}\geq\frac{1}{C}>0$, u being bounded from above, we easily obtain (34). □

Proof of Theorem 1 Put together Lemmas 9–11 and 13. □

Proof of Theorem 2 (i) ⇒ (iii) is obvious, while (iii) ⇒ (i) follows from the argument of [23].

(iii) ⇔ (iv) follows from Theorem 1.

(iv) ⇒ (ii') ⇒ (ii). Assume that deg $p=0$. Then by Theorem 1,

$$\lim_{|x|\to\infty}\Delta^j u(x)=\lim_{|x|\to|\infty}\Delta^j p(x)=0,\quad 1\leq j\leq m-1.$$

(ii) ⇒ (iv). By Theorem 1, $\sup_{\mathbb{R}^{2m}} p<\infty$ and

$$\lim_{|x|\to\infty}\Delta p(x)=\lim_{|x|\to\infty}\Delta u=0,$$

hence $\Delta p\equiv 0$ and, by Liouville's theorem, p is constant.

(i) ⇔ (v) follows from Lemma 16.

(i) ⇒ (vi) Given a conformal diffeomorphism φ of $\mathbb{R}^{2m}$, $\widetilde{\varphi}:=\pi^{-1}\circ\varphi\circ\pi$ is a conformal diffeomorphism of S^{2m}. Any metric of the form $g_u=e^{2u}g_{\mathbb{R}^{2m}}$, with u standard solution of (2), can be easily written as φ^*g_1, for some conformal diffeomorphism φ of $\mathbb{R}^{2m}$,where g_1 is as in (35). Then

$$\pi^*g_u=\pi^*\varphi^*g_1=(\varphi\circ\pi)^*g_1=(\pi\circ\widetilde{\varphi})^*g_1=\widetilde{\varphi}^*\pi^*g_1=\widetilde{\varphi}^*g_{S^2},$$

and clearly $\widetilde{\varphi}^*g_{S^2}$ is a smooth Riemannian metric on S^{2m}.

(vi) ⇒ (i). Assume u is non-standard. Then $u=v+p$, deg $p\geq 2$. Considering that $\sup_{\mathbb{R}^{2m}} p<+\infty$, we infer that p goes to $-\infty$ at least quadratically in some directions. Let $S=(0,\dots,0,1)\in S^{2m}$ be the South Pole, and

$$\pi:S^{2m}\backslash\{S\}\to\mathbb{R}^{2m},\quad \pi(\xi):=\frac{(\xi_1,\dots,\xi_{2m})}{1+\xi_{2m+1}}$$

be the stereographic projection from S. Then

$$(\pi^{-1})^*g_{S^{2m}}=\rho_0 g_{\mathbb{R}^{2m}},\quad \rho_0(x):=\frac{4}{(1+|x|^2)^2},$$

and

$$\pi^*g_u=\rho_1 g_{S^{2m}},\quad \rho_1:=\frac{e^{2u}}{\rho_0}\circ\pi\in C^\infty(S^{2m}\backslash\{S\}).$$

Since $e^{2u(x)}\to 0$ more rapidly than $|x|^{-4}$ in some directions, we have

$$\liminf_{\xi\to S}\rho_1(\xi)=\liminf_{|x|\to\infty}\frac{e^{2u(x)}}{\rho_0(x)}=0,$$

hence $\rho_1 g_{S^{2m}}$ does not extend to a Riemannian metric on S^{2m}.

To prove (7), let j be the largest integer such that $\Delta^j p \neq 0$. Then $\Delta^{j+1} p \equiv 0$ and from Theorem 6, we infer that $\deg p \leq 2j$. In fact $\deg p = 2j$ and $\Delta^j p \equiv C_0 \neq 0$. From Pizzetti's formula (10), we have

$$2m \sum_{i=0}^{j} b_i R^{2i} \Delta^i p(0) = \fint_{\partial B_R} 2mp dS$$

Exponentiating and using Jensen's inequality and Lemma 9, we infer

$$\exp\left(2m \sum_{i=0}^{j} b_i R^{2i} \Delta^i p(0)\right) \leq \fint_{\partial B_R} e^{2mp} dS \leq CR^{4m\alpha} \fint_{\partial B_R} e^{2mu} dS,$$

for $R \geq 4$. Therefore

$$\varphi(R) := R^{-4m\alpha+2m-1} \exp\left(2m \sum_{i=0}^{j} b_i R^{2i} \Delta^i p(0)\right) \in L^1([4, +\infty)),$$

and this is not possible if $C_0 = \Delta^j p > 0$, hence $C_0 < 0$. □

4 Examples

Following an argument of [7], we now see that solutions of the kind $v + p$ actually exist, even among radially symmetric functions, with $\deg p = 2m - 2$, and with $\deg p = 2$. For simplicity, we only treat the case when m is even; if m is odd, the proof is similar. We need the following lemma.

