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# On finite energy solutions of the KP-I equation

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**Abstract** We prove that the flow map of the Kadomtsev–Petviashvili-I (KP-I) equation is not uniformly continuous on bounded sets of the natural energy space.

**Keywords** Dispersive equations · Non-linear waves

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## 1 Introduction

The understanding of solutions to dispersive equations has considerably deepened during the last decade. Much of the progress is based on the idea of Bourgain of using $L^2$ based function spaces adapted to the linear operator. This technique establishes often a very non-trivial domination of the non-linearity by the linear operator, at least on small scales. It is connected to objects in harmonic analysis: restriction theorems, local smoothing, maximal functions and multilinear estimates. If applicable it leads to existence and uniqueness via Picard iteration or the implicit function theorem, and hence to uniform continuity and even differentiability of the flow map.

Despite the amazing success of this approach there are several problems where it failed completely. Two of the most interesting of them are the Benjamin–Ono and the Kadomtsev–Petviashvili-I (KP-I) equation. The Benjamin–Ono equation has been intensively studied during the last 3 years, and a very precise understanding has emerged: in the same way as for Burgers equation the low frequency part leads to a change of the speed of waves ([7]), which in turn contradicts uniform continuity of

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the flow map. This, however, does not imply illposedness. The transport effect can be controlled by a gauge transform, see Tao [10]. Ionescu and Kenig [5] approached the transformed problem by adapted function spaces and bilinear estimates and obtained well-posedness for initial data in $L^2$. The non-linear Schrödinger equation on compact manifolds has related properties which have been studied by different techniques by Burq, Gérard and the second author [1,2] and by Colliander, Christ and Tao [3]. We refer to [12] for a survey on the notion of well-posedness for dispersive PDE's. The result we obtain in this paper essentially answers a conjecture in [12].

In this paper we study the KP-I equation

$$u_t + u_{xxx} - \partial_x^{-1} u_{yy} + u u_x = 0, \tag{1.1}$$

where $(t,x,y) \in \mathbb{R}^3$, $u$ is real valued function and $\partial_x^{-1}$ formal notation for the antiderivative, which always exists for tempered distributions, but whose uniqueness requires further considerations. In this paper we only deal with antiderivatives in $L^2$ of $L^2$ functions $f$. In this case the antiderivative is uniquely defined and its Fourier transform can be defined through multiplication of $\widehat{f}$ by $(i\xi)^{-1}$. It is not hard to see that, if $f$ is in addition compactly supported, then it is integrable with mean zero, and hence the antiderivative could also be defined through the indefinite integral from $-\infty$.

The KP-I equation appears as an asymptotic model for the propagation of long, essentially one directional, small amplitude surface waves when the surface tension is bigger than some critical value. For smaller values of the surface tension we get the KP-II model. The KP-I equation can be written in the Lax pair form (see [13]) and thus it shares many features with the "integrable PDE's". One also has a family of particular solitary waves solutions called lump solutions. The study of the KP-I flow close to the lumps is a challenging issue.

A simple calculation shows that the solutions of (1.1) satisfy, at least formally, the conservation of the $L^2$ norm

$$N(u) = \int_{\mathbb{R}^2} u^2(t,x,y)\,dxdy = \text{const}$$

and the conservation of the energy

$$E(u) = \frac{1}{2}\int_{\mathbb{R}^2} \Big[(\partial_x u)^2 + (\partial_x^{-1} u_y)^2 - \frac{1}{3}u^3\Big](t,x,y)\,dxdy = \text{const}\,.$$

Taking into account the anisotropic Sobolev inequality (see e.g., [11] and Lemma 4.1),

$$\|u\|_{L^3(\mathbb{R}^2)}^3 \le C\|u\|_{L^2(\mathbb{R}^2)}^{\frac{3}{2}} \|u_x\|_{L^2(\mathbb{R}^2)} \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}^{\frac{1}{2}}$$

we deduce that the subspace of $L^2$ of finite energy is a Banach space $X$, which we call *energy space*, equipped with the norm

$$\|u\|_X = \|u\|_{L^2(\mathbb{R}^2)} + \|u_x\|_{L^2(\mathbb{R}^2)} + \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)}\,.$$

This provides a natural framework to study the non-linear problem (1.1). The Cauchy problem for the KP-I equation is known to be globally well-posed in spaces smaller than the energy space (see [6,8]). More precisely, the Cauchy problem associated to (1.1) is globally well-posed for data in the space $Z$ equipped with the norm

$$\|u\|_Z = \|u\|_{L^2(\mathbb{R}^2)} + \|u_{xx}\|_{L^2(\mathbb{R}^2)} + \|\partial_x^{-2} u_{yy}\|_{L^2(\mathbb{R}^2)}\,.$$

Notice that $u \in Z$ implies $u_x \in L^2(\mathbb{R}^2)$, $u_y \in L^2(\mathbb{R}^2)$, $\partial_x^{-1} u_y \in L^2(\mathbb{R}^2)$. Moreover, the Lax pair formulation of the KP-I equation implies that if the initial data

$$u_0 = u|_{t=0} \in Z$$

of (1.1) is smooth [i.e., in the intersection of all $H^s(\mathbb{R}^2)$, $s \in \mathbb{N}$], and if $\partial_x^{-1} u_0$ is also smooth then the global solution of (1.1) with data $u_0$ satisfies a third conservation law. Namely

$$F(u) = \text{const},$$

where

$$\begin{aligned} F(u) =& \frac{3}{2}\int_{\mathbb{R}^2} u_{xx}^2 + 5\int_{\mathbb{R}^2} u_y^2 + \frac{5}{6}\int_{\mathbb{R}^2} (\partial_x^{-2} u_{yy})^2 \\ & - \frac{5}{6}\int_{\mathbb{R}^2} u^2(\partial_x^{-2} u_{yy}) - \frac{5}{6}\int_{\mathbb{R}^2} u\,(\partial_x^{-1} u_y)^2 + \frac{5}{4}\int_{\mathbb{R}^2} u^2\, u_{xx} + \frac{5}{24}\int_{\mathbb{R}^2} u^4 . \end{aligned} \tag{1.2}$$

There is in fact an infinite sequence of formal conservation laws associated to the KP-I equation (see [13]). However, as noticed in [8], it is hard to find a suitable framework of distributions on $\mathbb{R}^2$ where these conservation laws make sense.

