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On the geometry of model almost complex manifolds with boundary

Abstract We study some special almost complex structures on strictly pseudoconvex domains in $\mathbb{R}^{2n}$. They appear naturally as limits under a nonisotropic scaling procedure and play a role of model objects in the geometry of almost complex manifolds with boundary. We determine explicitly some geometric invariants of these model structures and derive necessary and sufficient conditions for their integrability. As applications we prove a boundary extension and a compactness principle for some elliptic diffeomorphisms between relatively compact domains.

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Introduction and main results

In the past twenty years, symplectic geometry has been the field of many developments. For instance, M.Gromov proved the Nonsqueezing theorem, stating that there is no symplectic embedding of a ball into a "complex" cylinder with smaller radius, and A.Floer proved Arnold's conjecture on the number of fixed points for a symplectic diffeomorphism in certain manifolds, developing Morse theory on infinite-dimensional spaces. A main step in most of the recent developments in symplectic geometry relies on the existence of holomorphic discs. Given a symplectic form, the set of compatible almost complex structures is a non-empty contractible oriented manifold. As observed by M.Gromov, the space of complex curves

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in an almost complex manifold tells much information about the structure of the manifold. Symplectic invariants of the manifold appear as invariants of the cobordism class of the moduli space of holomorphic curves for any compatible almost complex structure. Underlying almost complex structures in symplectic geometry are involved, in the issue of Nijenhuis-Woolf's work [12], by geometric properties of elliptic operators. Fredholm theory provides the moduli space of holomorphic curves or spheres with a structure of an oriented manifold, and with a cobordism between moduli space of two distinct almost complex structures. One views therefore almost complex manifolds as natural manifolds for deformation theory (both of the structure and of the associated complex curves). The pertinence of this point of view relies on some compactness principle for associated complex curves. These compactness phenomena are based mainly on the Sobolev theory.

The present paper is dedicated to the study of strictly pseudoconvex domains in almost complex manifolds. This is a natural development of the previous work [3] where we proved the boundary regularity of some diffeomorphisms between strictly pseudoconvex domains in almost complex manifolds of real dimension four. The general regularity result, treated in the present paper, relies on the study of complex invariants of some special almost complex structures. Our approach is based on some deformation of almost complex manifolds with boundary. Inspired by the well-known methods of complex analysis and geometry [14], we perform non isotropic dilations, naturally associated with the geometric study of strictly convex domains in the euclidean space. The cluster set of deformed structures forms a smooth non trivial manifold of *model* almost complex structures on the euclidean space, containing the standard structure. Such nonisotropic deformations are relevant for several problems of geometric analysis on almost complex manifolds. Model structures naturally appear in the analytic study of strictly pseudoconvex domains in almost complex manifolds and also have other applications in almost complex manifolds.

In a previous paper [6] we used this method to obtain lower estimates of the Kobayashi-Royden infinitesimal metric near the boundary of a strictly pseudoconvex domain. These estimates are one of our main technical tools in the present paper. In the present paper we consider two distinct problems. The first problem affects the elliptic boundary regularity of diffeomorphisms. In the spirit of Fefferman's theorem [4] on the smooth extension of biholomorphisms between smooth strictly pseudoconvex domains, Eliashberg raised the following question. How does a symplectic diffeomorphism of the ball effect the contact structure of the sphere ? One approach consists in considering a compatible almost complex structure on the ball and study the extension of the push forward structure under the action of the diffeomorphism. This leads to an elliptic boundary regularity problem. Generically, this structure does not extend up to the sphere, since there exist symplectic diffeomorphisms which do not extend up to the sphere. However we prove that under some natural curvature conditions, the extension of this structure implies the smooth extension of the diffeomorphism up to the boundary. More precisely we have:

Theorem 0.1 *Let D and D' be two smooth relatively compact domains in real manifolds. Assume that D admits an almost complex structure J smooth on $\bar{D}$ and such that (D, J) is strictly pseudoconvex. Then a smooth diffeomorphism $f : D \to D'$ extends to a smooth diffeomorphism between $\bar{D}$ and $\bar{D}'$ if and only*

if the direct image $f_*(J)$ of J under f extends smoothly on $\bar{D}'$ and $(D', f_*(J))$ is strictly pseudoconvex.

Theorem 0.1 was proved in real dimension four in our work collaboration with B.Coupet [3]. In that situation, one can find a normalization of the structure such that the cluster set for dilated structures (note that dilations depend deeply on a choice of coordinates) is reduced to the standard integrable structure. In the general case, the manifold of model structures is non trivial, making the geometric study of model structures consistent. Thus in the present paper we give a definitive result, generalizing Fefferman's theorem (dealing with the case where D and D' are equipped with the standard structure of $\mathbb{C}^n$). Theorem 0.1 gives a criterion for the boundary extension of a diffeomorphism between two smooth manifolds, under the assumption that the source manifold admits an almost complex structure. So it can be viewed as a geometric version of the elliptic regularity.

The second problem concerns a compactness phenomenon for some diffeomorphisms. As this should be expected from the above general presentation, the study of the compactness of diffeomorphisms is transformed into the study of the compactness of induced almost complex structures, and consequently to an elliptic problem. We prove the following compactness principle:

Theorem 0.2 *Let (M, J) be an almost complex manifold, not equivalent to a model domain. Let $D = \{r < 0\}$ be a relatively compact domain in a smooth manifold N and let $(f^v)_v$ be a sequence of diffeomorphisms from M to D. Assume that*

(i) *the sequence $(J_v := f_*^v(J))_v$ extends smoothly up to $\bar{D}$ and is compact in the C^2 convergence on $\bar{D}$,*

(ii) *the Levi forms of ∂D, $\mathcal{L}^{J_v}(\partial D)$ are uniformly bounded from below (with respect to v) by a positive constant.*

Then the sequence $(f^v)_v$ is compact in the compact-open topology on M.

The paper is organized as follows. In the preliminary section one, we recall some basic notions of almost complex geometry. Section two is crucial. We introduce model almost complex structures and study their geometric properties. Section three contains a technical background necessary for the proof of Theorem 0.1. It mainly concerns properties of lifts of almost complex structures to tangent and cotangent bundles of a manifold. We use it to prove the boundary regularity of pseudoholomorhic discs attached to a totally real submanifold by means of geometric bootstrap arguments. In section four we describe nonisotropic deformations of strictly pseudoconvex almost complex manifolds with boundary. This allows to reduce the study of these manifolds to model structures of section one. In section five we prove Theorem 0.1. Our approach is inspired by the approach of Nirenberg-Webster-Yang [13], [15], [5], [18,19]. Finally in section six we prove Theorem 0.2.

1 Preliminaries

An almost complex structure on a smooth (C^∞) real $(2n)$-dimensional manifold M is a C^∞-field J of complex linear structures on the tangent bundle TM of M. We call the pair (M, J) an *almost complex manifold*. We denote by J_{st} the standard

structure in $\mathbb{R}^{2n}$ and by $\mathbb{B}$ the unit ball in $\mathbb{R}^{2n}$. An important special case of an almost complex manifold is a bounded domain D in $\mathbb{C}^n$ equipped with an almost complex structure J, defined in a neighborhood of $\bar{D}$, and sufficiently close to the standard structure J_{st} in the $\mathcal{C}^2$ norm on $\bar{D}$ and every almost complex manifold may be represented locally in such a form. More precisely, we have the following Lemma.

Lemma 1.1 *Let* (M, J) *be an almost complex manifold. Then for every point* $p \in M$ *and every* $\lambda_0 > 0$ *there exist a neighborhood* U *of* p *and a coordinate diffeomorphism* $z : U \to \mathbb{B}$ *such that* $z(p) = 0$, $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$ *and the direct image* $\hat{J} = z_*(J)$ *satisfies* $||\hat{J} - J_{st}||_{\mathcal{C}^2(\bar{\mathbb{B}})} \leq \lambda_0$.

Proof There exists a diffeomorphism z from a neighborhood U' of $p \in M$ onto $\mathbb{B}$ satisfying $z(p) = 0$ and $dz(p) \circ J(p) \circ dz^{-1}(0) = J_{st}$. For $\lambda > 0$ consider the dilation $d_\lambda : t \mapsto \lambda^{-1}t$ in $\mathbb{C}^n$ and the composition $z_\lambda = d_\lambda \circ z$. Then $\lim_{\lambda \to 0} ||(z_\lambda)_*(J) - J_{st}||_{\mathcal{C}^2(\bar{\mathbb{B}})} = 0$. Setting $U = z_\lambda^{-1}(\mathbb{B})$ for $\lambda > 0$ small enough, we obtain the desired statement. $\square$

Every complex one form w on M may be uniquely decomposed as $w = w_{(1,0)} + w_{(0,1)}$, where $w_{(1,0)} \in T^*_{(1,0)}M$ and $w_{(0,1)} \in T^*_{(0,1)}M$, with respect to the structure J. This enables to define the operators ∂_J and $\bar{\partial}_J$ on the space of smooth functions defined on M : given a complex smooth function u on M, we set $\partial_J u = du_{(1,0)}$ and $\bar{\partial}_J u = du_{(0,1)}$.

1.1 Real submanifolds in an almost complex manifold

Let Γ be a real smooth submanifold in M. We denote by $H^J(\Gamma)$ the J-holomorphic tangent bundle $T\Gamma \cap JT\Gamma$. Then Γ is *totally real* if $H^J(\Gamma) = \{0\}$ and Γ is *J-complex* if $T\Gamma = H^J(\Gamma)$.

If Γ is a real hypersurface in M defined by $\Gamma = \{r = 0\}$ and $p \in \Gamma$ then by definition

$$H^J_p(\Gamma) = \{v \in T_pM : dr(p)(v) = dr(p)(J(p)v) = 0\}$$
$$= \{v \in T_pM : \partial_J r(p)(v - iJ(p)v) = 0\}.$$

We recall the notions of the Levi form:

Definition 1.2 Let $\Gamma = \{r = 0\}$ be a smooth real hypersurface in M (r is any smooth defining function of Γ) and let $p \in \Gamma$.

(i) The *Levi form* of Γ at p is the map defined on $H^J_p(\Gamma)$ by $\mathcal{L}^J(\Gamma)(X_p) = J^\star dr[X, JX]_p$, where the vector field X is any section of the J-holomorphic tangent bundle $H^J\Gamma$ such that $X(p) = X_p$.

(ii) A real smooth hypersurface $\Gamma = \{r = 0\}$ in M is *strictly J-pseudoconvex* if its Levi form $\mathcal{L}^J(\Gamma)$ is positive definite on $H^J(\Gamma)$.

(iii) If r is a $\mathcal{C}^2$ function on M then the Levi form of r is defined on TM by $\mathcal{L}^J(r)(X) := -d(J^\star dr)(X, JX)$.

