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Weighted norm inequalities for heat-diffusion Laguerre's semigroups

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Abstract We consider three systems of Laguerre functions and their corresponding heat diffusion semigroups. For the associate maximal operators, we give necessary and sufficient conditions in order to obtain strong type, weak type and restricted weak type (p,p), with respect to a power weight x^δ, for $1 \le p \le \infty$. We also obtain sufficient conditions for more general weights.

Keywords Heat diffusion semigroups · Laguerre functions · Weights · Local maximal function

Mathematics Subject Classification (2000) 42C10 · 42A45 · 43A15 · 42B20 · 42B25

1 Introduction

For a given $\alpha > -1$, the Laguerre polynomials are defined by

$$e^{-x}x^\alpha L_n^\alpha(x) = \frac{1}{n!}\frac{d^n}{dx^n}\left(e^{-x}x^{n+\alpha}\right)$$

for $x \in (0,\infty)$. They form an orthogonal system with respect to the measure $e^{-x}x^\alpha\, dx$. More precisely

$$\int_0^\infty L_k^\alpha(x)L_j^\alpha(x)e^{-x}x^\alpha dx = \frac{\Gamma(k+\alpha+1)}{\Gamma(k+1)}\delta_{kj}.$$

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Moreover, when properly normalized, they constitute a basis for $L^2(\mathbb{R}^+, \mathrm{e}^{-x}x^\alpha\,\mathrm{d}x)$. From this system of polynomials, three different sets of Laguerre functions may be derived, namely

$$\mathcal{L}_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x)\mathrm{e}^{-x/2}x^{\alpha/2} \tag{1.1}$$

$$\varphi_n^\alpha(x) = \left(\frac{2n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x^2)\mathrm{e}^{-x^2/2}x^{\alpha+1/2} = \mathcal{L}_n^\alpha(x^2)(2x)^{1/2} \tag{1.2}$$

$$\ell_n^\alpha(x) = \left(\frac{n!}{\Gamma(n+\alpha+1)}\right)^{1/2} L_n^\alpha(x)\mathrm{e}^{-x/2} = \mathcal{L}_n^\alpha(x)x^{-\alpha/2} \tag{1.3}$$

which turn to be orthonormal basis for $L^2(\mathbb{R}^+)$ with respect to the Lebesgue measure, in the two first cases, and with respect to $x^\alpha\,\mathrm{d}x$, in the last one. Moreover, each of these systems is the set of eigenfunctions of a second order differential operator L, positive and self-adjoint with respect to the corresponding measure ($\mathrm{d}x$ or $x^\alpha\,\mathrm{d}x$). More precisely, those operators are

$$L_{\mathcal{L}}^\alpha = -x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - \frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{4} + \frac{\alpha^2}{4x}$$
$$L_{\varphi}^\alpha = \frac{1}{4}\left\{-\frac{\mathrm{d}^2}{\mathrm{d}x^2} + x^2 + \frac{1}{x^2}\left(\alpha^2 - \frac{1}{4}\right)\right\}$$
$$L_{\ell}^\alpha = -x\frac{\mathrm{d}^2}{\mathrm{d}x^2} - (\alpha+1)\frac{\mathrm{d}}{\mathrm{d}x} + \frac{x}{4}$$

and the sequences of eigenvalues are

$$\left\{n + \frac{\alpha+1}{2}\right\}_{n=0}^\infty$$

in each of the three cases.

Let us remind that in such situation, when we have that kind of differential operator, L, with a discrete set of eigenfunctions, say ψ_n, with eigenvalues λ_n, expanding $L^2(\mathrm{d}\mu)$, the *heat-diffusion semigroup* may be defined as

$$\mathrm{T}_t f = \sum_n \langle f, \psi_n\rangle\, \psi_n\, \mathrm{e}^{-t\lambda_n}, \quad t > 0$$

and $u(x,t) = \mathrm{T}_t f(x)$ will supply, at least formally, a solution to the *heat equation associated to L with initial data f*, namely,

$$\begin{cases} \dfrac{\partial u}{\partial t} = -Lu \\ u(x,0) = f(x) \end{cases}$$

Clearly, when f is, say, a finite linear combination of ψ_n, the semigroup admits the expression

$$\mathrm{T}_t f(x) = \int K(t,x,y)f(y)\mathrm{d}\mu(y), \quad t > 0 \tag{1.4}$$

where

$$K(t,x,y) = \sum \mathrm{e}^{-t\lambda_n}\psi_n(x)\psi_n(y)$$

is the heat diffusion kernel.

In our situation, there is a well known formula that allow us to find a suitable expression for the corresponding kernels (see the beginning of Sect. 5).

As it is well known, the *a.e.* convergence of the solution to the initial data amounts to study the behavior of the *associate maximal operator*

$$\mathrm{W}^*f(x) = \sup_{t>0} |\mathrm{T}_t f(x)|$$

The aim of this paper is to establish weighted boundedness results of the maximal operators related to the three systems of Laguerre functions on L^p spaces. For each of the systems, we give necessary and sufficient conditions on the exponent of a power weight in order to get strong type, weak type or restricted weak type (p,p), $1 \le p \le \infty$. Also, as a consequence of our estimates, we are able to obtain sufficient conditions to get strong type (p,p), $1 < p < \infty$, and weak type $(1,1)$, for more general type of weights.

Stempak, in the pioneer work [16], provided answers to these kind of questions, proving that for the unweighted case the corresponding maximal operators were of weak type $(1,1)$ for the systems of functions (1.1), (1.2) and (1.3) when $\alpha \ge 0$, $\alpha \ge -1/2$ and $\alpha > -1$, respectively. He also shows that the integral expression (1.4) gives in fact a $\mathcal{C}^\infty$ solution to the corresponding heat equation when the data is in $L^p(\mathbb{R}^+)$, for $1 \le p \le \infty$. Stempak's results of the associate maximal operator were later extended by Macías, Segovia and Torrea in [7] to the case of negative values of the parameter α for the system (1.1). In [8] the authors continue their research about the system $\{\mathcal{L}_n^\alpha\}$, $\alpha > -1$, providing a full description of the boundedness of W^* with power weights x^δ, assuming the exponent δ to be grater than -1. This restriction seems unnatural in the measure space $(\mathbb{R}^+, \mathrm{d}x)$, where x^δ is a locally integrable function for any real number δ, since any compact subset stays away from zero. Nevertheless, we use the underlying ideas and refine some of their estimates in order to majorize by operators more related to the structure of $\mathbb{R}^+$. More precisely, our bounds turn out to be in terms of a local maximal function on the "local" region $x \sim y$, modified Hardy operators and maximal operators associated to the natural convolution in $\mathbb{R}^+$ on the "global" region (see Sect. 3 for the precise definitions).

For the other two cases we do not need to go over to the same kind of estimates. We rather use some appropriate changes of variables that allow us to transfer the pointwise estimates already obtained for the first case to the other kernels. Such kind of connection between the systems has also been exploited in [1]. For these last two systems, Nowak in [12] obtained some results including more general weights. However, for the system $\{\varphi_n^\alpha\}$, the class he considers, $A_p(\mathrm{d}x)$, does not depend on the parameter α, as expected. We present some results in this direction at the end of the paper.

Finally, let us point out that questions related to convergence of these Laguerre functions expansions have been considered by several authors. See for example [6,9, 10,12,16–21]. Recently, we become aware of some new results of Nowak and Sjögren in [14] concerning the weak type $(1,1)$ for the maximal operators associated to heat diffusion semigroups for all Laguerre function systems in higher dimensions.

The organization of the paper is the following. In Sect. 2 we state our results with power weights concerning the maximal heat operator for all of the three systems. In Sect. 3 we present some known facts about the operators we are going to use to majorize the ones we are concerned with, while in Sect. 4 we establish and prove boundedness properties on $L^p(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ for a general operator controlled by a combination of the operators given in Sect. 3. In Sect. 5 we show that our maximal operators satisfy all the requirements of the proposition given in Sect. 4, completing the proof of the main results. Finally, in Sect. 6, we give sufficient conditions to obtain some more general weighted inequalities for the three systems of Laguerre's functions.

2 Statement of the results

Consider a measure space $(E, d\mu)$. For $1 \leq p < \infty$ and $1 \leq q \leq \infty$, the *Lorentz space* $L^{p,q}(E, \mathrm{d}\mu)$ consists of all measurable functions f on $(E, \mathrm{d}\mu)$ for which the quasi-norm

$$\|f\|_{L^{p,q}(E,\mathrm{d}\mu)} = \begin{cases} \left(\int_0^\infty \left[t\mu_f(t)^{1/p}\right]^q \frac{\mathrm{d}t}{t}\right)^{1/q}, & q < \infty, \\ \sup_{t>0} \left[t\mu_f(t)^{1/p}\right], & q = \infty, \end{cases} \tag{2.1}$$

is finite, where

$$\mu_f(t) = \mu(\{x \in E : |f(x)| > t\}).$$

We also consider

$$L^{\infty,\infty}(E, \mathrm{d}\mu) \doteq L^\infty(E, \mathrm{d}\mu)$$

with

$$\|f\|_{L^\infty(E,\mathrm{d}\mu)} = \operatorname{ess\,sup} |f|.$$

The Lorentz spaces $L^{p,q}$ satisfy, for $1 \leq p \leq \infty$,

$$L^{p,1} \hookrightarrow L^{p,p} \hookrightarrow L^{p,\infty}, \tag{2.2}$$

and

$$L^{p,p} = L^p, \tag{2.3}$$

with

$$\|f\|_{L^p(E,\mathrm{d}\mu)} = \left(\int_E |f(x)|^p \,\mathrm{d}\mu\right)^{1/p}.$$

For a sublinear operator R and $1 \leq p \leq \infty$, we say that R is of *strong type* (p, p) *on* $(E, \mathrm{d}\mu)$ when

$$\mathrm{R} : L^p(E, \mathrm{d}\mu) \longrightarrow L^p(E, \mathrm{d}\mu)$$

continuously. Also, we say that R is of *weak type* (p, p) *on* $(E, \mathrm{d}\mu)$ when

$$\mathrm{R} : L^p(E, \mathrm{d}\mu) \longrightarrow L^{p,\infty}(E, \mathrm{d}\mu)$$

and of *restricted weak type* (p, p) *on* $(E, \mathrm{d}\mu)$ when

$$\mathrm{R} : L^{p,1}(E, d\mu) \longrightarrow L^{p,\infty}(E, d\mu), \tag{2.4}$$

continuously in both cases.