Lemma 17 *Let $u(r)$ be a smooth radially symmetric function on $\mathbb{R}^n$, $n \geq 1$. Then for $m \geq 0$ we have*

$$\Delta^m u(0) = \frac{n}{c_m(n+2m)(2m)!} u^{(2m)}(0), \tag{37}$$

where the c_i's are the constants in Pizzetti's formula, and $u^{(2m)} := \frac{\partial^{2m} u}{\partial r^{2m}}$. In particular $\Delta^m u(0)$ has the sign of $u^{(2m)}(0)$.

Proof We first prove that

$$c_m \Delta^m u(0) = \frac{1}{R^{2m}} \fint_{B_R(0)} \frac{r^{2m}}{(2m)!} u^{(2m)}(0) dx. \tag{38}$$

Then, observing that

$$\fint_{B_R(0)} \frac{r^{2m}}{(2m)!} dx = \frac{nR^{2m}}{(n+2m)(2m)!}, \tag{39}$$

(37) follows at once. We prove (38) by induction. The case $m = 0$ reduces to $u(0) = u(0)$. Let us now assume that (38) has been proven for $i = 0, \dots, m-1$ and let us prove it for m. Since u is smooth, we have $u^{(i)}(0) = 0$ for any odd i, hence Taylor's formula reduces to

$$u(r) = \sum_{i=0}^{m} \frac{r^{2i}}{(2i)!} u^{(2i)}(0) + o(r^{2m+1}).$$

We now divide by R^{2m} in (13), take the limit as $R \to 0$ and, observing that $\Delta^{m+1}u(\xi)$ remains bounded as $R \to 0$, we find

$$\lim_{R\to 0} \frac{\fint_{B_R}\left(u - \sum_{i=0}^{m-1} c_i R^{2i}\Delta^i u(0)\right)dx}{R^{2m}} = c_m \Delta^m u(0).$$

Substituting Taylor's formula and using the inductive hypothesis, we see that most of the terms on the left-hand side cancel out (before taking the limit) and we are left with

$$\lim_{R\to 0} \frac{1}{R^{2m}} \fint_{B_R} \left(\frac{r^{2m}u^{(2m)}(0)}{(2m)!} + o(r^{2m+1}) \right) dx = c_m \Delta^m u(0).$$

Finally, to deduce (38), observe that, $\frac{1}{R^{2m}} \fint_{B_R(0)} o(r^{2m+1})dx \to 0$ as $R \to 0$, while $\frac{1}{R^{2m}} \fint_{B_R} \frac{r^{2m}u^{(2m)}(0)}{(2m)!}dx$ does not depend on R thanks to (39). □

Proposition 18 *For every $m \geq 2$ even, there exists a radially symmetric function u solving* (2), (4) *with $u(x) = -C|x|^{2m-2} + O(|x|^{2m-4})$.*

Proof Set $w_0 = \log \frac{2}{1+r^2}$. Then $\Delta^m w_0 = (2m-1)!e^{2mw_0}$. Define $u = u(r)$ to be the unique solution to the following ODE

$$\begin{cases} \Delta^m u = (2m-1)!e^{2mu} & \\ u(0) = \log 2 & \\ u^{(2j+1)}(0) = 0 & j = 0, \ldots, m-1 \\ u^{(2j)}(0) = \alpha_j \leq w_0^{(2j)}(0) & j = 1, \ldots, m-2 \\ u^{(2m-2)}(0) = \alpha_{m-1} < w_0^{(2m-2)}(0) & \end{cases}$$

where the α_j's are fixed. We shall first see that $w_0 \geq u$. Set $g := w_0 - u$. Then $g(r) > 0$ for $r > 0$ small enough, hence also $\Delta^m g > 0$ for small $r > 0$. From Lemma 17 we get

$$\Delta^j g(0) \geq 0, \quad j = 1, \ldots, m-2; \quad \Delta^{m-1} g(0) > 0. \tag{40}$$