(iv) A $\mathcal{C}^2$ real valued function r on M is *J-plurisubharmonic* on M (resp. strictly J-plurisubharmonic) if and only if $\mathcal{L}^J(r)(X) \geq 0$ for every $X \in TM$ (resp. $\mathcal{L}^J(r)(X) > 0$ for every $X \in TM\setminus\{0\}$).

1.2 Local representation of holomorphic discs

A smooth map f between two almost complex manifolds (M', J') and (M, J) is holomorphic if its differential satisfies the following holomorphy condition : $df \circ J = J' \circ df$ on TM. In case $(M', J') = (\Delta, J_{st})$ the map f is called a J-holomorphic disc. We denote by ζ the complex variable in $\mathbb{C}$. In view of Lemma 1.1, the holomorphy condition is usually written as

$$\frac{\partial f}{\partial \bar{\zeta}} + Q_J(f)\frac{\partial f}{\partial \zeta} = 0,$$

where $Q = (J_{st} + J)^{-1}(J_{st} - J)$ (see [16]). However, in view of Lemma 1.1 a basis $w := (w^1, \ldots, w^n)$ of $(1, 0)$ forms on M may be locally written as $w^j = dz^j + \sum_{k=1}^{n} A_{j,k}(z, \bar{z})d\bar{z}$ where $A_{j,k}$ is a smooth function. The disc f being J-holomorphic if $f^*(w^j)$ is a $(1, 0)$ form for $j = 1, \ldots, n$ (see [1]), f satisfies the following equation on Δ:

$$\frac{\partial f}{\partial \bar{\zeta}} + A(f)\frac{\overline{\partial f}}{\partial \zeta} = 0, \tag{1.1}$$

where $A = (A_{j,k})_{1 \le j,k \le n}$. We will use this second equation to characterize the J-holomorphy in the paper.

2 Model almost complex structures

The scaling process in complex manifolds deals with deformations of domains under holomorphic transformations called dilations. The usual nonisotropic dilations in complex manifolds, associated with strictly pseudoconvex domains, provide the unit ball (after biholomorphism) as the limit domain. In almost complex manifolds dilations are generically no more holomorphic with respect to the ambiant structure. The scaling process consists in deforming both the structure and the domain. This provides, as limits, a quadratic domain and a linear deformation of the standard structure in $\mathbb{R}^{2n}$, called *model structure*. We study some invariants of such structures. Let $(x^1, y^1, \ldots, x^n, y^n) = (z^1, \ldots, z^n) = ('z, z^n)$ denote the canonical coordinates of $\mathbb{R}^{2n}$.

Definition 2.1 Let J be an almost complex structure on $\mathbb{C}^n$. We call J a model structure if $J(z) = J_{st} + L(z)$ where L is given by a linear matrix $L = (L_{j,k})_{1 \le j,k \le 2n}$ such that $L_{j,k} = 0$ for $1 \le j \le 2n - 2$, $1 \le k \le 2n$, $L_{j,k} = 0$ for $j, k = 2n - 1, 2n$ and $L_{j,k} = \sum_{l=1}^{n-1} \left(a_l^{j,k} z^l + \bar{a}_l^{j,k} \bar{z}^l \right), a_l^{j,k} \in \mathbb{C}$, for $j = 2n - 1, 2n$ and $k = 1, \cdots, 2n - 2$.

The complexification $J_{\mathbb{C}}$ of a model structure J can be written as a $(2n \times 2n)$ complex matrix

$$J_{\mathbb{C}} = \begin{pmatrix} i & 0 & 0 & 0 & \cdots & 0 & 0 \\ 0 & -i & 0 & 0 & \cdots & 0 & 0 \\ 0 & 0 & i & 0 & \cdots & 0 & 0 \\ 0 & 0 & 0 & -i & \cdots & 0 & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & \tilde{L}_{2n-1,2} & 0 & \tilde{L}_{2n-1,4} & \cdots & i & 0 \\ \tilde{L}_{2n,1} & 0 & \tilde{L}_{2n,3} & 0 & \cdots & 0 & -i \end{pmatrix}, \qquad (2.1)$$

where $\tilde{L}_{2n-1,2k}(z,\bar{z}) = \sum_{l=1,\; l\neq k}^{n-1}(\alpha_l^k z^l + \beta_l^k \bar{z}^l)$ with $\alpha_l^k,\; \beta_l^k \in \mathbb{C}$. Moreover, $\tilde{L}_{2n,2k-1} = \overline{\tilde{L}_{2n-1,2k}}$.

With a model structure we associate model domains.

Definition 2.2 Let J be a model structure on $\mathbb{C}^n$ and $D = \{z \in \mathbb{C}^n : Re z^n + P_2('z, '\bar{z}) < 0\}$, where P_2 is homogeneous second degree real polynomial on $\mathbb{C}^{n-1}$. The pair (D, J) is called a *model domain* if D is strictly J-pseudoconvex in a neighborhood of the origin.

The aim of this Section is to define the complex hypersurfaces for model structures in $\mathbb{R}^{2n}$.

Let J be a model structure on $\mathbb{R}^{2n}$ and let N be a germ of a J-complex hypersurface in $\mathbb{R}^{2n}$.

Propostion 2.3

(i) *The model structure J is integrable if and only if $\tilde{L}_{2n-1,j}$ satisfies the compatibility conditions*

$$\frac{\partial \tilde{L}_{2n-1,2k}}{\partial \bar{z}^j} = \frac{\partial \tilde{L}_{2n-1,2j}}{\partial \bar{z}^k}$$

for every $1 \leq j, k \leq n-1$.
In that case there exists a global diffeomorphism of $\mathbb{R}^{2n}$ which is (J, J_{st}) holomorphic. In that case the germs of any J-complex hypersurface are given by one of the two following forms:
   (a) $N = A \times \mathbb{C}$ *where A is a germ of a J_{st}-complex hypersurface in $\mathbb{C}^{n-1}$,*
   (b) $N = \{('z, z^n) \in \mathbb{C}^n : z^n = \frac{i}{4}\sum_{j=1}^{n-1}\bar{z}^j \tilde{L}_{2n-1,2j}('z, '\bar{z}) + \frac{i}{4}\sum_{j=1}^{n-1}\bar{z}^j$
   $\tilde{L}_{2n-1,2j}('z, 0)) + \tilde{\varphi}('z)\}$ *where $\tilde{\varphi}$ is a holomorphic function locally defined in $\mathbb{C}^{n-1}$.*
(ii) *If J is not integrable then $N = A \times \mathbb{C}$ where A is a germ of a J_{st}-complex hypersurface in $\mathbb{C}^{n-1}$.*

Proof of Proposition 2.3 Let N be a germ of a J-complex hypersurface in $\mathbb{R}^{2n}$. If $\pi : \mathbb{R}^{2n} \to \mathbb{R}^{2n-2}$ is the projection on the $(2n-2)$ first variables, it follows from Definition 2.1, or similarly from condition (2.1) that $\pi(T_z N)$ is a J_{st}-complex hypersurface in $\mathbb{C}^{n-1}$.

It follows that either $dim_{\mathbb{C}}\pi(N) = n-1$ or $dim_{\mathbb{C}}\pi(N) = n-2$.
Case one : $dim_{\mathbb{C}}\pi(N) = n-1$. We prove the following Lemma:

Lemma 2.4 *There is a local holomorphic function $\tilde{\varphi}$ in $\mathbb{C}^{n-1}$ such that $N = \{('z, z^n) : z^n = \frac{i}{4}\sum_{j=1}^{n-1}\bar{z}^j \tilde{L}_{2n-1,2j}('z, '\bar{z}) + \frac{i}{4}\sum_{j=1}^{n-1}\bar{z}^j \tilde{L}_{2n-1,2j}('z, 0)) + \tilde{\varphi}('z)\}$.*

Proof of Lemma 2.4 A germ N can be represented as a graph $N = \{z^n = \varphi('z, '\bar{z})\}$ where φ is a smooth local complex function. Hence $T_z N = \left\{v_n = \sum_{j=1}^{n-1} \left(\frac{\partial \varphi}{\partial z^j}('z)v_j + \frac{\partial \varphi}{\partial \bar{z}^j}('z)\bar{v}_j\right)\right\}$. A vector $v = (x^1, y^1, \ldots, x^n, y^n)$ belongs to $T_z N$ if and only if the complex components $v^1 := x^1 + iy^1, \ldots, v^n := x^n + iy^n$ satisfy

$$i v_n = i \sum_{j=1}^{n-1} \left(\frac{\partial \varphi}{\partial z^j}('z)v_j + \frac{\partial \varphi}{\partial \bar{z}^j}('z)\bar{v}_j\right). \tag{2.2}$$

Similarly, the vector $J_z v$ belongs to $T_z N$ if and only if

$$\sum_{j=1}^{n-1} \tilde{L}_{2n,2j-1}('z)\bar{v}_j + i v_n = i \left(\sum_{j=1}^{n-1} \frac{\partial \varphi}{\partial z^j}('z)v_j - \sum_{j=1}^{n-1} \frac{\partial \varphi}{\partial \bar{z}^j}('z)\bar{v}_j\right). \tag{2.3}$$

It follows from (2.2) and (2.3) that N is J-complex if and only if

$$\sum_{j=1}^{n-1} \left(\tilde{L}_{2n,2j-1}('z)\bar{v}_j + 2i \frac{\partial \varphi}{\partial \bar{z}^j}(z)\bar{v}_j\right) = 0$$

for every $'v \in \mathbb{C}^{n-1}$, or equivalently if and only if

$$\tilde{L}_{2n,2j-1} = -2i \frac{\partial \varphi}{\partial \bar{z}^j}$$

for every $j = 1, \ldots, n - 1$. This last condition is equivalent to the compatibility conditions

$$\frac{\partial \tilde{L}_{2n,2j-1}}{\partial \bar{z}^k} = \frac{\partial \tilde{L}_{2n,2k-1}}{\partial \bar{z}^j} \text{ for } j, k = 1, \ldots, n - 1. \tag{2.4}$$

In that case there exists a local holomorphic function $\tilde{\varphi}$ in $\mathbb{C}^{n-1}$ such that

$$\varphi\left('z, '\bar{z}\right) = \frac{i}{2} \sum_{j=1}^{n-1} \bar{z}^j \left(\sum_{k \neq j} \alpha_k^j z^k\right) - \frac{i}{2} \sum_{j=1}^{n-2} \bar{z}^j \left(\sum_{k > j} \beta_k^j \bar{z}^k\right) + \tilde{\varphi}('z),$$

meaning that such J-complex hypersurfaces are parametrized by holomorphic functions in the variables $'z$. Moreover we can rewrite φ as

$$\varphi('z, '\bar{z}) = \frac{i}{4} \sum_{j=1}^{n-1} \bar{z}^j \tilde{L}_{2n-1,2j}('z, '\bar{z}) + \frac{i}{4} \sum_{j=1}^{n-1} \bar{z}^j \tilde{L}_{2n-1,2j}('z, 0)) + \tilde{\varphi}('z).$$

$\square$

We also have the following

Lemma 2.5 *The* $(1, 0)$ *forms of* J *have the form* $\omega = \sum_{k=1}^{n} c_k dz^k - \frac{i}{2} c_n \sum_{k=1}^{n-1} \tilde{L}_{2n-1,2k} d\bar{z}^k$ *with complex numbers* $c_1, \ldots, c_n$.

