

Global existence of null-form wave equations in exterior domains

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Abstract We provide a proof of global existence of solutions to quasilinear wave equations satisfying the null condition in certain exterior domains. In particular, our proof does not require estimation of the fundamental solution for the free wave equation. We instead rely upon a class of Keel–Smith–Sogge estimates for the perturbed wave equation. Using this, a notable simplification is made as compared to previous works concerning wave equations in exterior domains: one no longer needs to distinguish the scaling vector field from the other admissible vector fields.

1 Introduction

Inspired by the approach of Sideris and Tu [26] in the boundaryless case and the application of such techniques in Sideris [25], we prove global existence for multi-speed systems of quasilinear wave equations satisfying the null condition in certain exterior domains without using estimation of the fundamental solution for the free wave equation. To do so, we use a Keel–Smith–Sogge estimate for the perturbed equation established previously by the authors in [23]. In the previous works on nonlinear wave equations in exterior domains, [12], [21, 22], and [19, 20], it was necessary to use estimates that involved relatively few occurrences of the scaling vector field $L = t\partial_t + r\partial_r$. A notable innovation in the new approach allows us to no longer distinguish between the scaling vector field L and the other “admissible” invariant vector fields $Z = \{\Omega_{ij} = x_i\partial_j - x_j\partial_i, \partial_k : 1 \leq i, j \leq 3, 0 \leq k \leq 3\}$. This is accomplished

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by introducing modified vector fields that preserve the boundary condition, as was in part done in [21].

The main existence result is an analog of the classic results of Christodoulou [3] and Klainerman [14] in the boundaryless case. In the multiple speed boundaryless case, related results were established, e.g., by Sideris and Tu [26], Sogge [28], Agemi and Yokoyama [1], Kubota and Yokoyama [16], Hidano [4], Yokoyama [31], and Katayama [7–9]. In exterior domains, null form quasilinear wave equations were previously studied by Keel et al. [10], the authors [21], and Metcalfe et al. [19,20]. We note that the main theorem of this paper was previously established (in a more general context) in [19]. We, however, believe that the new techniques are of independent interest, and we are hopeful about their potential use in applications.

Our proof uses Klainerman's method of commuting vector fields [14] as was adapted to exterior domains by Keel et al. [12]. In particular, we restrict to the class of "admissible" vector fields that was mentioned above. Notably absent in this set are the hyperbolic rotations $\Omega_{0j} = t\partial_j + x_j\partial_t$ which do not seem appropriate for problems in exterior domains as they have unbounded normal component on the boundary. Moreover, in the multiple speed setting, these vector fields have an associated speed and only commute with the d'Alembertian of the same speed.

This approach relies upon a weighted space-time L^2 estimate, which will be referred to as the Keel–Smith–Sogge estimate or KSS estimate. Such estimates were established in [11] where they were first used to show long-time existence of solutions to nonlinear equations. With this estimate, existence is established using $O(1/|x|)$ decay rather than the more standard, but quite difficult to prove when there is boundary, $O(1/t)$ decay. An earlier, related estimate is due to Strauss [30] (Lemma 3). The proof in [11] is easily modified to establish these bounds in all dimensions $n \geq 3$ as is done in Metcalfe [18] and Hidano and Yokoyama [5] and has been used in, e.g., [5], [18], and [22] to study nonlinear equations. Recently, using techniques of Rodnianski [29], the authors [23] have established an analogous estimate for the perturbed equation. This inequality is essential to the approach presented in this article. It is worth noting that Alinhac [2] simultaneously obtained a related KSS-type estimate for the perturbed equation and for wave equations on curved backgrounds. The assumptions made on the perturbation in [2] are, however, not as favorable in the current setting.

The KSS estimate and energy estimates will be coupled with some well-known decay estimates in order to get global existence. These decay estimates are variants of those of Klainerman and Sideris [15] and are known to be rather widely applicable, including e.g. to the equations of elasticity. This will, of course, be used in combination with the extra decay afforded to us by the null condition.