It is presently not known whether (1.1) is well-posed in the energy space $X$, but we hope this problem will be given an affirmative answer in the near future. The goal of this paper is to show that, whatever the answer is, the flow map of (1.1) can not be uniformly continuous on bounded sets of the energy space $X$. Recall that, if one solves (1.1) in $X$ by the Picard iteration, then the flow map is automatically uniformly continuous on bounded subsets of $X$. Our result thus implies that the solution to (1.1) cannot by constructed by Picard iteration scheme, in sharp contrast to many other dispersive models, the KP-II equation [where (1.1) $-\partial_x^{-1} u_{yy}$ is replaced by $\partial_x^{-1} u_{yy}$], the KdV equation, etc. This feature of the KP-I equation was already observed in [8,9]. In the present paper we construct some solutions of (1.1) which are "responsible" for this phenomenon. Here is the precise statement of our result.

**Theorem 1** *There exist two positive constants c and C and two sequences $(u_n)$ and $(\widetilde{u}_n)$ of solutions of (1.1) such that for every $t \in [-1, 1]$,*

$$\sup_n \|u_n(t,\cdot)\|_X + \sup_n \|\widetilde{u}_n(t,\cdot)\|_X \le C\,,$$

*$(u_n)$ and $(\widetilde{u}_n)$ satisfy initially*

$$\lim_{n\to\infty} \|u_n(0,\cdot) - \widetilde{u}_n(0,\cdot)\|_X = 0,$$

*but, for every $t \in [-1, 1]$,*

$$\liminf_{n\to\infty} \|u_n(t,\cdot) - \widetilde{u}_n(t,\cdot)\|_X \ge c\,|t|\,.$$

In a previous paper [7], we proved a similar result for the Benjamin–Ono equation. The analysis in the KP-I context is more involved since we use an additional cancellation in the construction of the approximate solutions, related to the existence of zero speed waves in the $x$ direction for the linear KP-I equation, by which we mean that the $x$ component of the gradient of the symbol of the spatial operators vanishes for suitable large frequencies. For the KP-II equation this speed is never zero outside the origin and similar waves do not exist there. In addition, our analysis uses the Burgers type cancellation which was the only cancellation involved in the construction of [7]. Let us also notice that a technical modification of the proof of Theorem 1 is likely to show that in Theorem 1 one can replace the energy space by the Sobolev spaces $H^s(\mathbb{R}^2)$, $s > 0$ or the spaces $Y_s$ considered in [6] equipped with the norm

$$\|u\|_{Y_s} = \|u\|_{L^2(\mathbb{R}^2)} + \|D_x^s u\|_{L^2(\mathbb{R}^2)} + \|\partial_x^{-1} u_y\|_{L^2(\mathbb{R}^2)} .$$

## 2 Outline of the proof of Theorem 1

We decompose the proof into three parts:

(1) We construct a family of approximate solutions $u_{ap}$ depending on parameters $\omega$, $|\omega| \leq 1$ and $\lambda \gg 1$. Changing $\omega$ leads to a phase shift in the high frequency part for positive $t$, but for $t = 0$ the variation of $\omega$ is uniformly smooth. We show that the residual terms are small uniformly in all parameters, see Lemma 3.1.
(2) We study the bounds of the solutions $u$ with initial datum $u_{ap}(0)$ for $t \leq 1$ in many $L^2$ based spaces. This part relies on the well-posedness for smooth data as well as on conserved quantities.
(3) Energy arguments control $\|u_{ap} - u\|_{L^2}$. Interpolation with the bounds for $u$ and $u_{ap}$ yields that $u$ is close to $u_{ap}$ in suitable function spaces.

This yields the desired conclusion because $u_{ap}$ depends in a transparent way on $\omega$, which contradicts uniform continuity.

Let us explain the idea of the construction of the approximate solution. We denote $i$ times the symbol of the spatial part of the linear equation by

$$p(\xi, \eta) = \xi^3 + \xi^{-1}\eta^2$$

Let $\lambda \gg 1$ be a large parameter. The function

$$\cos(\lambda x + 4\lambda^3 t + \sqrt{3}\lambda^2 y) \tag{2.1}$$

is a solution to the linear equation. Its velocity vector is

$$\nabla p(\pm\lambda, \pm\sqrt{3}\lambda^2) = \begin{pmatrix} 0 \\ \pm 2\sqrt{3}\lambda \end{pmatrix}. \tag{2.2}$$

In particular the velocity of the plane wave (2.1) in the $x$ direction vanishes, which is the reason for choosing these points in the frequency space.

We fix for the sequel two constants

$$\frac{1}{2} < \alpha < 1 < \beta, \quad \alpha + \beta < 2 \tag{2.3}$$

and a function $\varphi \in C_0^\infty(\mathbb{R})$ supported in $[-2, 2]$ and identically 1 in $[-1, 1]$. Since the Fourier transform of

$$\varphi(x/\lambda^\alpha)\varphi(x/\lambda^\beta)\cos(\lambda x + 4\lambda^3 t + \sqrt{3}\lambda^2 y) \tag{2.4}$$

is very small outside a small neighborhood of $(\lambda, \sqrt{3}\lambda)$ the corresponding velocity vectors [given by $\nabla p$ evaluated in a small neighborhood of $(\lambda, \sqrt{3}\lambda)$] are close to (2.2), and hence (2.4) defines an approximate solution.