Proof of Lemma 2.5 Let $X = \sum_{k=1}^{n} \left(x_k \frac{\partial}{\partial z^k} + y_k \frac{\partial}{\partial \bar{z}^k} \right)$ be a $(0, 1)$ vector field. In view of (2.1), we have:

$$J_{\mathbb{C}}(X) = -iX \Leftrightarrow \begin{cases} x_k = 0, \quad \text{for } k = 1, \ldots, n-1 \\ x_n = \frac{i}{2} \sum_{k=1}^{n-1} y_k \tilde{L}_{2n-1,2k}. \end{cases}$$

Hence the $(0, 1)$ vector fields are given by

$$X = \sum_{k=1}^{n} y_k \frac{\partial}{\partial \bar{z}^k} + \frac{i}{2} \frac{\partial}{\partial z^n} \sum_{k=1}^{n-1} y_k \tilde{L}_{2n-1,2k}.$$

A $(1, 0)$ form $\omega = \sum_{k=1}^{n} (c_k dz^k + d_n d\bar{z}^k)$ satisfying $\omega(X) = 0$ for every $(0, 1)$ vector field X it satisfies $d_n = 0$ and $d_k + (i/2)c_n \tilde{L}_{2n-1,2k} = 0$ for every $k = 1, \ldots, n-1$. This gives the desired form for the $(1, 0)$ forms on $\mathbb{C}^n$. $\square$

Consider now the global diffeomorphism of $\mathbb{C}^n$ defined by

$$F('z, z^n) = \left('z, z^n - \frac{i}{4} \sum_{j=1}^{n-1} \bar{z}^j \tilde{L}_{2n-1,2j}('z,'\bar{z}) - \frac{i}{4} \sum_{j=1}^{n-1} \bar{z}^j \tilde{L}_{2n-1,2j}('z, 0) \right).$$

The map F is (J, J_{st}) holomorphic if and only if $F^*(dz^k)$ is a $(1, 0)$ form with respect to J, for every $k = 1, \ldots, n$.

Then $F^*(dz^k) = dz^k$ for $k = 1, \ldots, n-1$ and

$$F^*(dz^n) = dz^n + \sum_{k=1}^{n-1} \frac{\partial F_n}{\partial z^k} dz^k + \sum_{k=1}^{n-1} \frac{\partial F_n}{\partial \bar{z}^k} d\bar{z}^k$$

$$= dz^n + \sum_{k=1}^{n-1} \frac{\partial F_n}{\partial z^k} dz^k$$

$$- \frac{i}{4} \sum_{k=1}^{n-1} (\tilde{L}_{2n-1,2k}('z,'\bar{z})$$

$$+ \sum_{j \neq k} \bar{z}^j \frac{\partial \tilde{L}_{2n-1,2j}}{\partial \bar{z}^k} ('z,'\bar{z}) + \tilde{L}_{2n-1,2k}('z,'0)) d\bar{z}^k.$$

By the compatibility condition (2.4) we have

$$F^*(dz^n) = dz^n + \sum_{k=1}^{n-1} \frac{\partial F_n}{\partial \bar{z}^k} dz^k - \frac{i}{4} \sum_{k=1}^{n-1} (\tilde{L}_{2n-1,2k}('z,'\bar{z})$$

$$+ \tilde{L}_{2n-1,2k}('0,'\bar{z}) + \tilde{L}_{2n-1,2k}('z,'0)) d\bar{z}^k$$

$$= dz^n - \frac{i}{2} \sum_{k=1}^{n-1} \tilde{L}_{2n-1,2k}('z,'\bar{z}) d\bar{z}^k + \sum_{k=1}^{n-1} \frac{\partial F_n}{\partial z^k} dz^k.$$

These equalities mean that F is a local (J, J_{st})-biholomorphism of $\mathbb{C}^n$, and so that J is integrable.

Case two : $dim_{\mathbb{C}}\pi(N) = n-2$. In that case we may write $N = \pi(N) \times \mathbb{C}$, meaning that J-complex hypersurfaces are parametrized by J_{st}-complex hypersurfaces of $\mathbb{C}^{n-1}$.

We can conclude now the proof of Proposition 2.3. We proved in Case one that if there exists a J-complex hypersurface in $\mathbb{C}^n$ such that $dim\pi(N) = n-1$ (this is equivalent to the compatibility conditions (2.4)) then J is integrable. Conversely, it is immediate that if J is integrable then there exists a J-complex hypersurface whose form is given by Lemma 2.4 and hence that the compatibility conditions (2.4) are satisfied. This gives part (i) of Proposition 2.3.

To prove part (ii), we note that if J is not integrable then in view of part (i) the form of any J-complex hypersurface is given by Case two. □

3 Almost complex structures and totally real submanifolds

We recall that if (M, J) is an almost complex manifold then a submanifold N of M is *totally real* if $TN \cap J(TN) = \{0\}$.

3.1 Conormal bundle of a submanifold in (M, J)

The conormal bundle of a strictly J-pseudoconvex hypersurface in M provides an important example of a totally real submanifold in the cotangent bundle T^*M. More precisely, following Sato (see [21]) let $\widehat{J}$ denote the complete lift of J. If J has components $J_i{}^h$ then, in the matrix form we have

$$\widehat{J} = \begin{pmatrix} J_i{}^h & 0 \\ p_a \left(\dfrac{\partial J_i{}^a}{\partial x^j} - \dfrac{\partial J_j{}^a}{\partial x^i} \right) & J_h{}^i \end{pmatrix},$$

relative to local canonical coordinates (x, p) on T^*M.

Let N denote the Nijenhuis tensor and let $\gamma(NJ)$ be the $(1, 1)$ tensor on T^*M defined in coordinates (x, p) by:

$$\gamma(NJ) = \begin{pmatrix} 0 & 0 \\ p_a(NJ)_{ji}^a & 0 \end{pmatrix}.$$

Then the $(1, 1)$ tensor defined on T^*M by $\tilde{J} := \widehat{J} + (1/2)\gamma(NJ)$ defines an almost complex structure on the cotangent bundle T^*M. Moreover, if f is a biholomorphism between (M, J) and (M', J') then the cotangent map $\tilde{f} := (f, {}^t df^{-1})$ is a biholomorphism between $(T^*M, \tilde{J})$ and $(T^*M', \tilde{J}')$.

If Γ is a real submanifold in M, the conormal bundle $\Sigma_J(\Gamma)$ of Γ is the real subbundle of $T^*_{(1,0)}M$ defined by $\Sigma_J(\Gamma) = \left\{ \phi \in T^*_{(1,0)}M : Re\,\phi|_{T\Gamma} = 0 \right\}$. One can identify the conormal bundle $\Sigma_J(\Gamma)$ of Γ with any of the following subbundles of $T^*M : N_1(\Gamma) = \left\{ \varphi \in T^*M : \varphi|_{T\Gamma} = 0 \right\}$ and $N_2(\Gamma) = \left\{ \varphi \in T^*M : \varphi|_{JT\Gamma} = 0 \right\}$.

Propostion 3.1 *Let Γ be a $\mathcal{C}^2$ real hypersurface in (M, J). If Γ is strictly J-pseudoconvex, then the bundles $N_1(\Gamma)$ and $N_2(\Gamma)$ (except the zero section) are totally real submanifolds of dimension $2n$ in T^*M equipped with $\tilde{J}$.*

Proposition 3.1 is due to A.Tumanov [19] in the integrable case. The question whether a similar result was true in the almost complex case was asked by the second author to A.Spiro who gave a positive answer [17]. For completeness we give an alternative proof of this fact.

Proof of Proposition 3.1 Let $x_0 \in \Gamma$. We consider local coordinates (x, p) for the real cotangent bundle T^*M of M in a neighborhood of x_0. The fiber of $N_2(\Gamma)$ is given by $c(x)J^*d\rho(x)$, where c is a real nonvanishing function. In what follows we denote $-J^*d\rho$ by $d_J^c\rho$. For every $\varphi \in N_2(\Gamma)$ we have $\varphi|_{J(T\Gamma)} \equiv 0$. It is equivalenty to prove that $N_1(\Gamma)$ is totally real in $(T^*M, \tilde{J})$ or that $N_2(\Gamma)$ is totally real in $(T^*M, \tilde{J})$. We recall that if $\Theta = p_i dx^i$ in local coordinates then $d\Theta$ defines the canonical symplectic form on T^*M. If $V, W \in T(N_2(\Gamma))$ then $d\Theta(V, W) = 0$. Indeed the projection $pr_1(V)$ of V (resp. W) on M is in $J(T\Gamma)$ and the projection of V (resp. W) on the fiber annihilates $J(T\Gamma)$ by definition. It follows that $N_2(\Gamma)$ is a Lagrangian submanifold of T^*M for this symplectic form.

