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Primes dividing the degrees of the real characters

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Abstract Let G be a finite group and let $\mathrm{Irr}(G)$ denote the set of all complex irreducible characters of G. The Ito–Michler Theorem asserts that if a prime p does not divide the degree of any $\chi \in \mathrm{Irr}(G)$ then a Sylow p-subgroup P of G is normal in G. We prove a real-valued version of this theorem, where instead of $\mathrm{Irr}(G)$ we only consider the subset $\mathrm{Irr}_{\mathrm{rv}}(G)$ consisting of all real-valued irreducible characters of G. We also prove that the character degree graph associated to $\mathrm{Irr}_{\mathrm{rv}}(G)$ has at most 3 connected components. Similar results for the set of real conjugacy classes of G have also been obtained.

1 Introduction

Let G be a finite group, let $\mathrm{Irr}(G)$ be the set of irreducible complex characters of G, and let p be a prime number. One of the fundamental theorems in the Character Theory of Finite

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Groups is the Ito–Michler Theorem that asserts that if p does not divide the degree of any $\chi \in \mathrm{Irr}(G)$ then a Sylow p-subgroup P of G is normal in G. In this paper, we only consider a much smaller (and quite important) subset of the irreducible characters of G: the real-valued characters. Since non-trivial real characters only appear in groups of even order, the case $p = 2$ is the essential case here.

Theorem A *Let G be a finite group, and let P be a Sylow* 2*-subgroup of G. If all real-valued irreducible characters of G have odd degree, then* $P \lhd G$.

In fact, we have proved (see Theorem 4.2) that all real-valued irreducible characters of G are of odd degree if and only if G has a normal Sylow 2-subgroup of *Chillag–Mann type*. (This interesting class of 2-groups, studied in [5], are those in which every real-valued irreducible character is linear).

As pointed out by the referee, Theorem A might even be more general. As is well-known, real characters come into two types, according to their Frobenius–Schur indicator. It seems to be the case that in Theorem A it suffices to consider only +1 type real characters (i.e. the complex irreducible characters afforded by representations realizable over the field of real numbers). A proof of this fact, however, seems very hard and requires a considerable amount of work in addition to this already long paper.

The proof of Theorem A relies on the classification of finite simple groups. Theorem A does not remain true for odd primes p, even if we assume our group to have even order. We defer the study of when all real valued irreducible characters of a finite group have degree not divisible by p to some other place.

What are the primes dividing the degrees of the real-valued irreducible characters of G? In simple groups, as we will show, these are exactly all the primes dividing the order of G. But of course, this does not happen in general, specially, in solvable groups. There is a convenient way to study the primes dividing the character degrees of a finite group by using a natural graph. Let $\Gamma_{\mathrm{rv}}(G)$ be the graph of the finite group G whose vertices are the primes dividing the degrees of the real-valued irreducible characters of G, where we join two vertices p and q if pq divides $\chi(1)$ for some real-valued $\chi \in \mathrm{Irr}(G)$.

Theorem B *Let G be a finite group. Then* $\Gamma_{\mathrm{rv}}(G)$ *has at most three connected components. If G is solvable, then* $\Gamma_{\mathrm{rv}}(G)$ *has at most two.*

It is interesting to point out that in general the graph $\Gamma_{\mathrm{rv}}(G)$ is not an induced subgraph of $\Gamma(G)$ (the graph having vertices the primes dividing the degrees of the irreducible characters of G, where p and q are joined if pq divides some $\chi \in \mathrm{Irr}(G)$.) Analogues of Theorems A and B for the real conjugacy classes have also been obtained (see Theorems 6.1 and 6.2).

Finally, Theorem A is not true if we restrict our attention to the rational characters. For instance, the groups $PSL_2(3^{2f+1})$ provide an infinite family of counterexamples. In fact, this is the only family of examples among finite simple groups (see Theorem 2.7). On the other hand, it is very easy to find solvable examples: for instance, the semidirect product of a cyclic group of odd prime order with a cyclic group of order 2.

2 Even degree real-valued characters of almost simple groups

Throughout the paper, a character of a finite group is *real, resp. rational*, if it is real-valued, resp. rational-valued. If G is a finite group, we denote by $\mathrm{Irr}_{\mathrm{rv}}(G)$ the set of complex irreducible characters of G which are real-valued. C_n denotes a cyclic group of order n. If N is an integer and p a prime, then $N_{p'}$ denotes the p'-part of N; furthermore, $\pi(N)$ is the set of all prime divisors of N. Finally, $\Phi_m(t)$ is the m^{th} cyclotomic polynomial.

The hardest part toward the proof of Theorem A is the following.

Theorem 2.1 *Let S be a normal non-abelian finite simple subgroup of G, where G/S is a 2-group, and $\mathbf{C}_G(S) = 1$. Then there exists $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$ of even degree not containing S in its kernel.*

Before proving Theorem 2.1, we establish some auxiliary statements.

Lemma 2.2 *Let $a, n \geq 2$ be integers. Then $a^n - 1$ has a primitive prime divisor (PPD), that is a prime divisor of $a^n - 1$ that does not divide $\prod_{i=1}^{n-1}(a^i - 1)$, unless either $(a, n) = (2, 6)$ or $n = 2$ and $a + 1$ is a 2-power. Any such a PPD is at least $n + 1$.*

Proof See [29]. □

Lemma 2.3 *Assume S is a normal subgroup of a finite group G such that G/S is a 2-group with a unique involution γ. Assume $\alpha \in \mathrm{Irr}(S)$ is a nontrivial character such that $\alpha^\gamma = \bar{\alpha} \neq \alpha$. Then $\mathrm{Irr}(G)$ contains a real character χ of even degree which is nontrivial on S.*

Proof By our assumptions, the inertia group of α in G is just S. It follows that $\chi := \alpha^G$ is irreducible of even degree and nontrivial at S. Take the subgroup R between G and S such that $R/S = \langle\gamma\rangle$. Then α^R vanishes on $R \setminus S$ and equals $\alpha + \bar{\alpha}$ on S, so it is real. It follows that $\chi = (\alpha^R)^G$ is also real. □

Lemma 2.4 *Assume S is a normal subgroup of 2-power index of a finite group G such that $G/S \leq C_2 \times C_f$, where $C_2 = \langle\tau\rangle = H/S$ and σ is the unique involution in C_f. Assume $\alpha \in \mathrm{Irr}(S)$ is a real nontrivial τ-invariant character of even degree, not σ-invariant; moreover, it extends to a real character $\beta \in \mathrm{Irr}(H)$ in the case $\tau \in G/S$. Then $\mathrm{Irr}(G)$ contains a real character χ of even degree whose restriction to S contains α.*

Proof First we consider the case $\tau \notin G/S$. By our assumptions, α is fixed by τ, but not by σ nor by $\tau\sigma$. Hence the inertia group of α in G is just S. It follows that $\chi := \alpha^G$ is real, irreducible of even degree, and χ_S contains α. Next assume that $\tau \in G/S$, i.e. $G \geq H$. Since $\alpha^\sigma \neq \alpha$, β is not σ-invariant. Since $G/H \leq C_f$ and σ is the unique involution in C_f, the inertia group of β in G is just H. Now we can take $\chi := \beta^G$. □

The following lemmas use the Deligne–Lusztig theory of complex characters of finite groups of Lie type, cf. [4,7]. If $\mathcal{G}$ is a simple algebraic group, let $\pi_1(\mathcal{G})$ denote the fundamental group of $\mathcal{G}$.

Lemma 2.5 *Let $\mathcal{G}$ be a simple algebraic group in characteristic p, F a Frobenius map on $\mathcal{G}$, and let $G := \mathcal{G}^F$. Let the pair $(\mathcal{G}^*, F^*)$ be dual to $(\mathcal{G}, F)$ and $G^* := (\mathcal{G}^*)^{F^*}$. Assume $s \in G^*$ is a semisimple element of order r which is coprime to both $|\pi_1(\mathcal{G}^*)|$ and $|Z(G)|$.*

(i) *Then G has an irreducible character χ_s of degree $(G^* : C_{G^*}(s))_{p'}$ which is trivial at $Z(G)$. Furthermore, χ_s is real if and only if s is real in G^*.*
(ii) *Let σ, resp. σ^* be an automorphism of the (abstract) group $\mathcal{G}$, resp. $\mathcal{G}^*$, induced by the field automorphism $x \mapsto x^q$ for some power q of p, and such that $\sigma \circ F = F \circ \sigma$. Assume in addition that r does not divide $|(\mathcal{G}^*)^{\sigma^*}|$. Then χ_s is not σ-invariant.*

Proof (i) Since r is coprime to $|\pi_1(\mathcal{G}^*)|$, by [2, Corollary E-II.4.6] $C_{\mathcal{G}^*}(s)$ is connected. By the Deligne–Lusztig theory, to the G^*-conjugacy class of s one can associate an irreducible (semisimple) character χ_s of the indicated degree. By its definition, χ_s is a linear combination of Deligne–Lusztig characters $R^{\mathcal{G}}_{\mathcal{T}}(\theta)$, where $\mathcal{T}$ is an F-stable maximal torus of $\mathcal{G}$ and θ is a linear character of order r of $\mathcal{T}^F$. By [4, Proposition 3.6.8], $Z(G) = Z(\mathcal{G})^F$, also, $Z(\mathcal{G}) \leq \mathcal{T}$. It follows that $Z(G) \leq \mathcal{T}^F$. Since r is coprime to $|Z(G)|$, θ is trivial at $Z(G)$, whence $R^{\mathcal{G}}_{\mathcal{T}}(\theta)(z) = R^{\mathcal{G}}_{\mathcal{T}}(\theta)(1)$ for all $z \in Z(G)$. We conclude that $\mathrm{Ker}(\chi_s) \geq Z(G)$. It is well-known, cf. e.g. the proof of [25, Lemma 9.1], that $\chi_{s^{-1}} = \bar{\chi}_s$. Moreover, χ_s belongs to the Lusztig series $\mathcal{E}(\mathcal{G}^F, s)$, and distinct Lusztig series are disjoint. Hence χ_s is real if and only if s is real in G^*.

(ii) By [26, Corollary (2.5)], it suffices to show that the G^*-conjugacy class of s is not σ^*-invariant. Assume the contrary: $\sigma^*(s) = gsg^{-1}$ for some $g \in G^*$. Since σ^* is a Frobenius map on $\mathcal{G}^*$, by the Lang–Steinberg Theorem, $g^{-1} = x^{-1}\sigma^*(x)$ for some $x \in \mathcal{G}^*$. Now

$$\sigma^*(xsx^{-1}) = \sigma^*(x)\sigma^*(s)\sigma^*(x)^{-1} = xg^{-1} \cdot gsg^{-1} \cdot gx^{-1} = xsx^{-1},$$

whence $xsx^{-1} \in (\mathcal{G}^*)^{\sigma^*}$ and so r divides $|(\mathcal{G}^*)^{\sigma^*}|$, a contradiction. □

Lemma 2.6 *Let $\mathcal{G}$ be a simple algebraic group in characteristic p and F a Frobenius map on $\mathcal{G}$. Assume that $\chi \in \mathrm{Irr}(\mathcal{G}^F)$ is the unique unipotent character of degree $\chi(1)$. Then χ is rational.*

Proof Since χ is unipotent, χ occurs as a component of the Deligne–Lusztig character $R^{\mathcal{G}}_{\mathcal{T}}(1)$ for some F-stable maximal torus $\mathcal{T}$. The explicit formula for $R^{\mathcal{G}}_{\mathcal{T}}(1)$, cf. [4, Theorem 7.2.8], shows that $R^{\mathcal{G}}_{\mathcal{T}}(1)$ is rational and so it is invariant under any Galois automorphism γ of $\overline{\mathbb{Q}}$. It follows that χ^γ also occurs as a component of $R^{\mathcal{G}}_{\mathcal{T}}(1)$, whence χ^γ is also a unipotent character. Now the uniqueness of χ implies that $\chi^\gamma = \chi$. □

The rest of this section is devoted to the proof of Theorem 2.1.

1. If S is a sporadic finite simple group, the statement can be checked directly using GAP [9] or [6]. On the other hand, assume $S = \mathrm{Alt}_n$ with $n \geq 5$. If $n \neq 6$, then $\mathrm{Alt}_n \leq G \leq \mathrm{Sym}_n$, and by [25, Lemma 9.3], G has rational characters of degree d and $d+1$ where $d = n(n-3)/2$ and so we can choose χ to be one of these two. Assume $n = 6$. Then $\mathrm{Out}(S) = C_2^2 = \langle j_1, j_2\rangle$ with $\mathrm{Sym}_6 = \langle S, j_1\rangle$. Now $\mathrm{Irr}(S)$ contains a character α of degree 8, which extends to a real irreducible character of $\langle S, j_2\rangle$ and which is not j_1-invariant, cf. [6]. Hence we are done by Lemma 2.4.