By (2.2) and (2.3), we have that strong type implies weak type, ant they are equivalent when $p = \infty$. Also, weak type implies restricted weak type, and they are equivalent when $p = 1$. For that reason, in the statements of the theorems we consider the weak type $(1, 1)$ together with the restricted weak type results.

We note that in [4], R is defined to be of restricted weak type (p, p) when there exists a positive constant C such that the inequality

$$\int_{\{x\in E:|\mathrm{R}f(x)|>\lambda\}} \mathrm{d}\mu \leq \frac{C}{\lambda^p}\int_E |f(x)|^p\,\mathrm{d}\mu$$

holds for any $\lambda > 0$ and for all f characteristic functions of measurable sets. For R linear, or sublinear and nonnegative, as is shown in Theorem 5.3, Chap. 5 of [4], this definition is equivalent to the given in (2.4), which, by the way, is the definition of weak type in [4].

We now state the main results of this paper.

Let $\{\psi_n\}$ be any of the three systems of Laguerre functions given in (1.1), (1.2) and (1.3) and consider the heat diffusion kernel

$$K(t,x,y) \doteq \sum_{n=0}^{\infty} \mathrm{e}^{-t(n+\frac{\alpha+1}{2})}\psi_n(x)\psi_n(y),$$

the heat diffusion integral

$$K^t f(x) \doteq \int_0^{\infty} K(t,x,y)f(y)\,\mathrm{d}y$$

and the associate maximal operator

$$\mathrm{W}^* f(x) = \sup_{t>0} |K^t f(x)|.$$

We have the following results:

Theorem 2.1 *Let $\alpha > -1$ and $1 \leq p \leq \infty$. Associated to the Laguerre system $\{\mathcal{L}_n^\alpha\}$ given in (1.1), we have the following results:*
For any $t > 0$, the heat diffusion integral $K^t_{\mathcal{L}^\alpha} f(x)$ is finite a.e. for all $f \in L^p(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$, $1 < p \leq \infty$, if and only if $\delta < (\frac{\alpha}{2}+1)p - 1$, and for all $f \in L^{p,1}(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$, $1 \leq p < \infty$, if and only if $\delta \leq (\frac{\alpha}{2}+1)p-1$.

Moreover, the associate maximal operator $\mathrm{W}^*_{\mathcal{L}^\alpha}$ satisfies:

(a) *For $1 < p < \infty$ and $\delta < \frac{\alpha}{2}p + p - 1$, $\mathrm{W}^*_{\mathcal{L}^\alpha}$ is of strong type (p,p) on $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ if and only if $\delta > -\frac{\alpha}{2}p - 1$.*
(b) *For all real δ, $\mathrm{W}^*_{\mathcal{L}^\alpha}$ is of strong type (∞,∞) on $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ if and only if $\alpha \geq 0$.*
(c) *For $1 < p < \infty$ and $\delta < \frac{\alpha}{2}p + p - 1$, $\mathrm{W}^*_{\mathcal{L}^\alpha}$ is of weak type (p,p) on $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ if and only if $\delta \geq -\frac{\alpha}{2}p - 1$ when $\alpha \neq 0$, or $\delta > -1$ when $\alpha = 0$.*
(d) *For $1 \leq p < \infty$ and $\delta \leq \frac{\alpha}{2}p + p - 1$, $\mathrm{W}^*_{\mathcal{L}^\alpha}$ is of restricted weak type (p,p) on $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ if and only if $\delta \geq -\frac{\alpha}{2}p - 1$ when $\alpha \neq 0$, or $\delta > -1$ when $\alpha = 0$.*

Theorem 2.2 *Let $\alpha > -1$ and $1 \leq p \leq \infty$. Associated to the Laguerre system $\{\varphi_n^\alpha\}$ given in (1.2), we have the following results:*

For any $t > 0$, the heat diffusion integral $K^t_{\varphi^\alpha} f(x)$ is finite a.e. for all $f \in L^p(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$, $1 < p \leq \infty$, if and only if $\delta < (\alpha + \frac{3}{2})p - 1$, and for all $f \in L^{p,1}(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$, $1 \leq p < \infty$, if and only if $\delta \leq (\alpha+\frac{3}{2})p - 1$.

Moreover, the associate maximal operator $\mathrm{W}^*_{\varphi^\alpha}$ satisfies:

(a) *For $1 < p < \infty$ and $\delta < (\alpha+\frac{1}{2})p + p - 1$, $\mathrm{W}^*_{\varphi^\alpha}$ is of strong type (p,p) on $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ if and only if $\delta > -(\alpha+\frac{1}{2})p - 1$.*
(b) *For all real δ, $\mathrm{W}^*_{\varphi^\alpha}$ is of strong type (∞,∞) on $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ if and only if $\alpha \geq -\frac{1}{2}$.*

(c) *For* $1 < p < \infty$ *and* $\delta < (\alpha+\frac{1}{2})p+p-1$, $\mathrm{W}^*_{\varphi^\alpha}$ *is of weak type* (p,p) *on* $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ *if and only if* $\delta \geq -(\alpha+\frac{1}{2})p-1$ *when* $\alpha \neq -\frac{1}{2}$, *or* $\delta > -1$ *when* $\alpha = -\frac{1}{2}$.

(d) *For* $1 \leq p < \infty$ *and* $\delta \leq (\alpha+\frac{1}{2})p+p-1$, $\mathrm{W}^*_{\varphi^\alpha}$ *is of restricted weak type* (p,p) *on* $(\mathbb{R}^+, x^\delta\,\mathrm{d}x)$ *if and only if* $\delta \geq -(\alpha+\frac{1}{2})p-1$ *when* $\alpha \neq -\frac{1}{2}$, *or* $\delta > -1$ *when* $\alpha = -\frac{1}{2}$.

Theorem 2.3 *Let* $\alpha > -1$ *and* $1 \leq p \leq \infty$. *Associated to the Laguerre system* $\{\ell_n^\alpha\}$ *given in* (1.3), *we have the following results:*

For any $t > 0$, *the heat diffusion integral* $K^t_{\ell^\alpha}f(x)$ *is finite a.e. for all* $f \in L^p(\mathbb{R}^+, x^\delta x^\alpha\,\mathrm{d}x)$, $1 < p \leq \infty$, *if and only if* $\delta < (\alpha+1)p-1$, *and for all* $f \in L^{p,1}(\mathbb{R}^+, x^\delta x^\alpha\,\mathrm{d}x)$, $1 \leq p < \infty$, *if and only if* $\delta \leq (\alpha+1)p-1$.

Moreover, the associate maximal operator $\mathrm{W}^*_{\ell^\alpha}$ *satisfies:*

(a) *For* $1 < p < \infty$ *and* $\delta < (\alpha+1)(p-1)$, $\mathrm{W}^*_{\ell^\alpha}$ *is of strong type* (p,p) *on* $(\mathbb{R}^+, x^\delta x^\alpha\,\mathrm{d}x)$ *if and only if* $\delta > -\alpha-1$.

(b) $\mathrm{W}^*_{\ell^\alpha}$ *is of strong type* (∞,∞) *on* $(\mathbb{R}^+, x^\delta x^\alpha\,\mathrm{d}x)$ *for all real* δ.

(c) *For* $1 < p < \infty$ *and* $\delta < (\alpha+1)(p-1)$, $\mathrm{W}^*_{\ell^\alpha}$ *is of weak type* (p,p) *on* $(\mathbb{R}^+, x^\delta x^\alpha\,\mathrm{d}x)$ *if and only if* $\delta > -\alpha-1$.

(d) *For* $1 \leq p < \infty$ *and* $\delta \leq (\alpha+1)(p-1)$, $\mathrm{W}^*_{\mathcal{L}^\alpha}$ *is of restricted weak type* (p,p) *on* $(\mathbb{R}^+, x^\delta x^\alpha\,\mathrm{d}x)$ *if and only if* $\delta > -\alpha-1$.

Remark 2.1 Let us note that the restrictions on δ for the boundedness of the maximal operator in Theorem 2.1 become natural after we observe that they are the exact conditions for the Laguerre functions $\mathcal{L}_n^\alpha$ to belong to the $L^p(x^\delta\,\mathrm{d}x)$ spaces, $1 < p \leq \infty$, or to the $L^{p,1}(x^\delta\,\mathrm{d}x)$ spaces, $1 \leq p < \infty$, for the restricted weak type case. In fact, since $\mathcal{L}_n^\alpha(x) \sim x^{\alpha/2}$ when $x \to 0^+$ and $\mathcal{L}_n^\alpha(x) \sim O(\mathrm{e}^{-\epsilon x})$ when $x \to \infty$ (see [19, page 27]), $\mathcal{L}_n^\alpha$ will belong to $L^p(x^\delta\,\mathrm{d}x)$ when $\delta > -\frac{\alpha}{2}p-1$ ($\alpha \geq 0$ when $p=\infty$) and to $L^{p,1}(x^\delta\,\mathrm{d}x)$ when $\delta \geq -\frac{\alpha}{2}p-1$. Analogous observations hold for the systems $\{\varphi_n^\alpha\}$ and $\{\ell_n^\alpha\}$.

3 Some preliminary results

In this section we introduce our basic operators which, as we shall show in Sect. 4, majorize our maximal operators for some special values of the parameters. We state as lemmas their boundedness properties on weighted L^p and we either outline the proof or else give a reference.

We also include some estimates on the behavior of the modified Bessel functions which we will use frequently in Sect. 5 and also refer to a result of Landau needed in Sect. 4.

For $\beta > -1$ and $\eta > -1$, let H_0^β and H_∞^η denote the modified Hardy Operators

$$\mathrm{H}_0^\beta f(x) = x^{-\beta-1}\int_0^x f(y)y^\beta\,\mathrm{d}y$$

$$\mathrm{H}_\infty^\eta f(x) = x^\eta\int_x^\infty f(y)y^{-\eta-1}\,\mathrm{d}y,$$

for a measurable f defined on $\mathbb{R}^+$ and $x \in \mathbb{R}^+$.

Note that Hölder's inequality on Lorentz spaces implies that $\mathrm{H}_0^\beta f(x)$ is finite a.e. for any $f \in L^p(\mathbb{R}^+, x^\gamma\,\mathrm{d}x)$ if $1 < p \le \infty$ and $\gamma < (\beta+1)p - 1$, and for any $f \in L^{p,1}(\mathbb{R}^+, x^\gamma\,\mathrm{d}x)$ if $1 \le p < \infty$ and $\gamma \le (\beta+1)p-1$.

