Math. Z. (2009) 262:389–409
DOI 10.1007/s00209-008-0380-z

Mathematische Zeitschrift

Local Łojasiewicz exponents, Milnor numbers and mixed multiplicities of ideals

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Received: 17 October 2007 / Published online: 12 June 2008

Abstract Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite analytic map. We give an expression for the local Łojasiewicz exponent and for the multiplicity of g when the component functions of g satisfy certain condition with respect to a set of n monomial ideals $I_1, \dots, I_n$. We give an effective method to compute Łojasiewicz exponents based on the computation of mixed multiplicities. As a consequence of our study, we give a numerical characterization of a class of functions that includes semi-weighted homogenous functions and Newton non-degenerate functions.

Keywords Milnor number · Łojasiewicz exponents · Integral closure of ideals · Mixed multiplicities of ideals · Newton polyhedra

Mathematics Subject Classification (2000) Primary 32S05; Secondary 13H15

1 Introduction

One of the most known invariants of a germ of analytic function $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ with an isolated singularity at the origin is the Milnor number $\mu(f)$ of f. Kouchnirenko expressed in [18] the Milnor number of f in terms of the Newton polyhedron $\Gamma_+(f)$ of f. Another important invariant in singularity theory that has also been studied via Newton polyhedra is the local Łojasiewicz exponent $\mathcal{L}_0(f)$ of f. It is defined as the infimum of those real numbers $\alpha > 0$ such that

$$\|x\|^\alpha \leq C\|\nabla f(x)\|,$$

Work supported by DGICYT Grant MTM2006-06027.

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for some constant $C > 0$ and all x belonging to some open neighbourhood of the origin in $\mathbb{C}^n$, where ∇f denotes the gradient map of f. It is known that $\mathcal{L}_0(f)$ exists if and only if f has an isolated singularity at the origin, and that $\mathcal{L}_0(f)$ is a rational number in this case [19]. Moreover, by a result of Teissier, the degree of topological determinacy of f in $\mathcal{O}_n$ is equal to the smallest integer r such that $\mathcal{L}_0(f) < r$ (see [33, p. 281]). The computation or estimation from above of $\mathcal{L}_0(f)$ is not straightforward at all. We refer to [6,14] or [20] for results about this problem that consider the information supplied by the Newton polyhedron of f.

In this paper we study the number $\mathcal{L}_0(f)$ for all functions f contained in a class ampler than the class of Newton non-degenerate functions studied by Kouchnirenko and with a given Newton polyhedron. In order to give this expression we will look at Milnor numbers and local Łojasiewicz exponents of functions as special cases of the analogous invariants that are defined for arbitrary (not necessarily gradient) maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g^{-1}(0) = \{0\}$.

Let $\mathcal{O}_n$ denote the ring of analytic functions $(\mathbb{C}^n, 0) \to \mathbb{C}$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ denote an analytic map germ such that $g^{-1}(0) = \{0\}$ and let I be the ideal of $\mathcal{O}_n$ generated by the component functions of g. Then the colength $\dim_{\mathbb{C}} \mathcal{O}_n/I$ is also known as the *Milnor number* of g [11,29]. Let us remark that the Milnor number $\mu(g)$ of g, with g being regarded as an isolated complete intersection singularity, is given by $\mu(g) = \dim_{\mathbb{C}} \mathcal{O}_n/I - 1$ (see [22, p. 78]). We denote the colength $\dim_{\mathbb{C}} \mathcal{O}_n/I$ by $m_0(g)$ and we will refer to this number as the *multiplicity* of g. We remark that $m_0(g)$ is equal to the Poincaré–Hopf index of g at 0 (see [29] where an upper bound for $m_0(g)$ is given in terms of the degree of $\mathcal{K}$-determinacy of g). The definition of the Łojasiewicz exponent of g is analogous to that of a function $f \in \mathcal{O}_n$ by substituting the gradient ∇f by the component functions of g. It is known that $m_0(g) \leq [\mathcal{L}_0(g)]^n$, where $[a]$ denotes the integer part of a real number a (see [12] or [26]). We refer to [12,13,23] for important applications of the number $\mathcal{L}_0(g)$ in complex function theory on domains in $\mathbb{C}^n$.

Our study of $m_0(g)$ and of $\mathcal{L}_0(g)$ is based on a concept that we studied in [4] and that we call *Rees' multiplicity of ideals*. This is an integer that is associated to certain families of n ideals, not assumed to have finite colength, in a Noetherian local ring of dimension n (see Definition 2.1 and Remark 2.3). This number extends the notion mixed multiplicity of ideals defined by Teissier and Risler in [31]. We expose the definition of and basic results about Rees' multiplicities in Sect. 2.

Let us fix a family $I_1, \dots, I_n$ of monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$, where $\sigma(I_1, \dots, I_n)$ denotes the Rees' multiplicity of $I_1, \dots, I_n$. In Sect. 2 we recall the main result of [4] on the characterization of those analytic maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$, where $g_i \in I_i$, for all $i = 1, \dots, n$, such that $m_0(g) = \sigma(I_1, \dots, I_n)$. The set of such maps is denoted by $\mathcal{R}(I_1, \dots, I_n)$. This characterization is expressed via the respective Newton polyhedra of $I_1, \dots, I_n$. The elements of $\mathcal{R}(I_1, \dots, I_n)$ are called *strongly non-degenerate* maps with respect to $I_1, \dots, I_n$.

In Sect. 3 we show a formula expressed in terms of $I_1, \dots, I_n$ for the Łojasiewicz exponent of any map $g \in \mathcal{R}(I_1, \dots, I_n)$ such that $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \dots, n$ (see Corollary 3.4). This expression will arise as a consequence of a result about Rees' multiplicities (see Theorem 3.2) and the relation of Łojasiewicz exponents with the integral closure of ideals proven by Lejeune and Teissier [19]. Thus, we define the Łojasiewicz exponent of $I_1, \dots, I_n$, denoted by $\mathcal{L}_0(I_1, \dots, I_n)$, as $\mathcal{L}_0(g)$, where g is any of those maps.

We describe in Sect. 4 an effective method to compute $\mathcal{L}_0(I_1, \dots, I_n)$ via our result of [8] on the computation of the multiplicity of a monomial ideal and an equality proven by

Rees [27] relating the computation of mixed multiplicities with the computation of Samuel multiplicities.

In Sect. 5 we prove that the Newton number of a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$, as defined by Kouchnirenko in [18], is equal to the Rees' multiplicity of certain n ideals attached to Γ_+. This fact leads to a short proof of the monotonicity of Newton numbers with respect to reverse inclusion of Newton polyhedra (see Corollary 5.6). Moreover, in Sect. 5 we also show that the notion of strongly non-degenerate map (see Definition 2.8), when applied to gradient maps, determines a class of functions $f \in \mathcal{O}_n$ that includes semi-weighted homogeneous functions and Newton non-degenerate functions. We give a numerical characterization of these functions via their Milnor number in Corollary 5.8. We also give in Sect. 5 a converse for the result of Kouchnirenko in [18] on the computation of the Milnor number of an analytic function (see Theorem 5.7).

2 Mixed multiplicities of ideals

In this section we show the results of commutative algebra that we will need in order to expose our work. Let (R, m) be a Noetherian local ring and let I be an ideal of R. We denote by $e(I)$ the Samuel multiplicity of I. If we suppose that $\dim R = n$ and that $I_1, \dots, I_n$ are ideals of R of finite colength, we denote by $e(I_1, \dots, I_n)$ the mixed multiplicity of $I_1, \dots, I_n$ defined by Teissier and Risler in [31]. We refer to [17, Sect. 17] for fundamental results about mixed multiplicities of ideals.

Let us suppose that the residue field $k = R/m$ is infinite. Let $I_1, \dots, I_n$ be ideals of R. Let $a_{i1}, \dots, a_{is_i}$ be a generating system of I_i, where $s_i \geq 1$, for $i = 1, \dots, n$. Let $s = s_1 + \cdots + s_n$. We say that a property holds for *sufficiently general* elements of $I_1 \oplus \cdots \oplus I_n$ if there exists a non-empty Zariski-open set U in k^s such that the said property holds for all elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$ such that $g_i = \sum_j u_{ij} a_{ij}$, $i = 1, \dots, n$, where $(u_{11}, \dots, u_{1s_1}, \dots, u_{n1}, \dots, u_{ns_n}) \in U$.

If the ideals $I_1, \dots, I_n$ have finite colength, then we recall that, by virtue of a result of Rees (see [27] or [17, p. 335]), the mixed multiplicity of $I_1, \dots, I_n$ is obtained as $e(I_1, \dots, I_n) = e(g_1, \dots, g_n)$, for a sufficiently general element $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$.

We recall that, if the ideals $I_1, \dots, I_n$ are equal to a given ideal, say I, then $e(I_1, \dots, I_n) = e(I)$. If I and J are two ideals of finite colength of R and $i \in \{0, 1, \dots, n\}$, then $e_i(I, J)$ denotes the mixed multiplicity $e(I, \dots, I, J, \dots, J)$, where I is repeated $n - i$ times and J is repeated i times.

Now we show the definition, introduced by the author in [4], of a number associated to a family of ideals that generalizes the notion of mixed multiplicity. This number is fundamental in the results of this paper. We denote by $\mathbb{Z}_+$ the set of non-negative integers.

Definition 2.1 Let (R, m) be a Noetherian local ring of dimension n. Let $I_1, \dots, I_n$ be ideals of R. Then we define

$$\sigma(I_1, \dots, I_n) = \max_{r \in \mathbb{Z}_+} e(I_1 + m^r, \dots, I_n + m^r), \tag{1}$$

when the number on the right-hand side is finite. If the set $\{e(I_1 + m^r, \dots, I_n + m^r) : r \in \mathbb{Z}_+\}$ is non-bounded then we set $\sigma(I_1, \dots, I_n) = \infty$.

We remark that the ideals $I_1, \dots, I_n$ are not assumed to have finite colength in the above definition. If I_i has finite colength, for all $i = 1, \dots, n$, then we observe that $\sigma(I_1, \dots, I_n)$ equals the mixed multiplicity $e(I_1, \dots, I_n)$, since some power of the maximal ideal is contained in I_i in this case, for all $i = 1, \dots, n$. Proposition 2.2 characterizes the finiteness of $\sigma(I_1, \dots, I_n)$. Obviously $\sigma(I_1, \dots, I_n)$ is not finite for an arbitrary family of ideals $I_1, \dots, I_n$ of R.