So we will assume that S is a finite group of Lie type in characteristic p. If $p = 2$ and $S \neq {}^2F_4(2)'$, then the Steinberg character of S extends to a rational character of G by [8] and so we are done. In the case $S = {}^2F_4(2)'$ we can choose χ of degree 78. So we may assume $p > 2$.

2. We will use rank 3 permutation actions, cf. [27] for instance, to produce χ for several classical groups.

First we consider the case of $S = PSU_n(q)$ with $n \geq 4$. Equip the space $V = \mathbb{F}^n_{q^2}$ with the standard Hermitian form which has the isometry group $GU_n(q)$. Then $\Gamma U_n(q)$ (see [17] for the definition of this extension of $GU_n(q)$) acts on the set Ω of the singular 1-spaces of V, with character say ρ. It is known that $\rho_S = 1_S + \alpha + \beta$, where $\alpha, \beta \in \mathrm{Irr}(S)$ are of degree $q^2(q^n - (-1)^n)(q^{n-3} - (-1)^{n-3})/(q+1)(q^2-1)$ and $q^3(q^{n-1} - (-1)^{n-1})(q^{n-2} - (-1)^{n-2})/(q+1)(q^2-1)$. Notice that $q\alpha(1) - \beta(1) = (-1)^{n-2}q^n$ is odd, so exactly one of α, β has even degree; denote that one by γ. Observe that G is a subgroup of $\Gamma U_n(q)/\mathrm{Ker}(\rho)$. It follows that $\chi_G = 1_G + \alpha' + \beta'$, where $\alpha', \beta' \in \mathrm{Irr}(G)$ are some extensions to G of α and

β, and they are rational as ρ is rational. Now we can take χ to be the irreducible constituent of ρ_G that lies above γ.

Assume $S = PSU_3(q)$. The cases $q = 3, 5$ can be checked directly using [6], so we will assume $q = p^f \geq 7$. The character table of $SU_3(q)$ is given in [10], and with the notation therein, we choose the character $\alpha := \chi^{(1,5)}_{(q-1)(q^2-q+1)}$ and $\beta := \chi_{q^2-q}$. Notice that β is rational of even degree, so we are done if $G = S$. Thus we may assume $G > S$. Observe that $\mathrm{Out}(S) = C_{(3,q+1)} : C_{2f}$ [12], so up to a conjugation in $\mathrm{Aut}(S) \geq G$, we may also assume that G/S has a unique involution γ which is induced by the field automorphism $x \mapsto x^q$. One can check that γ sends α to its complex conjugate $\bar{\alpha}$ and $\alpha \neq \bar{\alpha}$, so we are done by Lemma 2.3.

3. Assume $S = PSp_{2n}(q)$ with $n \geq 2$. Equip the space $V = \mathbb{F}_q^{2n}$ with the standard symplectic form which has the isometry group $Sp_{2n}(q)$. We will consider the group $\Gamma Sp_{2n}(q)$ (cf. [17, Sect. 2.1] for the definition of this extension of $Sp_{2n}(q)$), which induces the full automorphism group of S since q is odd. Notice that $\Gamma Sp_{2n}(q)$ acts on the set Ω of the 1-spaces of V, with character say ρ. It is known that $\rho_S = 1_S + \alpha + \beta$, where $\alpha, \beta \in \mathrm{Irr}(S)$ are of degree $q(q^n-1)(q^{n-1}+1)/2(q-1)$ and $q(q^n+1)(q^{n-1}-1)/2(q-1)$. Notice that $\alpha(1) - \beta(1) = q^n$ is odd, so exactly one of α, β has even degree. Now G is a subgroup of $\Gamma Sp_{2n}(q)/\mathrm{Ker}(\rho)$ and so we are done as in 2.

Next assume $S = \Omega_{2n+1}(q)$ with $n \geq 3$. Equip the space $V = \mathbb{F}_q^{2n+1}$ with the standard quadratic form which has the isometry group $GO_{2n+1}(q)$. Then $\Gamma O_{2n+1}(q)$ (see [17, Sect. 2.1] for the definition of this extension of $GO_{2n+1}(q)$) acts on the set Ω of the singular 1-spaces of V, with character say ρ. It is known that $\rho_S = 1_S + \alpha + \beta$, where $\alpha, \beta \in \mathrm{Irr}(S)$ are of degree $q(q^n-1)(q^{n-1}+1)/2(q-1)$ and $q(q^n+1)(q^{n-1}-1)/2(q-1)$. Notice that $\alpha(1) - \beta(1) = q^n$ is odd, so exactly one of α, β has even degree. Now G is a subgroup of $\Gamma O_{2n+1}(q)/\mathrm{Ker}(\rho)$ and so we are done as in 2.

Next assume $S = P\Omega^+_{2n}(q)$ with $n \geq 4$. Equip the space $V = \mathbb{F}_q^{2n} = \langle e_1, f_1, \ldots, e_n, f_n\rangle$ with the quadratic form $Q(\sum_{i=1}^n (x_i e_i + y_i f_i)) = \sum_{i=1}^n x_i y_i$ which has the isometry group $GO^+_{2n}(q)$. Then $\Gamma O^+_{2n}(q)$ (see [17, Sect. 2.1] for the definition of this extension of $GO^+_{2n}(q)$) acts on the set Ω of the singular 1-spaces of V, with character say ρ. It is known that $\rho_S = 1_S + \alpha + \beta$, where $\alpha, \beta \in \mathrm{Irr}(S)$ are of degree $q(q^n-1)(q^{n-2}+1)/(q^2-1)$ and $q^2(q^{n-1}+1)(q^{n-1}-1)/(q^2-1)$. Notice that $q\alpha(1) - \beta(1) = q^n$ is odd, so exactly one of α, β has even degree. Now G is a subgroup of $\Gamma O^+_{2n}(q)/\mathrm{Ker}(\rho)$ (possibly after a conjugation in $\mathrm{Aut}(S)$ when $n = 4$), and so we are done as in 2.

4. Assume $S = P\Omega^-_{2n}(q)$ with $n \geq 4$ and $q = p^f$. Equip the space $V = \mathbb{F}_q^{2n}$ with a quadratic form which has the isometry group $GO^-_{2n}(q)$. Then the conformal orthogonal group $CO^-_{2n}(q)$ and a subgroup $CO^-_{2n}(q)^\circ$ of it (see [4] for the definition of these extensions of $GO^-_{2n}(q)$) act on the set Ω of the singular 1-spaces of V. Modding out the scalars, we see that $PCO^-_{2n}(q)^\circ$ acts on Ω. Recall [12, Theorem 2.5.12] that $\mathrm{Aut}(S) = \mathrm{Inndiag}(S) : C_{2f}$, and Inndiag(S) $\simeq PCO^-_{2n}(q)^\circ$ can be viewed as $\mathcal{G}^F$ for a simple algebraic group $\mathcal{G}$ of adjoint type and a Frobenius map F on $\mathcal{G}$. In fact, if σ is the automorphism of $\mathcal{G}$ coming from the field automorphism $x \mapsto x^p$ and τ is the graph automorphism that switches the last two (branching) nodes of the Dynkin diagram (of type D_n) of $\mathcal{G}$, then $F = \sigma^f \circ \tau$. One can identify a point stabilizer P of $\mathcal{G}^F$ acting on Ω with $\mathcal{P}^F$, where $\mathcal{P}$ is a standard parabolic subgroup of $\mathcal{G}$ with Levi subgroup of type D_{n-1}, cf. [4, p. 63]. Clearly, $\mathcal{P}$ and $\mathcal{P}^F$ are σ-stable, and σ induces a generator of the complement C_{2f} in the semidirect product $\mathrm{Aut}(S) = \mathrm{Inndiag}(S) : C_{2f}$ that we will also denote by σ. This discussion shows that, inside $\mathrm{Aut}(S)$, σ normalizes P. So if ρ is the character afforded by the permutation action of $\mathrm{Aut}(S)$ on the cosets of $\langle P, \sigma\rangle$, then $\rho|_{\mathcal{G}^F}$ is the character of the permutation action of $\mathcal{G}^F$ on Ω. It is

known that $\rho_S = 1_S + \alpha + \beta$, where $\alpha, \beta \in \mathrm{Irr}(S)$ are of degree $q(q^n+1)(q^{n-2}-1)/(q^2-1)$ and $q^2(q^{n-1}+1)(q^{n-1}-1)/(q^2-1)$. Notice that $q\alpha(1) - \beta(1) = -q^n$ is odd, so exactly one of α, β has even degree. So we are done as in 2.

5. We will explore the method of 4. to handle the simple groups of type E_6 and ${}^2E_6$. Let $\mathcal{G}$ be a simple algebraic group of type E_6 over $\mathbb{F}$ and of adjoint type. Let σ be the automorphism of $\mathcal{G}$ coming from the field automorphism $x \mapsto x^p$ and τ be the (involutory) graph automorphism of $\mathcal{G}$. Consider a standard parabolic $\mathcal{P}$ of $\mathcal{G}$ with Levi subgroup of type A_5. First we consider the untwisted case $F = \sigma^f$. Then $H := \mathcal{G}^F = E_6(q)_{ad}$ for $q = p^f$ and $S = [H, H]$ is simple. As in 4. we observe that $P := \mathcal{P}^F$ is stable under σ and τ, and $\mathrm{Aut}(S) = H : (\langle\sigma\rangle \times \langle\tau\rangle)$. Let ρ be the character afforded by the permutation action of $\mathrm{Aut}(S)$ on the cosets of $\langle P, \sigma, \tau\rangle$. Then $\rho|_H$ is a sum of unipotent characters, of the same rank as of the character $(1_{W(A_5)})^{W(E_6)}$ which is 5. Using this information one can show that

$$\rho|_H = 1_H + \phi_{6,1} + \phi_{20,2} + \phi_{30,3} + \phi_{15,4}$$

in the notation of [4, p. 480]. All these five irreducible constituents have distinct degrees, and three of them have even degrees. Since $H \lhd \mathrm{Aut}(S)$, it follows that ρ is a sum of 5 irreducible constituents, all rational and three of even degrees.

Next we consider the twisted case $F = \sigma^f \circ \tau$. Then $H := \mathcal{G}^F = {}^2E_6(q)_{ad}$ for $q = p^f$ and $S = [H, H]$ is simple. As above we observe that $P := \mathcal{P}^F$ is stable under σ, and $\mathrm{Aut}(S) = H : \langle\sigma\rangle$ Let ρ be the character afforded by the permutation action of $\mathrm{Aut}(S)$ on the cosets of $\langle P, \sigma\rangle$. Then $\rho|_H$ is again a sum of unipotent characters. Computer computation using CHEVIE [11] (done by F. Lübeck) shows that

$$\rho|_H = 1_H + \phi'_{2,4} + \phi_{4,1} + \phi_{9,2} + \phi'_{8,3}$$

in the notation of [4, p. 481]. Again, all these five irreducible constituents have distinct degrees, and three of them have even degrees. Since $H \lhd \mathrm{Aut}(S)$, it follows that ρ is a sum of 5 irreducible constituents, all rational and three of even degrees, and so we are done.

6. For the remaining Lie-type groups, we will use the Deligne–Lusztig theory to produce χ. We will again consider a certain simple algebraic group $\mathcal{G}$, a Frobenius map F on $\mathcal{G}$, and the pair $(\mathcal{G}^*, F^*)$ dual to $(\mathcal{G}, F)$. Also let $\mathbb{F}$ be the algebraic closure of $\mathbb{F}_p$. To illustrate the main ideas of this framework, we first handle the case $S = PSL_2(q)$ with $q = p^f \geq 7$ (even though one may settle this case in a more elementary way). Choose $\mathcal{G} = PSL_2(\mathbb{F})$ and F such that $\mathcal{G}^F \simeq PGL_2(q)$. Since $q \geq 7$, we can find an odd prime $r_1 | (q^2-1)$ and a semisimple element $s_1 \in (\mathcal{G}^*)^{F^*} = SL_2(q)$ of order r_1. Observe that s_1 is real. So by Lemma 2.5(i), χ_{s_1} is a real character (of even degree $q \pm 1$) of $PGL_2(q)$ which restricts irreducibly to S. Thus we are done if f is odd, or if f is even but $G \leq PGL_2(q)$.