Lemma 3.1 *For* H_0^β, *we have*

(a) *For* $1 < p < \infty$, *if* $\gamma < \beta p + p - 1$ *then* H_0^β *is of strong type* (p,p) *on* $\mathbb{R}^+$ *with measure* $x^\gamma\,\mathrm{d}x$.
(b) *For* $p = \infty$, H_0^β *is of strong type* (∞,∞) *on* $\mathbb{R}^+$ *with measure* $x^\gamma\,\mathrm{d}x$ *for any real* γ.
(c) *For* $p = 1$, *if* $\gamma \le \beta$ *then* H_0^β *is of weak type* $(1,1)$ *with measure* $x^\gamma\,\mathrm{d}x$.
(d) *For* $1 < p < \infty$, *if* $\gamma \le \beta p + p - 1$ *then* H_0^β *is of restricted weak type* (p,p) *with measure* $x^\gamma\,\mathrm{d}x$.

Proof For (a) and (c), see Theorem A and Theorem 2, respectively, of [3]. Part (b) holds since $\beta > -1$ and $L^\infty((0,\infty), x^\gamma\,\mathrm{d}x) = L^\infty((0,\infty),\mathrm{d}x)$, with equality of norms. For (d), it is enough to prove that H_0^β is of restricted weak type (p,p), with the measure $x^\gamma\,\mathrm{d}x$, for $p = (\gamma+1)/(\beta+1)$. Let $f \in L^{p,1}(x^\gamma\,\mathrm{d}x)$. Since $p > 1$ and $\beta > -1$ we have that $\beta - \gamma < 0$ and $\gamma > -1$. This implies that $x^{\beta-\gamma} \in L^{p',\infty}(x^\gamma\,\mathrm{d}x)$, where $p' = (\gamma+1)/(\gamma-\beta)$. Then, using Hölder's inequality we get

$$|\mathrm{H}_0^\beta f(x)| \le x^{-\beta-1}\int_0^\infty |f(y)| y^\beta\,\mathrm{d}y \le x^{-\beta-1}\|f\|_{L^{p,1}(x^\gamma\,\mathrm{d}x)}\,\|x^{\beta-\gamma}\|_{L^{p',\infty}(x^\gamma\,\mathrm{d}x)}.$$

It is an easy calculation to show that $x^{-\beta-1} \in L^{p,\infty}(x^\gamma\,\mathrm{d}x)$ if and only if $\gamma = \beta p + p - 1$. Thus, we get the desired inequality

$$\|\mathrm{H}_0^\beta f\|_{L^{p,\infty}(x^\gamma\,\mathrm{d}x)} \le C\,\|f\|_{L^{p,1}(x^\gamma\,\mathrm{d}x)}.$$

□

Lemma 3.2 *For* H_∞^η *we have*

(a) *For* $1 < p < \infty$, *if* $\gamma > -\eta p - 1$ *then* H_∞^η *is of strong type* (p,p) *with measure* $x^\gamma\,\mathrm{d}x$.
(b) *For* $p = \infty$, *if* $\eta > 0$ *then* H_∞^η *is of strong type* (∞,∞) *with measure* $x^\gamma\,\mathrm{d}x$ *for all real* γ.
(c) *For* $p = 1$, *if* $\begin{cases} \gamma \ge -\eta - 1, & \eta \ne 0 \\ \gamma > -1, & \eta = 0 \end{cases}$ *then* H_∞^η *is of weak type* $(1,1)$ *with measure* $x^\gamma\,\mathrm{d}x$.

Proof For (a) see Theorem B and for (c) see Theorems 4 and 5, all from [3]. Part (b) holds just like Lemma 3.1.b, using this time that $\eta > 0$. □

Let us observe that if $\gamma = -\eta p - 1$, with $\eta \ne 0$ and $1 < p < \infty$, then H_∞^η is not of weak type (p,p) with respect to the measure $x^\gamma\,\mathrm{d}x$, as we may expect by the results obtained in [8], with $\eta = \alpha/2$. For this reason, we consider the slightly better operators $\{\mathrm{T}_s^\eta\}_{s\in(0,1)}$ defined by

$$\mathrm{T}_s^\eta f(x) = x^\eta \int_x^\infty \varphi(s,y) f(y) y^{-\eta-1}\,\mathrm{d}y,$$

for some nonnegative function φ that satisfies

$$\sup_{(s,y)\in(0,1)\times(0,\infty)} \varphi(s,y) < \infty, \tag{3.1}$$

$$\|\varphi(s,y)y^\delta\|_{L^q(x,\infty)} < \infty \tag{3.2}$$

for any δ, $0 < s < 1$, $x > 0$ and $1 \le q \le \infty$, and

$$\sup_{0<s<1} \int_0^\infty (\varphi(s,y))^q \frac{\mathrm{d}y}{y} < \infty \tag{3.3}$$

for any $1 \le q < \infty$. Observe that if, for each $s \in (0,1)$, $\varphi(s,y)$ is continuous and rapidly decreasing at infinity, then (3.2) holds.

By (3.2), $\mathrm{T}_s^\eta f(x)$ is finite a.e. for $f \in L^p(\mathbb{R}^+, x^\gamma \mathrm{d}x)$, for all p and all γ.

T^η, defined by

$$\mathrm{T}^\eta f(x) = \sup_{0<s<1} |\mathrm{T}_s^\eta f(x)|$$

satisfies, by (3.1),

$$\mathrm{T}^\eta f(x) \le C\mathrm{H}_\infty^\eta |f|(x) \quad \forall x \in (0,\infty) \tag{3.4}$$

and, by Hölder's inequality and (3.3),

$$\mathrm{T}^\eta f(x) \le Cx^\eta \|f\|_{L^p(x^{-\eta p-1}\mathrm{d}x)}. \tag{3.5}$$

Inequality (3.4) implies that all the H_∞^η properties of Lemma 3.2 also hold for T^η. By (3.2), T^η is of strong type (∞,∞) also when $\eta = 0$. Inequality (3.5) implies that T^η is of weak type (p,p) with measure $x^\gamma \mathrm{d}x$, for $\gamma = -\eta p - 1$, when $\eta \neq 0$, since in that case x^η belongs to $L^{p,\infty}(x^\gamma \mathrm{d}x)$.

Examples of such functions φ are the ones we will use in Sect. 4:

$$\varphi(s,y) = \left(\frac{y}{s}\right)^\epsilon \mathrm{e}^{-c\frac{y}{s}} \tag{3.6}$$

or

$$\varphi(s,y) = \left(\frac{y^2}{s}\right)^\epsilon \mathrm{e}^{-c\frac{y^2}{s}}, \tag{3.7}$$

for some positive constants ϵ and c. Therefore, we have obtained

Lemma 3.3 (About T^η) *Let* $\eta > -1$. *Then*

(a) *For* $1 < p < \infty$, *if* $\gamma > -\eta p - 1$ *then* T^η *is of strong type* (p,p) *with measure* $x^\gamma \mathrm{d}x$.
(b) *For* $p = \infty$, *if* $\eta \ge 0$ *then* T^η *is of strong type* (∞,∞) *with measure* $x^\gamma \mathrm{d}x$ *for all real* γ.
(c) *For* $1 \le p < \infty$, *if* $\begin{cases} \gamma \ge -\eta p - 1, & \eta \neq 0 \\ \gamma > -1, & \eta = 0 \end{cases}$ *then* T^η *is of weak type* (p,p) *with measure* $x^\gamma \mathrm{d}x$.

Local Maximal Function $\mathrm{M}_{\mathrm{loc}}^\kappa$**.** For $\kappa > 1$, the Local Maximal Function is defined as

$$\mathrm{M}_{\mathrm{loc}}^\kappa f(x) = \sup_{0<a<x<b<\kappa a} \frac{1}{b-a} \int_a^b |f(y)|\,\mathrm{d}y$$

Definition 3.4 For $1 \le p < \infty$, let A^p_{loc} denote the class of all nonnegative weights ω on $(0,\infty)$ satisfying

$$\sup_{0<a<b<2a} \frac{1}{b-a} \left(\int_a^b \omega(x) \mathrm{d}x \right)^{\frac{1}{p}} \left(\int_a^b \omega(x)^{-\frac{p'}{p}} \mathrm{d}x \right)^{\frac{1}{p'}} < \infty$$

when $1 < p < \infty$, and

$$\sup_{0<a<b<2a} \frac{1}{b-a} \left(\int_u^v \omega(x) \mathrm{d}x \right) \left(\operatorname*{ess\,sup}_{x \in (a,b)} \omega^{-1}(x) \right) < \infty$$

when $p = 1$.

From Propositions 6.1 and 6.3 of [13], we have that if $\omega \in A^p_{\mathrm{loc}}$ then, for any $\kappa > 1$, $\mathrm{M}^{\kappa}_{\mathrm{loc}}$ is of strong type (p,p) with measure $\omega(x)\mathrm{d}x$, if $1 < p < \infty$, and of weak type $(1,1)$ with measure $\omega(x)\mathrm{d}x$, for $p = 1$. Is not difficult to see that $\omega(x) = x^{\gamma} \in A^p_{\mathrm{loc}}$, for all real γ, and $1 \le p < \infty$. Therefore we have

Lemma 3.5 *Let* $\kappa > 1$*, then*

(a) $\mathrm{M}^{\kappa}_{\mathrm{loc}}$ *is of strong type* (p,p) *with measure* $x^{\gamma}\,\mathrm{d}x$ *for any real* γ *and* $1 < p \le \infty$*,*
(b) $\mathrm{M}^{\kappa}_{\mathrm{loc}}$ *is of weak type* $(1,1)$ *with measure* $x^{\gamma}\,\mathrm{d}x$ *for any real* γ*.*

We will also need the following results:

Lemma 3.6 (Estimates for I_α) *Let* $\alpha > -1$*. If* $I_\alpha(z) = \mathrm{i}^{-\alpha} J_\alpha(iz)$ *is the modified Bessel function* (*where* J_α *is the usual Bessel function*)*, then there exist two positive constants* c_α *and* C_α *such that*

1. *if* $0 \le z \le 1$ *then* $c_\alpha z^\alpha \le I_\alpha(z) \le C_\alpha z^\alpha$
2. *if* $z \ge 1$ *then* $c_\alpha \mathrm{e}^z z^{-1/2} \le I_\alpha(z) \le C_\alpha \mathrm{e}^z z^{-1/2}$

For a proof see [5, p. 5, 86].