Let V be a vector field in $T(N_2(\Gamma)) \cap \tilde{J}T(N_2(\Gamma))$. We wish to prove that $V = 0$. According to what preceeds we have $d\Theta(V, W) = d\Theta(JV, W) = 0$ for every $W \in T(N_2(\Gamma))$. We restrict to W such that $pr_1(W) \in T\Gamma \cap J(T\Gamma)$. Since Θ is defined over $x_0 \in \Gamma$ by $\Theta = cd_J^c\rho$, then $d\Theta = dc \wedge d_J^c\rho + cdd_J^c\rho$. Since $d_J^c\rho(pr_1(V)) = d_J^c\rho(Jpr_1(V)) = d_J^c\rho(pr_1(V)) = d_J^c\rho(Jpr_1(V)) = 0$ it follows that $dd_J^c\rho(pr_1(V), pr_1(\tilde{J}W)) = 0$. However, by the definition of $\tilde{J}$, we know that $pr_1(\tilde{J}W) = Jpr_1(W)$. Hence, choosing $W = V$, we obtain that $dd_J^c\rho(pr_1(V), Jpr_1(V)) = 0$. Since Γ is strictly J-pseudoconvex, it follows that $pr_1(V) = 0$. In particular, V is given in local coordinates by $V = (0, pr_2(V))$. It follows now from the form of $\tilde{J}$ that $JV = (0, Jpr_2(V))$ (we consider $pr_2(V)$ as a vector in $\mathbb{R}^{2n}$ and J defined on $\mathbb{R}^{2n}$). Since $N_2(\Gamma)$ is a real bundle of rank one, then $pr_2(V)$ is equal to zero. $\square$

3.2 Boundary regularity of J-holomorphic discs

Many geometric questions in complex analysis or in CR geometry reduce to the study of the properties of holomorphic discs. Among these the boundary regularity of holomorphic discs attached to a totally real submanifold appeared as one of the essential tools in the understanding of extension phenomena. In the almost complex setting, this is stated by H.Hofer [8] (refered to a bootstrap argument), and a weaker regularity is proved by E.Chirka [1] and by S.Ivashkovich-V.Shevchischin [9]. This can be formulated as follows:

Propostion 3.2 *Let N be a smooth $\mathcal{C}^\infty$ totally real submanifold in (M, J) and let $\varphi : \Delta^+ \to M$ be J-holomorphic. Assume that the cluster set of φ on the real interval $]-1, 1[$ is contained in N. Then φ is of class $\mathcal{C}^\infty$ on $\Delta^+ \cup]-1, 1[$.*

Here Δ denotes the unit disc in $\mathbb{C}$ and $\Delta^+ := \{\zeta \in \Delta : Im(\zeta) > 0\}$.

In case N has a weaker regularity then the exact regularity of φ, related to that of N, can be derived directly from the following proof of Proposition 3.2.

Proof of Proposition 3.2 We proceed in three steps, using a geometric bootstrap argument.

Step one : 1/2-Hölder continuity . Since N is totally real, using a partition of unity, we may represent N as the zero set of the positive, smooth, strictly J-plurisubharmonic function ρ (see the details in [3]).

As usual we denote by $C(\varphi,] - 1, 1[)$ the cluster set of φ on $] - 1, 1[$; this consists of points $p \in M$ such that $p = \lim_{k \to \infty} \varphi(\zeta_k)$ for a sequence $(\zeta_k)_k$ in Δ^+ converging to a point in $] - 1, 1[$. The 1/2-Hölder extension of φ to $\Delta^+ \cup] - 1, 1[$ is contained in the following proposition (see Proposition 4.1 in [3] for its proof).

Propostion 3.3 *Let G be a relatively compact domain in (M, J) and let ρ be a strictly J-plurisubharmonic function of class $\mathcal{C}^2$ on $\bar{G}$. Let $\varphi : \Delta^+ \to G$ be a J-holomorphic disc such that $\rho \circ \varphi \geq 0$ on Δ^+. Suppose that $C(g,] - 1, 1[)$ is contained in the zero set of ρ. Then φ extends as a Hölder 1/2-continuous map on $\Delta^+ \cup] - 1, 1[$.*

Step two : The disc φ is of class $\mathcal{C}^{1+1/2}$. The following construction of the reflection principle for pseudoholomorphic discs is due to Chirka [1]. For reader's convenience we give the details. Let $a \in] - 1, 1[$. Our consideration being local at a, we may assume that $N = \mathbb{R}^n \subset \mathbb{C}^n, a = 0$ and J is a smooth almost complex structure defined in the unit ball $\mathbb{B}_n$ in $\mathbb{C}^n$.

After a complex linear change of coordinates we may assume that $J = J_{st} + O(|z|)$ and N is given by $x + ih(x)$ where $x \in \mathbb{R}^n$ and $dh(0) = 0$. If Φ is the local diffeomorphism $x \mapsto x, y \mapsto y - h(x)$ then $\Phi(N) = \mathbb{R}^n$ and the direct image of J by Φ, still denoted by J, keeps the form $J_{st} + O(|z|)$. Then J has a basis of $(1, 0)$-forms given in the coordinates z by $dz^j + \sum_k a_{jk} d\bar{z}^k$; using the matrix notation we write it in the form $\omega = dz + A(z)d\bar{z}$ where the matrix function $A(z)$ vanishes at the origin. Writing $\omega = (I + A)dx + i(I - A)dy$ where I denotes the identity matrix, we can take as a basis of $(1, 0)$ forms : $\omega' = dx + i(I + A)^{-1}(I - A)dy = dx + iBdy$. Here the matrix function B satisfies $B(0) = I$. Since B is smooth, its restriction $B_{|\mathbb{R}^n}$ on $\mathbb{R}^n$ admits a smooth extension $\hat{B}$ on the unit ball such that $\hat{B} - B_{|\mathbb{R}^n} = O(|y|^k)$ for any positive integer k. Consider the diffeomorphism $z^* = x + i\hat{B}(z)y$. In the z^*-coordinates the submanifold N still coincides with $\mathbb{R}^n$ and $\omega' = dx + iBdy = dz^* + i(B - \hat{B})dy - i(d\hat{B})y = dz^* + \alpha$, where the coefficients of the form α vanish upto first order on $\mathbb{R}^n$. Therefore there is a basis of $(1, 0)$-forms (with respect to the image of J under the coordinate diffeomorphism $z \mapsto z^*$) of the form $dz^* + A(z^*)d\bar{z}^*$, where A vanishes to first order on $\mathbb{R}^n$ and $\|A\|_{\mathcal{C}^1(\bar{\mathbb{B}}_n)} << 1$.

Consider the continuous map ψ defined on Δ by

$$
\begin{cases}
\psi = \varphi & \text{on } \Delta^+ \\
\\
\psi(\zeta) = \overline{\varphi(\bar{\zeta})} & \text{for } \zeta \in \Delta^-.
\end{cases}
$$

Since, in view of (1.1) the map φ satisfies

$$\bar{\partial}\varphi + A(\varphi)\overline{\partial\varphi} = 0 \tag{3.1}$$

on Δ^+, the map ψ satisfies the equation

$$\bar{\partial}\psi(\zeta) + \overline{A(\varphi(\bar{\zeta}))}\,\overline{\partial\psi(\zeta)} = 0$$

for $\zeta \in \Delta^-$.

Hence ψ is a solution on Δ of the elliptic equation

$$\bar{\partial}\psi + \lambda(\cdot)\overline{\partial\psi} = 0 \tag{3.2}$$

where λ is defined by $\lambda(\zeta) = A(\varphi(\zeta))$ for $\zeta \in \Delta^+\cup] -1, 1[$ and $\lambda(\zeta) = \overline{A(\varphi(\bar{\zeta}))}$ for $\zeta \in \Delta^-$. According to Step one, the map λ is Hölder $1/2$ continuous on Δ and vanishes on $]-1, 1[$. This implies that ψ is of class $\mathcal{C}^{1+1/2}$ on Δ by equation (3.2) (see [16,20]).

Step three : geometric bootstrap. Let $v = (1, 0)$ in $\mathbb{R}^2$ and consider the disc φ^c defined on Δ^+ by

$$\varphi^c(\zeta) = (\varphi(\zeta), d\varphi(\zeta)(v)).$$

We endow the tangent bundle TM with the complete lift J^c of J (see [21] for its definition). We recall that J^c is an almost complex structure on TM. Moreover, if ∇ is any J complex connection on M (ie $\nabla J = 0$) and $\bar{\nabla}$ is the connection defined on M by $\bar{\nabla}_X Y = \nabla_Y X + [X, Y]$ then J^c is the horizontal lift of J with respect to $\bar{\nabla}$. Another definition of J^c is given in [10] where this is characterized by a deformation property. The equality between the two definitions given in [21] and in [10] is obtained by their (equal) expression in the local canonical coordinates on TM:

$$J^c = \begin{pmatrix} J_i^h & (0) \\ t^a \partial_a J_i^h & J_i^h \end{pmatrix}.$$

(Here t^a are fibers coordinates).

Lemma 3.4 *TN is a totally real submanifold in (TM, J^c).*

Proof of Lemma 3.4 Let $X \in T(TN) \cap J^c(T(TN))$. If $X = (u, v)$ in the trivialisation $T(TM) = TM \oplus TM$ then $u \in TN \cap J(TN)$, implying that $u = 0$. Hence $v \in TN \cap J(TN)$, implying that $v = 0$. Finally, $X = 0$. □

The cluster set $C(\varphi^c,]-1, 1[)$ is contained in the smooth submanifold TN of TM. Applying Step two to φ^c and TN we prove that the first derivative of φ with respect to x ($x + iy$ are the standard coordinates on $\mathbb{C}$) is of class $\mathcal{C}^{1+1/2}$ on $\Delta^+\cup]-1, 1[$. The J-holomorphy equation (3.1) may be written as

$$\frac{\partial\varphi}{\partial y} = J(\varphi)\frac{\partial\varphi}{\partial x}$$

on $\Delta^+\cup]-1, 1[$. Hence $\partial\varphi/\partial y$ is of class $\mathcal{C}^{1+1/2}$ on $\Delta^+\cup]-1, 1[$, meaning that φ is of class $\mathcal{C}^{2+1/2}$ on $\Delta^+\cup]-1, 1[$. We prove now that φ is of class $\mathcal{C}^{3+1/2}$ on

$\Delta^+\cup] - 1, 1[$. The reader will conclude, repeating the same argument that φ is of class $\mathcal{C}^\infty$ on $\Delta^+\cup] - 1, 1[$.

Replace now the data (M, J) and φ by (TM, J^c) and φ^c in Step three. The map $^2\varphi^c$ defined on Δ^+ by $^2\varphi^c(\zeta) = (\varphi^c(\zeta), d\varphi^c(\zeta)(v))$ is $^2J^c$-holomorphic on Δ^+ ($^2J^c$ is the complete lift of J^c to the second tangent bundle $T(TM)$). According to Step two, its first derivative $\partial(^2\varphi^c)/\partial x$ is of class $C^{1+1/2}$ on $\Delta^+\cup] - 1, 1[$. This means that the second derivatives $\dfrac{\partial^2\varphi}{\partial x^2}$ and $\dfrac{\partial^2\varphi}{\partial x\partial y}$ are $C^{1+1/2}$ on $\Delta^+\cup] - 1, 1[$.

Differentiating equation (3.1) with respect to y, we prove that $\dfrac{\partial^2\varphi}{\partial y^2}$ is $C^{1+1/2}$ on $\Delta^+\cup] - 1, 1[$ and so that φ is $C^{3+1/2}$ on $\Delta^+\cup] - 1, 1[$. $\qquad\square$

3.3 Boundary regularity of diffeomorphisms in wedges

Let Γ and Γ' be two totally real maximal submanifolds in almost complex manifodls (M, J) and (M', J'). Let $W(\Gamma, M)$ be a wedge in M with edge Γ.