So we may assume that $2|f$ and $G \not\leq H := PGL_2(q)$. Consider a PPD r of $p^{2f}-1$, and a semisimple element $s \in (\mathcal{G}^*)^{F^*}$ of order r. By Lemma 2.5(i) we get a real irreducible character χ_s of degree $q-1$ of H which also restricts irreducibly to S. Since $\mathcal{G}$ is of adjoint type, by [1, Lemma 3.1], $\alpha := \chi_s|_S$ belongs to the principal r-block of S. It now follows that α is the semisimple character of $SL_2(q)$ corresponding to a semisimple element t in the dual group $PGL_2(q)$ of order divisible by r. Consider the automorphism σ of $SL_2(\mathbb{F})$ and $PSL_2(\mathbb{F})$ induced by the field automorphism $x \mapsto x^p$. The choice of r ensures that r does not divide $|(\mathcal{G}^*)^{\sigma^{f/2}}|$. So α is not $\sigma^{f/2}$-invariant by Lemma 2.5 (ii), where we denote the action on S induced by σ also by σ. Recall that $\mathrm{Out}(S) = C_2 \times C_f$, where $C_2 = H/S$ and $\sigma^{f/2}$ is the unique involution in C_f. Now we can apply Lemma 2.4 to α, with $\beta := \chi_s$.

7. Next we consider the case $S = PSL_n(q)$ with $q = p^f$ and $n \geq 3$. Choose $\mathcal{G} = PSL_n(\mathbb{F})$ and F such that $H := \mathcal{G}^F \simeq PGL_n(q)$. Consider the unipotent characters χ_λ of H labeled by the partitions $\lambda = (n-2, 2)$, $(n-2, 1^2)$ of n if $n \geq 4$, and $\lambda = (2, 1)$

if $n = 3$, cf. [4, p. 465]. If $n = 3$ then $\chi_{(2,1)}(1) = q(q+1)$ is even. If $n \geq 4$ then $\chi_{(n-2,1^2)}(1) - q\chi_{(n-2,2)}(1) = q^n$ is odd, and so exactly one of these two characters has even degree. It is well known that unipotent characters of H are rational and restrict irreducibly to S. So we may assume that $G \not\leq H$. Recall [12, Theorem 2.5.12] that $\mathrm{Aut}(S) = H : (\langle\sigma\rangle \times \langle\tau\rangle)$, where σ is the automorphism of S induced by the field automorphism $x \mapsto x^p$ and τ is the inverse-transpose. We will also consider $H^* := (\mathcal{G}^*)^{F^*} \simeq SL_n(q)$ and its natural module $V := \mathbb{F}_q^n$.

Assume f is odd. Then HG/H contains τH. If n is odd, choose r to be a PPD of $q^n - 1$ and $s \in H^*$ with eigenvalues λ^{q^i}, $0 \leq i \leq n-1$ on V. If n is even, choose r to be a PPD of $q^{n-1} - 1$ and $s \in H^*$ with eigenvalues 1 and λ^{q^i}, $0 \leq i \leq n-2$ on V. In both cases, we choose $\lambda \in \mathbb{F}$ with $|\lambda| = r$. By our choice, $r \geq n+1$ and so r satisfies the assumptions of Lemma 2.5. Claim that s is not real in H^*. (Assume the contrary. Then $\lambda^{-1} = \lambda^{q^i}$ for some i, $0 \leq i \leq m-1$, where $m = n$ if n is odd and $m = n-1$ if n is even. Since $r > 2$, $i > 0$. Also, $r|(q^{2i} - 1)$ and so $m|2i$. But this is a contradiction as m is odd and $0 < i < m$.) Thus χ_s is not real by Lemma 2.5. Consider an irreducible constituent μ of $\chi_s|_S$. By [1, Lemma 3.1], μ belongs to the principal r-block of S. Viewed as a character of H^*, μ also belongs to the principal r-block of H^*. Also, since $\mu(1)$ is coprime to p, μ is semisimple by [21, Lemma 7.2]. It now follows by the fundamental result of Broué and Michel [3] that μ is the semisimple character χ_t of H^* defined by an r-element $t \in H$. Notice that $\chi_s(1) > q-1$ and $|H/S| \leq q-1$, so $\mu(1) > 1$ and $|t| = r^a \geq r$. Arguing as in the proof of [23, Lemma 2.4], one can show that $|C_{H^*}(s)| = |C_H(t)|$ (and it is $(q^n-1)/(q-1)$ if n is odd and $q^{n-1} - 1$ if n is even). It follows that $\chi_s(1) = \mu(1)$ and so $\chi_s|_S$ is irreducible. Since r is coprime to $q-1$, there is a preimage $t_1 \in GL_n(q)$ of order r^a of t. Now arguing with eigenvalues of t_1 on V as above, we see that t_1 is not real in $GL_n(q)$. But r is again coprime to $q-1$ and $H = GL_n(q)/C_{q-1}$, so t is not real in H. Thus $\chi_s|_S$ is not real by Lemma 2.5(i). Let $\alpha := \chi_s|_{G\cap H}$. Since $G \cap H \geq S$, we conclude that α is irreducible, non-real, and H-invariant. By our assumptions, $G = \langle G \cap H, h\tau\rangle$ for some $h \in H$. Since τ commutes with $F = \sigma^f$, by [26, Corollary (2.5)], $(\chi_s)^\tau = \chi_{\tau(s)}$. Observe that $\tau(s)$ is conjugate to s^{-1} in $\mathcal{G}^*$ and so in H^* as well by the Lang–Steinberg Theorem, since $C_{\mathcal{G}^*}(s)$ is connected. Thus $(\chi_s)^\tau = \bar{\chi}_s \neq \chi_s$. It follows that $\alpha^{h\tau} = \bar{\alpha} \neq \alpha$. Hence we are done by Lemma 2.3.

From now on we assume f is even. If n is even, choose r to be a PPD of $p^{nf} - 1$ and $s \in H^*$ with eigenvalues λ^{q^i}, $0 \leq i \leq n-1$ on V. If n is odd, choose r to be a PPD of $p^{(n-1)f} - 1$ and $s \in H^*$ with eigenvalues 1 and λ^{q^i}, $0 \leq i \leq n-2$ on V (notice that $(n-1)f \geq 4$ so r exists). In both cases, we choose $\lambda \in \mathbb{F}$ with $|\lambda| = r$. By our choice, $r \geq 2(n-1)+1 > n$ and so r satisfies the assumptions of Lemma 2.5. Claim that s, s^{-1}, and $\tau(s)$ are all conjugate in H^*. (For, these three elements have the same characteristic polynomial on V which is at the same time also their minimal polynomial. Hence they are conjugate in $\mathcal{G}^* = SL_n(\mathbb{F})$. Since $C_{\mathcal{G}^*}(s)$ is connected, they are also conjugate in $(\mathcal{G}^*)^{F^*} = H^*$ by the Lang–Steinberg Theorem.) Thus χ_s is real by Lemma 2.5. Arguing with τ as above, we see that χ_s is τ- stable. Consider an irreducible constituent μ of $\chi_s|_S$. Arguing as in the case of odd f, we see that μ is the semisimple character χ_t of H^* defined by an r- element $t \in H$ with $|t| = r^a \geq r$. Moreover, $|C_{H^*}(s)| = |C_H(t)|$, and it is $(q^n-1)/(q-1)$ if n is even and $q^{n-1} - 1$ if n is odd. It follows that $\chi_s(1) = \mu(1)$ and so $\chi_s|_S$ is irreducible. Clearly, $\chi_s(1) = (H^* : C_{H^*}(s))_{p'}$ is divisible by $q-1$ and so it is even. Also, the choice of r implies that $|t|$ does not divide the order of $\mathcal{G}^{\sigma^{f/2}} = PGL_n(q^{1/2})$. So $\chi_t = \chi_s|_S$ is not $\sigma^{f/2}$-invariant. Thus $\alpha := \chi_s|_{G\cap H}$ is irreducible, nontrivial, τ-invariant but not $\sigma^{f/2}$-invariant. Consider the extension $K := \langle H, \tau\rangle$ of index 2 of H. By the main result of [13], every irreducible character of K is real. Since χ_s is τ-invariant, it now extends to a real irreducible character γ of K. Furthermore, if there is any $h \in H$ such that $(\tau h)^2 \in G \cap H$, then, setting $R := \langle G \cap H, \tau h \rangle \leq K$ and $\beta := \gamma|_R$, we see that β is real, of even degree, and $\beta|_{G \cap H} = \alpha$. Recall that $G/(G \cap H) \simeq HG/H \leq \mathrm{Aut}(S)/H = \langle \tau \rangle \times C_f$ and $\sigma^{f/2}$ is the unique involution in C_f. Applying Lemma 2.4 to $(G, R, G \cap H, \alpha)$ in place of (G, H, S, α), we see that $\mathrm{Irr}(G)$ contains a real character of even degree whose restriction to S contains χ_t and so nontrivial on S.

8. Here we consider the case S has type E_7. Recall [12, Theorem 2.5.12] that $\mathrm{Aut}(S) = H : C_f$, where $H := \mathrm{Inndiag}(S) = \mathcal{G}^F = E_7(q)_{ad}$, $q = p^f$, and $\mathcal{G}$ is the simple algebraic group $E_7(\mathbb{F})$ of adjoint type. We will also consider $H^* := (\mathcal{G}^*)^{F^*} = E_7(q)_{sc}$; notice that $S = H^*/Z(H^*)$ and $H/S = C_2$. By [28, Proposition 3.1], every semisimple element in H and in H^* is real. Set $K := G \cap H$ and choose a PPD r of $p^{18f} - 1$ and a semisimple element s of order r in L, where $L := H^*$ if $K = H$, resp. $L := H$ if $K = S$. By [23, Lemma 2.3], $|C_L(s)| = \Phi_{18}(q)\Phi_2(q)$. By Lemma 2.5(i), χ_s is a real irreducible character of even degree of K. Observe that in both cases χ_s is H-invariant. This is obvious in the former case, and follows from [26, Corollary (2.5)] applied to the automorphism induced by the conjugation by some element $y \in H \setminus S$ in the latter case. Also, by Lemma 2.5(ii), χ_s is not invariant under any nontrivial element of the group C_f of field automorphisms. It follows that the inertia group of χ_s in G is exactly K and so we can take $\chi = \chi_s^G$.

9. Now assume that $S \in \{{}^2G_2(q), G_2(q), {}^3D_4(q), F_4(q), E_8(q)\}$ with $q = p^f$. Assume furthermore that $p \neq 3$ if $S = G_2(q)$. Since G/S is a 2- power, $G \leq S : C_f$ and in fact $G = S$ if $S = {}^2G_2(q)$. In all these cases, any semisimple element $s \in S$ is real by [28, Proposition 3.1]. View S as $\mathcal{G}^F$ for a suitable simple algebraic group $\mathcal{G}$ and a Frobenius map F on $\mathcal{G}$. Choose a PPD r of $p^{mf} - 1$ and a semisimple element $s \in (\mathcal{G}^*)^{F^*} \simeq S$ of order r, where $m = 6$, resp. 6, 12, 12, 30, if S is of type ${}^2G_2$, resp. G_2, ${}^3D_4$, F_4, E_8. By [23, Lemma 2.3], $|C_{(\mathcal{G}^*)^{F^*}}(s)| = \Phi_m(q)$, unless $S = {}^2G_2(q)$ in which case $|C_{(\mathcal{G}^*)^{F^*}}(s)| = q \pm \sqrt{3q} + 1$ is odd. It follows by Lemma 2.5(i) that $\chi_s \in \mathrm{Irr}(S)$ is a real character of even degree.

Claim that for any field automorphism $\gamma \in C_f$ of 2-power order $2^c > 1$, χ_s is not γ-invariant. By Lemma 2.5(ii), it suffices to show that r does not divide $|(\mathcal{G}^*)^{\gamma^*}|$. This last statement is vacuous if $S = {}^2G_2(q)$. It is also obvious if $S = X(q)$ and $X \in \{G_2, F_4, E_8\}$, since in these cases $(\mathcal{G}^*)^{\gamma^*} = X(p^{f_1})$ with $1 \leq f_1 < f$. Assume $S = {}^3D_4(q)$ and r divides $|(\mathcal{G}^*)^{\gamma^*}|$. Let σ be the automorphism of $\mathcal{G}$ defined by the field automorphism $x \mapsto x^p$ and τ is the triality graph automorphism of $\mathcal{G}$. Then $F = \sigma^f \circ \tau$. Assuming $\gamma = \sigma^j$ with $1 \leq j \leq 3f - 1$, we see that r divides $|D_4(p^j)|$ and so it divides $p^{4j} - 1$ or $p^{6j} - 1$. The first possibility cannot occur as r is a PPD of $p^{12f} - 1$. So $r|(p^{6j} - 1)$, whence $2f$ divides j. It follows that $\gamma^3 = \sigma^{6f}$ acts trivially on $S < D_4(p^{3f})$ and so the order of γ in $\mathrm{Aut}(S)$ is not a nontrivial 2-power, a contradiction.