Proposition 2.2 [4] *Let $I_1, \dots, I_n$ be ideals of a Noetherian local ring (R, m) such that the residue field $k = R/m$ is infinite. Then $\sigma(I_1, \dots, I_n) < \infty$ if and only if there exist elements $g_i \in I_i$, for $i = 1, \dots, n$, such that $\langle g_1, \dots, g_n \rangle$ has finite colength. In this case, we have that $\sigma(I_1, \dots, I_n) = e(g_1, \dots, g_n)$ for sufficiently general elements $(g_1, \dots, g_n) \in I_1 \oplus \cdots \oplus I_n$.*

Remark 2.3 As pointed out in [4], the previous result shows that if $\sigma(I_1, \dots, I_n) < \infty$, then $\sigma(I_1, \dots, I_n)$ is equal to the mixed multiplicity of $I_1, \dots, I_n$ defined by Rees in [28, p. 181] via the notion of general extension of a local ring. Therefore, we refer to $\sigma(I_1, \dots, I_n)$ as the *Rees' mixed multiplicity* of $I_1, \dots, I_n$. We remark that this multiplicity is not formulated in [28] as in (1).

We will need the following known result (see [17, p. 345] or [30, Lemma 2.4]).

Lemma 2.4 *Let R be a Noetherian local ring of dimension $n \geq 1$. Let $I_1, \dots, I_n$ be ideals of R of finite colength. Let $g_1, \dots, g_n$ be elements of R such that $g_i \in I_i$, for all $i = 1, \dots, n$, and that the ideal $\langle g_1, \dots, g_n \rangle$ has also finite colength. Then*

$$e(g_1, \dots, g_n) \geq e(I_1, \dots, I_n).$$

Corollary 2.5 *Let R be a Noetherian local ring of dimension $n \geq 1$. Let $I_1, \dots, I_n$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$. Let $J_1, \dots, J_n$ be ideals of R such that $J_i \subseteq I_i$, for all $i = 1, \dots, n$, and $\sigma(J_1, \dots, J_n) < \infty$. Then*

$$\sigma(J_1, \dots, J_n) \geq \sigma(I_1, \dots, I_n).$$

Proof It follows as a direct application of Proposition 2.2 and Lemma 2.4. □

Let $I_1, \dots, I_n$ be ideals in a local ring R such that $\sigma(I_1, \dots, I_n) < \infty$. Then we define

$$r(I_1, \dots, I_n) = \min\left\{r \in \mathbb{Z}_+ : \sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)\right\}. \tag{2}$$

If I is an ideal of R, then we denote by $\overline{I}$ the integral closure of I. The number $r(I_1, \dots, I_n)$ is characterized in Sect. 3 in terms of the notion of integral closure of ideals. The following lemma will be useful in Sect. 4.

Lemma 2.6 *Let (R, m) be a local ring of dimension n. Let $I_1, \dots, I_n$ be ideals of R such that $\sigma(I_1, \dots, I_n) < \infty$. Then $\sigma(I_1^{r_1}, \dots, I_n^{r_n}) < \infty$, for all $r_1, \dots, r_n \geq 1$, and*

$$\sigma(I_1^{r_1}, \dots, I_n^{r_n}) = r_1 \dots r_n \sigma(I_1, \dots, I_n),$$

for all $r_1, \dots, r_n \geq 1$.

Proof Let $r_1, \dots, r_n$ be positive integers. For a given $r \geq 1$ we have that

$$\begin{aligned} e(I_1^{r_1} + m^r, \dots, I_n^{r_n} + m^r) &\leq e(I_1^{r_1} + m^{rr_1}, \dots, I_n^{r_n} + m^{rr_n}) \\ &= e(\overline{I_1^{r_1} + m^{rr_1}}, \dots, \overline{I_n^{r_n} + m^{rr_n}}) \\ &= e(\overline{(I_1 + m^r)^{r_1}}, \dots, \overline{(I_n + m^r)^{r_n}}) \\ &= r_1 \dots r_n e(I_1 + m^r, \dots, I_n + m^r) \leq r_1 \dots r_n \sigma(I_1, \dots, I_n). \end{aligned}$$

Then $\sigma(I_1^{r_1}, \dots, I_n^{r_n}) < \infty$, for all $r_1, \dots, r_n \geq 1$, if $\sigma(I_1, \dots, I_n) < \infty$.

Let us fix integers $r_1, \dots, r_n \geq 1$. Let r and r' denote the numbers $r(I_1, \dots, I_n)$ and $r(I_1^{r_1}, \dots, I_n^{r_n})$, respectively. By an argument analogous to the previous discussion and considering the definitions of r and r', we have that if $p \geq \max\{r, r'\}$ then

$$\begin{aligned}\sigma(I_1^{r_1}, \dots, I_n^{r_n}) &= e(I_1^{r_1} + m^p, \dots, I_n^{r_n} + m^p) = e(I_1^{r_1} + m^{pr_1}, \dots, I_n^{r_n} + m^{pr_n}) \\ &= e\left((I_1 + m^p)^{r_1}, \dots, (I_n + m^p)^{r_n}\right) = r_1 \cdots r_n e(I_1 + m^p, \dots, I_n + m^p) \\ &= r_1 \dots r_n \sigma(I_1, \dots, I_n).\end{aligned}$$

□

Let $I_1, \dots, I_n$ be a family of monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. For the sake of completeness, we show the characterization given in [4] of the maps $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ such that $g_i \in I_i$, for all $i = 1, \dots, n$, and that $e(g_1, \dots, g_n) = \sigma(I_1, \dots, I_n)$. Therefore we introduce some preliminary notions.

A subset $\Gamma_+ \subseteq \mathbb{R}^n_+$ is said to be a *Newton polyhedron* when there exists a subset $A \subseteq \mathbb{Z}^n_+$ such that Γ_+ is equal to the convex hull in $\mathbb{R}^n_+$ of the set $\{k + v : k \in A, v \in \mathbb{R}^n_+\}$. In this case we also denote Γ_+ by $\Gamma_+(A)$. A Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$ is termed *convenient* when Γ_+ intersects each coordinate axis.

Let us fix a coordinate system $x_1, \dots, x_n$ in $\mathbb{C}^n$. If $k \in \mathbb{Z}^n_+$, $k \neq 0$, then we denote the monomial $x_1^{k_1} \dots x_n^{k_n}$ by x^k. Let $h \in \mathcal{O}_n$, $h \neq 0$, and let $h = \sum_k a_k x^k$ be the Taylor expansion of h around the origin. The *support* of h, denoted by $\mathrm{supp}(h)$, is defined as the set of those $k \in \mathbb{Z}^n_+$ such that $a_k \neq 0$. Then the *Newton polyhedron of h* is defined as $\Gamma_+(h) = \Gamma_+(\mathrm{supp}(h))$. We say that h is a *convenient* function when $\Gamma_+(h)$ is convenient. If $D \subseteq \mathbb{R}^n_+$ is a compact set of $\mathbb{R}^n_+$, then we denote the polynomial $\sum_{k \in D} a_k x^k$ by h_D. If $\mathrm{supp}(h) \cap D = \emptyset$, then we set $h_D = 0$.

If I is an ideal of $\mathcal{O}_n$ and $g_1, \dots, g_r$ is a generating system of I, then the *Newton polyhedron of I* is defined as the convex hull of $\Gamma_+(g_1) \cup \dots \cup \Gamma_+(g_r)$. As is easy to check, this definition does not depend on the chosen generating system of I.

Given a Newton polyhedron $\Gamma_+ \subseteq \mathbb{R}^n_+$ and a vector $v \in \mathbb{R}^n_+$, $v \neq 0$, we define

$$\begin{aligned}\ell(v, \Gamma_+) &= \min\{\langle v, k\rangle : k \in \Gamma_+\} \\ \Delta(v, \Gamma_+) &= \{k \in \Gamma_+ : \langle v, k\rangle = \ell(v, \Gamma_+)\}.\end{aligned}$$

The sets $\Delta(v, \Gamma_+)$, where $v \in \mathbb{R}^n_+ \setminus \{0\}$, are called *faces* of Γ_+. The union of the compact faces of Γ_+ is called the *Newton boundary* of Γ_+. We remark that $\Delta(v, \Gamma_+)$ is compact if and only if $v \in (\mathbb{R}_+ \setminus \{0\})^n$.

If I is an ideal of $\mathcal{O}_n$ then we define $\ell(v, I) = \ell(v, \Gamma_+(I))$ and $\Delta(v, I) = \Delta(v, \Gamma_+(I))$. If $h \in \mathcal{O}_n$, $h \neq 0$, we define $\ell(v, h)$ and $\Delta(v, h)$ analogously. Given a vector $v \in \mathbb{R}^n_+$, $v \neq 0$, if the Taylor expansion of h around the origin is given by $h = \sum_k a_k x^k$, then we denote by $p_v(h)$ the function obtained as the sum of those terms $a_k x^k$ such that $k \in \mathrm{supp}(h) \cap \Delta(v, h)$.

Let $v = (v_1, \dots, v_n) \in (\mathbb{Z}_+ \setminus \{0\})^n$. If $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is an analytic map, then we say that g is *semi-weighted homogeneous with respect to v* when $p_v(g) = (p_v(g_1), \dots, p_v(g_n)) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a finite map, that is, when $(p_v(g))^{-1}(0) = \{0\}$. It is known that, in this case, we have $m_0(g) = \ell(v, g_1) \dots \ell(v, g_n)/v_1 \dots v_n$ (see for instance [2, Sect. 12]).

Definition 2.7 [4] Let $I_1, \dots, I_p$ be monomial ideals in $\mathcal{O}_n$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $g_i \in I_i$ and $g_i \neq 0$, for all $i = 1, \dots, p$. Let $v \in \mathbb{R}^n_+ \setminus \{0\}$

and let $\Delta_i = \Delta(v, I_i)$, for all $i = 1, \dots, p$. We say that g satisfies the (K_v) *condition with respect to* $I_1, \dots, I_p$ when

$$\{x \in \mathbb{C}^n : (g_1)_{\Delta_1}(x) = \cdots = (g_p)_{\Delta_p}(x) = 0\} \subseteq \{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\}.$$

Then the map g is termed *non-degenerate with respect to* $I_1, \dots, I_p$ when g satisfies the (K_v) condition with respect to $I_1, \dots, I_p$ for all $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$.

Let $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$. Then we define $\mathbb{R}^n_L = \{x \in \mathbb{R}^n : x_i = 0, \text{ for all } i \notin L\}$. We define $\mathbb{C}^n_L$ analogously. If $h \in \mathcal{O}_n$, then h^L denotes the sum of all terms of the Taylor expansion of h whose support belongs to $\mathbb{R}^n_L$. If no such terms exist, then we set $h^L = 0$. We denote by $\mathcal{O}_{n,L}$ the subring of $\mathcal{O}_n$ generated by all functions of $\mathcal{O}_n$ depending at most on the variables x_i such that $i \in L$. We observe that the map $\mathcal{O}_n \to \mathcal{O}_{n,L}$ given by $h \mapsto h^L$ is a ring epimorphism.