A first guess for the approximate solutions is

$$\begin{aligned} u_{ap}(t,x,y) = &-\lambda^{-1-\frac{\alpha+\beta}{2}}\varphi(x/\lambda^\alpha)\varphi(y/\lambda^\beta)\cos(4\lambda^3 t+\lambda x+\sqrt{3}\lambda^2 y+\omega t)\\ &-\lambda^{-1}\omega\,\varphi(x/(2\lambda^\alpha))\varphi(y/(2\lambda^\beta)). \end{aligned} \tag{2.5}$$

The crucial point is the dependence of the first part on $\omega$. If we plug $u_{ap}$ into the equation then the time derivative leads to an additional term (compared to $\omega = 0$), which is linear in $\omega$, and which essentially cancels against the product of the first and the second term in the non-linearity.

The range of $\alpha$ and $\beta$ is dictated by the velocity of plane waves and the uncertainty principle. A plane wave with frequence $(\xi,\eta)$ has the velocity $\nabla p(\xi,\eta)$, and the Hessian of $p$ describes the dispersion, the change of the velocity under small changes of the frequencies. We obtain the following conditions.

(1) The low frequency part [the second term in (2.5)] has to converge to 0 in $L^2$ as $\lambda \to \infty$ uniformly in $|\omega| \le 1$. Its norm is a constant times

$$|\omega|\lambda^{-1+\frac{\alpha+\beta}{2}}$$

hence we need $\alpha+\beta < 2$.

(2) The velocity in $y$ direction is of size $\lambda$. Hence $\beta > 1$ is needed so that the high frequency part is confined up to time 1 to an interval of size $\lambda^\beta$ in $y$ direction.

(3) Let $\lambda^\alpha$ be the spatial scale. By the uncertainty principle the uncertainty in frequency is at least $\lambda^{-\alpha}$. Then the uncertainty in the velocity in $x$ direction is given by the second $x$ derivative of the symbol times the uncertainty in the frequency, hence it is at least of size $\lambda^{1-\alpha}$. We search for approximate solutions on a time scale of 1, and hence the spatial scale is at least $\lambda^{1-\alpha}$. We need thus $\alpha > \frac{1}{2}$.

This is essentially the construction we shall employ below, up to an important detail: we want to obtain an approximate solution in the energy space, which forces us to do technical modifications so that our functions are $x$ derivatives of suitable functions.

The function $u_{ap}$ is an approximate solution in the sense that

$$\partial_t u_{ap} + \partial^3_{xxx}u - \partial_x^{-1}\partial^2_{yy}u_{ap} + u_{ap}\partial_x u_{ap}$$

is small in $L^2$.

Because of the structure of $u_{ap}$ the difference between $u_{ap}$ and the solution $u$ with the same initial data converges to zero in $X$. This last part of the proof is a consequence of well-posedness results in more regular spaces and suitable conserved quantitities for solutions, immediate estimates of higher norms for the approximate solution and an energy estimate for the difference between $u$ and $u_{ap}$.

This type of failure of uniform continuity is typical for a certain type of interations between low and high frequencies. The approach of this paper seems to be flexible and and applicable to several dispersive problems. The basic strategy consists in finding high frequency waves which remain in a sufficiently small region up to time 1. We do not see a general principle how to find these waves, but in all cases we are aware of the construction is guided by the classical Hamiltonian motion combined with the uncertainty principle.

## 3 Construction of the approximate solution

We begin the construction by collecting several elementary technical observations needed to obtain good antiderivatives with respect to $x$. If $f \in C_0^\infty(\mathbb{R})$ is such that

$$\int_{-\infty}^{\infty} f(x)dx = 0, \tag{3.1}$$

then, for every $x \in \mathbb{R}$,

$$\left|(\partial_x^{-1} f)(x)\right| = \left|\int_{-\infty}^{x} f(y)dy\right| \leq \operatorname{mes}\big(\operatorname{supp}(f)\big)\big(\sup_{y\in\mathbb{R}} |f(y)|\big).$$

In particular, if for some $R > 0$, $\operatorname{supp}(f) \subset [-R, R]$ then for every $x \in \mathbb{R}$,

$$\left|(\partial_x^{-1} f)(x)\right| \leq 2R\big(\sup_{y\in\mathbb{R}} |f(y)|\big),$$

and if in addition

$$\int_{-\infty}^{\infty} x f(x)dx = 0,$$

then we also have

$$\left|(\partial_x^{-2} f)(x)\right| \leq 4R^2 \sup_{y\in\mathbb{R}} |f(y)|.$$

Let us also notice that if $f \in C_0^\infty(\mathbb{R})$ is such that (3.1) holds and for some $R > 0$, $\operatorname{supp}(f) \subset [-R, R]$ then

$$\operatorname{supp}\big(\partial_x^{-1}(f)\big) \subset [-R, R].$$

We recall (2.3), that $\varphi \in C^\infty$ is supported in $[-2, 2]$, identically 1 in $[-1, 1]$, $\lambda > 1$ and set

$$\psi_\lambda(x) = \varphi\big(\frac{x}{\lambda^\alpha}\big) - 2\varphi\big(\frac{x}{\lambda^\alpha} + c_\lambda\big) + \varphi\big(\frac{x}{\lambda^\alpha} + 2c_\lambda\big),$$

where

$$c_\lambda = \frac{2\pi[10\lambda^{1+\alpha}]}{\lambda^{1+\alpha}},$$

with $[s]$ denoting the largest integer $\leq s$. Notice that, $\psi_\lambda$ is supported in an interval of size $\sim \lambda^\alpha$. In addition, for every $\gamma \in \mathbb{R}$,

$$\int_{-\infty}^{\infty} \psi_\lambda(x)\cos(\lambda x + \gamma)dx = \int_{-\infty}^{\infty} x\,\psi_\lambda(x)\cos(\lambda x + \gamma)dx = 0\,.$$

Therefore $\partial_x^{-1}(\psi_\lambda(x)\cos(\lambda x + \gamma))$ and $\partial_x^{-2}(\psi_\lambda(x)\cos(\lambda x + \gamma))$ are well defined $C_0^\infty(\mathbb{R})$ functions.