We have shown that the inertia group of χ_s in G is just S. Now we can take $\chi = \chi_s^G$.

10. Finally, we consider the case $S = G_2(q)$ and $q = 3^f$. In view of [6] we may assume $f > 1$. According to [12, Theorem 2.5.12], $\mathrm{Aut}(S) = S : C_{2f}$. Here $C_{2f} = \langle \sigma_1 \rangle$, σ_1^2 is the automorphism of $\mathcal{G}$ defined by the field automorphism $x \mapsto x^3$, and $S = \mathcal{G}^F$ with $\mathcal{G}$ a simple algebraic group of type G_2 and $F = \sigma_1^{2f}$. Then $\tau := \sigma_1^f$ is the unique involution in C_{2f}.

Assume $s \in S$ is such that the S-conjugacy class s^S of s is τ-invariant. Then $\tau(s) = gsg^{-1}$ for some $g \in S$. Define $\varphi \in \mathrm{Aut}(S)$ via $\varphi(x) = g^{-1}\tau(x)g$ for $x \in S$. Then $\varphi(s) = g^{-1} \cdot gsg^{-1} \cdot g = s$ and so $s \in C_S(\varphi)$. Since φ has order 2 in $\mathrm{Out}(S)$, we can write the order of φ in $\mathrm{Aut}(S)$ as $2^k a$ for some $k \geq 1$ and some odd a. Setting $\psi := \varphi^{2^{k-1}a}$, we see that $\psi^2 = 1_S$ and $s \in C_S(\psi)$. There are two possibilities. If ψ is an inner automorphism of S, then it is the conjugation by an involution in S and so $C_S(\psi) \simeq (SL_2(q) * SL_2(q)) \cdot 2$

by [18]. Otherwise, as shown in [18], ψ is S-conjugate to a fixed automorphism ϕ_2 of S; furthermore, $C_S(\psi)$ is isomorphic to $G_2(\sqrt{q})$ if f is even and to ${}^2G_2(q)$ if f is odd.

Now we set $m := 6$ if f is even, resp. $m := 3$ if f is odd. Choose a PPD r of $p^{mf} - 1$ and a semisimple element $s \in (\mathcal{G}^*)^{F^*} \simeq S$ of order r. By [23, Lemma 2.3], $|C_{(\mathcal{G}^*)^{F^*}}(s)| = \Phi_m(q)$. It follows by Lemma 2.5(i) that $\chi_s \in \mathrm{Irr}(S)$ is a real character of even degree. Observe that r does not divide $|(SL_2(q) * SL_2(q)) \cdot 2|$; furthermore, it is coprime to $|G_2(\sqrt{q})|$ if f is even, resp. to $|{}^2G_2(q)|$ if f is odd. Hence the previous paragraph shows that s^S is not τ-invariant. But τ commutes with F, so [26, Corollary (2.5)] implies that χ_s is not τ-invariant. We have shown that the inertia group of χ_s in G is just S and can now take $\chi = \chi_s^G$.

Theorem 2.1 has been now proved completely. □

Theorem 2.7 *Let S be a finite non-abelian simple group. Then all rational irreducible characters of S are of odd degree if and only if $S \cong PSL_2(3^{2f+1})$ for some integer $f \geq 1$.*

Proof Observe that if S is any simple group considered in pp. 1 – 5 of the proof of Theorem 2.1, then the character χ produced in the proof (in the case $G = S$) is actually rational. Now we will consider the remaining groups of Lie type in odd characteristic p. The case of $PSL_2(q)$ is done in [25, Lemma 9.4]; in particular, if $q \neq 3^{2f+1}$ then χ can be chosen to be of degree $q \pm 1$. Furthermore, the case of $PSL_n(q)$ with $n \geq 3$ has also been done in page 7 of the proof of Theorem 2.1. Assume S is an exceptional group of Lie type, not of type ${}^2G_2(q)$. Then we can consider $S = \mathcal{G}^F/Z(\mathcal{G}^F)$ for a simple simply connected algebraic group $\mathcal{G}$ and a Frobenius map F on $\mathcal{G}$ and observe that any unipotent character of $\mathcal{G}^F$ is trivial at $Z(\mathcal{G}^F)$. By Lemma 2.6, it suffices to find a character of $\mathcal{G}^F$ that is a unique unipotent character of some even degree d. Using [4, Sect. 13.9], we can choose $d = q\Phi_2(q)^2\Phi_3(q)/6$, resp. $q^3\Phi_2(q)^2\Phi_6(q)^2/2$, $q\Phi_1(q)^2\Phi_3(q)^2\Phi_8(q)/2$, $q\Phi_8(q)\Phi_9(q)$, $q\Phi_8(q)\Phi_{18}(q)$, $q^3\Phi_1(q)^4\Phi_3(q)^2\Phi_5(q)\Phi_7(q)\Phi_9(q)\Phi_{14}(q)/2$, $q\Phi_4(q)^2\Phi_8(q)\Phi_{12}(q)\Phi_{20}(q)\Phi_{24}(q)$, if $\mathcal{G}^F = G_2(q)$, resp. ${}^3D_4(q)$, $F_4(q)$, $E_6(q)$, ${}^2E_6(q)$, $E_7(q)$, $E_8(q)$. Finally, assume $S = {}^2G_2(q)$ with $q = 3^{2f+1}$ and $f \geq 1$. Again consider $S = \mathcal{G}^F$ for a simple algebraic group $\mathcal{G}$ of type G_2. Observe that $S \simeq (\mathcal{G}^*)^{F^*}$ (cf. [4]; also notice that the existence of Jordan decomposition of irreducible characters in the case of Suzuki and Ree groups is proved in [19]), and $S > {}^2G_2(3) = SL_2(8) \cdot 3$ contains a rational element s of order 7. Moreover, $C_S(s)$ has even index in S as s is certainly inverted by some even-order element of S. By [25, Lemma 9.1], $\chi_s \in \mathrm{Irr}(S)$ is a rational character of even degree. (This also follows from the explicit description of irreducible characters of S in terms of Deligne–Lusztig characters given in [14, Sect. 8.2].) □

3 Primes dividing the degrees of real-valued characters of simple groups

The aim of this section is to prove the following theorem:

Theorem 3.1 *Let S be a finite non-abelian simple group and let $\Gamma_{\mathrm{rv}}(S)$ be the prime graph on the set of the degrees of real irreducible characters of S, with vertex set $V(S)$. Then*

(i) $V(S) = \pi(|S|)$*; and*
(ii) $\Gamma_{\mathrm{rv}}(S)$ *has at most three connected components.*

Lemma 3.2 *Theorem 3.1 holds for sporadic finite simple groups.*

Proof Routine computer computation using GAP [9]. □

Proposition 3.3 *Theorem 3.1 holds for alternating groups $S = \mathrm{Alt}_n$. In fact, if $n \geq 7$ then $\Gamma_{\mathrm{rv}}(S)$ is connected.*

Proof 1. The cases $5 \leq n \leq 8$ can be checked directly using [6], so we may assume $n \geq 9$. We will largely follow the ideas of [22, p. 33–35] to show that for any odd prime $p \leq n$, S has a real irreducible character χ such that $2p|\chi(1)$. As in [22], we choose χ to be an irreducible constituent of $\xi|_S$, where ξ is the irreducible character of Sym_n corresponding to the partition $\alpha = (n-r-s, s+1, 1^{r-1})$ with $0 \leq r-1, s, r+2s+1 \leq n$. In particular,

$$\xi(1) = \binom{n}{s}\binom{n-s-1}{r-1}\frac{n-2s-r}{r+s}.$$

Let α' denote the partition associated to α. If $\alpha \neq \alpha'$, then $\chi = \xi$ and so it is real. If α is self-associated, then $\chi(1) = \xi(1)/2$, and this happens exactly in the following cases:

(i) $s = 0$ and $n = 2r + 1$,
(ii) $s = 1$ and $n = 2r + 2$.

In the case of (i), the partition $h(\alpha)$ of the hook lengths of α is (n), and the value $\xi_{h(\alpha)}$ of ξ at an (even) permutation with cycle type $h(\alpha)$ is $(-1)^r$ by [16, Lemma 2.5.12]. Similarly, in the case of (ii), $h(\alpha) = (n-1, 1)$ and $\xi_{h(\alpha)} = (-1)^r$. This implies by [16, Theorem 2.5.13] that, in these two cases, χ is real if and only if $2|r$.

In what follows, the choices of α are made following [22], and necessary divisibility properties of $\xi(1)$ were proved therein. For the reader's convenience, we will recall these choices here.

2. Assume n is odd. Then one chooses $r \in 2\mathbb{Z} + 1$ and $s \in 2\mathbb{Z}$, which implies $2|\xi(1)$.

Case 1: $n \not\equiv 0, -1 (\mathrm{mod}\ p)$. If $n \neq 2p-3$, choose $(r, s) = (p-2, 0)$. If $n = 2p-3$, then $p \geq 7$ as $n \geq 9$; here we choose $(r, s) = (3, p-5)$. In both subcases, $\alpha \neq \alpha'$ and $2p|\xi(1) = \chi(1)$, and so we are done.

Case 2: $n \equiv 0 (\mathrm{mod}\ p)$. If $p > 3$, choose $(r, s) = (1, 2)$. If $p = 3$, choose $(r, s) = (3, 2)$. In both subcases, $\alpha \neq \alpha'$ and $2p|\xi(1) = \chi(1)$.

Case 3: $n \equiv -1 (\mathrm{mod}\ p)$; in particular, $n \geq 2p-1$. If $n \neq 2p-1, 4p-1$, choose $(r, s) = (2p-1, 0)$. If $n = 2p-1$ and $n \neq 9$, or if $n = 4p-1$ and $n \neq 11$, take $(r, s) = (p, 2)$. In all these subcases, $\alpha \neq \alpha'$ and $2p|\xi(1) = \chi(1)$. If $(n, p) = (9, 5)$, then we choose $\alpha = (5, 2, 2)$, for which $\chi(1) = \xi(1) = 120$. Finally, if $(n, p) = (11, 3)$, then we choose $\alpha = (6, 5)$, for which $\chi(1) = \xi(1) = 132$.

3. Assume n is even. Then one chooses $r \in 2\mathbb{Z}$ and $s \in 2\mathbb{Z}+1$, which implies $2|\xi(1)$ and χ is real.

Case 1: $n \not\equiv 1, -2 (\mathrm{mod}\ p)$. If $p > 3$, choose $(r, s) = (p-3, 1)$. If $p = 3$, take $(r, s) = (6, 1)$. In both subcases, $4p|\xi(1)$ and so $2p|\chi(1)$.

Case 2: $n \equiv 1 (\mathrm{mod}\ p)$. If $n \geq 3p+1$, choose $(r, s) = (2, p)$. If $n = p+1$ (in particular, $p \geq 11$), choose $(r, s) = (2, 3)$. In both subcases, $\alpha \neq \alpha'$ and $2p|\xi(1) = \chi(1)$.

Case 3: $n \equiv -2 (\mathrm{mod}\ p)$ and $p > 3$; in particular, $n \geq 2p-2$. If $n \geq 4p-2$, choose $(r, s) = (2p, 1)$. If $n = 2p-2$ and $p > 7$, choose $(r, s) = (p-1, 3)$ In both subcases, $\alpha \neq \alpha'$ and $2p|\xi(1) = \chi(1)$. Finally, if $(n, p) = (12, 7)$, then we choose $\alpha = (9, 3)$, for which $\chi(1) = \xi(1) = 154$. □

The rest of the section is devoted to prove Theorem 3.1 in the case $S \in Lie(p)$. We will say that the characters $\chi_1, \ldots, \chi_k \in \mathrm{Irr}(S)$ *cover* S if any prime divisor of $|S|$ divides the degree of at least one χ_i. We aim to show that $\mathrm{Irr}_{\mathrm{rv}}(S)$ contains two or three characters that cover S. Let $\mathbb{F} := \overline{\mathbb{F}}_p$. We will consider a certain simple algebraic group $\mathcal{G}$ over $\mathbb{F}$ with a Frobenius map F on $\mathcal{G}$ and the pair $(\mathcal{G}^*, F^*)$ dual to $(\mathcal{G}, F)$. We will use Lemma 2.5 as well as certain permutation characters and unipotent characters to find desired real characters. The reality of considered semisimple elements is established in [28]. It turns out that the cases of $PSL_n(q)$ and $PSU_n(q)$ require attention the most.