We can prove inductively that $\Delta^{m-j} g \geq 0$, $j = 0, \ldots, m-1$ as long as $g(r) > 0$. Indeed

$$\int_{B_R(0)} \Delta^j g dx = \int_{\partial B_R(0)} \frac{\partial \Delta^{j-1} g}{\partial r} d\sigma, \tag{41}$$

hence, as long as $g(r) > 0$, we have $\frac{\partial \Delta^{j-1} g}{\partial r} > 0$, in particular $\frac{\partial g}{\partial r} > 0$, hence $g(r) > 0$ for all $r > 0$ for which it is defined. From (40) and (41) we inductively infer

$$\Delta^{m-j} g(r) \geq Cr^{2j-2},$$

and, since $\Delta w_0(r) \to 0$ as $r \to \infty$, there is $r_0 > 0$ such that

$$\Delta u \leq -Cr^{2m-4}, \quad \text{for } r \geq r_0,$$

integrating which, we find

$$u(r) \leq -Cr^{2m-2}, \quad \text{for } r \geq r_0. \tag{42}$$

To estimate u from below, we use the function

$$w_1(r) = \log 2 - C_1 r^2 - \cdots - C_{m-1} r^{2m-2},$$

where the constants C_i are chosen so that

$$\Delta^j u(0) \geq \Delta^j w_1(0).$$

Then we can proceed as above to prove that $u - w_1 \geq 0$. Hence the solution exists for all times and, thanks to (42) and Theorem 1, it has the asymptotic behaviour

$$u(r) = -Cr^{2m-2} + O(r^{2m-4}).$$

□

Remark Observe the abundance of solutions: we can choose the $(m-1)$-tuple of initial data $(\alpha_1, \dots, \alpha_{m-1})$ in a set containing an open subset of $\mathbb{R}^{m-1}$.

In the next example we show a radially symmetric solution in $\mathbb{R}^{2m}$, $m \geq 4$ even, of the form $u = v + p$, with $\deg p = 2$, thus showing that the hypothesis $u(x) = o(|x|^2)$ as $|x| \to \infty$ in Theorem 2 is sharp.

Proposition 19 *Let $w_0(r) := \log \frac{2}{1+r^2}$ and let $u = u(r)$ ($r = |x|$, $x \in \mathbb{R}^{2m}$ and m even) solve the following ODE*:

$$\begin{cases} \Delta^m u = (2m-1)! e^{2mu} \\ u(0) = \log 2 \\ u^{(2j+1)}(0) = 0 & j = 0, \dots, m-1 \\ u^{(2j)}(0) = w_0^{(2j)}(0) & j = 2, 3, \dots, m-1 \\ u''(0) = w_0''(0) - 1. \end{cases}$$

Then $u(r)$ is defined for all $r \geq 0$ and $u(r) = -Cr^2 + O(\log r)$ as $r \to +\infty$.

Proof As in the proof of Proposition 18, we can show that $g := w_0 - u \geq 0$ and $u(r) \leq -Cr^2$. To control u from below, we use the function $w_1(r) = w_0(r) - r^2$, so that redefining $g := u - w_1$, we have

$$g''(0) = 1, \quad g^{(j)}(0) = 0, \quad j = 0, 1, 3, 4, \dots, 2m-1.$$

and we can prove that $g \geq 0$ as before. Hence $u(r)$ exists for all $r \geq 0$, it is non-standard and $u(r) = -Cr^2 + O(\log r)$ as $r \to \infty$, as w_1 bounds it from below. □

Remark Using (36), we can easily compute that in the above examples

$$\lim_{|x| \to \infty} R_g(x) \to -\infty,$$

where $g = e^{2u} g_{\mathbb{R}^{2m}}$.

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Appendix

We prove here a few results used above.

Lemma 20 *Assume that $u : B_4 \to \mathbb{R}$ satisfies*

$$\|\Delta u\|_{W^{k,p}(B_4)} \leq C$$
$$\|u\|_{L^1(B_4)} \leq C,$$

for some $p \in (1, \infty)$. Then

$$\|u\|_{W^{k+2,p}(B_1)} \leq C.$$

Proof By Fubini's theorem we can choose $r > 0$ with $2 \leq r \leq 4$ such that

$$\|u\|_{L^1(\partial B_r)} \leq C\|u\|_{L^1(B_4)}.$$

Let's now write $u = u_1 + u_2$, where

$$\begin{cases} \Delta u_1 = 0 & \text{in } B_r \\ u_1 = u & \text{on } \partial B_r \end{cases} \qquad \begin{cases} \Delta u_2 = \Delta u & \text{in } B_r \\ u_2 = 0 & \text{on } \partial B_r \end{cases}$$