Next, to shorten the notation, we define for $|\omega| \leq 1$

$$\Phi_\lambda = \Phi_\lambda(t, x, y, \omega) = 4\lambda^3 t + \lambda x + \sqrt{3}\lambda^2 y + \omega t, \tag{3.2}$$

where we suppress $\omega$ in the notation of $\Phi_\lambda$, and we set

$$\widetilde{\psi}_\lambda(x) = \varphi\left(\frac{x}{2\lambda^\alpha}\right) - 2\varphi\left(\frac{x}{2\lambda^\alpha} + c_\lambda/2\right) + \varphi\left(\frac{x}{2\lambda^\alpha} + c_\lambda\right)$$

and

$$\varphi_\lambda(y) = \varphi(y/\lambda^\beta), \quad \widetilde{\varphi}_\lambda(y) = \varphi(y/(2\lambda^\beta)).$$

For $|\omega| \le 1$ and $\lambda \ge 1$, we define an approximate solution $u_{ap}$ of (1.1) by the formula

$$\begin{aligned} u_{ap}(t,x,y) = &- \lambda^{-1-\frac{\alpha+\beta}{2}} \psi_\lambda(x)\varphi_\lambda(y)\cos(4\lambda^3 t + \lambda x + \sqrt{3}\lambda^2 y + \omega t) \\ &- \lambda^{-1}\omega\, \widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\,. \end{aligned} \tag{3.3}$$

Notice that

$$\int_{-\infty}^{\infty} \widetilde{\psi}_\lambda(x)dx = \int_{-\infty}^{\infty} x\,\widetilde{\psi}_\lambda(x)dx = 0\,.$$

Therefore $\partial_x^{-1}(\widetilde{\psi}_\lambda)$ and $\partial_x^{-2}(\widetilde{\psi}_\lambda)$ are well defined $C_0^\infty(\mathbb{R})$ functions. Moreover, for $\lambda \gg 1$,

$$\psi_\lambda\widetilde{\psi}_\lambda = \psi_\lambda.$$

The main properties of $u_{ap}$ are collected in the following lemma.

**Lemma 3.1** *There exist $\delta > 0$, $c > 0$ and $C > 0$ such that for every $\omega \in [-1,1]$, every $\lambda \ge 1$,*

$$\left\|(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2)u_{ap} + u_{ap}\partial_x(u_{ap})\right\|_{L^2(\mathbb{R}^2)} \le C\lambda^{-1-\delta}\,. \tag{3.4}$$

*Moreover*

$$\|\partial_x^{-1}\partial_y u_{ap}(t)\|_{L^2(\mathbb{R}^2)} \le C, \tag{3.5}$$

$$\|\partial_x^{-2}\partial_y^2 u_{ap}(t)\|_{L^2(\mathbb{R}^2)} \le C\lambda \tag{3.6}$$

*and, for every $t, \omega, \omega' \in [-1,1]$,*

$$\|\partial_x(u_{ap,\omega}(t) - u_{ap,\omega'}(t))\|_{L^2(\mathbb{R}^2)} \ge c|\omega - \omega'||t| - C\lambda^{-\delta}. \tag{3.7}$$

*Remark 1* It is not hard to keep track of the size of $\delta$. Let $\varepsilon$ be a small positive constant, choose $\beta = 2\alpha = \frac{4}{3} - \varepsilon$. Then $\delta$ may be chosen to be $\frac{1}{3} - \varepsilon$.

*Proof* In the proof of this lemma, we denote by $o_{L^2}(\lambda^{-1})$ quantities having $L^2(\mathbb{R}^2)$ norm bounded by $C\lambda^{-1-\delta}$ for a suitable $\delta > 0$ uniformly for $t \in [-1,1]$, $\omega \in [-1,1]$ and what is the most important, $\lambda \ge 1$. The proof requires elementary but careful calculations.

It is easy to check, using integration by parts, that

$$\left\|\partial_x^{-1}\partial_y^2\big(\lambda^{-1}\omega\,\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\big)\right\|_{L^2(\mathbb{R}^2)} \le C\lambda^{-1+\frac{\alpha+\beta}{2}+\alpha-2\beta} = C\lambda^{-1+\frac{3}{2}(\alpha-\beta)}\,.$$

Next,

$$\left\|\partial_x^3\big(\lambda^{-1}\omega\,\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\big)\right\|_{L^2(\mathbb{R}^2)} \le C\lambda^{-1+\frac{\alpha+\beta}{2}-3\alpha}$$

and thus, thanks to the assumptions on $(\alpha,\beta)$, we obtain that

$$\left(\partial_x^3-\partial_x^{-1}\partial_y^2\right)\left(\lambda^{-1}\omega\,\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\right)=O_{L^2}\left(\lambda^{-1+\frac{\alpha+\beta}{2}+\max\{\alpha-2\beta,-3\alpha\}}\right)=o_{L^2}(\lambda^{-1}). \tag{3.8}$$

Coming back to the definition of $u_{ap}$, we can readily check that

$$u_{ap}\,\partial_x u_{ap}=-\omega\lambda^{-1-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin(\Phi_\lambda(t,x,y,\omega))+O_{L^2}\left(\lambda^{-2-\frac{\alpha-\beta}{2}}\right). \tag{3.9}$$

Notice that the leading term in (3.9) is coming from the product of the high frequency part of $\partial_x u_{ap}$ and the low frequency part of $u_{ap}$.