Lemma 3.4 *Theorem 3.1 holds true in the case* $S = PSL_n(q)$ *with* $n \geq 2$ *and* $q = p^f$.

Proof 1. Assume $n = 2$. In view of Proposition 3.3 we may assume $q \geq 7$. Then $\mathrm{Irr}_{\mathrm{rv}}(S)$ contains characters of degree $q, q-1$ and $q+1$, and so we are done. Thus we may assume $n \geq 3$. View S as $\mathcal{G}^F/Z(\mathcal{G}^F)$ for a suitable Frobenius map F on $\mathcal{G} := SL_n(\mathbb{F})$. Observe that $\mathrm{Irr}_{\mathrm{rv}}(SL_6(2))$ contains (unipotent) characters of degree $2^3 \cdot 5 \cdot 31$ and $2^2 \cdot 3 \cdot 7^2$, so we may assume $(n, q) \neq (6, 2)$.

2. Assume furthermore that n is even. Then we can find a PPD $r_n > n$ of $p^{nf} - 1$. Notice that Sylow r_n-subgroups of $(\mathcal{G}^*)^{F^*} = PGL_n(q)$ are cyclic and may be embedded in $PSp_n(q)$. Since any semisimple element of $PSp_n(q)$ is real by [28], we can find a real semisimple element $s_n \in (\mathcal{G}^*)^{F^*}$ of order r_n. By Lemma 2.5 and [23, Lemma 2.4], s_n defines a real character $\chi_n \in \mathrm{Irr}(S)$ of degree divisible by $\prod_{i=2}^{n-1}(q^i - 1)$. We also consider certain unipotent characters χ_λ of $PGL_n(q)$ labeled by partitions λ of n, cf. [4, p. 465]; any of them is rational and restricts irreducibly to S. In particular, $\chi_{(n-2,2)}(1) = (q^n - 1)(q^{n-1} - q^2)/(q^2 - 1)(q - 1)$, and so χ_n and $\chi_{(n-2,2)}$ cover S.

3. From now on we may assume n is odd. Suppose that we can find a PPD r_{n-1} of $p^{(n-1)f} - 1$ such that $(r_{n-1}, n) = 1$. Then Sylow r_{n-1}-subgroups of $SL_{n-1}(q)$ are cyclic and may be embedded in $Sp_{n-1}(q)$. Since any semisimple element of $Sp_{n-1}(q)$ is real by [28], we can find a real semisimple element $s_{n-1} \in SL_{n-1}(q)$ of order r_{n-1} By Lemma 2.5 and [23, Lemma 2.4], s_{n-1} then defines a real character $\chi_{n-1} \in \mathrm{Irr}(S)$ of degree divisible by $\prod_{2 \leq i \leq n,\ i \neq n-1}(q^i - 1)$. In particular, if $n \geq 5$ then χ_{n-1} and $\chi_{(n-1,1)}$ cover S. It remains to consider the cases where such an r_{n-1} does not exist.

4. Assume $n = 3$. The cases $q = 2, 3, 7, 8$ can be verified directly using [6]. Since we are assuming r_{n-1} does not exist, it remains to consider the two cases where either $q \geq 31$ is a Mersenne prime, or $q + 1 = 2^a \cdot 3^b$ for some $a, b \geq 1$. We show that in either case there is a real character $\chi_2 \in \mathrm{Irr}(S)$ of degree $q^3 - 1$. Choose $\epsilon \in \mathbb{F}^\times$ of order 4 in the former case, resp. $q + 1$ in the latter case, and take $s_2 \in (\mathcal{G}^*)^{F^*}$ with preimage $\mathrm{diag}(\epsilon, \epsilon^{-1}, 1)$ in $GL_3(\mathbb{F})$ (which we also denote by s_2). It is easy to check that

$$C_{PGL_3(\mathbb{F})}(s_2) = C_{GL_3(\mathbb{F})}(s_2)/Z(GL_3(\mathbb{F}))$$

and so it is connected, that

$$|C_{PGL_3(q)}(s_2)| = |C_{GL_3(q)}(s_2)/C_{q-1}| = |(C_{q^2-1} \times GL_1(q))/C_{q-1})| = q^2 - 1,$$

and that s_2 is real in $(\mathcal{G}^*)^{F^*}$ (as we can embed it in $SL_2(q) < (\mathcal{G}^*)^{F^*}$). Also, in the former case $|s| = 4$ is coprime to $|Z(\mathcal{G}^F)| = 3$, and in the latter case $|Z(\mathcal{G}^F)| = (3, q - 1) = 1$. By Lemma 2.5, s_2 defines a real character $\chi_2 \in \mathrm{Irr}(S)$ of degree $q^3 - 1$. Clearly, χ_2 and $\chi_{(2,1)}$ cover S.

5. Assume $n = 5$. Again choose $\epsilon \in \mathbb{F}^\times$ of order $q+1$ and take $s_2 \in (\mathcal{G}^*)^{F^*}$ with preimage $\mathrm{diag}(\epsilon, \epsilon^{-1}, 1, 1, 1)$ in $GL_5(\mathbb{F})$ (which we also denote by s_2). It is easy to check that

$$C_{PGL_5(\mathbb{F})}(s_2) = C_{GL_5(\mathbb{F})}(s_2)/Z(GL_5(\mathbb{F}))$$

and so it is connected, that

$$C_{PGL_5(q)}(s_2) = C_{GL_5(q)}(s_2)/C_{q-1} = (C_{q^2-1} \times GL_3(q))/C_{q-1},$$

and that s_2 is real in $(\mathcal{G}^*)^{F^*}$ (as we can embed it in $SL_2(q) < (\mathcal{G}^*)^{F^*}$). Also, the non-existence of r_{n-1} implies that $5|(q^2+1)$ and so $|Z(\mathcal{G}^F)| = (5, q-1) = 1$. By Lemma 2.5, s_2 defines a real character $\chi_2 \in \mathrm{Irr}(S)$ of degree $(q^5-1)(q^2+1)$. Now χ_2 together with $\chi_{(4,1)}$ (of degree $(q^5-q)/(q-1)$) and $\chi_{(3,1^2)}$ (of degree $q^3(q^2+1)(q^2+q+1)$) cover S.

6. Finally, we consider the case $n \geq 7$. Notice that $SL_7(2)$ has unipotent characters $\chi_{(5,2)}$ of degree $2^2 \cdot 5 \cdot 127$ and $\chi_{(5,1,1)}$ of degree $2^3 \cdot 3 \cdot 31$, while $\pi(|S|) = \{2, 3, 5, 7, 31, 127\}$. So we may assume $(n, q) \neq (7, 2)$. Then $p^{(n-1)f} - 1$ has primitive prime divisors, and any such PPD is larger than $(n-1)f$. So the non-existence of r_{n-1} implies that $f = 1$ and n is a prime. Now we can find a PPD r_{n-3} of $p^{(n-3)f} - 1$, and clearly $(r_{n-3}, n(q-1)) = 1$. Arguing as in 3), we see that there is a real semisimple element $s_{n-3} \in Sp_{n-3}(q) < GL_n(q)$ of order r_{n-3} with $C_{GL_n(q)}(s_{n-3}) = GL_3(q) \times C_{q^{n-3}-1}$. This element gives rise to a real semisimple element (which we also denote by s_{n-3}) of order r_{n-3} in $(\mathcal{G}^*)^{F^*} = PGL_n(q)$ with $|C_{PGL_n(q)}(s_{n-3})| = |(GL_3(q) \times C_{q^{n-3}-1})|/(q-1)$. By Lemma 2.5, s_{n-3} defines a real character $\chi_{n-3} \in \mathrm{Irr}(S)$ of degree divisible by $\prod_{4 \leq i \leq n,\ i \neq n-3}(q^i - 1)$. Now χ_{n-3} and $\chi_{(n-2,2)}$ (of degree $q^2(q^n-1)(q^{n-3}-1)/(q^2-1)(q-1)$) cover S. □

Lemma 3.5 *Theorem 3.1 holds true in the case* $S = PSU_n(q)$ *with* $n \geq 3$ *and* $q = p^f$.

Proof 1. Assume $n = 3$. The case $q = 3$ can be checked directly using [6], so we will assume that $q \geq 4$; in particular $(q-1)$ does not divide $q+1$. Hence, in the notation of [10], $\mathrm{Irr}_{\mathrm{rv}}(S)$ contains characters $\chi_{q(q-1)}$ of degree $q(q-1)$ and $\chi_{q^3+1}^{(q+1)}$ of degree q^3+1, and so we are done.

Thus we may assume $n \geq 4$. View S as $\mathcal{G}^F/Z(\mathcal{G}^F)$ for a suitable Frobenius map F on $\mathcal{G} := SL_n(\mathbb{F})$. Observe that $\mathrm{Irr}_{\mathrm{rv}}(PSU_6(2))$ contains characters of degree $2^3 \cdot 5 \cdot 11$ and $3 \cdot 7 \cdot 11$, while $\pi(|S|) = \{2, 3, 5, 7, 11\}$. Furthermore, $SU_7(2)$ has unipotent characters $\chi_{(5,2)}$ of degree $2^2 \cdot 5 \cdot 43$ and $\chi_{(5,1,1)}$ of degree $2^3 \cdot 3 \cdot 7 \cdot 11$, while $\pi(|S|) = \{2, 3, 5, 7, 11, 43\}$. So we may assume $(n, q) \neq (6, 2), (7, 2)$.

2. Assume furthermore that n is even. Then we can find a PPD $r_n > n$ of $p^{nf} - 1$. Let $V := \mathbb{F}_{q^2}^n$ denote the natural module for $SU_n(q)$. We distinguish two cases.

Suppose $n \equiv 0 \pmod 4$. Then Sylow r_n-subgroups of $GU_n(q)$ are cyclic and may be embedded in $Sp_n(q)$. Since any semisimple element of $Sp_n(q)$ is real by [28], we can find a real semisimple element $s_n \in Sp_n(q)$ of order r_n and then embed it in $SU_n(q)$. We may assume that $\mathrm{Spec}(s_n, V)$ contains an eigenvalue α with $|\alpha| = r_n$. By the choice of r_n, α has a minimal polynomial $f(t)$ of degree $n/2$ over $\mathbb{F}_{q^2}$, whose roots are $\alpha^{q^{2i}}$, $0 \leq i < n/2$. Since $s_n \in SU_n(q)$, it also has an eigenvalue α^{-q} which is not a root of $f(t)$ as $4|n$. Now if $\check{f}$ is the minimal polynomial of α^{-q} over $\mathbb{F}_{q^2}$, then the characteristic polynomial of s_n on V is $f \cdot \check{f}$. Standard computations show that $C_{GU_n(q)}(s_n) = GL_1(q^n)$.

Suppose $n \equiv 2 (\bmod\ 4)$. Again choose $\alpha \in \mathbb{F}^\times$ of order r_n and consider its minimal polynomial $f(t)$ over $\mathbb{F}_{q^2}$. The roots of $f(t)$ are $\alpha^{q^{2i}}$, $0 \leq i < n/2$ and they do not include α^{-1} since $n/2$ is odd. Notice that the Sylow r_n-subgroups of $GU_{n/2}(q)$ are cyclic. So we can find a semisimple element $s' \in GU_{n/2}(q)$ having eigenvalues $\alpha^{q^{2i}}$, $0 \leq i < n/2$, on $\mathbb{F}_{q^2}^{n/2}$. Via the embeddings $GU_{n/2}(q) < Sp_n(q) < SU_n(q)$, s' gives rise to a real element s_n of order r_n in $SU_n(q)$ and having eigenvalues $\alpha^{\pm q^{2i}}$, $0 \leq i < n/2$ on V. The explicit spectrum of s_n on V yields that $C_{GU_n(q)}(s_n) = GU_1(q^{n/2})^2$.