Lemma 3.7 (Landau) *Let X be a Banach function space* (*see* [4]) *and g a measurable function; then*

$$\int_E |gf|\, \mathrm{d}\mu < \infty \quad \forall\ f \in X$$

if and only if g belongs to the associate dual space X'*.*

For a proof, see [4, page 10].

Remark 3.1 The space $L^{p,q}(d\mu)$, with $1 \le p < \infty$ and $1 \le q \le \infty$, or $p = q = \infty$, is a Banach function space, and his associate dual space is $L^{p',q'}(d\mu)$, where $\frac{1}{p} + \frac{1}{p'} = 1$ and $\frac{1}{q} + \frac{1}{q'} = 1$, and moreover

$$\|g\|_{(L^{p,q})'} = \sup \left\{ \left| \int fg \right| : f \in L^{p,q} \right\} = \|g\|_{L^{p',q'}}.$$

4 A general result

Based on the results quoted in the previous section we establish sharp strong type, weak type and restricted weak type (p,p) inequalities with respect to a power weight for a general operator controlled by the operators of Lemmas 3.1, 3.3 and 3.5.

Proposition 4.1 *Let* $\{\mathrm{R}_s\}_{s\in I}$ *be a family of integral operators with nonnegative and measurable kernels* $R_s(x,y)$ *on* $(0,\infty)\times(0,\infty)$. *Let us assume that there exist constants* $\beta > -1$, $\eta > -1$, $\kappa > 1$ *and* C *such that*

$$\mathrm{R}_s f \leq C\,(\mathrm{M}^{\kappa}_{\mathrm{loc}} f + \mathrm{H}^{\beta}_0 f + \mathrm{T}^{\eta}_s f) \tag{4.1}$$

for all $s \in I$ *and for any nonnegative and measurable function* f *on* $(0,\infty)$. *Suppose further that for some* $s_0 \in I$ *there exists* $c > 0$ *such that*

$$R_{s_0}(x,y) \geq c\,x^{\eta} y^{\beta} \tag{4.2}$$

for all $(x,y) \in (0,1)\times(0,1)$.

Then we have

(i) *For* $1 < p \leq \infty$, $\mathrm{R}_s f(x)$ *is finite a.e. for any* $s \in I$ *and for all* $f \in L^p(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ *if and only if* $\gamma < (\beta+1)p - 1$

(ii) *For* $1 \leq p < \infty$, $\mathrm{R}_s f(x)$ *is finite a.e. for any* $s \in I$ *and for all* $f \in L^{p,1}(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ *if and only if* $\gamma \leq (\beta+1)p - 1$.

For the associate maximal operator

$$\mathrm{R}f(x) = \sup_{s\in I} |\mathrm{R}_s f(x)|$$

we have:

(a) *For* $1 < p < \infty$ *and* $\gamma < \beta p + p - 1$, R *is of strong type* (p,p) *on* $(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ *if and only if* $\gamma > -\eta p - 1$

(b) *For any real* γ, R *is of strong type* (∞,∞) *on* $(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ *if and only if* $\eta \geq 0$.

(c) *For* $1 < p < \infty$ *and* $\gamma < \beta p + p - 1$, R *is of weak type* (p,p) *on* $(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ *if and only if* $\gamma \geq -\eta p - 1$ *when* $\eta \neq 0$, *or* $\gamma > -1$ *when* $\eta = 0$.

(d) *For* $1 \leq p < \infty$ *and* $\gamma \leq \beta p + p - 1$, R *is of restricted weak type* (p,p) *on* $(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ *if and only if* $\gamma \geq -\eta p - 1$ *when* $\eta \neq 0$, *or* $\gamma > -1$ *when* $\eta = 0$.

Proof From our assumptions on R_s, the sufficient conditions on the domain of R_s for any $s \in I$ arise from the domain of each operator on the right hand side of inequality (4.1). Let us note that the domains of $\mathrm{M}^{\kappa}_{\mathrm{loc}}$ and T^{η}_s contain all $L^p(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ for any value of γ, and the domain of H^{β}_0 contains $L^p(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$ (or $L^{p,1}(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$) if $1 < p \leq \infty$ and $\gamma < (\beta+1)p - 1$ (respectively, $1 \leq p < \infty$ and $\gamma \leq (\beta+1)p - 1$).

For the necessary conditions in i) and ii), let assume first that $\mathrm{R}_{s_0} f(x) < \infty$ a.e. for all $f \in L^p(\mathbb{R}^+, x^{\gamma}\,\mathrm{d}x)$. Then, by (4.2)

$$\int_0^1 |f(y)| y^{\beta-\gamma} y^{\gamma}\,\mathrm{d}y < \infty \tag{4.3}$$

for all $f \in L^p((0,1), x^\gamma \mathrm{d}x)$. Then, by Lemma 3.7 and the Remark 3.1 below, $y^{\beta-\gamma} \in L^{p'}((0,1), x^\gamma \mathrm{d}x)$. If $1 < p \leq \infty$, this means that

$$\int_0^1 y^{(\beta-\gamma)p'} y^\gamma \, \mathrm{d}y < \infty$$

which implies $(\beta - \gamma)p' + \gamma > -1$ and hence $\gamma < (\beta + 1)p - 1$. For $p = \infty$, this holds for all γ since $\beta > -1$.

Now assume that $\mathrm{R}_{s_0} f(x) < \infty$ a.e. for all $f \in L^{p,1}(\mathbb{R}^+, x^\gamma \mathrm{d}x)$, with $1 \leq p < \infty$. By (4.2) we have that (4.3) holds for all $f \in L^{p,1}((0,1), x^\gamma \mathrm{d}x)$ and therefore Lemma 3.7 gives

$$y^{\beta-\gamma} \in L^{p',\infty}((0,1), x^\gamma \mathrm{d}x)$$

which implies, for $p = 1$, that $\gamma \leq \beta$, and for $1 < p < \infty$, that

$$\lambda^{p'} \int_{\{y \in (0,1): y^{\beta-\gamma} > \lambda\}} y^\gamma \, \mathrm{d}y \leq C, \tag{4.4}$$

for all $\lambda > 0$. We may assume $\beta < \gamma$, otherwise, the inequality $\gamma \leq \beta p + p - 1$ holds since $\beta > -1$ and $p > 1$. In this case

$$\{y \in (0,1) : y^{\beta-\gamma} > \lambda\} = \left(0, \lambda^{\frac{1}{\beta-\gamma}}\right) \quad \forall \lambda > 1.$$

Then (4.4) implies

$$\lambda^{p'} \int_0^{\lambda^{\frac{1}{\beta-\gamma}}} y^\gamma \, \mathrm{d}y \leq C \quad \forall \lambda > 1$$

that is

$$\lambda^{p' + (\gamma+1)(\beta-\gamma)^{-1}} \leq C \quad \forall \lambda > 1,$$

and consequently

$$p' + (\gamma + 1)(\beta - \gamma)^{-1} \leq 0$$

which gives $\gamma \leq \beta p + p - 1$.

Now, we consider the associate maximal operator R. The sufficient conditions for the strong, weak and restricted weak type (p,p), $1 \leq p \leq \infty$, of this operator come out from the hypothesis (4.1) together with Lemmas 3.1, 3.3 and 3.5.

Let us prove the necessary conditions.

Case (*b*) Let $f = \chi_{(\frac{1}{2},1)}(x)$, then $f \in L^\infty(\mathbb{R}^+, \mathrm{d}x) = L^\infty(\mathbb{R}^+, x^\gamma \mathrm{d}x)$. Assume R is of strong type (∞, ∞) with measure $x^\gamma \mathrm{d}x$. Then $\mathrm{R}f \in L^\infty(\mathbb{R}^+, x^\gamma \mathrm{d}x) = L^\infty(\mathbb{R}^+, \mathrm{d}x)$, which implies $\int_{1/2}^1 R_{s_0}(x,y)\, \mathrm{d}y \leq C$ for a.e. $x \in \mathbb{R}^+$. Therefore, hypothesis (4.2) gives

$$x^\eta \int_{\frac{1}{2}}^1 y^\beta \, \mathrm{d}y \leq C \text{ for a.e. } x \in (0,1)$$

and then

$$x^\eta \leq C \text{ for a.e. } x \in (0,1)$$

which implies $\eta \geq 0$.

For $1 \leq p < \infty$, consider again $f = \chi_{(\frac{1}{2},1)}$; then

$$f \in L^p(\mathbb{R}^+, x^\gamma \, \mathrm{d}x), \quad 1 \leq p < \infty,$$

and moreover

$$f \in L^{p,1}(\mathbb{R}^+, x^\gamma \, \mathrm{d}x), \quad 1 \leq p < \infty.$$

Also, by (4.2),

$$\mathrm{R}_{s_0} f(x) \geq c\chi_{(0,1)}(x) x^\eta \tag{4.5}$$

for some positive constant c.

Case (a) Assume first that R is of strong type (p,p), with measure $x^\gamma \, \mathrm{d}x$, $1 < p < \infty$. Then, $\mathrm{R}f \in L^p(\mathbb{R}^+, x^\gamma \, \mathrm{d}x)$, which implies $\mathrm{R}_{s_0} f \in L^p(\mathbb{R}^+, x^\gamma \, \mathrm{d}x)$, and by (4.5) we have

$$\int_0^1 x^{\eta p} x^\gamma \, \mathrm{d}x < \infty$$

which implies

$$\gamma > -\eta p - 1.$$

Cases (c) *and* (d) Suppose now that R is of weak type or restricted weak type (p,p) with measure $x^\gamma \, \mathrm{d}x$, $1 \leq p < \infty$. In any case, (4.5) implies that $x^\eta \in L^{p,\infty}((0,1), x^\gamma \, \mathrm{d}x)$, and therefore

$$\lambda^p \int_{E_\lambda} x^\gamma \, \mathrm{d}x \leq M \quad \forall \lambda > 0 \tag{4.6}$$

for some constant M, where

$$E_\lambda = \left\{ x \in (0,1) : x^\eta > \lambda \right\}.$$

Suppose first $\eta > 0$. Then $E_\lambda = (\lambda^{\frac{1}{\eta}}, 1)$ for all $0 < \lambda < 1$, and inequality (4.6) implies

$$\lambda^p \int_{\lambda^{\frac{1}{\eta}}}^1 x^\gamma \, \mathrm{d}x \leq M \quad \forall\, 0 < \lambda < 1,$$

which gives

$$\lambda^{p + \frac{\gamma+1}{\eta}} \leq M \quad \forall\, 0 < \lambda < 1.$$

From this we obtain the desired inequality $\gamma \geq -\eta p - 1$.