Next, we compute integrating by parts

$$\partial_x^{-1}(\psi_\lambda\cos\Phi_\lambda)=\int_{-\infty}^{x}\psi_\lambda\cos\Phi_\lambda=\lambda^{-1}\psi_\lambda\sin\Phi_\lambda-\lambda^{-1}\int_{-\infty}^{x}\partial_x[\psi_\lambda]\sin\Phi_\lambda\,.$$

We integrate by parts two more times to arrive at

$$\begin{aligned}\partial_x^{-1}(\psi_\lambda\cos\Phi_\lambda)=\lambda^{-1}\psi_\lambda\sin\Phi_\lambda+\lambda^{-2}\partial_x[\psi_\lambda]\cos\Phi_\lambda-&\\ -\lambda^{-3}\partial_x^2[\psi_\lambda]\sin\Phi_\lambda+\lambda^{-3}\int_{-\infty}^{x}\partial_x^3[\psi_\lambda]\sin\Phi_\lambda.&\end{aligned} \tag{3.10}$$

Using the Leibniz rule, since $\beta>1$, we infer that

$$\lambda^{-1-\frac{\alpha+\beta}{2}}\partial_y^2\left([\lambda^{-1}\psi_\lambda\sin\Phi_\lambda]\varphi_\lambda(y)\right)=-3\lambda^{2-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin\Phi_\lambda+O_{L^2}\left(\lambda^{-\beta}\right).$$

Similarly

$$\begin{aligned}\lambda^{-1-\frac{\alpha+\beta}{2}}\partial_y^2\left([\lambda^{-2}\partial_x[\psi_\lambda]\cos\Phi_\lambda]\varphi_\lambda(y)\right)=&-3\lambda^{1-\frac{\alpha+\beta}{2}}\partial_x[\psi_\lambda(x)]\varphi_\lambda(y)\cos\Phi_\lambda\\&+O_{L^2}\left(\lambda^{-1-\alpha-\beta}\right)\end{aligned}$$

and,

$$\lambda^{-1-\frac{\alpha+\beta}{2}}\partial_y^2\left([\lambda^{-3}\partial_x^2[\psi_\lambda]\sin\Phi_\lambda]\varphi_\lambda(y)\right)=O_{L^2}\left(\lambda^{-2\alpha}\right).$$

We recall that $\alpha>\frac{1}{2}$. Similarly

$$\lambda^{-4-\frac{\alpha+\beta}{2}}\partial_y^2\int_{-\infty}^{x}\partial_x^3[\psi_\lambda]\sin\Phi_\lambda=O_{L^2}\left(\lambda^{-2\alpha}\right).$$

Summarizing, we can conclude that

$$\begin{aligned}\partial_x^{-1}\partial_y^2\left(\lambda^{-1-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\cos\Phi_\lambda\right)=&-3\lambda^{2-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin\Phi_\lambda\\&-3\lambda^{1-\frac{\alpha+\beta}{2}}\partial_x[\psi_\lambda(x)]\varphi_\lambda(y)\cos\Phi_\lambda+o_{L^2}\left(\lambda^{-1}\right).\end{aligned} \tag{3.11}$$

Using the Leibniz rule, we infer

$$\begin{aligned}\partial_x^3\left(\lambda^{-1-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\cos\Phi_\lambda\right)=&\lambda^{2-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin\Phi_\lambda\\&-3\lambda^{1-\frac{\alpha+\beta}{2}}\partial_x[\psi_\lambda(x)]\varphi_\lambda(y)\cos\Phi_\lambda+o_{L^2}\left(\lambda^{-1}\right).\end{aligned}$$

Therefore using (3.8), we obtain that

$$\left(\partial_x^3-\partial_x^{-1}\partial_y^2\right)u_{ap}=-4\lambda^{2-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin\Phi_\lambda+o_{L^2}\left(\lambda^{-1}\right).$$

The cancellation of the term

$$3\lambda^{1-\frac{\alpha+\beta}{2}}\partial_x[\psi_\lambda(x)]\varphi_\lambda(y)\cos\Phi_\lambda$$

is the main new point in this paper. It is an analytic expression of the fact that the $x$ component of the velocity vector vanishes for the plane wave which we have chosen. Here, we essentially use that we are dealing with the KP-I equation, i.e., the sign in front of $\partial_x^{-1}\partial_y^2$ is crucial to achieve this cancellation. Since

$$\partial_t u_{ap}=4\lambda^{2-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin\Phi_\lambda+\lambda^{-1-\frac{\alpha+\beta}{2}}\omega\psi_\lambda(x)\varphi_\lambda(y)\sin\Phi_\lambda,$$

we obtain that

$$\left(\partial_t+\partial_x^3-\partial_x^{-1}\partial_y^2\right)u_{ap}=\omega\lambda^{-1-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin\Phi_\lambda+o_{L^2}\left(\lambda^{-1}\right).$$

This, together with (3.9), completes the proof of (3.4).

Using (3.10), arguing in the same way as there, we can write

$$\partial_x^{-1}\partial_y\left(\lambda^{-1-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\cos\Phi_\lambda\right)=\sqrt{3}\lambda^{-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\cos\Phi_\lambda+o_{L^2}\left(\lambda^{-1}\right).\tag{3.12}$$

Moreover

$$\left\|\partial_x^{-1}\partial_y\left(\lambda^{-1}\omega\,\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\right)\right\|_{L^2(\mathbb{R}^2)}\le C\lambda^{-1+\frac{\alpha+\beta}{2}+\alpha-\beta}\le C\tag{3.13}$$

which completes the proof of (3.5).