In both cases, we will denote the image of s_n in $(\mathcal{G}^*)^{F^*} = PGU_n(q)$ by the same symbol. Since $|s_n| = r_n$ is coprime to $q+1$, $C_{PGU_n(q)}(s_n) = C_{GU_n(q)}(s_2)/Z(GU_n(q))$. By Lemma 2.5, s_n defines a real character $\chi_n \in \mathrm{Irr}(S)$, of degree divisible by $\prod_{i=2}^{n-1}(q^i - (-1)^i)$ if $4|n$ and by $(q^{n/2}-1)\prod_{2\le i\le n-1,\ i\ne n/2}(q^i-(-1)^i)$ if $n \equiv 2(\bmod 4)$. We also consider certain unipotent characters χ_λ of $PGU_n(q)$ labeled by partitions λ of n, cf. [4, p. 465]; any of them is rational (since they are rational combinations of rational Deligne–Lusztig characters $R_{\mathcal{T}}^{\mathcal{G}}(1)$) and restricts irreducibly to S. In particular,

$$\chi_{(n-2,2)}(1) = \frac{q^2(q^n-(-1)^n)(q^{n-3}-(-1)^{n-3})}{(q^2-1)(q+1)}.$$

Clearly, χ_n and $\chi_{(n-2,2)}$ cover S.

3. From now on we may assume $n \ge 5$ is odd and $(n,q) \ne (7,2)$. Suppose that we can find a PPD r_{n-1} of $p^{(n-1)f}-1$ such that $(r_{n-1}, n) = 1$. Arguing as in 2), we can find a real semisimple element $s_{n-1} \in SU_{n-1}(q)$ of order r_{n-1} and then embed it in $(\mathcal{G}^*)^{F^*}$ such that $C_{(\mathcal{G}^*)^{F^*}}(s_{n-1})$ has order $q^{n-1}-1$ if $n \equiv 1(\bmod 4)$ and $(q^{(n-1)/2}+1)^2$ if $n \equiv 3(\bmod 4)$. By Lemma 2.5, s_{n-1} then defines a real character $\chi_{n-1} \in \mathrm{Irr}(S)$. Also, here we have

$$\chi_{(n-2,1^2)}(1) = \frac{q^3(q^{n-1}-1)(q^{n-2}+1)}{(q^2-1)(q+1)}.$$

In this case, χ_{n-1} and $\chi_{(n-2,1^2)}$ cover S. It remains to consider the cases where such an r_{n-1} does not exist; in particular $q = p$ and n is a prime dividing $q^{(n-1)/2}+1$.

4. Assume $n = 5$. The case $q = 2$ can be verified directly using [6]. Since we are assuming r_{n-1} does not exist, $5|(p^2+1)$ and so $p = 3$ or $p \ge 7$. If $p = 3$, then $\chi_{(3,2)}$ and $\chi_{(3,1^2)}$ cover $SU_5(3)$. Suppose $p \ge 7$ Choose $\epsilon \in \mathbb{F}^\times$ of order $q-1$ and take $s_2 \in (\mathcal{G}^*)^{F^*}$ with preimage $\mathrm{diag}(\epsilon, \epsilon^{-1}, 1, 1, 1)$ in $GU_5(\mathbb{F})$ (which we also denote by s_2). It is easy to check that

$$C_{PGL_5(\mathbb{F})}(s_2) = C_{GL_5(\mathbb{F})}(s_2)/Z(GL_5(\mathbb{F}))$$

and so it is connected, that

$$C_{PGU_5(q)}(s_2) = C_{GU_5(q)}(s_2)/C_{q+1} = (C_{q^2-1} \times GU_3(q))/C_{q+1},$$

and that s_2 is real in $(\mathcal{G}^*)^{F^*}$ (as we can embed it in $SU_2(q) < (\mathcal{G}^*)^{F^*}$). Also, $|Z(\mathcal{G}^F)| = (5, q+1) = 1$. By Lemma 2.5, s_2 defines a real character $\chi_2 \in \mathrm{Irr}(S)$ of degree divisible by $(q^5+1)(q^2+1)$. Now χ_2, $\chi_{(4,1)}$ (of degree $q(q-1)(q^2+1)$), and $\chi_{(3,1^2)}$ cover S.

5. Finally, we consider the case $n \ge 7$ is a prime. Then we can find a PPD r_{n-3} of $q^{n-3}-1$, and clearly $(r_{n-3}, n(q+1)) = 1$. Arguing as in 2), we see that there is a real semisimple element $s_{n-3} \in Sp_{n-3}(q) < GU_n(q)$ of order r_{n-3}, with $C_{GU_n(q)}(s_{n-3}) = GU_3(q) \times GL_1(q^{n-3})$ if $n \equiv 3(\bmod 4)$ and $C_{GU_n(q)}(s_{n-3}) = GU_3(q) \times GU_1(q^{(n-3)/2})^2$ if $n \equiv 1(\bmod 4)$. This element gives rise to a real semisimple element (which we also denote by s_{n-3}) of order r_{n-3} in $(\mathcal{G}^*)^{F^*} = PGU_n(q)$ with $|C_{PGU_n(q)}(s_{n-3})| = |C_{GU_n(q)}(s_{n-3})|/(q+1)$. By Lemma 2.5, s_{n-3} defines a real character $\chi_{n-3} \in \mathrm{Irr}(S)$. Now χ_{n-3} and $\chi_{(n-2,2)}$ (of degree $q^2(q^n+1)(q^{n-3}-1)/(q^2-1)(q+1)$) cover S. □

Now we will complete the proof of Theorem 3.1 for the remaining finite groups of Lie type.

1. Assume $S = PSp_{2n}(q)'$ with $n \ge 2$, resp. $\Omega_{2n+1}(q)$ with $n \ge 3$ and q odd. The cases of $PSp_4(2)'$ and $PSp_6(2)$ can be checked directly using [6], so we will assume $S \ne PSp_4(2)'$, $PSp_6(2)$. It follows that there is a PPD r of $q^{2n}-1$, and a semisimple element s of order

r in $(\mathcal{G}^*)^{F^*}$ (which is $SO_{2n+1}(q)$, resp. $PCSp_{2n}(q)$). Such an element is real by [28]. Furthermore, $|C_{(\mathcal{G}^*)^{F^*}}(s)| = q^n + 1$ by [23, Lemma 2.4]. Hence by Lemma 2.5, s defines a real character $\chi_1 \in \mathrm{Irr}(S)$ of degree divisible by $|S|_{p'}/(q^n + 1)$. On the other hand, S has a rank 3 permutation character with a rational irreducible constituent χ_2 of degree $q(q^n + 1)(q^{n-1} - 1)/2(q - 1)$, cf. [27]. Now χ_1, χ_2, and the Steinberg character of S if $q = 2$, cover S.

2. Assume $S = P\Omega^+_{2n}(q)$ with $n \geq 4$. The case of $\Omega^+_8(2)$ can be checked directly using [6], so we will assume $(n, q) \neq (4, 2)$. It follows that there is a PPD r of $q^{2n-2} - 1$. If n is even, then any semisimple element in $(\mathcal{G}^*)^{F^*} = P(CO^+_{2n}(q)^\circ)$ is real by [28], and so we can find a real semisimple element $s \in (\mathcal{G}^*)^{F^*}$ of order r. If n is odd, then any semisimple element in $P\Omega^-_{2n-2}(q)$ is real by [28], and so we can find a real semisimple element $s \in P\Omega^-_{2n-2}(q) < (\mathcal{G}^*)^{F^*}$ of order r. Notice that any Sylow r-subgroup of $P\Omega^-_{2n-2}(q)$ is cyclic and a Sylow subgroup for $(\mathcal{G}^*)^{F^*}$. In either case, $|C_{(\mathcal{G}^*)^{F^*}}(s)| = (q^{n-1} + 1)(q + 1)$ by [23, Lemma 2.4]. By Lemma 2.5, s defines a real character $\chi_1 \in \mathrm{Irr}(S)$ of degree divisible by $|S|_{p'}/(q^{n-1} + 1)(q + 1)$. On the other hand, S has a rank 3 permutation character with a rational irreducible constituent χ_2 of degree $q^2(q^{2n-2} - 1)/(q^2 - 1)$, cf. [27]. Now χ_1 and χ_2 cover S.

3. Assume $S = P\Omega^-_{2n}(q)$ with $n \geq 4$. If n is even, then we can find a PPD r of $q^{2n} - 1$ and a semisimple element $s \in (\mathcal{G}^*)^{F^*} = P(CO^-_{2n}(q)^\circ)$ of order r. Such an element s is real by [28] (since n is even), and $|C_{(\mathcal{G}^*)^{F^*}}(s)| = (q^n + 1)$ by [23, Lemma 2.4]. If n is odd, we can find a PPD r of $q^{2n-2} - 1$ and a semisimple element $s \in P\Omega^-_{2n-2}(q)$ of order r; any such element s is real by [28] (since $n - 1$ is even). Since $P\Omega^-_{2n-2}(q) < (\mathcal{G}^*)^{F^*}$, and any Sylow r-subgroup of $P\Omega^-_{2n-2}(q)$ is cyclic and a Sylow subgroup for $(\mathcal{G}^*)^{F^*}$, we can embed s in $(\mathcal{G}^*)^{F^*}$; furthermore, $|C_{(\mathcal{G}^*)^{F^*}}(s)| = (q^{n-1} + 1)(q + 1)$ by [23, Lemma 2.4]. By Lemma 2.5, s defines a real character $\chi_1 \in \mathrm{Irr}(S)$. Next, S has a rank 3 permutation character with rational irreducible constituents χ_2 of degree $q^2(q^{2n-2} - 1)/(q^2 - 1)$ and χ_3 of degree $q(q^n + 1)(q^{n-2} - 1)/(q^2 - 1)$, cf. [27]. Now χ_1, χ_2, and χ_3 cover S.

4. Assume S is of type $E_6(q)$, resp. ${}^2E_6(q)$. Then we can find a PPD r of $q^{12} - 1$, and a semisimple element $s \in F_4(q)$ of order r; any such element s is real by [28]. Since $F_4(q) < (\mathcal{G}^*)^{F^*}$ and any Sylow r-subgroup of $F_4(q)$ is cyclic and a Sylow subgroup for $(\mathcal{G}^*)^{F^*}$, we can embed s in $(\mathcal{G}^*)^{F^*}$. Furthermore, $|C_{(\mathcal{G}^*)^{F^*}}(s)| = \Phi_{12}(q)\Phi_3(q)$, resp. $\Phi_{12}(q)\Phi_6(q)$ by [23, Lemma 2.3]. By Lemma 2.5, s defines a real character $\chi_1 \in \mathrm{Irr}(S)$. In p. 5 of the proof of Theorem 2.1 we have shown that $\mathrm{Irr}(S)$ also contains a real unipotent character χ_2: $\phi_{20,2}$ of degree $q^2\Phi_4(q)\Phi_5(q)\Phi_8(q)\Phi_{12}(q)$ in the case of $E_6(q)$, and $\phi_{4,1}$ of degree $q^2\Phi_4(q)\Phi_8(q)\Phi_{10}(q)\Phi_{12}(q)$ in the case of ${}^2E_6(q)$. Clearly, χ_1 and χ_2 cover S.

5. Assume S is of type $G_2(q)$, resp. $F_4(q)$, $E_7(q)$, $E_8(q)$ with $q = p^f$. The cases of $G_2(3)$ and $G_2(4)$ can be checked directly using [6], so we will assume that $q \geq 5$ if $S = G_2(q)$. It follows that there exist a PPD r_1 of $p^{m_1 f} - 1$ and a PPD r_2 of $p^{m_2 f} - 1$, where $(m_1, m_2) = (6, 3)$, resp. $(12, 8)$, $(18, 14)$, $(30, 24)$. Now we can find a semisimple element $s_i \in (\mathcal{G}^*)^{F^*}$ of order r_i for $i = 1, 2$; such elements are real by [28]. Furthermore, $|C_{(\mathcal{G}^*)^{F^*}}(s_i)| = \Phi_{m_i}(q)$ or $\Phi_{m_i}(q)\Phi_2(q)$ by [23, Lemma 2.3]. By Lemma 2.5, s_i defines a real character $\chi_i \in \mathrm{Irr}(S)$. Now χ_1, χ_2, and the Steinberg character cover S.

6. Assume $S = {}^3D_4(q)$; in particular, every character of S is real by [28, Theorem 1.2]. There is a PPD r of $q^{12} - 1$, and a semisimple element s of order r in $(\mathcal{G}^*)^{F^*} \simeq S$; furthermore, $|C_{(\mathcal{G}^*)^{F^*}}(s)| = \Phi_{12}(q)$ by [23, Lemma 2.3]. Hence by Lemma 2.5, s defines a real character $\chi_1 \in \mathrm{Irr}(S)$ of degree $|S|_{p'}/\Phi_{12}(q)$. On the other hand, $\mathrm{Irr}(S)$ contains a unique character χ_2 of smallest degree $q\Phi_{12}(q)$, cf. [20]. Now χ_1 and χ_2 cover S.