If $\eta < 0$ then $E_\lambda = (0, \lambda^{\frac{1}{\eta}})$ for all $\lambda > 1$ and (4.6) implies that

$$\lambda^p \int_0^{\lambda^{\frac{1}{\eta}}} x^\gamma \, \mathrm{d}x \leq M \quad \forall \lambda > 1,$$

which gives

$$\lambda^{p+\frac{\gamma+1}{\eta}} \leq M \quad \forall \lambda > 1$$

and therefore $\gamma \leq -\eta p - 1$.

Finally, if $\eta = 0$, then $E_\lambda = (0,1)$ for all $0 < \lambda < 1$ and we must have $\gamma > -1$. □

5 Proof of the theorems

Before proceeding with the proofs, let us consider the heat-diffusion kernel for the first system of Laguerre's functions,

$$K_{\mathcal{L}^\alpha}(t,x,y) = \sum_{n=0}^{\infty} \mathrm{e}^{-t\lambda_n} \mathcal{L}_n^\alpha(x) \mathcal{L}_n^\alpha(y)$$

where $\lambda_n = n + \dfrac{\alpha+1}{2}$, for $t > 0$, $0 < x < \infty$ and $0 < y < \infty$. Following [7] or [8], after performing a change in the parameter, the kernel can be written as

$$K_{\mathcal{L}^\alpha}(t,x,y) = W_{\mathcal{L}^\alpha}\left(\frac{1-\mathrm{e}^{-t/2}}{1+\mathrm{e}^{-t/2}}, x, y\right)$$

with

$$W_{\mathcal{L}^\alpha}(s,x,y) = \frac{1}{2}\frac{1-s^2}{2s}\mathrm{e}^{-\frac{1}{4}(s+\frac{1}{s})(x+y)} I_\alpha\left(\frac{1-s^2}{2s}(xy)^{1/2}\right) \tag{5.1}$$

for $0 < s < 1$, $0 < x < \infty$ and $0 < y < \infty$, where $I_\alpha(z) = i^{-\alpha} J_\alpha(iz)$ is the modified Bessel function (J_α being the usual Bessel function, see [5]). Therefore, the maximal operator $\mathrm{W}^*_{\mathcal{L}^\alpha}$ may be expressed in terms of $W_{\mathcal{L}^\alpha}(s,x,y)$ by

$$\mathrm{W}^*_{\mathcal{L}^\alpha} f(x) = \sup_{0<s<1} \left| \int_0^\infty W_{\mathcal{L}^\alpha}(s,x,y) f(y) \mathrm{d}y \right|$$

Regarding the two other Laguerre's systems, we can check that their kernels are related to the above case since, by (1.2) and (1.3), we have

$$K_{\varphi^\alpha}(t,x,y) = \sum_{n=0}^{\infty} \mathrm{e}^{-t\lambda_n} \varphi_n^\alpha(x) \varphi_n^\alpha(y) = 2(xy)^{1/2} K_{\mathcal{L}^\alpha}(t,x^2,y^2)$$

and

$$K_{\ell^\alpha}(t,x,y) = \sum_{n=0}^{\infty} \mathrm{e}^{-t\lambda_n} \ell_n^\alpha(x) \ell_n^\alpha(y) = (xy)^{-\alpha/2} K_{\mathcal{L}^\alpha}(t,x,y).$$

Again, $\lambda_n = n + \dfrac{\alpha+1}{2}$ in both cases. After performing the same change of parameters, we arrive to

$$\mathrm{W}^*_{\varphi^\alpha} f(x) = \sup_{0<s<1} \left| \int_0^\infty W_{\varphi^\alpha}(s,x,y) f(y) \mathrm{d}y \right|$$

and

$$\mathrm{W}^{*}_{\ell^\alpha} f(x) = \sup_{0<s<1} \left| \int_0^\infty W_{\ell^\alpha}(s,x,y) f(y) y^\alpha \, \mathrm{d}y \right|,$$

with

$$W_{\varphi^\alpha}(s,x,y) = 2(xy)^{1/2} W_{\mathcal{L}^\alpha}(s,x^2,y^2) \tag{5.2}$$

and

$$W_{\ell^\alpha}(s,x,y) = 2(xy)^{-\alpha/2} W_{\mathcal{L}^\alpha}(s,x,y) \tag{5.3}$$

for $0 < s < 1$.

We proceed now with the proof of the theorems of Sect. 2.

Proof of Theorem 2.1 Let $\alpha > -1$. We will prove that the family of integral operators corresponding to the kernels (5.1), where $s \in (0,1)$, satisfies hypotheses (4.1) and (4.2) of Proposition 4.1, with $\beta = \eta = \frac{\alpha}{2}$ and $\kappa = 16$.

In order to obtain (4.2), let $s_0 = \sqrt{2} - 1$. Then $0 < s_0 < 1$ and $\frac{1-{s_0}^2}{2s_0} = 1$. For $0 < x < 1$ and $0 < y < 1$ we have $0 < \frac{1-{s_0}^2}{2s_0}(xy)^{1/2} \leq 1$ which implies, by Lemma 3.6, that the inequality

$$I_\alpha\left(\frac{1-{s_0}^2}{2s_0}(xy)^{1/2}\right) \geq c\,(xy)^{\alpha/2}$$

holds for some constant c. Also, for $0 < x < 1$ and $0 < y < 1$ we have

$$\mathrm{e}^{-\frac{1}{4}(s_0+\frac{1}{s_0})(x+y)} \geq c,$$

and then

$$W_{\mathcal{L}^\alpha}(\sigma,x,y) \geq c\,(xy)^{\alpha/2}.$$

Now, we will obtain (4.1). Let f be a nonnegative and measurable function on $\mathbb{R}^+$. Let denote with $D_s(x)$ the set $\left\{y \in (0,\infty) : 0 \leq \frac{1-s^2}{2s}(xy)^{1/2} \leq 1\right\}$. By Lemma 3.6, if $y \in D_s(x)$ then

$$I_\alpha\left(\frac{1-s^2}{2s}(xy)^{1/2}\right) \sim \left(\frac{1-s^2}{2s}\right)^{\alpha} (xy)^{\alpha/2}$$

and if $y \notin D_s(x)$ then

$$I_\alpha\left(\frac{1-s^2}{2s}(xy)^{1/2}\right) \sim \left(\frac{1-s^2}{2s}\right)^{-1/2} (xy)^{-1/4}\,\mathrm{e}^{\frac{1-s^2}{2s}(xy)^{1/2}}.$$

Thus, if we denote with $W^1_{\mathcal{L}^\alpha}(s,x,y)$ and $W^2_{\mathcal{L}^\alpha}(s,x,y)$ the restrictions of the kernel $W_{\mathcal{L}^\alpha}(s,x,y)$ to $D_s(x)$ and to $D_s(x)^c$, respectively, from (5.1) we obtain

$$W^1_{\mathcal{L}^\alpha}(s,x,y) \sim \left(\frac{1-s^2}{2s}\right)^{\alpha+1} (xy)^{\alpha/2} \mathrm{e}^{-\frac{1}{4}(s+\frac{1}{s})(x+y)} \tag{5.4}$$

and

$$W^2_{\mathcal{L}^\alpha}(s,x,y) \sim \left(\frac{1-s^2}{2s}\right)^{1/2} (xy)^{-1/4} \mathrm{e}^{-\frac{s}{4}|x^{1/2}+y^{1/2}|^2} \mathrm{e}^{-\frac{|x^{1/2}-y^{1/2}|^2}{4s}} \tag{5.5}$$

since

$$-\frac{1}{4}(s+\frac{1}{s})(x+y)+\frac{1-s^2}{2s}(xy)^{1/2}=-\frac{s}{4}(x^{1/2}+y^{1/2})^2-\frac{1}{4s}(x^{1/2}-y^{1/2})^2.$$

We consider separately three cases.

Local case: We fix $0<s<1$ and $0<x<\infty$, and we consider those y such that $x/4<y<4x$. Our aim is to obtain

$$\int_{x/4}^{4x} W_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y \le C\,\mathrm{M}^{\kappa}_{\mathrm{loc}} f(x) \tag{5.6}$$

for some $\kappa>1$ and some $C>0$ only dependent of α. Observe that $x/4<y<4x$ implies that $(xy)^{\alpha/2}\sim x^\alpha$ and $x+y\sim x$. Therefore the kernel in (5.4) is bounded by

$$W^1_{\mathcal{L}^\alpha}(s,x,y)\le C\frac{1}{x}\left(\frac{x}{s}\right)^{\alpha+1}\mathrm{e}^{-\frac{x}{s}}.$$

Since $\alpha>-1$, $t^{\alpha+1}\mathrm{e}^{-t}$ is bounded for all positive t, and we get

$$W^1_{\mathcal{L}^\alpha}(s,x,y)\le C\frac{1}{x},$$

consequently

$$\begin{aligned}\int_{x/4}^{4x} W^1_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y &\le C\,\frac{1}{4x-x/4}\int_{x/4}^{4x} f(y)\mathrm{d}y\\ &\le C\,\mathrm{M}^{16}_{\mathrm{loc}}f(x).\end{aligned}$$

Consider now the kernel (in 5.5). For every integer k we define the disjoint sets

$$B_k(x)=\left\{y:2^k\le\frac{|x^{1/2}-y^{1/2}|}{2s^{1/2}}<2^{k+1}\right\}. \tag{5.7}$$

Let k_0 be the unique integer that satisfies

$$2^{k_0+3}\le\left(\frac{x}{s}\right)^{1/2}<2^{k_0+4}.$$

It is easy to check that for $k\le k_0$

$$B_k(x)=(a_k,a_{k-1}]\cup[b_{k-1},b_k)$$

with

$$a_k=(x^{1/2}-2^{k+2}s^{1/2})^2,\quad b_k=(x^{1/2}+2^{k+2}s^{1/2})^2. \tag{5.8}$$

From this expression is clear that $a_k\nearrow x$ and $b_k\searrow x$ when $k\to-\infty$. Also

$$x/4\le a_{k_0}\le a_k<x<b_k\le b_{k_0}\le 4x$$

and

$$(a_{k_0},b_{k_0})\setminus\{x\}=\bigcup_{k\le k_0}B_k(x).$$

Consequently we may write

$$\int_{x/4}^{4x} W^2_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y = \int_{x/4}^{a_{k_0}} W^2_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y + \sum_{k\le k_0}\int_{B_k(x)} W^2_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y + \int_{b_{k_0}}^{4x} W^2_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y = \mathrm{I}+\mathrm{II}+\mathrm{III}$$

If $y \in B_k(x)$ then from (5.5), (5.7) and since $y \sim x$, we get

$$W^2_{\mathcal{L}^\alpha}(s,x,y) \le \frac{C}{(sx)^{1/2}}\,\mathrm{e}^{-2^{2k}}.$$

Also from (5.8) we have that $b_k - a_k = 2^{k+4}(sx)^{1/2}$. Then we obtain

$$\begin{aligned}\int_{B_k(x)} W^2_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y &\le \frac{C}{(sx)^{1/2}}\,\mathrm{e}^{-2^{2k}}\int_{a_k}^{b_k} f(y)\mathrm{d}y\\ &\le C\,2^k\mathrm{e}^{-2^{2k}}\frac{1}{b_k-a_k}\int_{a_k}^{b_k} f(y)\mathrm{d}y\\ &\le C\,2^k\mathrm{e}^{-2^{2k}}\mathrm{M}^{16}_{\mathrm{loc}}f(x).\end{aligned}$$

The last inequality holds since $b_k \le 4x = 16(x/4) \le 16a_k$ and $x \in (a_k, b_k)$. Using now that

$$\sum_{-\infty}^{\infty} 2^k\mathrm{e}^{-2^{2k}} < \infty$$

we obtain the desired estimate for II.