Let us now turn to the proof of (3.6). The low frequency part of $u_{ap}$ can be estimated as

$$\left\|\partial_x^{-2}\partial_y^2\left(\lambda^{-1}\omega\,\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\right)\right\|_{L^2(\mathbb{R}^2)}\le C\lambda^{-1+\frac{\alpha+\beta}{2}+2\alpha-2\beta}\le C.$$

We next estimate the high frequencies and repeat the calculation of (3.10),

$$\begin{aligned}\partial_x^{-1}(\psi_\lambda\sin\Phi_\lambda)&=\int_{-\infty}^{x}\psi_\lambda\sin\Phi_\lambda=-\lambda^{-1}\psi_\lambda\cos\Phi_\lambda+\lambda^{-1}\int_{-\infty}^{x}\partial_x[\psi_\lambda]\cos\Phi_\lambda\\&=-\lambda^{-1}\psi_\lambda\cos\Phi_\lambda+\lambda^{-2}\partial_x[\psi_\lambda]\sin\Phi_\lambda-\lambda^{-2}\int_{-\infty}^{x}\partial_x^2[\psi_\lambda]\sin\Phi_\lambda\,.\end{aligned}$$

Next, we estimate each term in the right hand-side of the above equality. First

$$\left\|\lambda^{-2}\psi_\lambda\cos\Phi_\lambda\right\|_{L^2(\mathbb{R}_x)}\le C\lambda^{-2}\lambda^{\frac{\alpha}{2}}\tag{3.14}$$

and then

$$\left\|\lambda^{-3}\partial_x[\psi_\lambda]\sin\Phi_\lambda\right\|_{L^2(\mathbb{R}_x)}\le C\lambda^{-3}\lambda^{-\alpha}\lambda^{\frac{\alpha}{2}}\le C\lambda^{-2}\lambda^{\frac{\alpha}{2}}\tag{3.15}$$

and finally

$$\left\|\lambda^{-3}\int_{-\infty}^{x}\partial_x^2[\psi_\lambda]\sin\Phi_\lambda\right\|_{L^2(\mathbb{R}_x)}\le C\lambda^{-3}\lambda^{-2\alpha}\lambda^{\alpha}\lambda^{\frac{1}{2}}\le C\lambda^{-2}\lambda^{\frac{\alpha}{2}}.$$

Notice that (3.14) and (3.15) imply that

$$\left\|\partial_x^{-1}(\lambda^{-2}\partial_x[\psi_\lambda]\cos\Phi_\lambda)\right\|_{L^2(\mathbb{R}_x)} \le C\lambda^{-2}\lambda^{\frac{\alpha}{2}}$$

which is the relevant bound for the second term in the right hand-side of (3.10). It remains to estimate the last two terms in the right hand-side of (3.10). We can write

$$\left\|\partial_x^{-1}[\lambda^{-3}\partial_x^2[\psi_\lambda]\sin\Phi_\lambda]\right\|_{L^2(\mathbb{R}_x)} \le C\lambda^{-3}\lambda^{-2\alpha}\lambda^{\alpha}\lambda^{\frac{1}{2}} \le C\lambda^{-2}\lambda^{\frac{\alpha}{2}},$$

since $\alpha > 1/2$. For the last term in the right hand-side of (3.10), we can write

$$\begin{aligned}\left\|\partial_x^{-1}\left[\lambda^{-3}\int_{-\infty}^{x}\partial_x^3[\psi_\lambda]\sin\Phi_\lambda\right]\right\|_{L^2(\mathbb{R}_x)} &= \left\|\partial_x^{-2}[\lambda^{-3}\partial_x^3[\psi_\lambda]\sin\Phi_\lambda]\right\|_{L^2(\mathbb{R}_x)}\\ &\le C\lambda^{-3}\lambda^{-3\alpha}\lambda^{\frac{1}{2}}\lambda^2 \le C\lambda^{-2}\lambda^{\frac{\alpha}{2}}\end{aligned}$$

since $\alpha > 1/2 > 3/7$.

Summarizing, we infer that the high frequencies of $u_{ap}$ can be estimated as

$$\left\|\partial_x^{-2}(\psi_\lambda\cos\Phi_\lambda)\right\|_{L^2(\mathbb{R}_x)} \le C\lambda^{-2}\lambda^{\frac{\alpha}{2}}$$

and thus, using that $\partial_y^2$ is causing at most an amplification factor $\lambda^4$, we conclude that

$$\left\|\partial_x^{-2}\partial_y^2\left(\lambda^{-1-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\cos\Phi_\lambda\right)\right\|_{L^2(\mathbb{R}^2)} \le C\lambda^{-1-\frac{\alpha+\beta}{2}}\left(\lambda^{-2}\lambda^{\frac{\alpha}{2}}\right)\left(\lambda^4\lambda^{\frac{\beta}{2}}\right) = C\lambda.$$

This proves (3.6).

Finally, we give the proof of (3.7). Notice that

$$\partial_x u_{ap} = \lambda^{-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\sin(\Phi_\lambda) + O_{L^2}\left(\lambda^{-1+\frac{\beta-\alpha}{2}}\right). \tag{3.16}$$

With $a = 4\lambda^3 t + \lambda x + \sqrt{3}\lambda^2 y$, we may write

$$\sin(a+\omega t) - \sin(a+\omega' t) = 2\sin(t(\omega-\omega')/2)\cos(a+t(\omega+\omega')/2),$$

and after a sequence of integrations by parts, we get

$$\left\|\lambda^{-\frac{\alpha+\beta}{2}}\psi_\lambda\varphi_\lambda\left\{\sin\Phi_{\omega,\lambda} - \sin\Phi_{\omega,\lambda'}\right\}\right\|_{L^2}^2 \ge c(|t||\omega-\omega'|)^2\left\|\lambda^{-\frac{\alpha+\beta}{2}}\psi_\lambda\varphi_\lambda\right\|_{L^2}^2 - C\lambda^{-2}.$$

Using the choice of $c_\lambda$, we can minorate $\|\psi_\lambda\|_{L^2(\mathbb{R})}$ and thus

$$\left\|\lambda^{-\frac{\alpha+\beta}{2}}\psi_\lambda\varphi_\lambda\left\{\sin\Phi_{\omega,\lambda} - \sin\Phi_{\omega',\lambda}\right\}\right\|_{L^2} \ge c|\omega-\omega'||t| - C\lambda^{-1}$$

which proves (3.7). This completes the proof of Lemma 3.1. □

## 4 Bounds for the exact solution

Let $u_{\omega,\lambda}(t,x,y)$ be a solution of the KP-I equation with data

$$u_{\omega,\lambda}(0,x,y) = -\lambda^{-1-\frac{\alpha+\beta}{2}}\psi_\lambda(x)\varphi_\lambda(y)\cos\left(\lambda x + \sqrt{3}\lambda^2 y\right) - \lambda^{-1}\omega\,\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y).$$

Thanks to the properties of $\psi_\lambda(x)$ and $\widetilde{\psi}_\lambda(x)$, we can apply the global well-posedness result of [6] to obtain that $u_{\omega,\lambda}(t,x,y)$ is globally defined and satisfies the conservation laws mentioned in the introduction (see also [8, Proposition 4]). Moreover, for every $t$, $\xi^{-1}\widehat{u_{\omega,\lambda}}(t,\xi,\eta)$ belongs to $L^2(\mathbb{R}^2)$ (see [4]).