Assume $S = {}^2F_4(q)'$. The case of ${}^2F_4(2)'$ can be checked directly using [6], so we will assume $q \geq 8$. It follows that there is a PPD r of $q^6 - 1$, and a semisimple element s of order r in $(\mathcal{G}^*)^{F^*} \simeq S$. Such an element is real by [28]; furthermore, $|C_{(\mathcal{G}^*)^{F^*}}(s)| = \Phi_6(q)$ by [23, Lemma 2.3]. Hence by Lemma 2.5, s defines a real character $\chi_1 \in \mathrm{Irr}(S)$ of degree $|S|_{p'}/\Phi_6(q)$. On the other hand, $\mathrm{Irr}(S)$ contains a unique, hence rational, character χ_2 of third smallest degree $q\Phi_6(q)\Phi_{12}(q)$, cf. [20]. Now χ_1 and χ_2 cover S.

Assume S is of type ${}^2B_2(q)$, resp. ${}^2G_2(q)$, with $q = p^f$ and $f \geq 3$ is odd. There exist a PPD r_1 of $p^{m_1 f} - 1$ and a PPD r_2 of $p^{m_2 f} - 1$, where $(m_1, m_2) = (1, 4)$, resp. $(1, 6)$. Now we can find a semisimple element $s_i \in (\mathcal{G}^*)^{F^*} \simeq S$ of order r_i for $i = 1, 2$; such elements are real by [28]. By [23, Lemma 2.3], $|C_{(\mathcal{G}^*)^{F^*}}(s_i)| = q - 1$ for $i = 1$ and $q \pm \sqrt{pq} + 1$ for $i = 2$. By Lemma 2.5, s_i defines a real character $\chi_i \in \mathrm{Irr}(S)$. Now χ_1, χ_2, and the Steinberg character cover S.

Theorem 3.1 has been proved completely. □

4 Proof of Theorem A

We shall need the following result.

Lemma 4.1 *Let G be a finite group and let $N \lhd G$ be with G/N of odd order. If $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$, then all irreducible constituents of χ_N are real-valued. If $\theta \in \mathrm{Irr}(N)$ is real-valued, then there exists a unique $\psi \in \mathrm{Irr}_{\mathrm{rv}}(G)$ over θ.*

Proof The second part is [25, Corollary 2.2]. The first part is easier. If $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$ and $\theta \in \mathrm{Irr}(N)$ lies under χ, then $\bar{\theta}$ and θ are G-conjugate by Clifford's theorem. Thus $\theta^g = \bar{\theta}$ for some $g \in G$ and g^2 stabilizes θ. Since $\langle g^2 N\rangle = \langle gN\rangle$, it follows that g stabilizes θ. Hence $\bar{\theta} = \theta$. □

Recall that a 2-group P is of *Chillag–Mann type* if $\mathrm{Irr}_{\mathrm{rv}}(P) = \mathrm{Irr}(P/\Phi(P))$.

Theorem 4.2 *Let G be a finite group and let $P \in \mathrm{Syl}_2(G)$. Then all $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$ have odd degree if and only if $P \lhd G$ of Chillag–Mann type.*

Proof Suppose first that $P \lhd G$ and that all characters in $\mathrm{Irr}_{\mathrm{rv}}(P)$ are linear. Let $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$ and let $\theta \in \mathrm{Irr}(P)$ be under χ, which we know by Lemma 4.1 that it is linear. Hence $\chi(1)$ divides $|G : P|$ which is odd.

Now, we assume that every character in $\mathrm{Irr}_{\mathrm{rv}}(G)$ has odd degree, and prove by induction on $|G|$ that $P \lhd G$. If $1 < N \lhd G$, then $\mathrm{Irr}_{\mathrm{rv}}(G/N) \subseteq \mathrm{Irr}_{\mathrm{rv}}(G)$, and by induction we have that $PN \lhd G$. Now, let $\theta \in \mathrm{Irr}_{\mathrm{rv}}(PN)$. By Lemma 4.1, there exists $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$ over θ. Since χ has odd degree, θ has odd degree. If $PN < G$, then $P \lhd PN$ by induction hypothesis, and so $P \lhd G$. Hence, we may assume that $PN = G$ for every $1 < N \lhd G$. Arguing by contradiction, we may assume that $P \neq 1$ is not normal in G. In particular, we deduce that G has a unique minimal normal subgroup, say M.

Suppose first that M is abelian. Hence, M is an odd order normal subgroup of G and $G = MP$. We also may assume that $M > 1$. In particular, $\mathbf{C}_P(M) = 1$ because M is the unique minimal normal subgroup of G. Since $P \neq 1$, we can choose an involution $x \in P$. If x fixes all $\mathrm{Irr}(M)$, then x fixes all conjugacy classes of M by Brauer's lemma (6.32) of [15]. Since $|M|$ is odd, it follows that $[x, M] = 1$, which is impossible. Therefore, there exists $\lambda \in \mathrm{Irr}(M)$ such that $1 \neq \nu = \lambda^{-1}\lambda^x$. Now, $\nu^x = \bar{\nu}$. If T is the stabilizer of ν in G and $\eta \in \mathrm{Irr}(T)$ is the *canonical extension* of ν to T (see Corollary (6.28) of [15]), we have that

$\eta^x = \bar{\eta}$ because $(\bar{\eta})^x$ and η are two *canonical extensions* of ν to T. Now, if $\chi = \eta^G \in \mathrm{Irr}(G)$, then

$$\bar{\chi} = (\bar{\eta})^G = (\eta^x)^G = \eta^G = \chi\,.$$

Thus χ is real and has odd degree. But $|G/M|$ is a 2-power, so χ_M is a multiple of ν. Since $\nu^x = \bar{\nu}$, this implies that $\nu = \bar{\nu}$ and this is not possible (since $\nu \neq 1$).

Thus we may assume that M is a direct product of non-abelian simple groups which are transitively permuted by P. Write $M = S_1 \times \cdots \times S_n$, where each S_i is simple non-abelian. Now, write $S = S_1$ and let $H = \mathbf{N}_G(S)$. Since $\mathbf{C}_G(S) \cap S = 1$, it easily follows that the group $\bar{H} = H/\mathbf{C}_H(S)$ has a normal simple group $\bar{S} = S\mathbf{C}_H(S)/\mathbf{C}_H(S)$ such that $\mathbf{C}_{\bar{H}}(\bar{S}) = 1$. Since $M \subseteq S\mathbf{C}_G(S)$, it follows that $\bar{H}/\bar{S}$ is a 2-group. Hence, by Theorem 2.1 there exists $\delta \in \mathrm{Irr}_{\mathrm{rv}}(\bar{H})$ of even degree such that $[\delta_S, 1_S] = 0$ when we inflate δ to H. Let $1 \neq \theta \in \mathrm{Irr}(S)$ be an irreducible constituent of δ_S and let

$$\psi = \theta \otimes 1_{S_2} \otimes \cdots \otimes 1_{S_n}\,.$$

Notice that δ lies over ψ. Now, let T be the inertia group of ψ in G. We claim that $T \leq H$. Perhaps, the easiest way to see this is the following. Let $t \in T$, and notice that $\mathrm{Ker}(\psi) = S_2 \times \cdots \times S_n$. Hence $\{S_2, \ldots, S_n\}$ is t-invariant, which implies that S_1 is t-invariant and so $t \in H$. Now, if ξ is the Clifford correspondent of δ over ψ, then we have that $\xi^G \in \mathrm{Irr}(G)$. Thus δ^G is an irreducible real character of even degree, and this contradiction finally proves that $P \lhd G$. Finally, notice that if $\theta \in \mathrm{Irr}_{\mathrm{rv}}(P)$, then θ lies under some irreducible $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$ by Lemma 4.1. In particular, we deduce that every real irreducible character of P is linear. □

5 The graph of real-valued characters

Let $\Gamma_{\mathrm{rv}}(G)$ be the prime graph on the set of the degrees of the irreducible real characters of the group G. Namely, $\Gamma_{\mathrm{rv}}(G)$ is the graph with vertex set

$$\mathrm{V}(G) = \bigcup_{\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)} \pi(\chi(1))$$

and edge set

$$\mathrm{E}(G) = \{\{p, q\} \mid p \neq q,\ \ \{p, q\} \subseteq \pi(\chi(1)) \text{ for some } \chi \in \mathrm{Irr}_{\mathrm{rv}}(G)\}\,.$$

For a nonlinear $\chi \in \mathrm{Irr}_{\mathrm{rv}}(G)$, we denote by $\Delta_G(\chi)$ the connected component of $\Gamma_{\mathrm{rv}}(G)$ that contains the set of vertices $\pi(\chi(1))$. In the same way, if $q \in \mathrm{V}(G)$ we denote by $\Delta_G(q)$ the connected component of $\Gamma_{\mathrm{rv}}(G)$ which q belongs to.

Let $n(\Gamma_{\mathrm{rv}}(G)) = |\{\Delta_G(\chi) \mid \chi \in \mathrm{Irr}_{\mathrm{rv}}(G)\}|$ be the number of connected components of the graph $\Gamma_{\mathrm{rv}}(G)$.

Observe that if $N \lhd G$, then $\Gamma_{\mathrm{rv}}(G/N)$ is a subgraph of $\Gamma_{\mathrm{rv}}(G)$. If $|G : N|$ is odd, then by Lemma 4.1 also $\Gamma_{\mathrm{rv}}(N)$ is a subgraph of $\Gamma_{\mathrm{rv}}(G)$.

We are now going to prove Theorem B. Observe that the bounds on the number of connected components of $\Gamma_{\mathrm{rv}}(G)$ are sharp, as $n(\Gamma_{\mathrm{rv}}(S_3)) = 2$ and $n(\Gamma_{\mathrm{rv}}(A_5)) = 3$.

Theorem 5.1 *Let G be a finite group. Then*

(i) $n(\Gamma_{\mathrm{rv}}(G)) \leq 3$;
(ii) $n(\Gamma_{\mathrm{rv}}(G)) \leq 2$ *if G is solvable.*

Proof Let G be a counterexample of minimal order. Then, $n(\Gamma_{rv}(G)) \geq 3$ if G is solvable and $n(\Gamma_{rv}(G)) \geq 4$ if G is non-solvable.

Let N be a maximal normal subgroup of G and assume that G/N is non-solvable. Then G/N is a non-abelian simple group and $\mathrm{V}(G/N) = \pi(|G/N|)$ by Theorem 3.1(i). Observe also that, by Theorem 3.1(ii), $N \neq 1$. Thus, $n(\Gamma_{rv}(G/N)) \leq 3 < n(\Gamma_{rv}(G))$ and hence there is a $\chi \in \mathrm{Irr}_{rv}(G)$ such that $\Delta_G(\chi) \cap \mathrm{V}(G/N) = \emptyset$. Then $(\chi(1), |G/N|) = 1$ and hence by [15, Corollary (11.29)] $\chi_N \in \mathrm{Irr}_{rv}(N)$. Therefore, by [15, Corollary (6.17)] for any $\psi \in \mathrm{Irr}_{rv}(G/N)$, we have $\chi\psi \in \mathrm{Irr}_{rv}(G)$. Taking $\psi \in \mathrm{Irr}_{rv}(G/N)$ with $\psi(1) \neq 1$, it follows that $\Delta_G(\chi) = \Delta_G(\chi\psi)$ intersects nontrivially $\mathrm{V}(G/N)$, a contradiction.

Assume now that G/N is a cyclic group of prime order $p \neq 2$ for some $N \lhd G$.

As $n(\Gamma_{rv}(N)) < n(\Gamma_{rv}(G))$, there exists a $\chi \in \mathrm{Irr}_{rv}(G)$ such that $\Delta_G(\chi) \cap \mathrm{V}(N) = \emptyset$. By Lemma 4.1, then $\chi_N = \sum_{i=1}^{p} \phi_i$ where each $\phi_i \in \mathrm{Irr}_{rv}(N)$ is linear and nontrivial. Hence $K = \mathrm{Ker}(\chi) \leq N$ and N/K is an elementary abelian 2-group, as $|N : \mathrm{Ker}(\phi_i)| = 2$. Take now $\psi \in \mathrm{Irr}_{rv}(G)$ such that $\Delta_G(\psi) \neq \Delta_G(2)$ and $\Delta_G(\psi) \neq \Delta_G(p)$ (it exists as $n(\Gamma_{rv}(G)) \geq 3$). Then $(\psi(1), |G/K|) = 1$ and hence $\psi_K \in \mathrm{Irr}_{rv}(K)$. Therefore, $\psi\chi \in \mathrm{Irr}_{rv}(G)$ and, as $p|\chi(1)$, it follows that $\Delta_G(\psi) = \Delta_G(\chi\psi) = \Delta_G(p)$, a contradiction.