The major innovations of this paper regard the variable-coefficient KSS estimate. First, we expand upon the proof in [23] and carefully prove the KSS estimates for the perturbed wave equation in the multiple speed setting. Having such estimates for the perturbed equation allows us to apply the KSS estimates even for terms of the highest order. This was not previously possible for quasilinear equations as there was a loss of regularity resulting from the occurrence of second derivatives. As such, we may now prove global existence using only energy methods. In particular, the decay estimates that we shall employ do not require direct estimation of the fundamental solution of the linear wave equation, and in particular, such estimates are known to hold for some related applications. We note, e.g., that the only obstacle to using similar techniques to study the equations of isotropic elasticity in exterior domains is deriving the existence of a KSS-type estimate for the perturbed linearized equations. This, however, is more

delicate than in the current setting due to the off-diagonal terms and is currently an open problem of interest.

A key obstacle to using only energy methods in previous studies was the boundary terms that arise in the Klainerman-Sideris estimates. In particular, there is a term localized near the boundary which has significantly less decay in t . To alleviate this, we shall use the additional decay in x , which was largely ignored in previous works, from the Klainerman-Sideris estimates. When combined with the KSS estimates, this decay permits us the necessary control over the boundary term.

As the coefficients of the scaling vector field can be large in an arbitrarily small neighborhood of the obstacle, previous works in exterior domains using the adapted method of commuting vector fields use estimates that required relatively few occurrences of L , and during the proofs of long-time existences, the scaling vector fields must be carefully tracked. This, at best, complicated these arguments. Using the variable coefficient KSS estimates, we show that it is not strictly necessary to differentiate L from the other admissible vector fields. In particular, we use a modified scaling vector field which preserves the Dirichlet boundary conditions as in [21], but instead of proving boundary term estimates, such as [21, Lemma 2.9], we are now able to control the resulting commutators using the variable coefficient KSS estimates.

Finally, we mention that our proof, unlike many of the previous works in exterior domains, does not directly use the decay of local energy, such as that of Lax, Morawetz, and Phillips [17]. Our hypotheses on the obstacle are, however, sufficient to guarantee said decay. It is conceivable that the techniques contained herein could be important in other applications where the rate of decay of local energy is slower, say e.g. in even dimensions.

We now more precisely describe the main result of this article. We begin by fixing a bounded obstacle $\mathcal{K} \subset \mathbb{R}^3$ with smooth boundary. Moreover, we shall assume that \mathcal{K} is star-shaped with respect to the origin. As we shall see, scaling will allow us to assume without loss that $\mathcal{K} \subset \{|x| \leq 1\}$, and this assumption is made throughout. The star-shapedness assumption is used to see that certain boundary terms in our energy estimates and KSS estimates have a favorable sign. This is reminiscent of arguments from Morawetz [24].

In the exterior of \mathcal{K} , we shall study systems of quasilinear wave equations of the form

$$\begin{cases} \partial_t^2 u^I - c_I^2 \Delta u^I = B_{K,\gamma}^{I,\alpha\beta} \partial_\gamma u^K \partial_\alpha \partial_\beta u^I, & (t, x) \in \mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K}, \quad I = 1, 2, \dots, D, \\ u(t, \cdot)|_{\partial\mathcal{K}} = 0, \\ u(0, \cdot) = f, \quad \partial_t u(0, \cdot) = g. \end{cases} \tag{1.1}$$

Here and throughout we use the Einstein summation convention. Repeated Greek indices α, β, γ , and δ are implicitly summed from 0 to 3. Repeated lowercase Latin indices a, b are summed from 1 to 3, and repeated uppercase Latin indices I, J, K are summed from 1 to D . In the sequel, we will use $\square = (\square_{c_1}, \dots, \square_{c_D})$ to denote the vector-valued d'Alembertian, where $\square_{c_I} = \partial_t^2 - c_I^2 \Delta$. For simplicity, we shall study the nonrelativistic case where

$$c_1 > \dots > c_D > 0.$$

Straightforward modifications will allow for repeated wave speeds.

The $B_{K,\gamma}^{J,\alpha\beta}$ in (1.1) are real constants satisfying the symmetry conditions

$$B_{K,\gamma}^{J,\alpha\beta} = B_{K,\gamma}^{J,\alpha\beta} = B_{K,\gamma}^{J,\beta\alpha}. \tag{1.2}$$

In order to get global existence, we shall also need to assume that (1.1) satisfies the null condition. In the nonrelativistic case, this says that the self-interactions among the quasilinear terms satisfy the standard null condition. That is,

$$B_{J,\gamma}^{J,\alpha\beta} \xi_\alpha \xi_\beta \xi_\gamma = 0, \quad \text{whenever } \frac{\xi_0^2}{2} - \xi_1^2 - \xi_2^2 - \xi_3^2 = 0, \quad J = 1, \dots, D. \tag{1.3}$$