In order to bound $u_{\omega,\lambda}$ in $Z$, we will use the following anisotropic Sobolev inequality.

**Lemma 4.1** *For $2 \le p \le 6$ there exists $C > 0$ such that for every $u \in X$,*

$$\|u\|_{L^p(\mathbb{R}^2)} \le C\|u\|_{L^2(\mathbb{R}^2)}^{\frac{6-p}{2p}} \ \|u_x\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{p}} \ \|\partial_x^{-1}u_y\|_{L^2(\mathbb{R}^2)}^{\frac{p-2}{2p}} \,. \tag{4.1}$$

We refer to [11] for a proof of (4.1). The $L^2$ conservation law yields,

$$\|u_{\omega,\lambda}(t,\cdot)\|_{L^2(\mathbb{R}^2)} = \|u_{\omega,\lambda}(0,\cdot)\|_{L^2(\mathbb{R}^2)} = \|u_{ap}(0,\cdot)\|_{L^2(\mathbb{R}^2)} \le C\,.$$

The energy conservation, (3.5) and (4.1) with $p = 3$ yield

$$E(u_{\omega,\lambda}(t,\cdot)) = E(u_{\omega,\lambda}(0,\cdot)) = E(u_{ap}(0,\cdot)) \le C\,.$$

Another use of (4.1) with $p = 3$ then gives the following bound for the leading part of the energy,

$$\|\partial_x(u_{\omega,\lambda})(t,\cdot)\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-1}\partial_y(u_{\omega,\lambda})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} \le C. \tag{4.2}$$

We now establish several bounds for the cubic and quartic terms of the functional $F$ of (1.2). We can write, by invoking (4.1) with $p = 3, 4$

$$\begin{aligned}\Big|\frac{5}{6}\int_{\mathbb{R}^2} u^2(\partial_x^{-2}u_{yy})\Big| &\le \|u\|_{L^4(\mathbb{R}^2)}^2\left\|\partial_x^{-2}u_{yy}\right\|_{L^2(\mathbb{R}^2)}\\ &\le C\|u\|_{L^2(\mathbb{R}^2)}^{\frac12}\|u_x\|_{L^2(\mathbb{R}^2)}\left\|\partial_x^{-1}u_y\right\|_{L^2(\mathbb{R}^2)}^{\frac12}\left\|\partial_x^{-2}u_{yy}\right\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

and

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} u\,(\partial_x^{-1}u_y)^2\Big| &\le \|u\|_{L^3(\mathbb{R}^2)}\left\|\partial_x^{-1}u_y\right\|_{L^3(\mathbb{R}^2)}^2\\ &\le C\|u\|_{L^2(\mathbb{R}^2)}^{\frac12}\|u_x\|_{L^2(\mathbb{R}^2)}^{\frac13}\left\|\partial_x^{-1}u_y\right\|_{L^2(\mathbb{R}^2)}^{\frac76}\|u_y\|_{L^2(\mathbb{R}^2)}^{\frac23}\left\|\partial_x^{-2}u_{yy}\right\|_{L^2(\mathbb{R}^2)}^{\frac13}\,.\end{aligned}$$

Next,

$$\begin{aligned}\Big|\int_{\mathbb{R}^2} u^2\,u_{xx}\Big| &\le \|u\|_{L^4(\mathbb{R}^2)}^2\|u_{xx}\|_{L^2(\mathbb{R}^2)}\\ &\le C\|u\|_{L^2(\mathbb{R}^2)}^{\frac12}\|u_x\|_{L^2(\mathbb{R}^2)}\left\|\partial_x^{-1}u_y\right\|_{L^2(\mathbb{R}^2)}^{\frac12}\|u_{xx}\|_{L^2(\mathbb{R}^2)}\end{aligned}$$

and finally

$$\Big|\int_{\mathbb{R}^2} u^4\Big| \le C\|u\|_{L^2(\mathbb{R}^2)}\|u_x\|_{L^2(\mathbb{R}^2)}^2\left\|\partial_x^{-1}u_y\right\|_{L^2(\mathbb{R}^2)}\,.$$

Using the above bounds, estimates (3.6), (4.2), and the conservation of $F$, we obtain that for $\lambda \ge 1$,

$$F(u_{\omega,\lambda}(t,\cdot)) = F(u_{\omega,\lambda}(0,\cdot)) = F(u_{ap}(0,\cdot)) \le C\lambda^2\,.$$

Using again the estimates for the cubic and the quartic terms of $F$, we obtain that the leading part of $F$ satisfies for $\lambda \geq 1$, $t \in [-1, 1]$,

$$\left\|\partial_x^2(u_{\omega,\lambda})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} + \|\partial_y(u_{\omega,\lambda})(t,\cdot)\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-2}\partial_y^2(u_{\omega,\lambda})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} \leq C\lambda.$$

## 5 The difference between approximate and exact solution

We begin by controlling the size of $u_{ap}$. Using (3.6) and the definition of $u_{ap}$, we infer that for $\lambda \geq 1$, $t \in [-1, 1]$,

$$\left\|\partial_x^2(u_{ap})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} + \|\partial_y(u_{ap})(t,\cdot)\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-2}\partial_y^2(u_{ap})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} \leq C\lambda$$

and thus, with

$$v_{\omega,\lambda} = u_{\omega,\lambda} - u_{ap}, \tag{5.1}$$

$$\left\|\partial_x^2(v_{\omega,\lambda})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} + \|\partial_y(v_{\omega,\lambda})(t,\cdot)\|_{L^2(\mathbb{R}^2)} + \left\|\partial_x^{-2}\partial_y^2(v_{\omega,\lambda})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} \leq C\lambda.$$

In particular

$$\left\|\partial_x^2(v_{\omega,\lambda})(t,\cdot)\right\|_{L^2(\mathbb{R}^2)} \leq C\lambda. \tag{5.2}$$

We next bound the $L^2$ norm of $v_{\omega,\lambda}$.