We can finally assume that, $|G/N| = 2$ for every maximal normal subgroup N of G.

Let $M = O^2(G)$ and consider $L \leq M$, L normal in G, such that M/L is a chief factor of G (observe that M is non-trivial).

We claim that there is a $\chi \in \mathrm{Irr}_{rv}(G)$ such that $\Delta_G(\chi) \cap \pi(|G/L|) = \emptyset$. This is clear if M/L is solvable, since then $\pi(|G/L|) = \{2, q\}$ for some prime q. If M/L is nonsolvable, then M/L is a direct product of isomorphic non-abelian simple groups and, by Theorem 3.1 $\mathrm{V}(M/L) = \pi(|M/L|) = \pi(|G/L|)$. But as $n(\Gamma_{rv}(M/L)) \leq 3 < n(\Gamma_{rv}(G))$, there is a $\chi \in \mathrm{Irr}_{rv}(G)$ such that $\Delta_G(\chi) \cap \mathrm{V}(M/L) = \emptyset$ and the claim is proved.

In particular, $(\chi(1), |G/L|) = 1$ and hence $\chi_L \in \mathrm{Irr}_{rv}(L)$. Observe now that G/L has no normal Sylow 2-subgroup, as otherwise there is a maximal normal subgroup N of G such that $|G/N| = |M/L| \neq 2$. Therefore, by Theorem A there is a $\psi \in \mathrm{Irr}_{rv}(G/L)$ such that $\psi(1)$ is even. Since $\chi\psi \in \mathrm{Irr}_{rv}(G)$, it follows that $\Delta_G(\chi) = \Delta_G(\chi\psi)$ intersects nontrivially $\pi(|G/L|)$, against the choice of χ. □

6 Real conjugacy classes

Recall that an element g of a finite group G is said to be *real* in G if there is an $x \in G$ such that $g^{-1} = g^x$. In the following we denote by $\mathrm{Re}(G)$ the set of the real elements in G. A conjugacy class $g^G = \{g^x \mid x \in G\}$ is said to be a *real class* when g is a real element of G or, equivalently, when $g^G = (g^{-1})^G$.

It has been observed that the set of the sizes of the conjugacy classes of a finite group shows strong similarities with the set of the degrees of the irreducible characters. Usually, results for conjugacy classes are somewhat stronger and also easier to prove than the corresponding results for character degrees.

This is the case for real classes and real characters as well. The following Theorem 6.1, which is an analogue for conjugacy classes of Theorem A, does not require the use of the classification of the finite simple groups.

Theorem 6.1 *Let G be a finite group, and let P be a Sylow 2-subgroup of G. Then all real classes of G have odd size if and only if $P \lhd G$ and* $\mathrm{Re}(P) \subseteq \mathbf{Z}(P)$.

We recall that the 2-groups P such that $\mathrm{Re}(P) \subseteq \mathbf{Z}(P)$ have been studied by Chillag and Mann in [5].

We also have an analogue of Theorem B. Let $\Gamma^*(G)$ be the prime graph on the set of the sizes of the real conjugacy classes of G. Precisely, $\Gamma^*(G)$ is the graph with vertex set

$$\mathrm{V}^*(G) = \bigcup_{g \in \mathrm{Re}(G)} \pi(|g^G|)$$

and edge set

$$\mathrm{E}^*(G) = \left\{ \{p, q\} \mid p \neq q, \ \ \{p, q\} \subseteq \pi(|g^G|) \text{ for some } g \in \mathrm{Re}(G) \right\}.$$

Let us denote by $n(\Gamma^*(G))$ the number of connected components of the graph $\Gamma^*(G)$.

Theorem 6.2 *For any finite group G,*

$$n(\Gamma^*(G)) \leq 2.$$

We start now working towards the proof of Theorems 6.1 and 6.2.

Lemma 6.3 (a) *If $x \in \mathrm{Re}(G)$ and $|x^G|$ is odd, then $x^2 = 1$.*

(b) *If $x, y \in \mathrm{Re}(G)$, $xy = yx$ and $(|x^G|, |y^G|) = 1$, then $xy \in \mathrm{Re}(G)$. If further $(o(x), o(y)) = 1$, then $\pi(|x^G|) \cup \pi(|y^G|) \subseteq \pi(|(xy)^G|)$.*

(c) *Let $M \lhd G$ be a 2-subgroup and let $x \in G$ be an element of odd order. If $xM \in \mathrm{Re}(G/M)$, then $x \in \mathrm{Re}(G)$.*

(d) *If $N \lhd G$ and $|G/N|$ is odd, then $\mathrm{Re}(G) = \mathrm{Re}(N)$.*

Proof (a) The inversion map, acting on the real class x^G, must have at least a fixed point. Hence $y = y^{-1}$ for all $y \in x^G$.

(b) Let $u, v \in G$ such that $x^{-1} = x^u$ and $y^{-1} = y^v$. Since $[G : C_G(x)]$ and $[G : C_G(y)]$ are coprime, then $G = C_G(x)C_G(y)$ and we can write $uv^{-1} = a^{-1}b$ for suitable $a \in C_G(x)$ and $b \in C_G(y)$. It follows that

$$(xy)^{-1} = y^{-1}x^{-1} = x^{-1}y^{-1} = x^u y^v = x^{au} y^{bv} = (xy)^g$$

where $g = au = bv$. Finally, if x and y have coprime orders, then $C_G(xy) = C_G(x) \cap C_G(y)$ and hence $\pi(|(xy)^G|) \supseteq \pi(|x^G|) \cup \pi(|y^G|)$.

(c) This is a special case of the Lemma 2.2(d) in [24]. For reader's convenience, we give a proof of it anyway. Let $g \in G$ such that $(xM)^g = x^{-1}M$ and write $H = \langle x \rangle$. By the Frattini argument, g normalizes some conjugate H_0 of H in MH and hence g acts as the inversion on H_0. It follows that $H_0 \subseteq \mathrm{Re}(G)$ and then that $x \in \mathrm{Re}(G)$.

(d) Let $g \in \mathrm{Re}(G)$ and let x be an element of G such that $g^x = g^{-1}$. Observe that gN is a real element of G/N and, since by (a) a group of odd order has no nontrivial real element, it follows $g \in N$. Consider now $y = x^m$, where $m = o(x)_{2'}$ is the $2'$-part of the order of x. Then $g^y = g^{-1}$ and, as y is a 2-element, $y \in N$, which implies that $g \in \mathrm{Re}(N)$. Hence, $\mathrm{Re}(G) \subseteq \mathrm{Re}(N)$. The other inclusion is trivial. □

We next describe the groups with no nontrivial real elements of odd order.

Proposition 6.4 *The following are equivalent*:

(a) *Every nontrivial element in* $\mathrm{Re}(G)$ *has even order;*
(b) *Every element in* $\mathrm{Re}(G)$ *is a 2-element;*
(c) *G has normal Sylow 2-subgroup.*

Proof Observing that powers of real elements are real as well, it is clear that (a) and (b) are equivalent.

We show now that (a) implies (c). By Lemma 6.3(c), if (a) holds in G then also in $G/O_2(G)$ there is no nontrivial real element of odd order. Working by induction on $|G|$, we can hence assume that $O_2(G) = 1$. Assume by contradiction that $|G|$ is even and let $x \in G$ be an involution. For every $g \in G$, the subgroup $D = \langle x, x^g \rangle$ is dihedral and then $D = \mathrm{Re}(D) \subseteq \mathrm{Re}(G)$ and D is a 2-group. By a theorem of Baer, it follows that $x \in O_2(G) = 1$, a contradiction.

Finally, if G has a normal Sylow 2-subgroup P, then by Lemma 6.3(d) $\mathrm{Re}(G) = \mathrm{Re}(P)$ and hence (c) implies (b). □

Proof of Theorem 6.1 Let P be a Sylow 2-subgroup of G.

Assume first that $|g^G|$ is odd for all $g \in \mathrm{Re}(G)$. Then by Lemma 6.3(a) $g^2 = 1$ for all $g \in \mathrm{Re}(G)$ and hence by Proposition 6.4 P is normal in G. Further, if $g \in \mathrm{Re}(P)$ then clearly $g \in \mathrm{Re}(G)$ and hence g centralizes P.

Conversely, assume that $P \lhd G$ and that $\mathrm{Re}(P) \subseteq \mathbf{Z}(P)$. By Lemma 6.3(d) $\mathrm{Re}(G) = \mathrm{Re}(P)$ and hence $|g^G|$ is odd for all $g \in \mathrm{Re}(G)$. □

We now come to the proof of Theorem 6.2. For $g \in \mathrm{Re}(G)$, $g \notin Z(G)$, we denote by $\Delta^*_G(g)$ the connected component of $\Gamma^*(G)$ that contains the set of vertices $\pi(|g^G|)$. In the same way, if $q \in \mathrm{V}^*(G)$ we denote by $\Delta^*_G(q)$ the connected component of $\Gamma^*(G)$ which q belongs to.

Proof of Theorem 6.2 Working by contradiction, let G be a group of minimal order such that $n(\Gamma^*(G)) \geq 3$.

Assume first that there exists a nontrivial element of odd order $x \in \mathrm{Re}(G)$. By taking a suitable power, we can assume that $o(x) = p$ with p an odd prime. By Lemma 6.3(a), then $\Delta^*_G(x) = \Delta^*_G(2)$. Since $n(\Gamma^*(G)) \geq 3$, there is a connected component Δ^* of $\Gamma^*(G)$ with $\Delta^* \neq \Delta^*_G(x)$ and $\Delta^* \neq \Delta^*_G(p)$. Let y be a noncentral real element of G such that $\Delta^* = \Delta^*_G(y)$. Because $2 \notin \Delta^*$, y is an involution. Further, as $p \notin \Delta^*$, y commutes with a Sylow p-subgroup of G and, up to conjugation, we can assume that x and y commute. Hence by Lemma 6.3(b) we get the contradiction $\Delta^*_G(x) = \Delta^*_G(xy) = \Delta^*_G(y)$.

Therefore, every real element in G is a 2-element and hence by Proposition 6.4 G has a normal Sylow 2-subgroup P. In particular, by the Feit–Thompson theorem G is solvable. Let N be a maximal normal subgroup of G containing P. Then $[G : N] = q$ an odd prime and $\mathrm{Re}(G) = \mathrm{Re}(N)$ by Lemma 6.3(d). If $x \in \mathrm{Re}(G)$, then $\pi(|x^N|) \subseteq \pi(|x^G|) \subseteq \pi(|x^N|) \cup \{q\}$.

By minimality, $n(\Gamma^*(N)) \leq 2$ and this implies that a connected component of $\Gamma^*(G)$ must consist of the single prime q. Thus, if $g \in \mathrm{Re}(G)$ and $q \in \pi(|g^G|)$, then $|g^G|$ is a q-power and $g \in Z(N)$. Further, g is an involution by Lemma 6.3(a).

Let $Z = \mathbf{Z}(P) \lhd G$. As P lies in the kernel of the action of N on Z, we have the decomposition $Z = C_Z(N) \times [Z, N]$, with $C_Z(N), [Z, N] \lhd G$. Let $x \in G$ with $\Delta^*_G(x) = \{q\}$. Then $x \in Z \cap C_G(N) = C_Z(N)$. Observe now that $[Z, N] \neq 1$, as otherwise every real class of odd size would have q-power size and hence $n(\Gamma^*(G)) \leq 2$. Let $y \in [Z, N]$ be an involution. Then $y \in \mathrm{Re}(G)$ and $(|x^G|, |y^G|) = 1$, so by Lemma 6.3(b) it follows $xy \in \mathrm{Re}(G)$. Since x and y lie in normal subgroups of G which intersect trivially, we have $C_G(xy) = C_G(x) \cap C_G(y)$ and hence $\pi(|(xy)^G|) \supseteq \pi(|x^G|) \cup \pi(|y^G|)$. This gives $\Delta^*_G(x) = \Delta^*_G(xy) = \Delta^*_G(y)$, a contradiction. □

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