To solve (1.1), the data must be assumed to satisfy the relevant compatibility conditions. Letting $J_k u = \{\partial_x^\alpha u : 0 \leq |\alpha| \leq k\}$, we know that for a fixed m and formal H^m solution u of (1.1), we can write $\partial_t^k u(0, \cdot) = \psi_k(J_k f, J_{k-1} g)$, $0 \leq k \leq m$, for compatibility functions ψ_k depending on the nonlinearity, $J_k f$, and $J_{k-1} g$. For $(f, g) \in H^m \times H^{m-1}$, the compatibility condition simply requires that the ψ_k vanish on $\partial\mathcal{K}$ for $0 \leq k \leq m - 1$. For smooth (f, g) , we say that the compatibility condition is satisfied to infinite order if this vanishing condition holds for all m . We refer the reader to [10] for a more thorough exposition on these compatibility conditions.

We are now prepared to state our main theorem.

Theorem 1.1 *Let \mathcal{K} be a fixed compact obstacle with smooth boundary that is star-shaped with respect to the origin. Assume that the $B_{K,\gamma}^{J,\alpha\beta}$ are as above. Then, there is a constant $\varepsilon_0 > 0$ and an integer constant $N > 0$ so that if the data $(f, g) \in C^\infty(\mathbb{R}^3 \setminus \mathcal{K})$ satisfy the compatibility condition to infinite order and the smallness condition*

$$\sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial_x^\alpha \partial_x f\|_2 + \sum_{|\alpha| \leq N} \|\langle x \rangle^{|\alpha|} \partial_x^\alpha g\|_2 \leq \varepsilon_0, \tag{1.4}$$

then (1.1) has a unique global solution $u \in C^\infty([0, \infty) \times \mathbb{R}^3 \setminus \mathcal{K})$.

This paper is organized as follows. In the next section, we provide derivations of the energy estimates and KSS estimates for the perturbed wave equations. We also show that an appropriate variant of these holds when u is replaced by $\Gamma^\mu u$. Here $\Gamma = \{L, Z\}$ is the set of admissible vector fields. As mentioned previously, the fact that we no longer need to distinguish between L and Z is a significant innovation in this paper. In the third section, we present the decay estimates that we will require. These are fairly well-known, but in the interest of making this paper somewhat self-contained, the proofs are sketched. In the last section, we prove the main result, Theorem 1.1.

2 Energy estimates and Keel–Smith–Sogge estimates

In this section, we establish the energy and KSS estimates for the perturbed wave equation that we shall require in the sequel. We must take care to insure that our estimates will not destroy the null structure.

We will be concerned with solutions $u^I \in C^\infty(\mathbb{R}_+ \times \mathbb{R}^3 \setminus \mathcal{K})$ of the Dirichlet-wave equation

$$\begin{cases} (\square_h u)^I = F^I, \\ u|_{\partial\mathcal{K}} = 0 \end{cases} \tag{2.1}$$

where

$$(\square_h u)^I = (\partial_t^2 - c_I^2 \Delta)u^I + \sum_{J=1}^D \sum_{\alpha, \beta=0}^3 h^{IJ, \alpha\beta}(t, x) \partial_\alpha \partial_\beta u^J. \tag{2.2}$$

We shall assume that the $h^{IJ, \alpha\beta}$ satisfy the symmetry conditions

$$h^{IJ, \alpha\beta} = h^{JI, \alpha\beta} = h^{IJ, \beta\alpha}, \tag{2.3}$$

as well as the size condition

$$|h| = \sum_{I, J=1}^D \sum_{\alpha, \beta=0}^3 |h^{\alpha\beta}(t, x)| \leq \delta \ll 1. \tag{2.4}$$

We denote $u = (u^1, \dots, u^D)$. Here, we are working in the Euclidean metric, and indices are raised with this metric.