**Lemma 5.1** *There exist* $\delta > 0$ *such that*

$$\|v_{\omega,\lambda}(t,\cdot)\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{-1-\delta} \tag{5.3}$$

*uniformly in* $\lambda \geq 1$, $|\omega| \leq 1$ *and* $|t| \leq 1$.

*Proof* The function $v_{\omega,\lambda}$ solves the equation

$$\left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right)v_{\omega,\lambda} + v_{\omega,\lambda}\partial_x(v_{\omega,\lambda}) + \partial_x(u_{ap}v_{\omega,\lambda}) + G = 0, \tag{5.4}$$

where $v_{\omega,\lambda}(0, x, y) = 0$ and

$$G = \left(\partial_t + \partial_x^3 - \partial_x^{-1}\partial_y^2\right)u_{ap} + u_{ap}\partial_x(u_{ap})\,.$$

Thanks to (3.4),

$$\|G(t,\cdot)\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{-1-\delta}, \quad \delta > 0.$$

Multiplying (5.4) by $v_{\omega,\lambda}$ and an integration over $\mathbb{R}^2$ gives

$$\begin{aligned}\frac{d}{dt}\|v_{\omega,\lambda}(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} &\lesssim \|\partial_x u_{ap}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)}\|v_{\omega,\lambda}(t,\cdot)\|^2_{L^2(\mathbb{R}^2)} \\ &\quad + \|v_{\omega,\lambda}(t,\cdot)\|_{L^2(\mathbb{R}^2)}\|G(t,\cdot)\|_{L^2(\mathbb{R}^2)}\,.\end{aligned}$$

From the definition of $u_{ap}$, we infer that

$$\|\partial_x u_{ap}(t,\cdot)\|_{L^\infty(\mathbb{R}^2)} \leq C\lambda^{-1}\,.$$

Therefore, by Gronwall's inequality for $t \in [-1, 1]$,

$$\|v_{\omega,\lambda}(t,\cdot)\|_{L^2(\mathbb{R}^2)} \leq \sup_{t\in[-1,1]} \|G(t,\cdot)\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{-1-\delta}\,.$$

This completes the proof of Lemma 5.1. □

Interpolation between (5.2) and (5.3) gives that for $\lambda \geq 1$,

$$\|\partial_x(v_{\omega,\lambda})(t,\cdot)\|_{L^2(\mathbb{R}^2)} \leq C\lambda^{-\delta/2}. \tag{5.5}$$

After these preparations we turn to the proof of Theorem 1. Consider the two families of solutions $(u_{1,\lambda})$ and $(u_{-1,\lambda})$, $\lambda \gg 1$. Write for $\lambda \geq 1$,

$$\begin{aligned}\|u_{1,\lambda}(0,\cdot) - u_{-1,\lambda}(0,\cdot)\|_X &= 2\|\lambda^{-1}\,\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\|_X \\ &\leq 2\lambda^{-1}\|\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y)\|_{L^2(\mathbb{R}^2)} \\ &\quad +2\lambda^{-1}\|\partial_x[\widetilde{\psi}_\lambda(x)]\widetilde{\varphi}_\lambda(y)\|_{L^2(\mathbb{R}^2)} \\ &\quad +2\lambda^{-1}\|\partial_x^{-1}[\widetilde{\psi}_\lambda(x)]\partial_y[\widetilde{\varphi}_\lambda(y)]\|_{L^2(\mathbb{R}^2)} \\ &\leq C\lambda^{-1}\lambda^{\frac{\alpha+\beta}{2}} + C\lambda^{-1}\lambda^{\alpha}\lambda^{-\beta}\lambda^{\frac{\alpha+\beta}{2}}\,.\end{aligned}$$

Thanks to the assumptions on $(\alpha, \beta)$, we obtain that

$$\lim_{\lambda\to\infty} \|u_{1,\lambda}(0,\cdot) - u_{-1,\lambda}(0,\cdot)\|_X = 0\,.$$

To conclude we provide a non-trivial lower bound on

$$\liminf_{\lambda\to\infty} \|\partial_x(u_{1,\lambda} - u_{-1,\lambda})\|_{L^2}.$$

Equation (5.5) reduces this to the corresponding statement for $u_{ap}$, which is inequality (3.7). This completes the proof of Theorem 1.

*Remark 5.2* Actually, we proved a stronger statement than Theorem 1. We obtained the existence of two families of solutions of the KP-I equation which remain bounded in the energy space, such that their difference tend to zero in the energy space but such that for $t \in [-1, 1]$, $t \neq 0$ the $x$ derivative of their difference in $L^2(\mathbb{R}^2)$, which is only a part of the energy norm, does not tend to zero.

*Remark 5.3* As in [7], if one is interested to show the failure of uniform continuity of the flow of KP-I on $H^s(\mathbb{R}^2)$ for large $s$ a modification of the low frequency part of the approximate solution is needed. Namely one should replace

$$\omega\lambda^{-1}\widetilde{\psi}_\lambda(x)\widetilde{\varphi}_\lambda(y) \tag{5.6}$$

in $u_{ap}$ by the solution of the KP-I equation with initial data (5.6). We refer to [7] for the details of this construction.

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