For I and III, we first check that

$$a_{k_0} \le \frac{9}{16}x < \frac{25}{16}x \le b_{k_0}.$$

Then if either $y \in (x/4, a_{k_0})$ or $y \in (b_{k_0}, 4x)$, there is a positive constant c such that

$$\mathrm{e}^{-\frac{|x^{1/2}-y^{1/2}|^2}{4s}} \le \mathrm{e}^{-c\frac{x}{s}}.$$

This estimate in (5.5), together with $y \sim x$, gives

$$\begin{aligned}\int_{x/4}^{a_{k_0}} W^2_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y &\le \left(\frac{x}{s}\right)^{1/2}\mathrm{e}^{-c\frac{x}{s}}\frac{1}{x}\int_{x/4}^{4/x} f(y)\mathrm{d}y\\ &\le C\,\mathrm{M}^{16}_{\mathrm{loc}}f(x)\end{aligned}$$

and also

$$\int_{b_{k_0}}^{4x} W^2_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y \le C\,\mathrm{M}^{16}_{\mathrm{loc}}f(x).$$

Thus we have obtained the desired estimate (5.6) with $\kappa = 16$.

The other two cases are the **global cases**, at zero (when $0 < y < x/4$) and at infinity (when $4x < y < \infty$). In both situations we have

$$\mathrm{e}^{-\frac{|x^{1/2}-y^{1/2}|^2}{4s}} \le \mathrm{e}^{-c\frac{x}{s}}\mathrm{e}^{-d\frac{y}{s}}$$

for some positive constants c and d. Consequently we get

$$W^1_{\mathcal{L}^\alpha}(s,x,y) \le C\left(\frac{1-s^2}{2s}\right)^{\alpha+1}(xy)^{\alpha/2}\mathrm{e}^{-c\frac{x}{s}}\mathrm{e}^{-d\frac{y}{s}} \tag{5.9}$$

and

$$W^2_{\mathcal{L}^\alpha}(s,x,y) \le C\,\frac{1-s^2}{2s}\left(\frac{1-s^2}{2s}(xy)^{1/2}\right)^{-1/2}\mathrm{e}^{-c\frac{x}{s}}\mathrm{e}^{-d\frac{y}{s}}. \tag{5.10}$$

Global case at zero: We consider $0 < y < x/4$. For the kernel $W^1_{\mathcal{L}^\alpha}(s,x,y)$ we have, from (5.9), that

$$\begin{aligned} W^1_{\mathcal{L}^\alpha}(s,x,y) &\le C\,\frac{1}{s^{\alpha+1}}x^{\alpha/2}y^{\alpha/2}\mathrm{e}^{-c\frac{x}{s}} \\ &= C\,x^{-\alpha/2-1}y^{\alpha/2}\left(\frac{x}{s}\right)^{\alpha+1}\mathrm{e}^{-c\frac{x}{s}} \\ &\le C\,x^{-\alpha/2-1}y^{\alpha/2}, \end{aligned}$$

since $\alpha > -1$. As for the kernel $W^2_{\mathcal{L}^\alpha}(s,x,y)$, we consider first $\alpha \ge -1/2$. In that case

$$\left(\frac{1-s^2}{2s}(xy)^{1/2}\right)^{-1/2} \le \left(\frac{1-s^2}{2s}(xy)^{1/2}\right)^{\alpha}$$

since $\frac{1-s^2}{2s}(xy)^{1/2} \ge 1$, and from (5.10) we have

$$W^2_{\mathcal{L}^\alpha}(s,x,y) \le C\,\frac{1}{s^{\alpha+1}}x^{\alpha/2}y^{\alpha/2}\mathrm{e}^{-c\frac{x}{s}} \le C\,x^{-\alpha/2-1}y^{\alpha/2}.$$

Also, if $-1 < \alpha < -1/2$, we have

$$\begin{aligned} W^2_{\mathcal{L}^\alpha}(s,x,y) &\le C\,\frac{1}{s^{1/2}}x^{-1/4}y^{-1/4}\mathrm{e}^{-c\frac{x}{s}}\mathrm{e}^{-d\frac{y}{s}} \\ &= C\,x^{-\alpha/2-1}y^{\alpha/2}\left(\frac{x}{s}\right)^{\alpha/2+3/4}\mathrm{e}^{-c\frac{x}{s}}\left(\frac{y}{s}\right)^{-\alpha/2-1/4}\mathrm{e}^{-d\frac{y}{s}} \\ &\le C\,x^{-\alpha/2-1}y^{\alpha/2}, \end{aligned}$$

where the last inequality arises since both exponents $\alpha/2 + 3/4$ and $-\alpha/2 - 1/4$ are positive.

Then, for both kernels, we obtain the same estimate leading to

$$W_{\mathcal{L}^\alpha}(s,x,y) \le C\,x^{-\alpha/2-1}y^{\alpha/2},$$

and we may conclude that

$$\int_0^{x/4} W_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y \leq C\,\mathrm{H}_0^{\alpha/2}f(x).$$

Global case at infinity: We consider $4x < y < \infty$. Analogously to the global case at zero, from (5.9) and (5.10) we obtain

$$W^1_{\mathcal{L}^\alpha}(s,x,y) \leq C\,x^{\alpha/2}y^{-\alpha/2-1}\varphi_1(s,y)$$

where

$$\varphi_1(s,y) = \left(\frac{y}{s}\right)^{\alpha+1} \mathrm{e}^{-c\frac{y}{s}},$$

and

$$W^2_{\mathcal{L}^\alpha}(s,x,y) \leq C\,x^{\alpha/2}y^{-\alpha/2-1}\varphi_1(s,y)$$

where

$$\varphi_2(s,y) = \left(\frac{y}{s}\right)^{\epsilon} \mathrm{e}^{-c\frac{y}{s}},$$

with $\epsilon = \max\{\alpha+1, \alpha/2+3/4\}$. These estimates allow us to conclude that

$$\int_{4x}^{\infty} W_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y \leq C\,\mathrm{T}_s^{\alpha/2}f(x),$$

where the function φ involved in the last operator is $\varphi_1+\varphi_2$, that clearly satisfies the requirements asked in Sect. 3, as we remarked in (3.6). □

Remark 5.1 If we keep track of the factor $1-s^2$ in the above estimates of the kernel, we would get that

$$\int_0^{\infty} W_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y \leq C(1-s)^\sigma \left(\mathrm{H}_0^{\alpha/2}f(x) + \mathrm{M}^{16}_{\mathrm{loc}}f(x) + \mathrm{T}^{\alpha/2}f(x)\right),$$

for all $0 < s < 1$, where $\sigma = \min\{\alpha+1, 1/2\}$. From the change of parameters $s = (1+\mathrm{e}^{-t/2})/(1-\mathrm{e}^{-t/2})$ we get that $1-s^2$ is equivalent to $\mathrm{e}^{-t/2}$. Thus, for all $t > 0$, we obtain

$$K^t_{\mathcal{L}^\alpha}f(x) \leq C\mathrm{e}^{-t\frac{\sigma}{2}} \left(\mathrm{H}_0^{\alpha/2}f(x) + \mathrm{M}^{16}_{\mathrm{loc}}f(x) + \mathrm{T}^{\alpha/2}f(x)\right).$$

Similar estimates for the semigroup were obtained in [7,16,8].

Proof of Theorem 2.2 We will prove that the family of integral operators with kernels $W_{\varphi^\alpha}(s,x,y)$, given by (5.2), satisfies the Proposition hypotheses (4.1) and (4.2) with $\eta = \beta = \alpha + 1/2$ and $\kappa = 4$.

In order to use the relationship (5.2), we will need some estimates obtained in the proof of Theorem 2.1. More precisely, we will use that

$$W_{\mathcal{L}^\alpha}(\sqrt{2}-1,x,y) \geq C\,(xy)^{\alpha/2}, \tag{5.11}$$

for $0 < x < 1$ and $0 < y < 1$. Also

$$\int_{x/4}^{4x} W_{\mathcal{L}^\alpha}(s,x,y)f(y)\mathrm{d}y \leq C\,\mathrm{M}^{16}_{\mathrm{loc}}f(x), \tag{5.12}$$

and for the global region

$$W_{\mathcal{L}^\alpha}(s,x,y) \leq C\,x^{-\alpha/2-1}y^{\alpha/2}, \tag{5.13}$$

with $0 < y < x/4$ and

$$W_{\mathcal{L}^\alpha}(s,x,y) \leq C\,x^{\alpha/2}y^{-\alpha/2-1}\varphi(s,y), \tag{5.14}$$

with $4x < y < \infty$, where φ is the sum of two functions like (3.6).

From relation (5.2) and inequality (5.11), we easily obtain

$$W_{\varphi^\alpha}(\sqrt{2}-1,x,y) \geq C(xy)^{\alpha+1/2}$$

for $0 < x < 1$ and $0 < y < 1$. Thus, hypothesis (4.2) holds.

To obtain (4.1), let f a be nonnegative and measurable function on $\mathbb{R}^+$. We consider, as in Theorem 2.1, three cases.