If $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ is an analytic map germ, we denote by g^L the map $(g_1^L, \dots, g_p^L) : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^p, 0)$. Moreover, if I is a monomial ideal of $\mathcal{O}_n$, then I^L will denote the ideal of $\mathcal{O}_{n,L}$ generated by all elements h^L, where h varies in I.

Definition 2.8 [4] Let $I_1, \dots, I_p$ be monomial ideals of $\mathcal{O}_n$ such that $I_1 + \cdots + I_p$ is an ideal of finite colength in $\mathcal{O}_n$. Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ be an analytic map germ such that $g_i \in I_i$, for all $i = 1, \dots, p$. We say that g is *strongly non-degenerate with respect to* $I_1, \dots, I_p$ when for all $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$, the map $g^L : (\mathbb{C}^n_L, 0) \to (\mathbb{C}^p, 0)$ is non-degenerate with respect to the non-zero ideals of the sequence $I_1^L, \dots, I_p^L$.

Under the conditions of the previous definition, we denote by $\mathcal{R}(I_1, \dots, I_p)$ the set of all maps $g = (g_1, \dots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$ such that $g_i \in I_i$, for all $i = 1, \dots, p$, and such that g is strongly non-degenerate with respect to $I_1, \dots, I_p$. We remark that, since we assume that $I_1 + \cdots + I_p$ is an ideal of finite colength, then the family of non-zero ideals in the sequence $I_1^L, \dots, I_p^L$ is non-empty, for all $L \subseteq \{1, \dots, n\}$, $L \neq \emptyset$. We denote by $\mathcal{R}_0(I_1, \dots, I_p)$ the family of maps $(g_1, \dots, g_p) \in \mathcal{R}(I_1, \dots, I_p)$ such that $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \dots, p$. As will be seen, we are mainly concerned with the case $p = n$.

We observe that, under the conditions of Definition 2.7, if the ideal I_1 is an ideal generated by a single monomial, say x^k, and $g_1 = x^k$, then the map g is automatically non-degenerate with respect to $I_1, \dots, I_p$. This fact lead us to introduce Definition 2.8 in [4]. However, Proposition 2.10 shows that Definitions 2.7 and 2.8 are equivalent when I_i has finite colength, for $i = 1, \dots, p$.

Example 2.9 Let us consider the ideals of $\mathcal{O}_3$ given by $I_1 = \langle x^5, x^2y, y^5\rangle$, $I_2 = \langle y^7, x^2y^3\rangle$ and $I_3 = \langle z\rangle$. Let $g : (\mathbb{C}^3, 0) \to (\mathbb{C}^3, 0)$ be the map given by

$$g(x, y, z) = \left(x^5 + y^5 + x^2y - 2xy^3, y^7 + x^2y^3 - 2xy^5, z\right).$$

Let us denote by g_1, g_2, g_3 the respective coordinate functions of g. The map g is non-degenerate with respect to I_1, I_2, I_3, since I_3 is generated by a single monomial which is equal to g_3.

Let $L = \{1, 2\}$, we have that $I_1^L = I_1, I_2^L = I_2, I_3^L = 0$. Let $v = (2, 1)$ and let $\Delta_i = \Delta(v, I_i)$, for $i = 1, 2$. The polynomials g_1^L and g_2^L vanish along the curve $y^2 - x = 0$. Then we have that $(g_1^L, g_2^L) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is degenerate with respect to I_1^L, I_2^L, and therefore g is not strongly non-degenerate with respect to I_1, I_2, I_3. However if we replace g_2 by $g_2' = y^7 + x^2y^3$ then $(g_1, g_2', g_3) \in \mathcal{R}(I_1, I_2, I_3)$.

Proposition 2.10 [4] *Let $I_1, \ldots, I_p$ be monomial ideals of finite colength of $\mathcal{O}_n$. Let $g_i \in I_i$, for $i = 1, \ldots, p$, and let us consider the map $g = (g_1, \ldots, g_p) : (\mathbb{C}^n, 0) \to (\mathbb{C}^p, 0)$. Then $g \in \mathcal{R}(I_1, \ldots, I_p)$ if and only if g is non-degenerate with respect to $I_1, \ldots, I_p$.*

The following result gives a numerical characterization of the elements of $\mathcal{R}(I_1, \ldots, I_n)$.

Theorem 2.11 [4] *Let $I_1, \ldots, I_n$ be monomial ideals of $\mathcal{O}_n$. Suppose that $\sigma(I_1, \ldots, I_n) < \infty$. Let $g_1, \ldots, g_n \in \mathcal{O}_n$ such that $g_i \in I_i$, for all $i = 1, \ldots, n$. Then the following conditions are equivalent*:

(1) *the ideal $\langle g_1, \ldots, g_n \rangle$ has finite colength and $\sigma(I_1, \ldots, I_n) = e(g_1, \ldots, g_n)$;*
(2) *$g \in \mathcal{R}(I_1, \ldots, I_n)$.*

Definitions 2.7 and 2.8 are motivated by the notion of Newton non-degenerate function introduced by Kouchnirenko [18]. This notion motivated in turn the definition of Newton non-degenerate ideal (see [8,10] or [32]). Let I be an ideal of $\mathcal{O}_n$ and let $g_1, \ldots, g_r$ be a generating system of I. Then we recall that the ideal I is said to be *Newton non-degenerate* when for each compact face Δ of $\Gamma_+(I)$ we have

$$\left\{x \in \mathbb{C}^n : (g_1)_\Delta(x) = \cdots = (g_r)_\Delta(x) = 0\right\} \subseteq \left\{x \in \mathbb{C}^n : x_1 \cdots x_n = 0\right\}.$$

It is straightforward to see that this definition does not depend on the generating system of I. Then a function $f \in \mathcal{O}_n$ is termed *Newton non-degenerate* when the ideal generated by $x_1 \frac{\partial f}{\partial x_1}, \ldots, x_n \frac{\partial f}{\partial x_n}$ is Newton non-degenerate.

We observe that any monomial ideal is Newton non-degenerate. Moreover, it is clear that an ideal I of $\mathcal{O}_n$ is Newton non-degenerate if and only if I admits a generating system $g_1, \ldots, g_r$ such that the map $(g_1, \ldots, g_r) : (\mathbb{C}^n, 0) \to (\mathbb{C}^r, 0)$ is non-degenerate with respect to $I^0, \ldots, I^0$, with I^0 repeated r times, where I^0 is the monomial ideal of $\mathcal{O}_n$ generated by all x^k such that $k \in \Gamma_+(I)$. If I is an ideal of finite colength, then I is Newton non-degenerate if and only if I admits a generating system $g_1, \ldots, g_r$ such that $(g_1, \ldots, g_r) \in \mathcal{R}(I^0, \ldots, I^0)$, by Proposition 2.10. Hence, we observe that Lemma 2.4 and Theorem 2.11 constitute a generalization of the following theorem (which in turn is extended to modules via the notion of Buchsbaum-Rim multiplicity in [7]).

If $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a convenient Newton polyhedron, then we denote by $\mathrm{V}_n(\Gamma_+)$ the n-dimensional volume of $\mathbb{R}^n_+ \smallsetminus \Gamma_+$.

Theorem 2.12 [8] *Let I be an ideal of $\mathcal{O}_n$ of finite colength. Then $e(I) \geq n!\mathrm{V}_n(\Gamma_+(I))$ and equality holds if and only if I is Newton non-degenerate.*

Corollary 2.13 *Let $I_1, \ldots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \ldots, I_n) < \infty$. Then*

$$\sigma(I_1, \ldots, I_n) \geq e(I_1 + \cdots + I_n)$$

and equality holds if and only if the ideal $\langle g_1, \ldots, g_n \rangle$ is Newton non-degenerate, for all $(g_1, \ldots, g_n) \in \mathcal{R}_0(I_1, \ldots, I_n)$.

Proof It follows as a consequence of Theorems 2.11 and 2.12. □

3 An expression for the Łojasiewicz exponent

If I is an arbitrary ideal of $\mathcal{O}_n$ of finite colength and $g_1, \ldots, g_r$ is a generating system of I, then the *Łojasiewicz exponent of I*, denoted by $\mathcal{L}_0(I)$, is defined as the infimum of those

$\alpha > 0$ such that there exists an open neighbourhood U of 0 in $\mathbb{C}^n$ and a constant $C > 0$ such that

$$\|x\|^{\alpha} \leq C \sup_{1\leq i\leq r} |g_i(x)|, \tag{3}$$

for all $x \in U$.

By a result of Lejeune and Teissier (see [19, p. 55]), we have that $\mathcal{L}_0(g)$ is a rational number and that $\mathcal{L}_0(I)$ satisfies the above Łojasiewicz-type inequality (that is, $\mathcal{L}_0(I)$ is the minimum of the set of $\alpha > 0$ satisfying (3)). By [19, p. 55] we also have

$$\mathcal{L}_0(I) = \min\left\{r/s : m^r \subseteq \overline{I^s}\right\}$$

and that $\mathcal{L}_0(I)$ is equal to the number $\tau^*(I)$ defined by D'Angelo in [12, p. 21]. That is

$$\mathcal{L}_0(I) = \sup_{\gamma\in\mathcal{P}} \left(\inf_{h\in I} \frac{\mathrm{ord}(h\circ\gamma)}{\mathrm{ord}(\gamma)}\right), \tag{4}$$

where $\mathcal{P}$ denotes the set of analytic maps $(\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$. The number on the right of (4) is also known as the *order of contact of* I, for a given ideal I of $\mathcal{O}_n$ [23]. It is proven in [23] that, in the case $n = 2$, the computation of $\mathcal{L}_0(I)$ via relation (4) reduces to considering a finite number of analytic curves $\gamma : (\mathbb{C}, 0) \to (\mathbb{C}^n, 0)$.

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite analytic map germ, that is, a map such that 0 is isolated in $g^{-1}(0)$. Then the *Łojasiewicz exponent of* g is defined as the Łojasiewicz exponent of the ideal generated by the component functions of g. We denote this number by $\mathcal{L}_0(g)$. In this section we express the Łojasiewicz exponent of a map $g \in \mathcal{R}_0(I_1, \dots, I_n)$ in terms of $I_1, \dots, I_n$.