Propostion 3.5 *If $F : W(\Gamma, M) \to M'$ is (J, J')-holomorphic and if the cluster set of Γ is contained in Γ' then F extends as a $\mathcal{C}^\infty$ map up to Γ.*

Proof of Proposition 3.5 In view of Proposition 3.2 the proof is classical (see [3]). $\qquad\square$

As a direct application of Proposition 3.5 we obtain the following partial version of Fefferman's Theorem:

Corollary 3.6 *Let D and D' be two smooth relatively compact domains in real manifolds. Assume that D admits an almost complex structure J smooth on $\bar{D}$ and such that (D, J) is strictly pseudoconvex. Let f be a smooth diffeomorphism $f : D \to D'$, extending as a $\mathcal{C}^1$ diffeomorphism (still called f) between $\bar{D}$ and $\bar{D}'$. Then f is a smooth $\mathcal{C}^\infty$ diffeomorphism between $\bar{D}$ and $\bar{D}'$ if and only if the direct image $f_*(J)$ of J under f extends smoothly on $\bar{D}'$ and $(D', f_*(J))$ is strictly pseudoconvex.*

Proof of Corollary 3.6 The cotangent lift f^* of f to the cotangent bundle over D, locally defined by $f^* := (f, {}^t(df)^{-1})$, is a $(\tilde{J}, \tilde{J}')$-biholomorphism from T^*D to T^*D', where $J' := f_*(J)$. According to Proposition 3.1, the conormal bundle $\Sigma(\partial D)$ (resp. $\Sigma(\partial D')$) is a totally real submanifold in T^*M (resp. T^*M'). We consider $\Sigma(\partial D)$ as the edge of a wedge $W(\Sigma(\partial D), M)$ contained in TD. Then we may apply Proposition 3.5 to $F = f^*$ to conclude. $\qquad\square$

4 Boundary estimates and the scaling process

Our further considerations rely deeply on the following estimates of the Kobayashi-Royden infinitesimal pseudometric obtained in [6] :

Proposition A. *Let $D = \{\rho < 0\}$ be a relatively compact domain in an almost complex manifold (M, J), where ρ is a C^2 defining function of D, strictly J-plurisubharmonic in a neighborhood of $\bar{D}$. Then there exist positive constants c and λ_0 such that for every almost complex structure J' defined in a neighborhood of $\bar{D}$ and such that $\|J' - J\|_{C^2(\bar{D})} \leq \lambda_0$ we have :*

$$K_{(D,J')}(p, v) \geq c \left[\frac{|\partial_J \rho(p)(v - iJ(p)v)|^2}{|\rho(p)|^2} + \frac{\|v\|^2}{|\rho(p)|} \right]^{1/2}, \qquad (4.1)$$

for every $p \in D$ and every $v \in T_p M$.

Let D (resp. D') be a strictly pseudoconvex domain in an almost complex manifold (M, J) (resp. (M', J')) and let f be a (J, J')-biholomorphism from D to D'. Fix a point $p \in \partial D$ and a sequence $(p^k)_k$ in D converging to p. After extraction we may assume that the sequence $(f(p^k))_k$ converges to a point p' in $\partial D'$. According to the Hopf lemma, f has the boundary distance property. Namely, there is a positive constant C such that

$$(1/A)\, dist(f(p^k), \partial D') \leq dist(p^k, \partial D) \leq A\, dist(f(p^k), \partial D'), \quad (4.2)$$

where A is independent of k (see [3]).

Since all our considerations are local we set $p = p' = 0 \in \mathbb{C}^n$. We may assume that $J(0) = J_{st}$ and $J'(0) = J_{st}$. Let U (resp. V) be a neighborhood of the origin in $\mathbb{C}^n$ such that $D \cap U = \{z \in U : \rho(z, \bar{z}) := z_n + \bar{z}_n + Re(K(z)) + H(z, \bar{z}) + \cdots < 0\}$ (resp. $D' \cap V = \{w \in V : \rho'(w, \bar{w}) := w_n + \bar{w}_n + Re(K'(w)) + H'(w, \bar{w}) + \cdots < 0\}$) where $K(z) = \sum k_{\nu\mu} z^\nu z^\mu$, $k_{\nu\mu} = k_{\mu\nu}$, $H(z) = \sum h_{\nu\mu} z^\nu \bar{z}^\mu$, $h_{\nu\mu} = \bar{h}_{\mu\nu}$ and ρ is a strictly J-plurisubharmonic function on U (resp. $K'(z) = \sum k'_{\nu\mu} w^\nu w^\mu$, $k'_{\nu\mu} = k'_{\mu\nu}$, $H'(w) = \sum h'_{\nu\mu} w^\nu \bar{w}^\mu$, $h'_{\nu\mu} = \bar{h}'_{\mu\nu}$ and ρ' is a strictly J'-plurisubharmonic function on V).

4.1 Asymptotic behaviour of the tangent map of f

We wish to understand the limit behaviour (when $k \to \infty$) of $df(p^k)$. Consider the vector fields

$$v^j := (\partial\rho/\partial x^n)\partial/\partial x^j - (\partial\rho/\partial x^j)\partial/\partial x^n$$

for $j = 1, \ldots, n - 1$, and

$$v^n := (\partial\rho/\partial x^n)\partial/\partial y^n - (\partial\rho/\partial y^n)\partial/\partial x^n.$$

Restricting U if necessary, the vector fields $X^1, \ldots, X^{n-1}$ defined by $X^j := v^j - iJ(v^j)$ form a basis of the J-holomorphic tangent space to $\{\rho = \rho(z)\}$ at any $z \in U$. Moreover, if $X^n := v^n - iJv^n$ then the family $X := (X^1, \ldots, X^n)$ forms a basis of $(1, 0)$ vector fields on U. Similarly we define a basis $X' := (X'^1, \ldots, X'^n)$ of $(1, 0)$ vector fields on V such that $(X'^1(w), \ldots, X'^{n-1}(w))$ defines a basis of the J'-holomorphic tangent space to $\{\rho' = \rho'(w)\}$ at any $w \in V$. We denote by $A(p^k) := (A(p^k)_{j,l})_{1 \leq j,l \leq n}$ the matrix of the map $df(p^k)$ in the basis $X(p^k)$ and $X(f(p^k))$.

Remark 4.1 In sake of completeness we should write X_0 and X_0' to emphasize that the structure was normalized by the condition $J(0) = J_{st}$ and $A(0, p^k)$ for $A(p^k)$. The same construction is valid for any boundary point of D. The corresponding notations will be used in Proposition 4.5.

Propostion 4.2 *The matrix $A(p^k)$ satisfies the following estimates :*

$$A(p^k) = \begin{pmatrix} O_{n-1,n-1}(1) & O_{n-1,1}(dist(p^k, \partial D)^{-1/2}) \\ O_{1,n-1}(dist(p^k, \partial D)^{1/2}) & O_{1,1}(1) \end{pmatrix}.$$

The matrix notation means that the following estimates are satisfied: $A(p^k)_{j,l} = O(1)$ for $1 \le j, l \le n - 1$, $A(p^k)_{j,n} = O(dist(p^k, \partial D)^{-1/2})$ for $1 \le j \le n - 1$, $A(p^k)_{n,l} = O(dist(p^k, \partial D)^{1/2})$ for $1 \le l \le n - 1$ and $A(p^k)_{n,n} = O(1)$.

The proof of Proposition 4.2 is given in [3] (Proposition 3.5) in dimension two but is valid without any modification in any dimension. This is based on Proposition A. We note that the asymptotic behaviour of $A(p^k)$ depends only on the distance from the point to ∂D, not on the choice of the sequence $(p^k)_k$.

4.2 Scaling process and model domains

The following construction is similar to the two dimensional case. For every k denote by q^k the projection of p^k to ∂D and consider the change of variables α^k defined by

$$\begin{cases} (z^j)^* = \dfrac{\partial \rho}{\partial \bar{z}^n}(q^k)(z^j - (q^k)^j) - \dfrac{\partial \rho}{\partial \bar{z}^j}(q^k)(z^n - (q^k)^n), & \text{for } 1 \le j \le n - 1, \\ (z^n)^* = \sum_{j=1}^n \dfrac{\partial \rho}{\partial z^j}(q^k)(z^j - (q^k)^j). \end{cases}$$

If $\delta_k := dist(p^k, \partial D)$ then $\alpha^k(p^k) = (0, -\delta_k)$ and $\alpha^k(D) = \{2Rez^n + O(|z|^2) < 0\}$ near the origin. Moreover, the sequence $(\alpha^k)_*(J)$ converges to J as $k \to \infty$, since the sequence $(\alpha^k)_k$ converges to the identity map. Let $(L^k)_k$ be a sequence of linear automorphisms of $\mathbb{R}^{2n}$ such that $(T^k := L^k \circ \alpha^k)_k$ converges to the identity, and $D^k := T^k(D)$ is defined near the origin by $D^k = \{\rho_k(z) = Rez^n + O(|z|^2) < 0\}$. The sequence of almost complex structures $(J_k := (T^k)_*(J))_k$ converges to J as $k \to \infty$ and $J_k(0) = J_{st}$. Furthermore $\tilde{p}_k := T^k(p^k)$ satisfies $\tilde{p}_k = (o(\delta_k), \delta_k'' + io(\delta_k))$ with $\delta_k'' \sim \delta_k$.

We proceed similarly on D'. Denote by s^k the projection of $f(p^k)$ onto $\partial D'$ and define the transformation β^k by

$$\begin{cases} (w^j)^* = \dfrac{\partial \rho'}{\partial \bar{w}^n}(s^k)\left(w^j - (s^k)^j\right) - \dfrac{\partial \rho'}{\partial \bar{w}^j}(s^k)(w^n - (s^k)^n), & \text{for } 1 \le j \le n - 1, \\ (w^n)^* = \sum_{j=1}^n \dfrac{\partial \rho'}{\partial w^j}(s^k)(w^j - (s^k)^j). \end{cases}$$

We define a sequence $(T'^k)_k$ of linear transformations converging to the identity and satisfying the following properties. The domain $(D^k)' := T'^k(D')$ is defined near the origin by $(D^k)' = \{\rho_k'(w) := Rew^n + O(|w|^2) < 0\}$, and $\tilde{f}(p_k) =$

$T'^k(f(p_k)) = (o(\varepsilon_k), \varepsilon_k'' + io(\varepsilon_k))$ with $\varepsilon_k'' \sim \varepsilon_k$, where $\varepsilon_k = dist(f(p_k), \partial D')$. The sequence of almost complex structures $(J_k' := (T'^k)_*(J'))_k$ converges to J' as $k \to \infty$ and $J_k'(0) = J_{st}$.

Finally, the map $f^k := T'^k \circ f \circ (T^k)^{-1}$ satisfies $f^k(\tilde{p}_k) = \tilde{f}(p_k)$ and is a (J_k, J_k')-biholomorphism between the domains D^k and $(D')^k$.