We will need to define the full energy-momentum tensor associated to (2.1). To begin, let

$$\begin{aligned} Q_{0\beta}[u] &= \partial_0 u^I \partial_\beta u^I - \frac{1}{2} \delta_{0\beta} [|\partial_0 u|^2 - c_I^2 |\nabla_x u^I|^2] \\ &\quad + \delta_{0\gamma} h^{IJ, \gamma\delta} \partial_\delta u^J \partial_\beta u^I - \frac{1}{2} \delta_{0\beta} h^{IJ, \gamma\delta} \partial_\gamma u^J \partial_\delta u^I, \end{aligned} \tag{2.5}$$

and

$$\begin{aligned} Q_{\alpha\beta}[u] &= -c_I^2 \partial_\alpha u^I \partial_\beta u^I - \frac{1}{2} \delta_{\alpha\beta} [|\partial_0 u|^2 - c_I^2 |\nabla_x u^I|^2] \\ &\quad + \delta_{\alpha\gamma} h^{IJ, \gamma\delta} \partial_\delta u^J \partial_\beta u^I - \frac{1}{2} \delta_{\alpha\beta} h^{IJ, \gamma\delta} \partial_\gamma u^J \partial_\delta u^I, \quad \alpha = 1, 2, 3. \end{aligned} \tag{2.6}$$

An elementary calculation yields

$$D^\alpha Q_{\alpha\beta}[u] = \partial_\beta u^I (\square_h u)^I + (\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J \partial_\beta u^I - \frac{1}{2} (\partial_\beta h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I. \tag{2.7}$$

2.1 Energy estimate

From (2.7), we are quickly able to obtain the well-known energy estimate for the perturbed wave equation.

Proposition 2.1 *Assume that \mathcal{K} is a bounded obstacle with C^1 -boundary. Assume also that the perturbation terms are as above. Suppose that $u \in C^\infty$ solves (2.1) and for every t , $u(t, x) = 0$ for large x . Then,*

$$\begin{aligned} \|u'(t, \cdot)\|_2^2 &\lesssim \|u'(0, \cdot)\|_2^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |(\square_h u)^I \partial_0 u^I| \, dx \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left[|(\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J \partial_0 u^I| + |(\partial_0 h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I| \right] \, dx \, ds. \end{aligned} \tag{2.8}$$

Here $u' = (\partial_t u, \nabla_x u)$ is used to denote the full space-time gradient.

Indeed, we need only examine the $\beta = 0$ components of (2.5) and (2.6). Integrating (2.7) over $S_t = [0, t] \times \mathbb{R}^3 \setminus \mathcal{K}$, it immediately follows that

$$\begin{aligned} \int_{\mathbb{R}^3 \setminus \mathcal{K}} Q_{00}[u](t, \cdot) \, dx &= \int_{\mathbb{R}^3 \setminus \mathcal{K}} Q_{00}[u](0, \cdot) \, dx + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \partial_0 u^I (\square_h u)^I \, dx \, ds \\ &+ \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} (\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J \partial_0 u^I \, dx \, ds - \frac{1}{2} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} (\partial_0 h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I. \end{aligned} \tag{2.9}$$

Here, we have used the fact that ∂_t preserves the Dirichlet boundary condition. Thus, the integrand of the boundary term that results in the application of the divergence theorem vanishes identically. If δ in (2.4) is sufficiently small, it follows that

$$(2 \max_I \{c_I^2, c_I^{-2}\})^{-1} |u'(t, x)|^2 \leq Q_{00}[u](t, x) \leq 2 \max_I \{c_I^2, c_I^{-2}\} |u'(t, x)|^2. \tag{2.10}$$

And, thus, (2.9) immediately yields (2.8).

2.2 Keel–Smith–Sogge estimates

As mentioned previously, a key estimate that allows long time existence to be deduced from decay in the spatial variables is a weighted mixed norm estimate of Keel et al. [11]. In a different context, Rodnianski [29] proved a variant of the KSS estimate using energy methods, and as was shown in [23], these methods are stable under small perturbations. The following proposition is essentially from [23]. Here, additional care is required to preserve the null structure of the equation, and we also explicitly examine the multiple speed system.