By standard L^p-estimates we have $\|u_2\|_{W^{k+2,p}(B_r)} \leq C\|\Delta u\|_{W^{k,p}(B_r)}$. From the representation formula of Poisson

$$u_1(x) = \int_{\partial B_r} u_1(y)\Gamma(x-y)dS(y),$$

we obtain $\|u_1\|_{C^k(B_1)} \leq C_k\|u_1\|_{L^1(\partial B_r)}$ for every $k \geq 0$. □

Proof of Proposition 4 Let $\|h\|_{L^1(B_4)} \leq C$, and let us assume $n > 2$. We proceed by steps. *Step 1*. We show by induction on j that

$$\|\Delta^{m-j}h\|_{L^\infty(B_2)} \leq C. \tag{43}$$

The step $j = 0$ is obvious, as $\Delta^m h \equiv 0$. Let us prove the step $j \geq 1$. Let

$$G_{2r}(x) := \frac{1}{(2-n)\omega_n}\left(\frac{1}{|x|^{n-2}} - \frac{1}{(2r)^{n-2}}\right)$$

be the Green function for the Laplace operator on B_{2r} with singularity at 0. Then

$$\Delta^{m-j}h(0) = \fint_{\partial B_{2r}} \Delta^{m-j}h dx + \int_{B_{2r}} G_{2r}\Delta^{m-j+1}h dx.$$

By inductive hypothesis and the scaling property of G_{2r}, the last term is bounded by Cr^2, hence

$$\Delta^{m-j}h(0) \leq \fint_{\partial B_{2r}} \Delta^{m-j}h dx + Cr^2,$$

and integrating with respect to r on $[1/2, 1]$, we obtain

$$\Delta^{m-j}h(0) \leq \fint_{B_2} \Delta^{m-j}h dx + C. \tag{44}$$

To estimate $\fint_{B_2} \Delta^{m-j} h dx$, we use Pizzetti's formula for h at $x \in B_2$,

$$c_{m-j}\Delta^{m-j}h(x) = -\sum_{i=0}^{m-j-1} c_i \Delta^i h(x) - \underbrace{\sum_{i=m-j+1}^{m} c_i \Delta^i h(x) + \fint_{B_1(x)} h dy}_{\leq C}$$

by the inductive hypothesis again, and the L^1-bound on h and get

$$c_{m-j}\Delta^{m-j}h(x) \leq -\sum_{i=0}^{m-j-1} c_i \Delta^i h(x) + C. \tag{45}$$

Averaging in (45) over B_2 and using (44), we find

$$c_{m-j}\Delta^{m-j}h(0) \leq -\sum_{i=0}^{m-j-1}\left(c_i \fint_{B_2} \Delta^i h(x) dx\right) + C.$$

and its scaled version

$$c_{m-j}\Delta^{m-j}h(0) \leq -\sum_{i=0}^{m-j-1}\left(c_i r^{2(i-m+j)} \fint_{B_{2r}} \Delta^i h(x) dx\right) + C r^{2(j-m)}. \tag{46}$$

Consider now a non-negative function $\varphi \in C_c^\infty((1,2))$, with $\int_1^2 \varphi(r) dr = 1$. From (46), we find

$$c_{m-j}\Delta^{m-j}h(0) \leq -\sum_{i=0}^{m-j-1} c_i \int_1^2 \left(r^{2(i-m+j)} \fint_{B_{2r}} \Delta^i h(x) dx \, \varphi(r)\right) dr + C.$$

Each term in the sum on the right-hand side can be written as

$$\begin{aligned}
&\left| C \int_1^2 r^{2(i-m+j)-n} \int_{\partial B_{2r}} \frac{\partial \Delta^{i-1} h}{\partial \nu} dS \varphi(r) dr \right| \\
&\leq C \left| \int_{B_2 \setminus B_1} r^{2(i-m+j)-n} \frac{\partial \Delta^{i-1} h(x)}{\partial \nu} \varphi(|x|) dx \right| \\
&= C \int_{B_2 \setminus B_1} |h(x)| \left| \frac{\partial}{\partial \nu} \Delta^{i-1}\left(r^{2(i-m+j)-n}\varphi(|x|)\right)\right| dx \\
&\leq C \fint_{B_2} |h(x)| dx.
\end{aligned}$$