Lemma 3.1 *Let $I_1, \dots, I_n$ be monomial ideals in $\mathcal{O}_n$ and let $g_i \in I_i$, $i = 1, \dots, n$. Let us suppose that $g = (g_1, \dots, g_n)$ is non-degenerate with respect to $(I_1, \dots, I_n)$ and that $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \dots, n$. Let r be a positive integer. Then there exist $\mathbb{C}$-linear combinations $h_1, \dots, h_n$ of $x_1^r, \dots, x_n^r$ such that $(g_1 + h_1, \dots, g_n + h_n)$ is non-degenerate with respect to $(I_1 + m^r, \dots, I_n + m^r)$.*

Proof Let us fix a vector $v = (v_1, \dots, v_n) \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and let $\Delta_i = \Delta(v, I_i + m^r)$, for all $i = 1, \dots, n$. Let $\Delta_i' = \Delta_i \cap \Gamma(I_i)$, for all $i = 1, \dots, n$, where $\Gamma(I_i)$ denotes the union of all compact faces of $\Gamma_+(I_i)$, for $i = 1, \dots, n$. Then Δ_i' is either a face of $\Gamma_+(I_i)$ or empty, for all $i = 1, \dots, n$.

Let us suppose that $h_1, \dots, h_n$ are $\mathbb{C}$-linear combinations of $x_1^r, \dots, x_n^r$. If $\Delta_i \cap \Delta(v, m^r) = \emptyset$, for all $i = 1, \dots, n$, then $(g_i + h_i)_{\Delta_i} = (g_i)_{\Delta_i'}$, for all $i = 1, \dots, n$. Thus no conditions on the polynomials $h_1, \dots, h_n$ are needed in order to ensure that the set of common zeros of $(g_i + h_i)_{\Delta_i}$, $i = 1, \dots, n$, is contained in $\{x \in \mathbb{C}^n : x_1 \dots x_n = 0\}$, since g is non-degenerate with respect to $I_1, \dots, I_n$.

Let $B = \{i : \Delta_i' \neq \emptyset\}$, let $v_0 = \min_i v_i$ and let $L = \{i : v_i = v_0\}$. Let $e_1, \dots, e_n$ denote the canonical basis of $\mathbb{R}_+^n$. Since $\Gamma_+(g_i) = \Gamma_+(I_i)$, for all $i = 1, \dots, n$, then we have $(g_i)_{\Delta_i} = (g_i)_{\Delta_i'} \neq 0$, for all $i \in B$. Moreover it is straightforward to see that

$$(h_i)_{\Delta_i} = \begin{cases} h_i^L, & \text{if } \Delta_i \cap \Delta(v, m^r) \neq \emptyset \\ 0, & \text{otherwise.} \end{cases}$$

Let $C = \{i : \Delta_i \cap \Delta(v, m^r) \neq \emptyset\}$. Let $i \in \{1, \dots, n\}$, then

$$(g_i + h_i)_{\Delta_i} = \begin{cases} (g_i)_{\Delta'_i}, & \text{if } i \in B \smallsetminus C \\ (g_i)_{\Delta'_i} + h_i^L, & \text{if } i \in B \cap C \\ h_i^L, & \text{if } i \notin B. \end{cases} \tag{5}$$

We have that $(g_i)_{\Delta'_i}$ is a non-zero weighted homogeneous polynomial with respect to $(v_1, \dots, v_n)$, for all $i \in B$. Therefore, by (5), we can choose the $\mathbb{C}$-linear combinations of $x_1^r, \dots, x_n^r$ defining the polynomials $h_1, \dots, h_n$ in such a way that the greatest common divisor of the set of non-zero polynomials $\{(g_i + h_i)_{\Delta_i} : i = 1, \dots, n\}$ is a monomial. Then the result follows, since the Newton polyhedron of $(I_1 + m^r) \cdots (I_n + m^r)$ has a finite number of faces. □

Theorem 3.2 *Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$. Let r be a positive integer. Then the following conditions are equivalent*:

(1) $\sigma(I_1, \dots, I_n) = e(I_1 + m^r, \dots, I_n + m^r)$;
(2) $m^r \subseteq \overline{\langle g_1, \dots, g_n \rangle}$, *for all* $g \in \mathcal{R}(I_1, \dots, I_n)$;
(3) $m^r \subseteq \overline{\langle g_1, \dots, g_n \rangle}$, *for some* $g \in \mathcal{R}_0(I_1, \dots, I_n)$.

Proof Let us see (1) ⇒ (2). Let $g = (g_1, \dots, g_n) \in \mathcal{R}(I_1, \dots, I_n)$ and let H denote the ideal of $\mathcal{O}_n$ generated by $g_1, \dots, g_n$. Then, let us suppose that $e(H) = e(I_1 + m^r, \dots, I_n + m^r)$. By Rees' multiplicity Theorem (see [17, p. 222]), we have that $m^r \subseteq \overline{H}$ if and only if $e(H) = e(H + m^r)$. We also have that $\overline{m^r} = \overline{\langle x_1^r, \dots, x_n^r \rangle}$ (see [17, Proposition 8.1.5]). Hence $e(H + m^r) = e(H + \langle x_1^r, \dots, x_n^r \rangle)$.

Let $J = \langle x_1^r, \dots, x_n^r \rangle$. From a result of Northcott and Rees (see [24, p. 153] or [17, p. 166]), the multiplicity of $e(H + \langle x_1^r, \dots, x_n^r \rangle)$ is equal to the multiplicity $e(f_1 + h_1, \dots, f_n + h_n)$, where $(f_1, \dots, f_n)$ and $(x_1^r, \dots, x_n^r)$ are sufficiently general elements of $H \oplus \cdots \oplus H$ and $J \oplus \cdots \oplus J$, respectively. Then, let D and G be squared matrices of size n with entries in $\mathbb{C}$ such that

$$[D\,|G\,]\, V^t = \begin{bmatrix} f_1 + h_1 \\ \vdots \\ f_n + h_n \end{bmatrix},$$

where V^t denotes the transpose of the $1 \times 2n$ matrix $V = [g_1 \ \cdots \ g_n \ x_1^r \ \cdots \ x_n^r]$ and $[D|G]$ denotes the juxtaposition of the matrices D and G. Since the coefficients of D are generic, we can suppose that D is invertible. In particular, we find that

$$\left[\mathbf{I}_n \,\middle|\, D^{-1}G\right] V^t = D^{-1} \begin{bmatrix} f_1 + h_1 \\ \vdots \\ f_n + h_n \end{bmatrix}, \tag{6}$$

where $\mathbf{I}_n$ stands for the identity matrix of size n. Therefore, the entries of the matrix on the left hand side of (6) are of the form $g_1 + h'_1, \dots, g_n + h'_n$, where h'_i is a $\mathbb{C}$-linear combination of $x_1^r, \dots, x_n^r$, for all $i = 1, \dots, n$. Relation (6) implies that

$$\langle f_1 + h_1, \dots, f_n + h_n \rangle = \langle g_1 + h'_1, \dots, g_n + h'_n \rangle.$$

Then the ideal $\langle g_1 + h'_1, \dots, g_n + h'_n \rangle$ has also finite colength and $e(g_1 + h'_1, \dots, g_n + h'_n) \geq e(I_1 + m^r, \dots, I_n + m^r)$ by Lemma 2.4. In particular, we have

$$\begin{aligned} e(H) &\geq e(H+m^r) = e(f_1+h_1,\dots,f_n+h_n) = e(g_1+h_1',\dots,g_n+h_n') \\ &\geq e(I_1+m^r,\dots,I_n+m^r) = \sigma(I_1,\dots,I_n) = e(H). \end{aligned}$$

Thus $e(H) = e(H+m^r)$ and consequently $m^r \subseteq \overline{H}$.

The implication (2) $\Rightarrow$ (3) is obvious. Let us see (3) $\Rightarrow$ (1). Let $g \in \mathcal{R}_0(I_1,\dots,I_n)$ such that $m^r \subseteq \overline{\langle g_1,\dots,g_n\rangle}$.

By Lemma 3.1 there exist $\mathbb{C}$-linear combinations $h_1,\dots,h_n$ of $x_1^r,\dots,x_n^r$ such that the map $(g_1+h_1,\dots,g_n+h_n)$ is non-degenerate with respect to $I_1+m^r,\dots,I_n+m^r$. In particular, we have that $e(g_1+h_1,\dots,g_n+h_n) = e(I_1+m^r,\dots,I_n+m^r)$, by Proposition 2.10 and Theorem 2.11. Let us suppose that $m^r \subseteq \overline{\langle g_1,\dots,g_n\rangle}$. Then $e(g_1+h_1,\dots,g_n+h_n) \geq e(g_1,\dots,g_n)$. Therefore

$$\begin{aligned} \sigma(I_1,\dots,I_n) &\geq e(I_1+m^r,\dots,I_n+m^r) = e(g_1+h_1,\dots,g_n+h_n) \\ &\geq e(g_1,\dots,g_n) = \sigma(I_1,\dots,I_n), \end{aligned}$$

where we have applied Theorem 2.11 in the last equality. Then $e(I_1+m^r,\dots,I_n+m^r) = \sigma(I_1,\dots,I_n)$. □

Let $I_1,\dots,I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1,\dots,I_n) < \infty$ and let $(g_1,\dots,g_n) \in \mathcal{R}_0(I_1,\dots,I_n)$. Then, from Theorem 3.2, we have

$$r(I_1,\dots,I_n) = \min\left\{r \geq 1 : m^r \subseteq \overline{\langle g_1,\dots,g_n\rangle}\right\}. \tag{7}$$

Despite the above equality, we remark that the ideals $\langle g_1,\dots,g_n\rangle$, where $(g_1,\dots,g_n)$ varies in $\mathcal{R}_0(I_1,\dots,I_n)$, do not have the same integral closure (it is easy to find some examples).

In the next example we show that relation (7) does not hold for an arbitrary $g \in \mathcal{R}(I_1,\dots,I_n)$.

Example 3.3 Let I_1 and I_2 be the ideals of $\mathcal{O}_2$ given by $I_1 = \langle x^5\rangle$ and $I_2 = \langle xy, y^3\rangle$. Let $g_1 = x^5$, $g_2 = y^3$ and let $I = \langle g_1, g_2\rangle$. Then we observe that $\sigma(I_1,I_2) = e(g_1,g_2) = 15$. Hence $(g_1,g_2) \in \mathcal{R}(I_1,I_2) \smallsetminus \mathcal{R}_0(I_1,I_2)$, since $\Gamma_+(g_2) \neq \Gamma_+(I_2)$. Moreover, the fact that I is a monomial ideal implies that $m^5 \subseteq \overline{I}$. However, a simple computation shows that $e(I_1+m^5, I_2+m^5) = 10 < \sigma(I_1,I_2)$.