Proposition 2.2 *Suppose that \mathcal{K} is a C^1 , bounded, star-shaped obstacle as above. Suppose, further, that the perturbation terms $h^{IJ, \alpha\beta}$ are as above. Then, if $u \in C^\infty$ solves (2.1) and for every t , $u(t, x) = 0$ for large x ,*

$$\begin{aligned} &\| \langle x \rangle^{-1/2} u' \|_{L_t^2 L_x^2(S_t)}^2 + (\log(2+t))^{-1} \| \langle x \rangle^{-1/2} u' \|_{L_t^2 L_x^2(S_t)}^2 \\ &\lesssim \| u'(0, \cdot) \|_2^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\partial_{t,x} u^I| + \frac{|u^I|}{r} \right) |(\square_h u)^I| \, dx \, ds \\ &+ \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J \right| \left(|\partial_{t,r} u^I| + \frac{|u^I|}{r} \right) \, dx \, ds \\ &+ \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_{t,r} h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I \right| \, dx \, ds \\ &+ \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |\nabla u| \left(|\nabla u| + \frac{|u|}{r} \right) \, dx \, ds \end{aligned} \tag{2.11}$$

for any $t \geq 0$.

In the proposition, we are using the notation $\langle x \rangle = \langle r \rangle = \sqrt{1 + |x|^2}$. We are also using the notation $\langle x \rangle^{-1/2-\delta}$ to indicate that the estimate holds with this weight replaced by $\langle x \rangle^{-1/2-\delta}$ for any $\delta > 0$. The implicit constant depends on this δ , and in practice, we will only require the estimate for a fixed, positive δ .

Here, we contract the energy momentum tensor, (2.5) and (2.6), with a radial vector field $X = f(r)\partial_r$ which allows us to define the momentum density

$$P_\alpha[u, X] = Q_{\alpha\beta}[u]X^\beta.$$

Computing the divergence, we have

$$\begin{aligned} -D^\alpha P_\alpha[u, X] &= -(\partial_r u^I)(\square_h u)^I f(r) - (\partial_\gamma h^{IJ, \gamma\delta})\partial_\delta u^J \partial_r u^I f(r) \\ &\quad + \frac{1}{2}(\partial_r h^{IJ, \gamma\delta})\partial_\gamma u^J \partial_\delta u^I f(r) \\ &\quad + c_I^2 f'(r)(\partial_r u^I)^2 + c_I^2 \frac{f(r)}{r} |\nabla u^I|^2 - \frac{1}{2} \text{tr} \pi \left[-|\partial_0 u|^2 + c_I^2 |\nabla_x u^I|^2 \right] \\ &\quad - \frac{x_a}{r} h^{IJ, a\delta} \partial_\delta u^J \partial_r u^I f'(r) + \frac{x_a}{r} h^{IJ, a\delta} \partial_\delta u^J \partial_r u^I \frac{f(r)}{r} \\ &\quad - h^{IJ, a\delta} \partial_a u^I \partial_\delta u^J \frac{f(r)}{r} + \frac{1}{2} (\text{tr} \pi) h^{IJ, \gamma\delta} \partial_\gamma u^J \partial_\delta u^I. \end{aligned}$$

Here π denotes the deformation tensor of X and, as can be checked,

$$\text{tr} \pi = f'(r) + 2 \frac{f(r)}{r}. \tag{2.12}$$

At this point, we define the modified momentum density

$$\tilde{P}_0[u, X] = P_0[u, X] + \frac{f(r)}{r} u^I \partial_0 u^I + \frac{f(r)}{r} h^{IJ, 0\beta} u^I \partial_\beta u^J, \tag{2.13}$$

$$\begin{aligned} \tilde{P}_\alpha[u, X] &= P_\alpha[u, X] - c_I^2 \frac{f(r)}{r} u^I \partial_\alpha u^I + \frac{c_I^2}{2} \partial_\alpha \left(\frac{f(r)}{r} \right) (u^I)^2 + \frac{f(r)}{r} h^{IJ, \alpha\beta} u^I \partial_\beta u^J, \\ &\hspace{25em} \alpha = 1, 2, 3 \end{aligned} \tag{2.14}$$