Let $\phi_k : ('z, z^n) \mapsto (\delta_k^{1/2}'z, \delta_k z^n)$ and $\psi_k(w', w^n) = (\varepsilon_k^{1/2} w', \varepsilon_k w^n)$ and set $\hat{f}^k = (\psi_k)^{-1} \circ f^k \circ \phi_k$. The map $\hat{f}^k$ is $(\hat{J}_k, \hat{J}_k')$-biholomorphic, where $\hat{J}_k := ((\phi_k)^{-1})_*(J_k)$ and $\hat{J}_k' := (\psi_k^{-1})_*(J_k')$. If $\hat{D}^k := \phi_k^{-1}(D^k)$ and $(\hat{D}')^k := \psi_k^{-1}((D')^k)$ then $\hat{D}^k = \{z \in \phi_k^{-1}(U) : \hat{\rho}_k(z) < 0\}$ where

$$\hat{\rho}_k(z) := \delta_k^{-1}\rho(\phi_k(z)) = 2Rez^n + \delta_k^{-1}[2ReK\left(\delta_k^{1/2}'z, \delta_k z^n\right)$$
$$+H\left(\delta_k^{1/2}'z, \delta_k z^n\right) + o\left(|\left(\delta_k^{1/2}'z, \delta_k z^n\right)|^2\right).$$

and $(\hat{D}')^k = \left\{w \in \phi_k^{-1}(V) : \hat{\rho}_k'(z) < 0\right\}$ where

$$\hat{\rho}_k'(w) := \varepsilon_k^{-1}\rho'\left(\psi_k(w)\right) = 2Rew^n + \varepsilon_k^{-1}[2ReK'(\varepsilon_k^{1/2}w', \varepsilon_k w^n)$$
$$+H'(\varepsilon_k^{1/2}w', \varepsilon_k w^n) + o(|(\varepsilon_k^{1/2}w', \varepsilon_k w^n)|^2).$$

Since U is a neighborhood of the origin, the pullbacks $\phi_k^{-1}(U)$ converge to $\mathbb{C}^n$ and the functions $\hat{\rho}_k$ converge to $\hat{\rho}(z) = 2Rez^n + 2ReK('z, 0) + H('z, 0)$ in the $\mathcal{C}^2$ norm on compact subsets of $\mathbb{C}^n$. Similarly, since V is a neighborhood of the origin, the pullbacks $\psi_k^{-1}(U')$ converge to $\mathbb{C}^n$ and the functions $\hat{\rho}_k'$ converge to $\hat{\rho}'(w) = 2Rew^n + 2ReK'('z, 0) + H'('z, 0)$ in the $\mathcal{C}^2$ norm on compact subsets of $\mathbb{C}^n$. If $\Sigma := \{z \in \mathbb{C}^n : \hat{\rho}(z) < 0\}$ and $\Sigma' := \{w \in \mathbb{C}^n : \hat{\rho}'(w) < 0\}$ the sequence of points $\hat{p}_k = \phi_k^{-1}(\tilde{p}_k) \in \hat{D}^k$ converges to the point $(0, -1) \in \Sigma$ and the sequence of points $\hat{f}(p_k) = \psi_k^{-1}(\tilde{f}(p_k)) \in \hat{D}'^k$ converges to $(0, -1) \in \Sigma'$. Finally $\hat{f}^k(\hat{p}_k) = \hat{f}(p_k)$.

The limit behaviour of the dilated objects is given by the following proposition.

Propostion 4.3 (i) *The sequences $(\hat{J}_k)$ and $(\hat{J}_k')$ of almost complex structures converge to model structures J_0 and J_0' uniformly (with all partial derivatives of any order) on compact subsets of $\mathbb{C}^n$.*

(ii) *(Σ, J_0) and (Σ', J_0') are model domains.*

(iii) *The sequence $(\hat{f}^k)$ (together with all derivatives) is a relatively compact family (with respect to the compact open topology) on Σ; every cluster point $\hat{f}$ is a (J_0, J_0')-biholomorphism between Σ and Σ', satisfying $\hat{f}(0, -1) = (0, -1)$ and $\hat{f}^n('0, z^n) = z^n$ on Σ.*

Proof of Proposition 4.3 Proof of (i). We focus on structures $\hat{J}_k$. Consider $J = J_{st} + L(z) + O(|z|^2)$ as a matrix valued function, where L is a real linear matrix. The Taylor expansion of J_k at the origin is given by $J_k = J_{st} + L^k(z) + O(|z|^2)$ on U, uniformly with respect to k. Here L^k is a real linear matrix converging to L at infinity. Write $\hat{J}_k = J_{st} + \hat{L}^k + O(\delta_k)$. If $L^k = (L_{j,l}^k)_{j,l}$ then $\hat{L}_{j,l}^k = L_{j,l}^k(\phi_k(z))$

for $1 \leq j \leq n - 1$, $1 \leq l \leq n$, $\hat{L}_{n,l}^k = \delta_k^{-1/2} L_{n,l}^k(\phi_k(z))$ for $1 \leq l \leq n - 1$ and $\hat{L}_{n,n}^k = L_{n,n}^k(\phi_k(z))$. This gives the conclusion.

Proof of (ii). We focus on (Σ, J_0). By the invariance of the Levi form we have $\mathcal{L}^{J_k}(\rho_k)(0)(\phi_k(v)) = \mathcal{L}^{\hat{J}_k}(\rho_k \circ \phi_k)(0)(v)$. Write $J_0 = J_{st} + L^{\infty}$. Since ρ_k is strictly J_k-plurisubharmonic uniformly with respect to k (ρ_k converges to ρ and J_k converges to J), multiplying by δ_k^{-1} and passing to the limit at the right side as $k \to \infty$, we obtain that $\mathcal{L}^{J_0}(\hat{\rho})(0)(v) \geq 0$ for any v. Now let $v = (v', 0) \in T_0(\partial \Sigma)$. Then $\phi_k(v) = \delta_k^{1/2} v$ and so $\mathcal{L}_k^J(\rho)(0)(v) = \mathcal{L}^{\hat{J}_k}(\rho_k)(0)(v)$. Passing to the limit as k tends to infinity, we obtain that $\mathcal{L}^{J_0}(\hat{\rho})(0)(v) > 0$ for any $v = (v', 0)$ with $v' \neq 0$.

Proof of (iii). This statement is a consequence of Proposition A. We refer to Section 7 of [3] for the existence and the biholomorphy of $\hat{f}$. We prove the identity on $\hat{f}^n$. Let t be a real positive number. Then we have:

Lemma 4.4 $\lim_{t \to \infty} \hat{\rho}'(\hat{f}('0, -t)) = \infty$.

Proof of Lemma 4.4 According to the boundary distance property (4.2) we have

$$|\rho'(f \circ (T^k)^{-1} \circ \phi_k)('0, -t)| \geq C \, dist\left(T_k^{-1}('0, -\delta_k t)\right).$$

Then

$$|\hat{\rho}_k'(\hat{f}^k('0, -t))| \geq C\varepsilon_k^{-1}\delta_k \, t.$$

Since $\hat{\rho}_k'$ converges to $\hat{\rho}'$ uniformly on compact subsets of Σ' and $\varepsilon_k \simeq \delta_k$ (by the boundary distance property (4.2)) we obtain:

$$|\hat{\rho}'(\hat{f}('0, -t))| \geq Ct.$$

This proves Lemma 4.4. $\qquad\square$

We turn back to the proof of part (iii) of Proposition 4.3. Assume first that J (and similarly J') are not integrable (see Proposition 2.3). Consider a J-complex hypersurface $A \times \mathbb{C}$ in $\mathbb{C}^n$ where A is a J_{st} complex hypersurface in $\mathbb{C}^{n-1}$. Since $f((A \times \mathbb{C}) \cap \mathbb{H}_{P_1}) = (A' \times \mathbb{C}) \cap \mathbb{H}_{P_2}$ where A' is a J_{st} complex hypersurface in $\mathbb{C}^{n-1}$, it follows that the restriction of $\hat{f}^n$ to $\{'z =' 0, Re(z^n) < 0\}$ is a J_{st} automorphism of $\{'z =' 0, Re(z^n) < 0\}$. Let $\phi : \zeta \mapsto (\zeta - 1)/(\zeta + 1)$. The function $\hat{g} := \phi^{-1} \circ \hat{f}^n \circ \phi$ is a J_{st} automorphism of the unit disc in $\mathbb{C}$. In view of Lemma 4.4 this satisfies $\hat{g}(0) = 0$ and $\hat{g}(1) = 1$. Hence $\hat{g} \equiv id$ and $\hat{f}^n('0, z^n) = z^n$ on Σ.

Assume now that J and J' are integrable. Let F (resp. F') be the diffeomorphism from Σ to $\mathbb{H}_P$ (resp. from Σ to $\mathbb{H}_{P'}$) given in the proof of Proposition 2.3. The diffeomorphism $g := F' \circ f \circ F^{-1}$ is a J_{st}-biholomorphism from $\mathbb{H}_P$ to $\mathbb{H}_{P'}$ satisfying $g('0, -1) = ('0, -1)$. Since (Σ, J) and (Σ', J') are model domains, the domains $\mathbb{H}_P$ and $\mathbb{H}_{P'}$ are strictly J_{st}-pseudoconvex. In particular, since P and P' are homogeneous of degree two, there are linear complex maps L, L' in $\mathbb{C}^{n-1}$ such that the map G (resp. G') defined by $G('z, z_n) = (L('z), z_n)$ (resp. $G'('z, z_n) = (L'('z), z_n)$) is a biholomorphism from $\mathbb{H}_P$ (resp. $\mathbb{H}_{P'}$) to $\mathbb{H}$. The map $G' \circ g \circ G^{-1}$ is an automorphism of $\mathbb{H}$ satisfying $G' \circ g \circ G^{-1}('0, -1) = ('0, -1)$.

Let Φ be the J_{st} biholomorphism from $\mathbb{H}$ to the unit ball $\mathbb{B}_n$ of $\mathbb{C}^n$ defined by $\Phi('z, z^n) = (\sqrt{2}'z/1 - z^n, (1 + z^n)/(1 - z^n))$. Let $\hat{g} := \Phi^{-1} \circ g \circ \Phi$. In view of lemma 4.2 this satisfies $\hat{g}(0) = 0$ and $\hat{g}('0, 1) = ('0, 1)$. Hence $\hat{g}^n \equiv id$ and $\hat{f}^n('z, z^n) = z^n$ for every z in Σ. $\qquad\square$

According to part (ii) of Proposition 4.3 and restricting U if necessary, one may view $D \cap U$ as a strictly J_0-pseudoconvex domain in $\mathbb{C}^n$ and J as a small deformation of J_0 in a neighborhood of $\bar{D} \cap U$. The same holds for $D' \cap V$.