Corollary 3.4 *Let $I_1,\dots,I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1,\dots,I_n) < \infty$. If $g \in \mathcal{R}_0(I_1,\dots,I_n)$, then $\mathcal{L}_0(g)$ depends only on $I_1,\dots,I_n$ and it is given by:*

$$\mathcal{L}_0(g) = \min_{s\geq 1} \frac{r(I_1^s,\dots,I_n^s)}{s}. \tag{8}$$

Proof By a result of Lejeune and Teissier [19], we have

$$\mathcal{L}_0(g) = \min\left\{r/s \in \mathbb{Q}_+ : m^r \subseteq \overline{\langle g_1,\dots,g_n\rangle^s}\right\}, \tag{9}$$

for any analytic map germ $g : (\mathbb{C}^n,0) \to (\mathbb{C}^n,0)$ such that $g^{-1}(0) = \{0\}$. Let us suppose that $g \in \mathcal{R}_0(I_1,\dots,I_n)$ and let r and s be positive integers. Then it is easy to see that $(g_1^s,\dots,g_n^s) \in \mathcal{R}_0(I_1^s,\dots,I_n^s)$. Hence, by Theorem 3.2, it follows that $m^r \subseteq \overline{\langle g_1^s,\dots,g_n^s\rangle}$ if and only if

$$\sigma(I_1^s,\dots,I_n^s) = e\left(I_1^s+m^r,\dots,I_n^s+m^r\right).$$

Moreover we have $\overline{\langle g_1, \dots, g_n \rangle^s} = \overline{\langle g_1^s, \dots, g_n^s \rangle}$ (see [17, Proposition 8.1.5]). Then, from (9) we conclude

$$\mathcal{L}_0(g) = \min \left\{ r/s \in \mathbb{Q}_+ : \sigma(I_1^s, \dots, I_n^s) = e\left(I_1^s + m^r, \dots, I_n^s + m^r\right) \right\}. \tag{10}$$

Then, applying (10) and the definition of $r(I_1^s, \dots, I_n^s)$, $s \geq 1$, the result follows. □

Remark 3.5 Under the conditions of the previous result, if we do not assume that $\Gamma_+(I_i) = \Gamma_+(g_i)$, for all $i = 1, \dots, n$, then $\mathcal{L}_0(g)$ depends only on the ideals $J_1, \dots, J_n$ where J_i is the ideal generated by all monomials x^k such that $k \in \operatorname{supp}(g_i)$, for $i = 1, \dots, n$, by Corollary 3.4.

Under the conditions of Corollary 3.4, we will denote the number on the right hand side of (8) by $\mathcal{L}_0(I_1, \dots, I_n)$ and we call this number the *Łojasiewicz exponent of* $I_1, \dots, I_n$. As we see in the next section, the computation of $\mathcal{L}_0(I_1, \dots, I_n)$ is not obvious.

Corollary 3.6 *Let* $I_1, \dots, I_n$ *be monomial ideals of* $\mathcal{O}_n$ *such that* $\sigma(I_1, \dots, I_n) < \infty$. *Then*

$$r(I_1, \dots, I_n) - 1 < \mathcal{L}_0(I_1, \dots, I_n) \leq r(I_1, \dots, I_n).$$

Proof Let $g \in \mathcal{R}_0(I_1, \dots, I_n)$ and let $r(g)$ denote the minimum of those $r \geq 1$ such that $m^r \subseteq \overline{\langle g_1, \dots, g_n \rangle}$. Then the result follows from (7) and the fact that $r(g)$ is the least integer bigger than or equal to $\mathcal{L}_0(g)$. □

The previous result can be seen as a extension to non-gradient maps of the main result of [1].

4 On the effective computation of Łojasiewicz exponents

If $I_1, \dots, I_n$ are monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$, then we show a method to compute $\mathcal{L}_0(I_1, \dots, I_n)$ that is based on the following result of Płoski [25]. In practise, this method requires a powerful computational tool.

We recall from the Introduction that, if $g = (g_1, \dots, g_n) : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ is a finite analytic map germ, then $m_0(g)$ denotes the multiplicity of g at the origin. That is $m_0(g) = e(g_1, \dots, g_n)$.

Theorem 4.1 [25, p. 358] *Let* $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ *be an analytic map germ such that* $g^{-1}(0) = \{0\}$. *Let us write* $\mathcal{L}_0(g) = \frac{p}{q}$, *where* p, q *are relative prime positive integers. Then* $1 \leq q \leq p \leq m_0(g)$.

In the remaining section let us fix n monomial ideals $I_1, \dots, I_n$ of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n)$ is finite. We will denote $\sigma(I_1, \dots, I_n)$ by σ. For each integer s such that $1 \leq s \leq \sigma$, let us define

$$r_s = \begin{cases} r(I_1^s, \dots, I_n^s), & \text{if } r(I_1^s, \dots, I_n^s) \leq \sigma \\ 0, & \text{otherwise.} \end{cases} \tag{11}$$

Corollary 4.2 *Under the above conditions we have*

$$\mathcal{L}_0(I_1, \dots, I_n) = \min_{\substack{1 \leq s \leq r_s \leq \sigma \\ (s, r_s) = 1}} \frac{r_s}{s}.$$

Proof Let us suppose, by Corollary 3.4, that $\mathcal{L}_0(I_1, \dots, I_n) = \frac{r}{s}$, where $s > 0$ and $r = r(I_1^s, \dots, I_n^s)$. Then

$$\sigma(I_1^s, \dots, I_n^s) = e(I_1^s + m^r, \dots, I_n^s + m^r). \tag{12}$$

Let us write $r = ar'$ and $s = as'$, for some positive integers r', s', where a is the greatest common divisor of r and s. Then r'/s' is an irreducible fraction. From Lemma 2.6 and the properties of mixed multiplicity (see [17, p. 152]) we have

$$\sigma(I_1^s, \dots, I_n^s) = a^n \sigma(I_1^{s'}, \dots, I_n^{s'})$$
$$e(I_1^s + m^r, \dots, I_n^s + m^r) = a^n e(I_1^{s'} + m^{r'}, \dots, I_n^{s'} + m^{r'}).$$

Then relation (12) is equivalent to

$$\sigma(I_1^{s'}, \dots, I_n^{s'}) = e(I_1^{s'} + m^{r'}, \dots, I_n^{s'} + m^{r'}).$$

Hence $\mathcal{L}_0(I_1, \dots, I_n)$ is equal to the minimum between the quotients $r(I_1^s, \dots, I_n^s)/s$, where $s \geq 1$ and $r(I_1^s, \dots, I_n^s)/s$ is irreducible. Moreover, the number $\mathcal{L}_0(I_1, \dots, I_n)$ is realized as the Łojasiewicz exponent of an analytic map germ $g \in \mathcal{R}_0(I_1, \dots, I_n)$, by Corollary 3.4. Then, the result follows easily from Theorem 4.1. □

In view of the preceding result, if σ is known then the computation of $\mathcal{L}_0(I_1, \dots, I_n)$ reduces to compute the numbers r_s defined in (11). That is, for each integer $s \in \{1, \dots, \sigma\}$, we need to compute the minimum between the integers $r \in \{s, \dots, \sigma\}$ such that

$$s^n \sigma = e(I_1^s + m^r, \dots, I_n^s + m^r). \tag{13}$$

Let us fix an integer $s \in \{1, \dots, \sigma\}$. In order to compute the multiplicity on the right of (13), we point out that, by a result of Rees [27, p. 409], it is known that if $J_1, \dots, J_n$ are ideals of finite colength in a Noetherian local ring of dimension n, then

$$e(J_1, \dots, J_n) = \frac{1}{n!} \sum_{\substack{L \subseteq \{1,\dots,n\} \\ L \neq \emptyset}} (-1)^{n-|L|} e\left(\prod_{i \in L} J_i \right). \tag{14}$$

If we suppose that $J_1, \dots, J_n$ are monomial ideals of $\mathcal{O}_n$ of finite colength then the multiplicities $e(\prod_{i\in L} J_i)$ that appear in (14) can be computed effectively through the method shown in [8] to compute the multiplicity of a monomial ideal. Let us explain this. Let J be a monomial ideal of $\mathcal{O}_n$ of finite colength and let h denote the sum of all monomials x^k such that k is a vertex of $\Gamma_+(J)$. Then by [8, Theorem 5.1] we have

$$e(J) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{\langle x_1 \frac{\partial h}{\partial x_1}, \dots, x_n \frac{\partial h}{\partial x_n} \rangle}. \tag{15}$$

Hence the mixed multiplicity $e(I_1^s + m^r, \dots, I_n^s + m^r)$ of (13) can be computed by taking $J_i = I_i^s + m^r$, for all $i = 1, \dots, n$, in relation (14) and then computing the multiplicities of the monomial ideals involved in (14) via the equality (15). Therefore the number r_s is computed effectively by testing the equality (13) for all $r \in \{s, \dots, \sigma\}$ and thus $\mathcal{L}_0(I_1, \dots, I_n)$ is obtained via Corollary 4.2.

Example 4.3 Let us consider the ideals of $\mathcal{O}_3$ given by $I_1 = \langle x^2, y^3, z \rangle$, $I_2 = \langle xy^2, z^2 \rangle$ and $I_3 = \langle z \rangle$. Given an analytic map $g \in \mathcal{R}_0(I_1, I_2, I_3)$ then it is straightforward to see that g is semi-weighted homogeneous with respect to $w = (3, 2, 6)$. Then $\sigma(I_1, I_2, I_3) = 7$. Applying Corollary 4.2 and the method to compute the numbers $r(I_1^s, I_2^s, I_3^s)$, for $1 \leq s \leq 7$, we obtain that $\mathcal{L}_0(I_1, I_2, I_3) = \frac{7}{2}$. The colengths involved in the computation of the integers $r(I_1^s, I_2^s, I_3^s)$ have been obtained with the aid of the program *Singular* [16].

Let $g : (\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ be a finite analytic map. Then it is known that $m_0(g) \leq [\mathcal{L}_0(g)]^n$ (see [12] or [26]). Therefore, from this fact and Theorem 4.1, if $\mathcal{L}_0(g)$ is not an integer then it follows that

$$\mathcal{L}_0(g) = N + \frac{b}{a}, \tag{16}$$

where N is an integer and a, b are relatively prime integers such that $0 < b < a < N^{n-1}$ (see also [25] or [26]). Then we obtain the following result.

Corollary 4.4 *Let $r = r(I_1, \dots, I_n)$ and let*

$$\theta = r - 1 + \frac{(r-1)^{n-1} - 2}{(r-1)^{n-1} - 1}. \tag{17}$$

Let us suppose that $\theta = \frac{c}{d}$, where c, d are relatively prime positive integers. Then

(1) *either $\mathcal{L}_0(I_1, \dots, I_n) = r$ or $\mathcal{L}_0(I_1, \dots, I_n) = r - 1 + \frac{b}{a}$, where a,b are relatively prime integers such that $0 < b < a < (r-1)^{n-1}$;*
(2) *we have $\mathcal{L}_0(I_1, \dots, I_n) < r$ if and only if*

$$d^n \sigma(I_1, \dots, I_n) = e(I_1^d + m^c, \dots, I_n^d + m^c). \tag{18}$$

Proof The first part follows easily from (16) and Corollary 3.6.