A tedious but elementary calculation yields

$$\begin{aligned} -D^\alpha \tilde{P}_\alpha[u, X] &= -\left(\partial_r u^I + \frac{u^I}{r}\right)(\square_h u)^I f(r) - (\partial_\gamma h^{IJ, \gamma\delta})\partial_\delta u^J \left(\partial_r u^I + \frac{u^I}{r}\right) f(r) \\ &\quad + \frac{1}{2}(\partial_r h^{IJ, \gamma\delta})\partial_\gamma u^J \partial_\delta u^I f(r) \\ &\quad + c_I^2 f'(r)(\partial_r u^I)^2 + c_I^2 \frac{f(r)}{r} |\nabla u^I|^2 - \frac{1}{2} f'(r) \left[-|\partial_0 u|^2 + c_I^2 |\nabla_x u^I|^2 \right] \\ &\quad - \frac{x_a}{r} h^{IJ, a\delta} \partial_\delta u^J \left(\partial_r u^I + \frac{u^I}{r}\right) f'(r) + \frac{x_a}{r} h^{IJ, a\delta} \partial_\delta u^J \left(\partial_r u^I + \frac{u^I}{r}\right) \frac{f(r)}{r} \\ &\quad - h^{IJ, a\delta} \partial_a u^I \partial_\delta u^J \frac{f(r)}{r} + \frac{1}{2} f'(r) h^{IJ, \gamma\delta} \partial_\gamma u^J \partial_\delta u^I - \frac{c_I^2}{2} \Delta \left(\frac{f(r)}{r} \right) (u^I)^2 \end{aligned} \tag{2.15}$$

Integrating both sides of (2.14) in a time strip S_t gives

$$\begin{aligned} & \int_{\mathbb{R}^3 \setminus \mathcal{K}} \tilde{P}_0[u, X](0) \, dx - \int_{\mathbb{R}^3 \setminus \mathcal{K}} \tilde{P}_0[u, X](t) \, dx + \int_0^t \int_{\partial \mathcal{K}} \tilde{P}_a[u, X] n^a \, d\sigma \, ds \\ &= - \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} D^\alpha \tilde{P}_\alpha[u, X] \, dx \, ds. \end{aligned} \tag{2.16}$$

Here $\vec{n} = (n^1, n^2, n^3)$ is the outward unit normal to \mathcal{K} , and $d\sigma$ is the surface measure on $\partial \mathcal{K}$.

At this point, as in [29] and [23], we choose

$$f(r) = \frac{r}{r + \rho}$$

for a positive constant ρ . Notice, in particular, that we have $|f(r)| \lesssim 1$ and $|f'(r)| \lesssim \frac{1}{r}$. Thus, it follows that

$$\begin{aligned} \left| \int_{\mathbb{R}^3 \setminus \mathcal{K}} \tilde{P}_0[u, X](0) \, dx \right| &= \left| \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(\partial_t u(0, x) \partial_r u(0, x) f(r) \right. \right. \\ &\quad \left. \left. - f(r) h^{IJ, 0\delta}(0, x) \partial_\delta u^J(0, x) \partial_r u^I(0, x) + \frac{f(r)}{r} u^I(0, x) \partial_t u^I(0, x) \right. \right. \\ &\quad \left. \left. + \frac{f(r)}{r} h^{IJ, 0\beta}(0, x) u^I(0, x) \partial_\beta u^J(0, x) \right) \, dx \right| \\ &\lesssim \|u'(0, \cdot)\|_2^2. \end{aligned} \tag{2.17}$$

For the last inequality, we are applying the Schwarz inequality and a Hardy inequality. We are also using (2.4).

A similar bound holds for $\tilde{P}_0[u, X](t)$, and thus, using the energy inequality (2.8),

$$\begin{aligned} \left| \int_{\mathbb{R}^3 \setminus \mathcal{K}} \tilde{P}_0[u, X](t) \, dx \right| &\lesssim \|u'(t, \cdot)\|_2^2 \\ &\lesssim \|u'(0, \cdot)\|_2^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |(\square_h u)^I \partial_0 u^I| \, dx \, ds \\ &\quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left[|(\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J \partial_0 u^I| + |(\partial_0 h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I| \right] \, dx \, ds. \end{aligned} \tag{2.18}$$

Since the Dirichlet boundary conditions allow us to write $\partial_a u^I = \partial_{\vec{n}} u^I n_a$ on $\partial \mathcal{K}$ and since $\langle x, \vec{n} \rangle > 0$ on $\partial \mathcal{K}$ for star-shaped \mathcal{K} , for the spatial boundary terms,

we have

$$\int_0^t \int_{\partial\mathcal{K}} \tilde{P}_a[u, X]n^a \, d\sigma \, ds \leq -\frac{c_I^2}{4} \int_0^t \int_{\partial\mathcal{K}} \frac{f(r)}{r} (\partial_{\vec{n}} u^I)^2 \langle x, \vec{n} \rangle \, d\sigma \, ds \leq 0. \tag{2.19}$$

Here, we have also used the smallness of the perturbation, (2.4).