Local case: $x/2 < y < 2x$. From (5.2) we have

$$\int_{x/2}^{2x} W_{\varphi^\alpha}(s,x,y)f(y)\mathrm{d}y = 2x^{1/2}\int_{x/2}^{2x} W_{\mathcal{L}^\alpha}(s,x^2,y^2)f(y)y^{1/2}\mathrm{d}y.$$

If we introduce the change of variable $z = y^2$ we get

$$\int_{x/2}^{2x} W_{\mathcal{L}^\alpha}(s,x^2,y^2)f(y)y^{1/2}\mathrm{d}y = \frac{1}{2}\int_{x^2/4}^{4x^2} W_{\mathcal{L}^\alpha}(s,x^2,z)g(z)\mathrm{d}z,$$

where $g(z) = f(z^{1/2})z^{-1/4}$. Applying inequality (5.12) we obtain

$$\int_{x/2}^{2x} W_{\varphi^\alpha}(s,x,y)f(y)\mathrm{d}y \leq Cx^{1/2}\mathrm{M}^{16}_{\mathrm{loc}}\,g(x^2).$$

Changing variables again we can see that

$$x^{1/2}\mathrm{M}^{16}_{\mathrm{loc}}\,g(x^2) = \sup_{0<a^{1/2}<x<b^{1/2}<4a^{1/2}} \frac{1}{b-a}\int_{a^{1/2}}^{b^{1/2}} f(y)(xy)^{1/2}\mathrm{d}y$$

Since

$$b-a = (b^{1/2}+a^{1/2})(b^{1/2}-a^{1/2}) \sim x(b^{1/2}-a^{1/2})$$

and $(xy)^{1/2} \sim x$, we have that

$$x^{1/2}\mathrm{M}^{16}_{\mathrm{loc}}\,g(x^2) \leq C\,\mathrm{M}^{4}_{\mathrm{loc}}f(x).$$

Therefore,

$$\int_{x/2}^{2x} W_{\varphi^\alpha}(s,x,y)f(y)\mathrm{d}y \le C\,\mathrm{M}^4_{\mathrm{loc}}f(x).$$

Global case at zero: $0 < y < x/2$.

Since $0 < y^2 < x^2/4$, we use relation (5.2) and inequality (5.13) to obtain

$$W_{\varphi^\alpha}(s,x,y) \le Cx^{-(\alpha+1/2)-1}y^{\alpha+1/2}$$

for $0 < y < x/2$. Thus we have

$$\int_0^{x/2} W_{\varphi^\alpha}(s,x,y)f(y)\mathrm{d}y \le C\,\mathrm{H}_0^{\alpha+1/2}f(x)$$

for any $s \in (0,1)$.

Global case at infinity: $2x < y < \infty$.

Analogously to previous case, and using inequality (5.14), we obtain

$$W_{\varphi^\alpha}(s,x,y) \le Cx^{\alpha+1/2}y^{-(\alpha+1/2)-1}\tilde{\varphi}(s,y)$$

for $0 < y < x/2$, where $\tilde{\varphi}(s,y)$ is the sum of two functions of the form

$$\left(\frac{y^2}{s}\right)^{\epsilon} \mathrm{e}^{-c\frac{y^2}{s}},$$

for some positive constants ϵ and c. Since those functions are like (3.7), we have that

$$\int_{2x}^{\infty} W_{\varphi^\alpha}(s,x,y)f(y)\mathrm{d}y \le C\,\mathrm{T}_s^{\alpha+1/2}f(x).$$

Therefore, hypothesis (4.1) of Proposition 4.1 is satisfied with $\eta = \beta = \alpha + 1/2$ and $\kappa = 16$. □

Proof of Theorem 2.3 Proceeding in an analogous way to the proof of Theorem 2.2, using this time (5.3), it is easy to check that the family of integral operators with kernels $W_{\ell^\alpha}(s,x,y)$, where $s \in (0,1)$, satisfies hypotheses (4.1) and (4.2) with $\eta = 0$, $\beta = \alpha$ and $\kappa = 16$. Therefore, we obtain the desired results if we replace γ by $\delta + \alpha$ in Proposition 4.1. □

6 General weighted inequalities

In the previous section we have bounded our maximal operators by modified Hardy operators and the local maximal function pointwisely; more precisely, we have obtained (omitting constants):

$$\mathrm{W}^*_{\mathcal{L}^\alpha} \lesssim \mathrm{H}_0^{\alpha/2} + \mathrm{M}^{16}_{\mathrm{loc}} + \mathrm{H}_\infty^{\alpha/2}, \tag{6.1}$$

$$\mathrm{W}^*_{\varphi^\alpha} \lesssim \mathrm{H}_0^{\alpha+\frac{1}{2}} + \mathrm{M}^{4}_{\mathrm{loc}} + \mathrm{H}_\infty^{\alpha+\frac{1}{2}}, \tag{6.2}$$

and

$$\mathrm{W}^*_{\ell^\alpha} \lesssim \mathrm{H}^\alpha_0 + \mathrm{M}^{16}_{\mathrm{loc}} + \mathrm{H}^0_\infty. \tag{6.3}$$

In fact, we have shown a little bit stronger estimates, using the operator T^η instead of H^η_∞. But, by (3.4), the above inequalities also hold. The reason for this choice is that now we will restrict our results to strong type (p,p) and weak type $(1,1)$ inequalities, and in such cases the operators H^η_∞ and T^η behave in the same way (see Lemmas 3.2 and 3.3).

For the operator $\mathrm{M}^\kappa_{\mathrm{loc}}$, the class A^p_{loc} given in Definition 3.4 provides a characterization of weights that gives strong type (p,p), for $1 < p < \infty$, and weak type $(1,1)$. For the operators H^β_0 and H^η_∞, with $\eta > -1$ and $\beta > -1$, such characterization of weights is also known. More precisely, we quote the following facts from [3], Theorems A, B, 2, 5 and 4, respectively:

– H^β_0 is of strong type (p,p), $1 \le p < \infty$, with respect to $\omega(x)\mathrm{d}x$ if and only if

$$\sup_{r>0}\left(\int_r^\infty \omega(x)x^{-p(\beta+1)}\mathrm{d}x\right)^{1/p}\left(\int_0^r \omega(x)^{-\frac{p'}{p}}x^{p'\beta}\mathrm{d}x\right)^{1/p'} < \infty \tag{6.4}$$

– H^η_∞ is of strong type (p,p), $1 \le p < \infty$, with respect to $\omega(x)\mathrm{d}x$ if and only if

$$\sup_{r>0}\left(\int_0^r \omega(x)x^{p\eta}\mathrm{d}x\right)^{1/p}\left(\int_r^\infty \omega(x)^{-\frac{p'}{p}}x^{-p'(\eta+1)}\mathrm{d}x\right)^{1/p'} < \infty \tag{6.5}$$

– H^β_0 is of weak type $(1,1)$ with respect to $\omega(x)\mathrm{d}x$ if and only if

$$\sup_{r>0}\left(\int_r^\infty \left(\frac{r}{x}\right)^\epsilon \omega(x)x^{-\beta-1}\mathrm{d}x\right)\left(\operatorname*{ess\,sup}_{x\in(0,r)} \omega(x)^{-1}x^\beta\right) < \infty \tag{6.6}$$

for some $\epsilon > 0$.

– H^η_∞ is of weak type $(1,1)$ with respect to $\omega(x)\mathrm{d}x$ if and only if

$$\sup_{r>0}\left(\int_0^r \left(\frac{x}{r}\right)^\epsilon x^\eta\omega(x)\mathrm{d}x\right)\left(\operatorname*{ess\,sup}_{x\in(r,\infty)} \frac{1}{x^{\eta+1}\omega(x)}\right) < \infty \tag{6.7}$$

for some positive ϵ, when $\eta > 0$, or

$$\sup_{r>0} r^\eta\left(\int_0^r \omega(x)\mathrm{d}x\right)\left(\operatorname*{ess\,sup}_{x\in(r,\infty)} \frac{1}{x^{\eta+1}\omega(x)}\right) < \infty \tag{6.8}$$

when $-1 < \eta \le 0$.

We consider the following class of weights.

Definition 6.1 Let $\eta > -1$ and $\beta > -1$ such that $\eta + \beta > -1$. For $1 < p < \infty$, we say that a nonnegative weight ω belongs to class $A^{\eta,\beta}_p$ if there exists a constant C such that

$$\left(\int_a^b \omega(x)x^{p\eta}\mathrm{d}x\right)^{\frac{1}{p}} \left(\int_a^b \omega(x)^{-\frac{p'}{p}} x^{p'\beta}\mathrm{d}x\right)^{\frac{1}{p'}} \leq C \int_a^b x^{\eta+\beta}\mathrm{d}x$$

for all $0 \leq a < b < \infty$.

For $p = 1$, we say that $\omega \in A_1^{\eta,\beta}$ if there exist positive constants C and γ such that

$$\left(\int_a^b \left(\frac{a}{x} + \frac{x}{b}\right)^{\gamma} \omega(x)x^{\eta}\mathrm{d}x\right) \operatorname*{ess\,sup}_{x\in(a,b)} \omega(x)^{-1}x^{\beta} \leq C \int_a^b x^{\eta+\beta}\mathrm{d}x$$

when $\eta \neq 0$, or

$$\left(\int_a^b \omega(x)\mathrm{d}x\right) \operatorname*{ess\,sup}_{x\in(a,b)} \omega(x)^{-1}x^{\beta} \leq C \int_a^b x^{\beta}\mathrm{d}x$$

when $\eta = 0$, for all $0 \leq a < b < \infty$.

Similar classes of weights were introduced in [2]. Note that the power weights that belong to $A_p^{\eta,\beta}$ are exactly the ones that satisfy the sufficient conditions of Proposition 4.1 (a) if $1 < p < \infty$ and (d) if $p = 1$, in order to obtain strong type (p,p) and weak type $(1,1)$, respectively. Indeed, $\omega(x) = x^{\delta} \in A_p^{\eta,\beta}$ if and only if $-1 - \eta p < \delta < \beta p + p - 1$ when $1 < p < \infty$, $-1 - \eta \leq \delta \leq \beta$ when $p = 1$ and $\eta \neq 0$, or $-1 < \delta \leq \beta$ when $p = 1$ and $\eta = 0$. In this more general setting, we have the following results.