As we saw in the proof of Corollary 3.4, we have

$$\mathcal{L}_0(I_1, \dots, I_n) = \min\left\{r/s \in \mathbb{Q}_+ : s^n \sigma(I_1, \dots, I_n) = e\left(I_1^s + m^r, \dots, I_n^s + m^r\right)\right\}. \tag{19}$$

It is straightforward to see that the greatest number of the form $\frac{b}{a}$ such that $0 < b < a < (r-1)^{n-1}$ is given by $\frac{(r-1)^{n-1}-2}{(r-1)^{n-1}-1}$. Then the second part of the corollary follows as a consequence of this fact and relation (19). □

We remark that condition (18) can be tested by using relations (14) and (15). We denote by $\theta(I_1, \dots, I_n)$ the number defined in (17), where $I_1, \dots, I_n$ are ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) < \infty$.

Example 4.5 Let us consider the ideals of $\mathcal{O}_3$ given by $I_1 = \langle x, y^3, z^3 \rangle$, $I_2 = \langle y^2, z^2 \rangle$, $I_3 = \langle z^4 \rangle$. We observe that, if $g \in \mathcal{R}_0(I_1, I_2, I_3)$, then g is semi-weighted homogeneous with respect to the weights $w = (3, 1, 1)$. Then $\sigma(I_1, I_2, I_3) = 8$ and, following the method described before Example 4.3, we find that $r(I_1, I_2, I_3) = 4$. As a consequence we have $\theta(I_1, I_2, I_3) = \frac{31}{8}$. Moreover

$$\sigma(I_1^8, I_2^8, I_3^8) = 8^3 \sigma(I_1, I_2, I_3) = 4096$$
$$e(I_1^8 + m^{31}, I_2 + m^{31}, I_3^8 + m^{31}) = 3968.$$

Since these numbers are not equal we conclude that $\mathcal{L}_0(I_1, I_2, I_3) = r(I_1, I_2, I_3) = 4$, by Corollary 4.4.

Example 4.6 Let us consider the ideals $I_1 = \langle x^5, x^2y^2, y^5 \rangle$ and $I_2 = \langle x^3y^3 \rangle$ of $\mathcal{O}_2$. Any element $g = (g_1, g_2) \in \mathcal{R}_0(I_1, I_2)$ verifies that g is non-degenerate with respect to the Newton filtration in $\mathcal{O}_2$ defined by $\Gamma_+(I_1)$ (see the details about this definition in [10]).

Then, as a consequence of [10, Theorem 3.3] have that $e(g_1, g_2) = 30$. Thus $\sigma(I_1, I_2) = 30$, by Theorem 2.11. Moreover we have $r(I_1, I_2) = 8$ and $\theta(I_1, I_2) = \frac{47}{6}$. We also obtain

$$\sigma(I_1^6, I_2^6) = 6^2\sigma(I_1, I_2) = 1080$$
$$e(I_1^6 + m^{47}, I_2^6 + m^{47}) = 1080.$$

Then $\mathcal{L}_0(I_1, I_2) < 8$. In fact, using Corollary 4.2, we deduce $\mathcal{L}_0(I_1, I_2) = \frac{15}{2}$.

Let $e_1, \dots, e_n$ denote the canonical basis of $\mathbb{R}^n_+$. Let $I_1, \dots, I_n$ be monomial ideals of $\mathcal{O}_n$ such that $\sigma(I_1, \dots, I_n) = e(I_1 + \cdots + I_n) < \infty$. Then, as a consequence of Corollary 2.13 and [5, Corollary 3.6], we have that $\mathcal{L}_0(I_1, \dots, I_n)$ is an integer and it is given by

$$\mathcal{L}_0(I_1, \dots, I_n) = \max\{P_1, \dots, P_n\},$$

where $P_i \in \mathbb{Z}_+$, for all $i = 1, \dots, n$, and $P_i e_i$ denotes the point where the Newton boundary of $\Gamma_+(I_1 + \cdots + I_n)$ intersects the x_i-axis, for $i = 1, \dots, n$.

We remark that if I_1, I_2 are two monomial ideals of $\mathcal{O}_2$ such that $\sigma(I_1, I_2) < \infty$ and if $g = (g_1, g_2) : (\mathbb{C}^2, 0) \to (\mathbb{C}^2, 0)$ is a finite analytic map such that $\Gamma_+(g_i) = \Gamma_+(I_i)$, for $i = 1, 2$, then g is non degenerate with respect to I_1, I_2 if and only if the map g satisfies the condition given by Lenarcik in [20, Definition 4.1]. Therefore [20, Theorem 4.2] shows an effective computation of $\mathcal{L}_0(g)$, for all $g \in \mathcal{R}_0(I_1, I_2)$ in terms of certain combinatorial aspects of $\Gamma_+(I_1)$ and $\Gamma_+(I_2)$ that are easily computable (see also [9, Theorem 4.3]). The techniques applied in the proof of the said result of Lenarcik for maps of two complex variables are based on the Newton–Puiseux theorem.

The next result helps in the understanding of the failure of semicontinuity of Łojasiewicz exponents [12,13,23].

Proposition 4.7 *Let (R, m) be a Noetherian local ring of dimension n. For each $i = 1, \dots, n$ let us consider ideals I_i and J_i such that $I_i \subseteq J_i$. Let suppose that $\sigma(I_1, \dots, I_n) < \infty$ and that $\sigma(I_1, \dots, I_n) = \sigma(J_1, \dots, J_n)$. Then*

$$\mathcal{L}_0(I_1, \dots, I_n) \leq \mathcal{L}_0(J_1, \dots, J_n). \tag{20}$$

Proof If r, s are positive integers then

$$s^n\sigma(I_1, \dots, I_n) = \sigma(I_1^s, \dots, I_n^s) \geq e(I_1^s + m^r, \dots, I_n^s + m^r) \geq e(J_1^s + m^r, \dots, J_n^s + m^r).$$

Since $\sigma(I_1, \dots, I_n) = \sigma(J_1, \dots, J_n)$ it follows that $r(I_1^s, \dots, I_n^s) \leq r(J_1^s, \dots, J_n^s)$. Therefore

$$\min_{s \geq 1} \frac{r(I_1^s, \dots, I_n^s)}{s} \leq \min_{s \geq 1} \frac{r(J_1^s, \dots, J_n^s)}{s}.$$

□

The strict inequality in (20) can hold, as we will see in the next example.

Example 4.8 This is inspired by the example (5.1) of [23]. Let us consider the ideals of $\mathcal{O}_2$ given by $I_1 = \langle x^3, y^8 \rangle$, $I_2 = \langle x^2, y^{101} \rangle$, $J_1 = \langle x, x^3, y^8 \rangle$ and $J_2 = I_2$. Let us define the functions $f_s = sx + x^3 + y^8$ and $g = x^2 - y^{101}$, where $s \geq 0$ is parameter.

We observe that (f_s, g) is strongly non-degenerate with respect to J_1, J_2 and that (f_0, g) is strongly non-degenerate with respect to I_1, I_2. Moreover $e(f_s, g) = e(f_0, g) = 16$, for all $s \geq 0$. Then

$$\sigma(I_1, I_2) = 16 = \sigma(J_1, J_2),$$

by Theorem 2.11. Then we can apply Proposition 4.7 to deduce that $\mathcal{L}_0(I_1, I_2) \leq \mathcal{L}_0(J_1, J_2)$. We remark that $\mathcal{L}_0(I_1, I_2) = \mathcal{L}_0(f_0, g)$ and that $\mathcal{L}_0(J_1, J_2) = \mathcal{L}_0(f_s, g)$, if $s > 0$, by Corollary 3.4. In fact, by [23] we have $\mathcal{L}_0(f_0, g) = 8$ and $\mathcal{L}_0(f_s, g) = 16$, if $s > 0$ (these computations can be done also via Corollary 4.4).

5 Mixed multiplicities of monomial ideals and Milnor numbers

In this section, we show that the Rees' mixed multiplicity of certain ideals attached to a Newton polyhedron Γ_+ is equal to the Newton number $\nu(\Gamma_+)$ defined by Kouchnirenko.

If $f \in \mathcal{O}_n$, we denote by ∇f the gradient map of f. Then ∇f is the map $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ given by

$$\nabla f(x) = \left(\frac{\partial f}{\partial x_1}(x), \dots, \frac{\partial f}{\partial x_n}(x)\right).$$

We denote by $J(f)$ the *Jacobian ideal* of f, that is, the ideal generated by the components of ∇f. We denote by $I(f)$ the ideal of $\mathcal{O}_n$ generated by

$$x_1 \frac{\partial f}{\partial x_1}, \dots, x_n \frac{\partial f}{\partial x_n}.$$

We will also write f_{x_i} instead of $\frac{\partial f}{\partial x_i}$, for all $i = 1, \dots, n$.

Let Γ_+ be a Newton polyhedron in $\mathbb{R}^n_+$. We denote by $\mathcal{O}(\Gamma_+)$ the set of all functions $f \in \mathcal{O}_n$ such that $\Gamma_+(f) = \Gamma_+$ and f has an isolated singularity at the origin, that is, $(\nabla f)^{-1}(0) = \{0\}$. We recall that if $f \in \mathcal{O}_n$ has an isolated singularity at the origin, then the *Milnor number* of f is defined as $\mu(f) = \dim_{\mathbb{C}} \mathcal{O}_n / J(f)$.

If $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a convenient Newton polyhedron, then Kouchnirenko defined in [18] the *Newton number* of Γ_+ as

$$\nu(\Gamma_+) = n!\mathrm{V}_n(\Gamma_+) - (n-1)!\mathrm{V}_{n-1}(\Gamma_+) + \cdots + (-1)^{n-1}\mathrm{V}_1(\Gamma_+) + (-1)^n,$$

where $\mathrm{V}_i(\Gamma_+)$ denotes the sum of the i-dimensional volumes of the intersection of $\mathbb{R}^n_+ \smallsetminus \Gamma_+$ with the coordinate planes of dimension i, for all $i = 1, \dots, n-1$.