Working with $-h$ and observing the local character of the above estimates, we obtain (43).
Step 2. Fix $\ell \geq m$. We can prove inductively that

$$\|\Delta^{\ell-j} h\|_{W^{2j,p}(B_2)} \leq C(p).$$

The step $j = 0$ is obvious, as $\Delta^\ell h \equiv 0$. For the inductive step, we see that by Lemma 20 applied to $\Delta^{\ell-j}h$ (and a simple covering argument to fix the radii), we have

$$\|\Delta^{\ell-j}h\|_{W^{2j,p}(B_1)} \leq C\|\Delta(\Delta^{\ell-j}h)\|_{W^{2j-2,p}(B_2)} + C\underbrace{\|\Delta^{\ell-j}h\|_{L^1(B_2)}}_{\leq C \text{ by Step 1}} \leq C,$$

for every $1 < p < \infty$, and the usual covering argument extends the estimate to B_2. Therefore $\|h\|_{W^{2\ell,p}(B_1)} \leq C(p,\ell)$, and we conclude applying Sobolev's theorem. □

Proposition 21 *Let $u \in C^{2m}(\overline{B}_1)$ such that*

$$\begin{cases} (-\Delta)^m u \leq C_1 & \textit{in } B_1 \\ (-\Delta)^j u \leq C_1 & \textit{on } \partial B_1 \textit{ for } 0 \leq j \leq m-1 \end{cases} \tag{47}$$

Then there exists a constant C independent of u such that

$$u \leq C \quad \textit{in } B_1.$$

If $C_1 = 0$ in (47), then $u < 0$ in B_1, unless $u \equiv 0$.

Proof By induction on m. The case $m = 1$ follows from the maximum principle, applied to the function $v(x) := u(x) - C|x|^2$, which is subharmonic for C large enough. Assume now that the case $m - 1$ has been dealt with and let us consider u satisfying (47). Then $v := -\Delta u$ satisfies $v \leq C$ in B_1 by inductive hypothesis. Applying the case $m = 1$ again we conclude. Similarly if $C_1 = 0$. □

Proposition 22 (Fundamental solution) *For $m \geq 1$, set*

$$\gamma_m := \omega_{2m} 2^{2m-2}[(m-1)!]^2, \tag{48}$$

where $\omega_{2m} := |S^{2m-1}| = \frac{(2\pi)^m}{(2m-2)!!}$. Then the function

$$K(x) := \frac{1}{\gamma_m} \log \frac{1}{|x|}$$

is a fundamental solution of $(-\Delta)^m$ in $\mathbb{R}^{2m}$, i.e. $(-\Delta)^m K = \delta_0$.

Proof The case $m = 1$ is well-known, so we shall assume $m \geq 2$. Set $r := |x|$. For radial functions we have $\Delta = \frac{\partial^2}{\partial r^2} + \frac{n-1}{r}\frac{\partial}{\partial r}$, hence for $j \geq 1$

$$-\Delta \log \frac{1}{r} = \frac{2(m-1)}{r^2}, \quad -\Delta \frac{1}{r^{2j}} = \frac{4j(m-1-j)}{r^{2j+2}}.$$

Then

$$(-\Delta)^j \log \frac{1}{r} = 2^{2j-1} \frac{(j-1)!(m-1)!}{(m-j-1)!} \frac{1}{r^{2j}} \tag{49}$$

$$(-\Delta)^{m-1} \log \frac{1}{r} = 2^{2m-3}(m-2)!(m-1)! \frac{1}{r^{2m-2}}. \tag{50}$$

Given a function $\varphi \in C_c^\infty(\mathbb{R}^{2m})$, we can apply the usual procedure of integrating by parts in $\mathbb{R}^{2m} \backslash B_\varepsilon(0)$ using

$$\lim_{\varepsilon\to 0}\int\limits_{\partial B_\varepsilon(0)} |D^k K| dS = 0, \quad 0 \le k \le 2m-2,$$

to obtain

$$\begin{aligned}\int\limits_{\mathbb{R}^{2m}} (-\Delta)^m \varphi K dx &= \lim_{\varepsilon\to 0}\int\limits_{\partial B_\varepsilon(0)} -\varphi \frac{\partial(-\Delta)^{m-1}K}{\partial\nu} dS \\ &= \fint\limits_{\partial B_\varepsilon(0)} \varphi dS \to \varphi(0).\end{aligned}$$

□

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