Theorem 6.1 *Let $\alpha > -1$, $1 \leq p < \infty$ and $\omega \in A_p^{\alpha/2,\alpha/2}$. Then $W^*_{\mathcal{L}^\alpha}$ is of strong type (p,p) for $p > 1$ and of weak type $(1,1)$ on $(\mathbb{R}^+, \omega(x)\mathrm{d}x)$.*

Theorem 6.2 *Let $\alpha > -1$, $1 \leq p < \infty$ and $\omega \in A_p^{\alpha+\frac{1}{2},\alpha+\frac{1}{2}}$. Then $W^*_{\varphi^\alpha}$ is of strong type (p,p) for $p > 1$ and of weak type $(1,1)$ on $(\mathbb{R}^+, \omega(x)\mathrm{d}x)$.*

Theorem 6.3 *Let $\alpha > -1$, $1 \leq p < \infty$ and $\omega(x)x^{\alpha} \in A_p^{0,\alpha}$ (or equivalently $\omega \in A_p(x^{\alpha}\mathrm{d}x)$). Then $W^*_{\ell^\alpha}$ is of strong type (p,p) for $p > 1$ and of weak type $(1,1)$ on $(\mathbb{R}^+, \omega(x)x^{\alpha}\mathrm{d}x)$.*

We shall prove the three Theorems simultaneously.

Proof We will prove that if $\omega \in A_p^{\eta,\beta}$ then ω satisfies the required conditions for the boundedness of the operators $\mathrm{M}^{\kappa}_{\mathrm{loc}}$, H^{β}_{0} and $\mathrm{H}^{\eta}_{\infty}$. Then, estimates (6.1), (6.2) and (6.3) would imply the conclusions.

Let $\omega \in A_p^{\eta,\beta}$. It is immediate to check that $\omega \in A^p_{\mathrm{loc}}$, $1 \leq p < \infty$, since $x \sim a \sim b$ on local intervals (a,b), where $0 < a < x < b < 2a$.

Next, assume $1 < p < \infty$. We will prove that ω satisfies (6.4) and (6.5). Note that $\omega \in A_p^{\eta,\beta}$ is equivalent to say $\omega(x)x^{p\eta-\eta-\beta} \in A_p(x^{\eta+\beta}\mathrm{d}x)$. Then, from the theory of A_p weights, we know that $\omega(x)x^{p\eta-\eta-\beta} \in A_q(x^{\eta+\beta}\mathrm{d}x)$, for some $1 < q < p$. This, together with Hölder's inequality, give

$$\left(\int_a^b \omega(x)x^{p\eta}\mathrm{d}x\right)\left(\int_a^b \upsilon(x)\mathrm{d}x\right)^{\frac{q}{q'}} \sim \left(\int_a^b x^{\eta+\beta}\mathrm{d}x\right)^{q} \tag{6.9}$$

for all $0 \leq a < b < \infty$, with $\upsilon(x) = \left(\omega(x)x^{p\eta-\eta-\beta}\right)^{-\frac{q'}{q}} x^{\eta+\beta}$.

Let $r > 0$. Breaking the integral into dyadic intervals, using (6.9) and that $(0, r) \subset (0, r2^k)$ for any $k \geq 0$, we obtain

$$\int_r^\infty \omega(x) x^{-p(\beta+1)} \mathrm{d}x \leq C \sum_{k=0}^\infty \left(r2^k\right)^{-p(\eta+\beta+1)} \int_0^{r2^{k+1}} \omega(x) x^{p\eta}\,\mathrm{d}x$$

$$\leq C \sum_{k=0}^\infty \left(r2^k\right)^{(q-p)(\eta+\beta+1)} \left(\int_0^r \upsilon(x)\mathrm{d}x\right)^{-\frac{q}{q'}}$$

$$\leq C\, r^{-p(\eta+\beta+1)} \int_0^r \omega(x) x^{\eta p}\,\mathrm{d}x,$$

where the last inequality arises since $\sum 2^{k(q-p)(\eta+\beta+1)} < \infty$. Therefore, using again that $\omega \in A_p^{\eta,\beta}$,

$$\left(\int_r^\infty \omega(x) x^{-p(\beta+1)} \mathrm{d}x\right)^{\frac{1}{p}} \left(\int_0^r \omega(x)^{-\frac{p'}{p}} x^{p'\beta}\,\mathrm{d}x\right)^{\frac{1}{p'}} \leq C$$

holds for all $r > 0$. Thus, condition (6.4) is satisfied.

On the other hand, let us note that also $\left(\omega(x) x^{p\eta-\eta-\beta}\right)^{-\frac{p'}{p}} \in A_{q'}(x^{\eta+\beta}\mathrm{d}x)$, for some $q' < p'$. Then we have

$$\int_a^b \omega(x)^{-\frac{p'}{p}} x^{\beta p'} \mathrm{d}x \left(\int_a^b \left(\omega(x) x^{p\eta-\eta-\beta}\right)^{\frac{p'q}{pq'}} x^{\eta+\beta}\mathrm{d}x\right)^{\frac{q'}{q}} \sim \left(\int_a^b x^{\eta+\beta}\mathrm{d}x\right)^{q'}$$

for all $0 \leq a < b < \infty$. We proceed analogously as we did before to obtain (6.5).

Consider now $p = 1$. In order to prove that (6.6), (6.7) (if $\eta > 0$) and (6.8) (if $-1 < \eta \leq 0$) hold, we will use the following facts:

If $\omega \in A_1^{\eta,\beta}$ then we have

$$b^{-\beta} \operatorname*{ess\,inf}_{x\in(b/2,b)} \omega(x) \leq C \operatorname*{ess\,inf}_{x\in(a,2a)} \omega(x) x^{-\beta} \tag{6.10}$$

and

$$a^{\eta+1} \operatorname*{ess\,inf}_{x\in(a,2a)} \omega(x) \leq C \operatorname*{ess\,inf}_{x\in(b/2,b)} \omega(x) x^{\eta+1} \tag{6.11}$$

for all positive a and b such that $b \geq 2a$.

Indeed,

$$\operatorname*{ess\,inf}_{x\in(b/2,b)} \omega(x) \leq C_\gamma\, b^{-\eta-1} \int_{b/2}^b \left(\frac{x}{b}\right)^\gamma \omega(x) x^\eta\,\mathrm{d}x$$

for any $\gamma \geq 0$. From Definition 6.9 we have

$$\int_a^b \left(\frac{x}{b}\right)^\gamma \omega(x) x^\eta\,\mathrm{d}x \leq C\, b^{\beta+\eta+1} \operatorname*{ess\,inf}_{x\in(a,2a)} \omega(x) x^{-\beta}$$

for some $\gamma > 0$ if $\eta \neq 0$ and $\gamma = 0$ when $\eta = 0$; then we get (6.10). Inequality (6.11) arises in a similar way.

We will prove now (6.6). Let $r > 0$ and $\epsilon > 0$, then

$$\int_r^\infty \left(\frac{r}{x}\right)^\epsilon \omega(x)x^{-\beta-1}\,dx \le C\sum_{k=0}^\infty 2^{-k\epsilon}(2^k r)^{-\beta-1}\int_{2^k r}^{2^{k+1}r}\omega(x)dx.$$

Since ω satisfies the A^1_{loc} condition and (6.10) holds, we have that for any $k \in \mathbb{N}_0$

$$\begin{aligned}(2^k r)^{-\beta-1}\int_{2^k r}^{2^{k+1}r}\omega(x)dx &\le C\,(2^k r)^{-\beta}\operatorname*{ess\,inf}_{x\in(2^k r,2^{k+1}r)}\omega(x)\\ &\le C\operatorname*{ess\,inf}_{x\in(s,2s)}\omega(x)x^{-\beta}\end{aligned}$$

for any $0 < s \le r$. Then, since $\epsilon > 0$ we get

$$\int_r^\infty \left(\frac{r}{x}\right)^\epsilon \omega(x)x^{-\beta-1}\,dx \le C\operatorname*{ess\,inf}_{x\in(0,r)}\omega(x)x^{-\beta}$$

and condition (6.6) is satisfied.

In order to prove (6.7), we write

$$\int_0^r \left(\frac{x}{r}\right)^\epsilon \omega(x)x^\eta\,dx \le C\sum_{k=0}^\infty 2^{-k\epsilon}\left(r2^{-k}\right)^\eta\int_{r2^{-k-1}}^{r2^{-k}}\omega(x)dx.$$

Using (6.11) and the A^1_{loc} condition we have

$$\left(r2^{-k}\right)^\eta\int_{r2^{-k-1}}^{r2^{-k}}\omega(x)dx \le C\operatorname*{ess\,inf}_{x\in(s/2,s)}\omega(x)x^{\eta+1}$$

for any $s \ge r$ and any $k \in \mathbb{N}_0$. Then we get

$$\int_0^r \left(\frac{x}{r}\right)^\epsilon \omega(x)x^\eta dx \le C\operatorname*{ess\,inf}_{x\in(r,\infty)}\omega(x)x^{\eta+1}$$

and condition (6.7) is satisfied for all η and in particular for $\eta > 0$.

For $-1 < \eta < 0$ we have, since $\omega \in A^1_{loc}$, that

$$\int_0^r \omega(x)dx \le C\sum_{k=0}^\infty r2^{-k}\operatorname*{ess\,inf}_{x\in(r2^{-k-1},r2^{-k})}\omega(x).$$

By (6.11), for all $s \ge r$ we have

$$r^\eta\int_0^r \omega(x)dx \le C\left(\sum_{k=0}^\infty 2^{k\eta}\right)\operatorname*{ess\,inf}_{x\in(s/2,s)}\omega(x)x^{\eta+1}$$

which implies

$$r^{\eta}\int_0^r \omega(x)\mathrm{d}x \le C \operatorname*{ess\,inf}_{x\in(r,\infty)} \omega(x)x^{\eta+1}$$

since $\eta < 0$. Thus, (6.8) holds for $-1 < \eta < 0$.

To prove that the above inequality also holds for $\eta = 0$, consider $r > 0$. For a.e. $x \in (r,\infty)$, there exists some $k \in \mathbb{N}_0$ such that $x \in (r2^k, r2^{k+1})$. Then

$$\begin{aligned}\omega(x)^{-1}x^{-1} &\le \left(r2^k\right)^{-\beta-1} \operatorname*{ess\,sup}_{y\in(0,r2^{k+1})} \omega(y)^{-1}y^{\beta}\\ &\le C\left(\int_0^r \omega(x)\mathrm{d}x\right)^{-1}.\end{aligned}$$

where the last inequality holds by Definition 6.9. Finally, taking ess sup over (r,∞) we obtain (6.8) for $\eta = 0$. □

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