Let us suppose that $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a Newton polyhedron that is not convenient. Let Q denote the set of indices $i \in \{1, \dots, n\}$ such that Γ_+ does not intersect the x_i-axis. Let Γ be the union of the compact faces of Γ_+ and let ρ_Γ denote the sum of the monomials x^k such that $k \in \Gamma$. Then the *Newton number* of Γ_+, also denoted by $\nu(\Gamma_+)$, is defined as

$$\nu(\Gamma_+) = \sup_{r \in \mathbb{Z}_+} \nu\left(\Gamma_+\left(\rho_\Gamma + \sum_{i \in Q} x_i^r\right)\right). \tag{21}$$

We observe that in this case we could have $\nu(\Gamma_+) = \infty$. Now we recall a celebrated result of Kouchnirenko.

Theorem 5.1 [18] *Let Γ_+ be a Newton polyhedron of $\mathbb{R}^n_+$ such that $\mathcal{O}(\Gamma_+) \neq \emptyset$. Then $\nu(\Gamma_+) < \infty$ and $\mu(f) \geq \nu(\Gamma_+)$, for all $f \in \mathcal{O}(\Gamma_+)$. Moreover, the equality $\mu(f) = \nu(\Gamma_+)$ holds for all Newton non-degenerate function $f \in \mathcal{O}(\Gamma_+)$.*

If Γ_+ is a Newton polyhedron of $\mathbb{R}^n_+$ such that Γ_+ is not convenient and Γ_+ has some face of dimension $n-1$, then it is shown in [18, p. 18] a constructive method to compute $\nu(\Gamma_+)$.

Definition 5.2 If Γ_+ is a Newton polyhedron in $\mathbb{R}^n_+$. For all $i = 1, \dots, n$, we define the *i-th Jacobian ideal of* Γ_+ as

$$J_i(\Gamma_+) = \left\langle x^\nu : \nu \in \Gamma_+(f_{x_i}),\ f \in \mathcal{O}_n,\ \Gamma_+(f) = \Gamma_+ \right\rangle.$$

We observe that $J_i(\Gamma_+)$ is generated by all monomials x^ν whose exponent ν belongs to the set $\{k - e_i : k \in \Gamma_+,\ k_i > 0\}$, for all $i = 1, \dots, n$.

If $\Gamma_+ \subseteq \mathbb{R}^n_+$ is a convenient Newton polyhedron then we remark that $J_i(\Gamma_+)$ is an integrally closed monomial ideal of finite colength, for all $i = 1, \dots, n$.

Let $\Gamma_+ \subseteq \mathbb{R}^n_+$ denote an arbitrary Newton polyhedron. If $f \in \mathcal{O}_n$ verifies that $\Gamma_+(f) = \Gamma_+$, then $\Gamma_+(f_{x_i}) \subseteq \Gamma_+(J_i(\Gamma_+))$, for all $i = 1, \dots, n$. If equality holds for all $i = 1, \dots, n$, then we say that the function f is Γ*-full*. We observe that the function ρ_Γ is not always a Γ-full function. However, a simple observation reveals that examples of Γ-full functions can be obtained as finite sums of a high enough amount of monomials x^k such that $k \in \Gamma_+$.

If Γ_+ is a Newton polyhedron in $\mathbb{R}^n_+$ such that $\mathcal{O}(\Gamma_+) \neq \emptyset$ then $\sigma(J_1(\Gamma_+), \dots, J_n(\Gamma_+)) < \infty$, by Lemma 2.4. We now will focus our attention to functions $f \in \mathcal{O}(\Gamma_+)$ such that ∇f is strongly non-degenerate with respect to $J_1(\Gamma_+), \dots, J_n(\Gamma_+)$.

Theorem 5.3 *Let $f : (\mathbb{C}^n, 0) \to (\mathbb{C}, 0)$ be an analytic function germ with an isolated singularity at the origin. Let $\Gamma_+ = \Gamma_+(f)$ and let $J_i = J_i(\Gamma_+)$, for all $i = 1, \dots, n$. Suppose that f is Newton non-degenerate. Then $\nabla f \in \mathcal{R}(J_1, \dots, J_n)$.*

Proof Let A denote the support of f and let $A_i = \{k \in A : k_i > 0\}$, for all $i = 1, \dots, n$. If $v \in \mathbb{R}^n_+ \smallsetminus \{0\}$, then a straightforward computation shows

$$p_v\left(\frac{\partial f}{\partial x_i}\right) = \frac{\partial}{\partial x_i} p_v(f_{A_i}),$$

for all $i = 1, \dots, n$. We also have, by similar computations, that

$$x_i\, p_v\left(\frac{\partial f}{\partial x_i}\right)(x) = \left(x_i \frac{\partial f_{A_i}}{\partial x_i}\right)_{\Delta(v, \Gamma_+(f_{A_i}))}. \tag{22}$$

Let Δ denote the face $\Delta(v, \Gamma_+(f))$ and let $i \in \{1, \dots, n\}$. Then, we observe

$$\left(x_i \frac{\partial f}{\partial x_i}\right)_\Delta = \begin{cases} \left(x_i \frac{\partial f_{A_i}}{\partial x_i}\right)_{\Delta(v, \Gamma_+(f_{A_i}))}, & \text{if } \Delta \cap A_i \neq \emptyset \\ 0, & \text{otherwise.} \end{cases} \tag{23}$$

Let Γ^i_+ denote the Newton polyhedron of J_i, for all $i = 1, \dots, n$.

Let us suppose first that Γ_+ is a convenient Newton polyhedron and that f is Γ-full. Let us suppose that ∇f is not non-degenerate with respect to $J_1, \dots, J_n$. Then there exists a vector $v \in (\mathbb{R}_+ \smallsetminus \{0\})^n$ and a point $x_0 \in (\mathbb{C} \smallsetminus \{0\})^n$ such that $(\partial f / \partial x_i)_{\Delta_i}(x_0) = 0$, for all $i = 1, \dots, n$, where $\Delta_i = \Delta(v, \Gamma^i_+)$, for all $i = 1, \dots, n$. Since we assume that f is Γ-full, then we have that the polynomials $p_v(\partial f / \partial x_i)$ and $(\partial f / \partial x_i)_{\Delta_i}$ coincide. Then

$$p_v\left(\frac{\partial f}{\partial x_i}\right)(x_0) = 0, \quad \text{for all } i = 1, \dots, n.$$

Hence relations (22) and (23) show

$$\left(x_i \frac{\partial f}{\partial x_i}\right)_\Delta (x) = 0, \quad \text{for all } i = 1, \dots, n,$$

where Δ denotes the compact face $\Delta(v, f)$. In particular, we deduce that f is not Newton non-degenerate, which contradicts our hypothesis. Then ∇f is non-degenerate with respect to $J_1, \dots, J_n$. Since Γ_+ is convenient, then all the ideals J_i have finite colength. Then ∇f is strongly non-degenerate with respect to $J_1, \dots, J_n$, by Proposition 2.10.

Let us suppose that f is not Γ-full. Then, we can consider a function $h \in \mathcal{O}_n$ such that the function f' given by $f' = f + h$ verify that $f' \in \mathcal{O}(\Gamma_+)$, f' is Newton non-degenerate and convenient and f' is Γ-full. Then $\mu(f) = \mu(f')$ by Theorem 5.1, since f and f' are Newton non-degenerate and they have the same Newton polyhedron. By the above discussion we deduce that $\nabla f'$ is strongly non-degenerate with respect to $J_1, \dots, J_n$. In particular we have

$$e\left(\frac{\partial f}{\partial x_1}, \dots, \frac{\partial f}{\partial x_n}\right) = e\left(\frac{\partial f'}{\partial x_1}, \dots, \frac{\partial f'}{\partial x_n}\right) = \sigma\,(J_1, \dots, J_n),$$

where the second equality follows by Theorem 2.11. But, also by Theorem 2.11, it follows that ∇f is strongly non-degenerate with respect to $J_1, \dots, J_n$.

Now let us suppose that Γ_+ is not a convenient Newton polyhedron. By an application of Nakayama's Lemma there exist an integer $r \geq 1$ and an homogeneous polynomial q of degree r such that $\mu(f) = \mu(f + q)$. Now let $f' = f + q$. Since we can take q as a generic linear combination of the set of monomials x^k of degree r, then we can assume that f' is convenient and Newton non-degenerate, by [18, Théorème 6.1]. Let $\Gamma'_+ = \Gamma_+(f + q)$. By the previous discussion we have that f' is strongly non-degenerate with respect to $J_1(\Gamma'_+), \dots, J_n(\Gamma'_+)$.

Since $\Gamma_+ \subseteq \Gamma'_+$, we have $J_i(\Gamma_+) \subseteq J_i(\Gamma'_+)$, for all $i = 1, \dots, n$. Then

$$\sigma\,(J_1, \dots, J_n) \leq \mu(f) = \mu(f') = \sigma\left(J_1(\Gamma'_+), \dots, J_n(\Gamma'_+)\right) \leq \sigma\,(J_1, \dots, J_n),$$

where the first and the last inequalities come from Lemma 2.4. Then, from Theorem 2.11, the map ∇f is strongly non-degenerate with respect to $J_1, \dots, J_n$. □

In order to simplify the notation, if Γ_+ is a Newton polyhedron in $\mathbb{R}^n_+$ such that $\mathcal{O}(\Gamma_+) \neq \emptyset$, then we denote by $\mathcal{R}(\Gamma_+)$ the set of those $f \in \mathcal{O}_n$ such that $\Gamma_+(f) = \Gamma_+$ and that $\nabla f \in \mathcal{R}(J_1(\Gamma_+), \dots, J_n(\Gamma_+))$. We denote by $\mathcal{R}_0(\Gamma_+)$ the set of Γ-full functions of $\mathcal{R}(\Gamma_+)$, that is, $\mathcal{R}_0(\Gamma_+) = \mathcal{R}_0(J_1(\Gamma_+), \dots, J_n(\Gamma_+))$.

We point out that the converse of Theorem 5.3 does not hold in general, as the next example shows (see also Example 5.9). If $f \in \mathcal{R}(\Gamma_+)$, then we will show in Theorem 5.7 a sufficient condition on f implying that f is Newton non-degenerate.

Example 5.4 Let us consider the function of $\mathcal{O}_3$ given by $f(x, y, z) = (x + y)^2 + xz + z^2$ (this is the function defined in [18, Remarque 1.21]). We have that $\Gamma_+(f) = \Gamma_+(x^2, y^2, z^2)$ and $J_i(\Gamma_+) = \langle x, y, z\rangle$, for all $i = 1, 2, 3$. Therefore

$$\sigma\,(J_1(\Gamma_+), J_2(\Gamma_+), J_3(\Gamma_+)) = 1 = \mu(f).$$

Hence $f \in \mathcal{R}(\Gamma_+)$, by Theorem 2.11; but f is not Newton non-degenerate, as is easy to check.