For $p \in \partial D$ and $z \in D$ let $X_p(z)$ and $X'_{f(p)}(f(z))$ be the basis of $(1, 0)$ vector fields defined in Subsection 3.2 (see Remark 4.1). The elements of the matrix of df_z in the bases $X_p(z)$ and $X'_{f(p)}(f(z))$ are denoted by $A_{js}(p, z)$. According to Proposition 4.2 the function $A_{n,n}(p, \cdot)$ is upper bounded on D.

Propostion 4.5 *We have:*

(a) *Every cluster point of the function $z \mapsto A_{n,n}(p, z)$ is real when z tends to $p \in \partial D$.*
(b) *For $z \in D$, let $p \in \partial D$ such that $|z - p| = dist(z, \partial D)$. There exists a constant A, independent of $z \in D$, such that $|A_{n,n}(p, z)| \geq A$.*

Proof of Proposition 4.5 (a) Suppose that there exists a sequence of points (p^k) converging to a boundary point p such that $A_{n,n}(p, \cdot)$ tends to a complex number a. Applying the above scaling construction, we obtain a sequence of maps $(\hat{f}^k)_k$. For $k \geq 0$ consider the dilated vector fields

$$Y_k^j := \delta_k^{1/2} \left((\phi_k^{-1}) \circ T^k \right) (X^j(p^k))$$

for $j = 1, \ldots, n - 1$, and

$$Y_k^n := \delta_k \left((\phi_k^{-1}) \circ T^k \right) (X_n(p^k)).$$

Similarly we define

$$Y_k^{'j} := \varepsilon_k^{-1/2} \left((\psi_k^{-1}) \circ T'^k \right) \left(X'^j(f(p^k)) \right)$$

for $j = 1, \ldots, n - 1$, and

$$Y_k^{'n} := \varepsilon_k^{-1}((\psi_k^{-1}) \circ T'^k) \left(X'_n(f(p^k)) \right).$$

For every k, the n-tuple $Y^k := (Y_k^1, \ldots, Y_k^n)$ is a basis of $(1, 0)$ vector fields for the dilated structure $\hat{J}^k$. In view of Proposition 4.3 the sequence $(Y^k)_k$ converges to a basis of $(1, 0)$ vector fields of $\mathbb{C}^n$ (with respect to J_0) as k tends to ∞. Similarly, the n-tuple $Y'^k := (Y_k^{'1}, \ldots, Y_k^{'n})$ is a basis of $(1, 0)$ vector fields for the dilated structure $\hat{J}'^k$ and $(Y'^k)_k$ converges to a basis of $(1, 0)$ vector fields of $\mathbb{C}^n$ (with respect to J_0') as k tends to ∞. In particular the last components Y_k^n and $Y_k^{'n}$ converge to the $(1, 0)$ vector field $\partial/\partial z^n$. Denote by $\hat{A}_{js}^k$ the elements of the matrix of $d\hat{f}^k(0, -1)$. Then $A_{n,n}^k$ converges to $(\partial \hat{f}^n/\partial z^n)(0, -1) = 1$, according

to Proposition 4.3. On the other hand, $A_{n,n}^k = \varepsilon_k^{-1}\delta_k A_{n,n}$ converges to a by the boundary distance preserving property (4.2). This gives the statement.

(b) Suppose that there is a sequence of points (p^k) converging to the boundary such that $A_{n,n}$ tends to 0. Repeating precisely the argument of (a), we obtain that $(\partial \hat{f}^n/\partial z^n)(0, -1) = 0$; this contradicts part (iii) of Proposition 4.3. $\qquad\square$

5 Proof of Theorem 0.1

From now on we are in the hypothesis of Theorem 0.1. The key point of the proof of Theorem 0.1 consists in the following claim :

Claim: The cluster set of the cotangent lift f^* on $\Sigma(\partial D)$ is contained in $\Sigma(\partial D')$.

Indeed, assume for the moment the claim satisfied. We recall that according to Proposition 3.1 the conormal bundle $\Sigma_J(\partial D)$ of ∂D is a totally real submanifold in the cotangent bundle T^*M. Consider the set $S = \{(z, L) \in \mathbb{R}^{2n} \times \mathbb{R}^{2n} : dist((z, L), \Sigma_J(\partial D)) \le dist(z, \partial D), z \in D\}$. Then, in a neighborhood U of any totally real point of $\Sigma_J(\partial D)$, the set S contains a wedge W_U with $\Sigma_J(\partial D) \cap U$ as totally real edge.

Then in view of Proposition 3.5 we obtain the following Proposition:

Propostion 5.1 *Restricting the aperture of the wedge W_U if necessary, the map f^* extends to $W_U \cup \Sigma(\partial D)$ as a $\mathcal{C}^\infty$-map.*

Proposition 5.1 implies immediately that f extends as a smooth $\mathcal{C}^\infty$ diffeomorphism from $\bar{D}$ to $\bar{D}'$.

Therefore the proof of Theorem 0.1 can be reduced to the proof of the claim.

Step one. We first reduce the problem to the following local situation. Let D and D' be domains in $\mathbb{C}^n$, Γ and Γ' be open $\mathcal{C}^\infty$-smooth pieces of their boundaries, containing the origin. We assume that an almost complex structure J is defined and $\mathcal{C}^\infty$-smooth in a neighborhood of the closure $\bar{D}$, $J(0) = J_{st}$. Similarly, we assume that $J'(0) = J_{st}$. The hypersurface Γ (resp. Γ') is supposed to be strictly J-pseudoconvex (resp. strictly J'-pseudoconvex). Finally, we assume that $f : D \to D'$ is a (J, J')-biholomorphic map. It follows from the estimates of the Kobayashi-Royden infinitesimal pseudometric given in [6] that f extends as a 1/2-Hölder homeomorphism between $D \cup \Gamma$ and $D' \cup \Gamma'$, such that $f(\Gamma) = \Gamma'$ and $f(0) = 0$. Finally Γ is defined in a neighborhood of the origin by the equation $\rho(z) = 0$ where $\rho(z) = 2Re z^n + 2ReK(z) + H(z) + o(|z|^2)$ and $K(z) = \sum K_{\mu\nu}z^{\mu\nu}$, $H(z) = \sum h_{\mu\nu}z^\mu \bar{z}^\nu$, $k_{\mu\nu} = k_{\nu\mu}, h_{\mu\nu} = \bar{h}_{\nu\mu}$. As we noticed at the end of Section 3 the hypersurface Γ is strictly $\hat{J}$-pseudoconvex at the origin. The hypersurface Γ' admits a similar local representation. In what follows we assume that we are in this setting.

Let $\Sigma := \{z \in \mathbb{C}^n : 2Re z^n + 2ReK('z, 0) + H('z, 0) < 0\}$, $\Sigma' := \{z \in \mathbb{C}^n : 2Re z^n + 2ReK'('z, 0) + H'('z, 0) < 0\}$. If (p^k) is a sequence of points in D converging to 0, then according to Proposition 4.3, the scaling procedure associates with the pair $(f, (p^k)_k)$ two linear almost complex structures J_0 and J_0', both defined on $\mathbb{C}^n$, and a (J_0, J_0')-biholomorphism $\hat{f}$ between Σ and Σ'. Moreover (Σ, J_0) and (Σ', J_0') are model domains. To prove that the cluster set of the cotangent lift of f at a point in $N(\Gamma)$ is contained in $N(\Gamma')$, it is sufficient to prove that $(\partial \hat{f}^n/\partial z^n)('0, -1) \in \mathbb{R}\backslash\{0\}$.

Step two. The proof of the Claim is given by the following Proposition.

Propostion 5.2 *Let K be a compact subset of the totally real part of the conormal bundle $\Sigma_J(\partial D)$. Then the cluster set of the cotangent lift f^* of f on the conormal bundle $\Sigma(\partial D)$, when (z, L) tends to $\Sigma_J(\partial D)$ along the wedge W_U, is relatively compactly contained in the totally real part of $\Sigma(\partial D')$.*

We recall that the totally real part of $\Sigma(\partial D')$ is the complement of the zero section in $\Sigma(\partial D')$.

Proof of Proposition 5.2 Let (z^k, L^k) be a sequence in W_U converging to $(0, \partial_J \rho(0)) = (0, dz^n)$. We shall prove that the sequence of linear forms $Q^k := {}^t df^{-1}(w^k)L^k$, where $w^k = f(z^k)$, converges to a linear form which up to a *real* factor (in view of Part (a) of Proposition 4.5) coincides with $\partial_J \rho(0) = dz^n$ (we recall that ^{t} denotes the transposed map). It is sufficient to prove that the $(n-1)$ first components of Q^k with respect to the dual basis $(\omega_1, \ldots, \omega_n)$ of X converge to 0 and the last one is bounded below from the origin as k goes to infinity. The map X being of class $\mathcal{C}^1$ we can replace $X(0)$ by $X(w^k)$. Since $(z^k, L^k) \in W_U$, we have $L^k = \omega_n(z^k) + O(\delta_k)$, where δ_k is the distance from z^k to the boundary. Since $|||df^{-1}_{w^k}||| = 0(\delta_k^{-1/2})$, we have $Q^k = {}^t df^{-1}_{w^k}(\omega_n(z^k)) + O(\delta_k^{1/2})$. By Proposition 4.3, the components of ${}^t df^{-1}_{w^k}(\omega_n(z^k))$ with respect to the basis $(\omega_1(z^k), \ldots, \omega_n(z^k))$ are the elements of the last line of the matrix $df^{-1}_{w^k}$ with respect to the basis $X'(w^k)$ and $X(z^k)$. So its $(n-1)$ first components are $0(\delta_k^{1/2})$ and converge to 0 as k tends to infinity. Finally the component $A^k_{n,n}$ is bounded below from the origin by Part (b) of Proposition 4.5. □

6 Compactness principle

In this section we prove Theorem 0.2.

We note that condition (ii) is equivalent to the existence, at each $p \in \partial D$, of a strictly J-plurisubharmonic local defining function for ∂D (consider the function $\rho + C\rho^2$ for a sufficiently large positive C).

We first recall the following result proved in [6]:

Proposition B. *(Localization principle) Let D be a domain in an almost complex manifold (M, J), let $p \in \bar{D}$, let U be a neighborhood of p in M (not necessarily contained in D) and let $z : U \to \mathbb{B}$ be the diffeomorphism given by Lemma 1.1. Let u be a $\mathcal{C}^2$ function on $\bar{D}$, negative and J-plurisubharmonic on D. We assume that $-L \leq u < 0$ on $D \cap U$ and that $u - c|z|^2$ is J-plurisubharmonic on $D \cap U$, where c and L are positive constants. Then there exist a positive constant s and a neighborhood $V \subset\subset U$ of p, depending on c and L only, such that for $q \in D \cap V$ and $v \in T_q M$ we have the following estimate:*

$$K_{(D,J)}(q, v) \geq s K_{(D \cap U, J)}(q, v). \tag{6.1}$$

We can now prove Theorem 0.2.