Corollary 5.5 *Let $\Gamma_+ \subseteq \mathbb{R}^n_+$ be a Newton polyhedron such that $\mathcal{O}(\Gamma_+) \neq \emptyset$. Let J_i denote the ideal $J_i(\Gamma_+)$, for all $i = 1, \dots, n$. Let $f \in \mathcal{O}(\Gamma_+)$ and let H_i denote the ideal generated by all x^k such that $k \in \Gamma_+(f_{x_i})$, for all $i = 1, \dots, n$. If f is Newton non-degenerate, then*

$$\nu(\Gamma_+) = \sigma(H_1, \dots, H_n) = \sigma(J_1, \dots, J_n).$$

Proof We have $H_i \subseteq J_i$, for all $i = 1, \ldots, n$. Then

$$\mu(f) \geq \sigma(H_1, \ldots, H_n) \geq \sigma(J_1, \ldots, J_n).$$

Therefore, the result follows as a consequence of Theorems 5.1 and 5.3. □

Corollary 5.6 *Let $\Gamma_+, \Gamma'_+ \subseteq \mathbb{R}^n_+$ be Newton polyhedra in $\mathbb{R}^n_+$ such that $\Gamma_+ \subseteq \Gamma'_+$. Let us suppose that $\nu(\Gamma_+)$ and $\nu(\Gamma'_+)$ are finite. Then*

$$\nu(\Gamma_+) \geq \nu(\Gamma'_+).$$

Proof Since $\Gamma_+ \subseteq \Gamma'_+$, we have $J_i(\Gamma_+) \subseteq J_i(\Gamma'_+)$, for all $i = 1, \ldots, n$. By Corollary 5.5 we have

$$\nu(\Gamma_+) = \sigma\left(J_1(\Gamma_+), \ldots, J_n(\Gamma_+)\right)$$
$$\nu(\Gamma'_+) = \sigma\left(J_1(\Gamma'_+), \ldots, J_n(\Gamma'_+)\right).$$

Therefore the result follows as a consequence of Lemma 2.5. □

The existence of an elementary proof of the previous result was posed as a problem by Arnold in [3, p. 48]. We remark that an elementary proof of Corollary 5.6 for the case $n = 2$ was given by Lenarcik in [21, Sect. 6] following a completely different approach.

As mentioned in the Introduction, if Γ_+ is a Newton polyhedron in $\mathbb{R}^n_+$ and $f \in \mathcal{O}(\Gamma_+)$, then Γ_+ has been used by many authors to estimate the Łojasiewicz exponent of ∇f. Let $J_i = J_i(\Gamma_+)$, for $i = 1, \ldots, n$. As a consequence of Corollaries 3.4 and 5.5, we have that if $f \in \mathcal{R}_0(\Gamma_+)$, then

$$\mathcal{L}_0(\nabla f) = \mathcal{L}_0(J_1, \ldots, J_n) = \min_{s \geq 1} \frac{r(J_1^s, \ldots, J_n^s)}{s}. \tag{24}$$

Therefore, the number $\mathcal{L}_0(\nabla f)$ depends only on Γ_+, for all $f \in \mathcal{O}(\Gamma_+)$ such that $\mu(f) = \nu(\Gamma_+)$ and f is Γ-full.

The next result can be seen as a converse of Theorem 5.1.

Theorem 5.7 *Let Γ_+ be a Newton polyhedron of $\mathbb{R}^n_+$ such that $\mathcal{O}(\Gamma_+) \neq \emptyset$ and Γ_+ is convenient. Let $f \in \mathcal{O}(\Gamma_+)$ such that the ideal $I(f)$ has finite colength in $\mathcal{O}_n$. Suppose that for all $L \subsetneq \{1, \ldots, n\}$, $L \neq \emptyset$, it holds that*

$$\left(\frac{\partial f}{\partial x_i}\right)_L = 0, \quad \textit{for all } i \notin L. \tag{25}$$

If $\mu(f) = \nu(\Gamma_+)$ then f is Newton non-degenerate.

Proof By Theorem 2.12, it suffices to prove that the colength of $I(f)$ in $\mathcal{O}_n$ equals the number $n!\mathrm{V}_n(\Gamma_+)$. Let us denote the ideal $J_i(\Gamma_+)$ by J_i, for all $i = 1, \ldots, n$.

If $L \subseteq \{1 \ldots, n\}$ and $i \in L$ then a straightforward computation shows

$$J_i\left((\Gamma_+)_L\right) = \left(J_i(\Gamma_+)\right)^L. \tag{26}$$

It is known (see [18, p. 17]) that

$$\dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(f)} = \sum_{L \subseteq \{1,\ldots,n\}} \mu(f_L), \tag{27}$$

where we define $\mu(f_\emptyset) = 1$.

Let $L \subseteq \{1, \dots, n\}$ and suppose that $L = \{i_1, \dots, i_p\}$, where $1 \leq i_1 < \cdots < i_p \leq n$. Then $(\partial f/\partial x_{i_j})_L = \partial f_L/\partial x_{i_j}$, for all $j = 1, \dots, p$. Moreover, since $\mu(f) = \nu(\Gamma_+)$ we have that ∇f is strongly non-degenerate with respect to $J_1, \dots, J_n$, by Theorem 2.11 and Corollary 5.5. This fact together with condition (25) shows that the map $(\partial f_L/\partial x_{i_1}, \dots, \partial f_L/\partial x_{i_p})$ is non-degenerate with respect to $(J_{i_1})^L, \dots, (J_{i_p})^L$. Therefore, by Theorem 2.11, we have

$$\mu(f_L) = \sigma\left((J_{i_1})^L, \dots, (J_{i_p})^L\right). \tag{28}$$

We will also denote the multiplicity on the right of (28) by $\sigma(J_i^L : i \in L)$.

Let $h \in \mathcal{O}(\Gamma_+)$ such that h is Newton non-degenerate. In particular the ideal $I(h)$ has finite colength. Then, by relation (27) and Lemma 2.4, we obtain

$$\begin{aligned} n!\mathrm{V}_n(\Gamma_+) &= \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(h)} = \sum_{L \subseteq \{1,\dots,n\}} \mu(h_L) \geq \sum_{L \subseteq \{1,\dots,n\}} \sigma\left((J_i)^L : i \in L\right) \\ &= \sum_{I \subseteq \{1,\dots,n\}} \mu(f_L) = \dim_{\mathbb{C}} \frac{\mathcal{O}_n}{I(f)} \geq n!\mathrm{V}_n(\Gamma_+). \end{aligned}$$

Then $n!\mathrm{V}_n(\Gamma_+)$ must be equal to the colength of $I(f)$ and the result follows. □

Let $w = (w_1, \dots, w_n) \in (\mathbb{Z}_+ \setminus \{0\})^n$. If $f \in \mathcal{O}_n$ then we say that f is *semi-weighted homogeneous with respect to* w when $p_w(f)$ has an isolated singularity at the origin. Let W_n denote the set of functions $f \in \mathcal{O}_n$ such that there exists some $w \in (\mathbb{R}_+ \setminus \{0\})^n$ such that f is semi-weighted homogeneous with respect to w and let K_n be the set of Newton non-degenerate functions of $\mathcal{O}_n$.

It is known that there is no inclusion relation between K_n and W_n. However, K_n and W_n are contained in the class of functions f such that there exists some Newton polyhedron Γ_+ such that $f \in \mathcal{R}(\Gamma_+)$, by virtue of Theorems 2.11 and 5.3, respectively. That is, we can see K_n and W_n as particular cases of the same property. This property is characterized numerically through the value of the Milnor number, as we see in Corollary 5.8. Therefore, the next result consists of a generalization of the main result of Furuya and Tomari [15] on the characterization of semi-weighted homogeneous functions (see also [10, Theorem 3.3], where non-degenerate maps $(\mathbb{C}^n, 0) \to (\mathbb{C}^n, 0)$ with respect to a Newton filtration are characterized).

We remark that C.T.C. Wall showed in [34] a different approach to the problem of seeking a theory considering simultaneously semi-weighted homogeneous functions and convenient Newton non-degenerate functions.

Corollary 5.8 *Let Γ_+ be a Newton polyhedron in $\mathbb{R}^n_+$ such that $\mathcal{O}(\Gamma_+) \neq \emptyset$. Let J_i denote the ideal $J_i(\Gamma_+)$, for all $i = 1, \dots, n$. Let $f \in \mathcal{O}_n$ with an isolated singularity at the origin. Suppose that $\Gamma_+(f) \subseteq \Gamma_+$. Then*

$$\mu(f) \geq \sigma(J_1, \dots, J_n), \tag{29}$$

and equality holds if and only if $f \in \mathcal{R}(\Gamma_+)$.

Proof Since $\Gamma_+(f) \subseteq \Gamma_+$, we have that $\Gamma_+(f_{x_i}) \subseteq \Gamma_+(J_i)$, for all $i = 1, \dots, n$, which is to say that $f_{x_i} \in J_i$, for all $i = 1, \dots, n$, since each ideal J_i is integrally closed. Then the result follows as an immediate application of Lemma 2.4 and Theorem 2.11. □

Example 5.9 Let us consider the function $f \in \mathcal{O}_3$ given by $f(x_1, x_2, x_3) = x_2^5 + x_1^2(x_1 - x_2)^2 + x_1^2 x_2 x_3 + x_3^4$. Let Γ_+ denote the Newton polyhedron of f. Using the program

Singular [16] we check that $\mu(f) = 30 = \nu(\Gamma_+)$. We observe that the function f is neither Newton non-degenerate nor semi-weighted homogeneous with respect to any $w \in (\mathbb{R}_+ \smallsetminus \{0\})^n$. However, by the previous corollary we deduce that $f \in \mathcal{R}(\Gamma_+)$.

Let H_i denote the ideal generated by the monomials x^k such that $k \in \Gamma_+(f_{x_i})$, for $i = 1, 2, 3$. Then $\mu(f) = \sigma(H_1, H_2, H_3) = 30$, by Corollary 5.5. Moreover $r(H_1, H_2, H_3) = 4$, $\theta(H_1, H_2, H_3) = \frac{31}{8}$ and

$$\sigma(H_1^8, H_2^8, H_3^8) = 8^3\sigma(H_1, H_2, H_3) = 15360$$
$$e(H_1^8 + m^{31}, H_2^8 + m^{31}, H_2^8 + m^{31}) = 15168.$$

Then $\mathcal{L}_0(\nabla f) = 4$, by Remark 3.5 and Corollary 4.4.

Acknowledgments Part of this paper was developed during the stay of the author at the Max-Planck-Institut für Mathematik in Bonn during March–April of 2007. The author wishes to express his deep gratitude to this institution for its support, hospitality and excellent working conditions.

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