Proof of Theorem 0.2 We assume that the assumptions of Theorem 0.2 are satisfied. We proceed by contradiction. Assume that there is a compact K_0 in M, points $p^\nu \in M$ and a point $q \in \partial D$ such that $\lim_{\nu \to \infty} f^\nu(p^\nu) = q$.

Lemma 6.1 *For every relatively compact neighborhood V of q there is ν_0 such that for $\nu \ge \nu_0$ we have : $\lim_{x \to q} \inf_{q' \in D \cap \partial V} d^K_{(D, J_\nu)} = \infty$.*

Proof of Lemma 6.1 Restricting U if necessary, we may assume that the function $\rho + C\rho^2$ is a strictly J_ν-plurisubharmonic function in a neighborhood of $\bar{D} \cap U$, for sufficiently large ν. Moreover, using Proposition B, we can focus on $K_{D \cap U}$. Smoothing $D \cap U$, we may assume that the hypothesis of Proposition A are satisfied on $D \cap U$, uniformly for sufficiently large ν. In particular, the inequality (4.1) is satisfied on $D \cap U$, with a positive constant c independent of ν. The result follows by a direct integration of this inequality. $\square$

The following Lemma is a corollary of Lemma 6.1.

Lemma 6.2 *For every $K \subset\subset M$ we have : $\lim_{\nu \to \infty} f^\nu(K) = q$.*

Proof of Lemma 6.2 Let $K \subset\subset M$ be such that $x^0 \in K$. Since the function $x \mapsto d^K_D(x^0, x)$ is bounded from above by a constant C on K, it follows from the decreasing property of the Kobayashi pseudodistance that

$$d^K_{(D, J_\nu)}(f^\nu(x^0), f^\nu(x)) \le C \tag{6.2}$$

for every ν and every $x \in K$. It follows from Lemma 6.1 that for every $V \subset\subset U$, containing p, we have :

$$\lim_{\nu \to \infty} d^K_{(D, J_\nu)}(f^\nu(x^0), D \cap \partial V) = +\infty. \tag{6.3}$$

Then from conditions (6.2) and (6.3) we deduce that $f^\nu(K) \subset V$ for every sufficiently large ν. This gives the statement. $\square$

Fix now a point $p \in M$ and denote by p^ν the point $f^\nu(p)$. We may assume that the sequence $(J_\nu := f^\nu_*(J))_\nu$ converges to an almost complex structure J' on $\bar{D}$ and according to Lemma 6.2 we may assume that $\lim_{\nu \to \infty} p^\nu = q$. We apply Subsection 4.3 to the domain D and the sequence $(q^\nu)_\nu$. We denote by T^ν the linear transformation $T^\nu := M^\nu \circ L^\nu \circ \alpha^\nu$, as in Subsection 4.3, and we consider $D^\nu := T^\nu(D)$, and $J^\nu := T^\nu_*(J_\nu)$. If ϕ_ν is the nonisotropic dilation $\phi_\nu : ('z, z^n) \mapsto (\delta_\nu^{1/2} \, 'z, \delta_\nu z^n)$ then we set $\hat{f}^\nu := \phi_\nu^{-1} \circ T^\nu \circ f$ and $\hat{J}^\nu := (\phi_\nu^{-1})_*(J^\nu)$. We also consider $\hat{\rho}_\nu := \delta_\nu^{-1} \circ \rho \circ \phi_\nu$ and $\hat{D}^\nu := \{\hat{\rho}_\nu < 0\}$. As proved in Subsection 4.3, the sequence $(\hat{D}^\nu)_\nu$ converges, in the local Hausdorff convergence, to a domain $\Sigma := \{z \in C^n : \hat{\rho}(z) := 2Rez^n + 2ReK('z, 0) + H('z, 0) < 0\}$, where K and H are homogeneous of degree two. According to Proposition 4.3 we have:

(i) The sequence $(\hat{J}^\nu)$ converges to a model almost complex structure J_0, uniformly (with all partial derivatives of any order) on compact subsets of C^n,
(ii) (Σ, J_0) is a model domain,
(iii) the sequence $(\hat{f}^\nu)_\nu$ converges to a (J, J_0) holomorphic map F from M to Σ.

To prove Theorem 0.2, it remains to prove that F is a diffeomorphism from M to Σ. We first notice that according to condition (ii) of Theorem 0.2 and Lemma 6.1, the domain D is complete J_ν-hyperbolic. In particular, since f^ν is a (J, J_ν) biholomorphism from M to D, the manifold M is complete J-hyperbolic. Consequently, for every compact subset L of M, there is a positive constant C such that for every $z \in L$ and every $v \in T_z M$ we have $K_{(M,J)}(z, v) \geq C\|v\|$. Consider the map $\hat{g}^\nu := (\hat{f}^\nu)^{-1}$. This is a $(\hat{J}^\nu, J)$ biholomorphism from $\hat{D}^\nu$ to M. Let K be a compact set in Σ. We may consider $\hat{g}^\nu(K)$ for sufficiently large ν. By the decreasing property of the Kobayashi distance, there is a compact subset L in M such that $\hat{g}^\nu(K) \subset L$ for sufficiently large ν. Then for every $w \in K$ and for every $v \in T_w \Sigma$ we obtain, by the decreasing of the Kobayashi-Royden infinitesimal pseudometric:

$$\|df^\nu(w)(v)\| \leq (1/C)\|v\|,$$

uniformly for sufficiently large ν. According to Ascoli Theorem, we may extract from $(\hat{g}^\nu)_\nu$ a subsequence, converging to a map G from Σ to M. Finally, on any compact subset K of M, by the equality $\hat{g}^\nu \circ \hat{f}^\nu = id$ we obtain $F \circ G = id$. This gives the result. □

As a corollary of Theorem 0.2 we obtain the following almost complex version of the Wong-Rosay Theorem in real dimension four:

Corollary 6.3 *Let (M, J) (resp. (M', J')) be an almost complex manifold of real dimension four. Let D (resp. D') be a relatively compact domain in M (resp. N). Consider a sequence $(f^\nu)_\nu$ of diffeomorphisms from D to D' such that the sequence $(J_\nu := f^\nu_*(J))_\nu$ extends to $\bar{D}'$ and converges to J' in the C^2 convergence on $\bar{D}'$.*

Assume that there is a point $p \in D$ and a point $q \in \partial D'$ such that $\lim_{\nu\to\infty} f^\nu(p) = q$ and such that D' is strictly J'-pseudoconvex at q. Then there is a (J, J_{st})-biholomorphism from M to the unit ball $\mathbb{B}^2$ in $\mathbb{C}^2$.

Proof of Corollary 6.3 The proof of Corollary 6.3 follows exactly the same lines. Indeed, by assumtion there is a fixed neighborood U of q such that $D' \cap U$ is strictly J_ν-pseudoconvex on U. According to Lemma 6.2, we know that for every compact subset K of D the set $f^\nu(K)$ is contained in V for sufficiently large ν. If we fix a point $p \in D$ we may therefore apply Subsection 4.4 to the sequence $(f^\nu(p))_\nu$ and to the domain D' (with V exactly as in Subsection 4.4). The proof is then identical to the proof of Theorem 0.2. □

References

1. Chirka, E.: Introduction to the almost complex analysis. Lecture notes (2003)
2. Coupet, B.: Precise regularity up to the boundary of proper holomorphic mappings. Ann. Scuola Norm. Sup. Pisa **20**, 461–482 (1993)
3. Coupet, B., Gaussier, H., Sukhov, A.: Fefferman's mapping theorem on almost complex manifolds in complex dimension two. Math. Z. **250**, 59–90 (2005)
4. Fefferman, Ch.: The Bergman kernel and biholomorphic mappings of pseudoconvex domains. Invent. math. **26**, 1–65 (1974)
5. Forstneric, F.: An elementary proof of Fefferman's theorem. Exposit. Math. **10**, 135–149 (1992)
6. Gaussier, H., Sukhov, A.: Estimates of the Kobayashi metric in almost complex manifolds. Bull. Soc. Math. France **133**, 259–273 (2005)

7. Gromov, M.: Pseudoholomorphic curves in symplectic manifolds. Invent. Math. **82-2**, 307–347 (1985)
8. Hofer, H.: Pseudoholomorphic curves in symplectizations with applications to the Weinstein conjecture in dimension three. Invent. Math. **114**, 515–563 (1993)
9. Ivashkovich, S., Shevchishin, V.: Reflection principle and J-complex curves with boundary on totally real immersions. Comm. in Contemp. Math. **4-1**, 65–106 (2002)
10. Lempert, L., Szöke, R.: The tangent bundle of an almost complex manifold. Canad. Math. Bull. **44**, 70–79 (2001)
11. Newlander, A., Nirenberg, L.: Complex analytic coordinates in almost complex manifolds. Ann. of Math. (2) **65**, 391–404 (1957)
12. Nijenhuis, A., Woolf, W.: Some integration problems in almost-complex and complex manifolds. Ann. Math. **77**, 429–484 (1963)
13. Nirenberg, L., Webster, S., Yang, P.: Local boundary regularity of holomorphic mappings. Comm. Pure Applied Math. **33**, 305–338 (1980)
14. Pinchuk, S.: The scaling method and holomorphic mappings. Several complex variables and complex geometry, Part 1 (Santa Cruz, CA, 1989), 151–161, Proc. Sympos. Pure Math., **52** Part 1, Amer. Math. Soc., Providence, RI, 1991
15. Pinchuk, S., Khasanov, S.: Asymptotically holomorphic functions and their applications. Math. USSR Sb. **62**, 541–550 (1989)
16. Sikorav, J.-C.: Some properties of holomorphic curves in almost complex manifolds. In: Holomorphic curves and symplectic geometry. Ed. M.Audin, J.Lafontaine, Birkhauser, (1994), pp. 165–189
17. Spiro, A.: Total reality of the conormal bundle of a real hypersurface in an almost complex manifold. Preprint, December 2003
18. Tumanov, A.: Analytic discs and the regularity of CR mappings in higher codimension. Duke Math. J. **76**, 793–807 (1994)
19. Tumanov, A.: Extremal discs and the regularity of CR mappings in higher codimension. Amer. J. Math. **123**, 445–473 (2001)
20. Vekua, N.P.: Systems of singular integral equations. Nordholf, Groningen 1967
21. Yano, K., Ishihara, Sh.: Tangent and cotangent bundles. Marcel Dekker NY, 1973