If we use that $\Delta(f(r)/r) \leq 0$ and (2.17)–(2.19) in (2.16), we see that

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} f'(r) c_I^2 (\partial_r u^I)^2 + \frac{f(r)}{r} c_I^2 |\nabla u^I|^2 - \frac{1}{2} f'(r) \left[-|\partial_0 u|^2 + c_I^2 |\nabla_x u^I|^2 \right] \, dx \, ds \\ & \lesssim \|u'(0, \cdot)\|_2^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\partial_{t,x} u^I| + \frac{|u^I|}{r} \right) |(\square_h u)^I| \, dx \, ds \\ & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |(\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J| \left(|\partial_{t,r} u^I| + \frac{|u^I|}{r} \right) \, dx \, ds \\ & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |(\partial_{t,r} h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I| \, dx \, ds + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |\nabla u| \left(|\nabla u| + \frac{|u|}{r} \right) \, dx \, ds. \end{aligned}$$

Since $f'(r) \leq \frac{f(r)}{r}$, this implies

$$\begin{aligned} & \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} f'(r) |\partial_t u|^2 + f'(r) c_I^2 (\partial_r u^I)^2 + \frac{f(r)}{r} c_I^2 |\nabla u^I|^2 \, dx \, ds \\ & \lesssim \|u'(0, \cdot)\|_2^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\partial_{t,x} u^I| + \frac{|u^I|}{r} \right) |(\square_h u)^I| \, dx \, ds \\ & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |(\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J| \left(|\partial_{t,r} u^I| + \frac{|u^I|}{r} \right) \, dx \, ds \\ & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |(\partial_{t,r} h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I| \, dx \, ds + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |\nabla u| \left(|\nabla u| + \frac{|u|}{r} \right) \, dx \, ds. \end{aligned} \tag{2.20}$$

By choosing $\rho = 1$ and $\rho = 2^k$ for an integer $k \geq 1$ respectively, we see that

$$\int_0^t \int_{|x| \leq 1} |u'|^2 \, dx \, ds$$

and

$$\int_0^t \int_{2^{k-1} \leq |x| \leq 2^k} \frac{|u'|^2}{r} \, dx \, ds$$

are bounded by the right side of (2.20). If we sum these resulting estimates over $k \geq 1$, we see immediately that the bound for the first term in (2.11) holds. The same argument yields the bound for the second term in the left of (2.11). Indeed, the estimate follows trivially from (2.8) when the spatial norm is over $|x| \geq t$. Thus, we need only sum over the $O(\log(2 + t))$ choices of k with $2^{k-1} \lesssim t$.

2.3 Main L^2 estimate

In this section, we show that higher order energy estimates also hold. In particular, we show that versions of (2.8) and (2.11) hold when u is replaced by $\Gamma^\mu u$. In order to do so, we introduce modified vector fields that preserve the boundary condition. This extends an idea initiated in [21].

Notice that, by combining the main results [(2.8) and (2.11)] of the preceding sections, we have

$$\begin{aligned} & \| \langle x \rangle^{-1/2-} u' \|_{L_t^2 L_x^2(S_t)}^2 + (\log(2 + t))^{-1} \| \langle x \rangle^{-1/2} u' \|_{L_t^2 L_x^2(S_t)}^2 + \| u'(t, \cdot) \|_2^2 \\ & \lesssim \| u'(0, \cdot) \|_2^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\partial_{t,x} u^I| + \frac{|u^I|}{r} \right) |F^I| \, dx \, ds \\ & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^I \right| \left(|\partial_{t,r} u^I| + \frac{|u^I|}{r} \right) \, dx \, ds \\ & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_{t,r} h^{IJ, \gamma\delta}) \partial_\gamma u^I \partial_\delta u^I \right| \, dx \, ds + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |\nabla u| \left(|\nabla u| + \frac{|u|}{r} \right) \, dx \, ds \end{aligned} \tag{2.21}$$

when u solves (2.1). Notice, in particular, that if F vanishes for $|x| > 2$, then we can bound the second term in the right side by

$$\| u' \|_{L_t^2 L_x^2(S_t \cap \{|x| \leq 2\})} \| F \|_{L_t^2 L_x^2(S_t)} \leq \varepsilon \| \langle x \rangle^{-1/2-} u' \|_{L_t^2 L_x^2(S_t)}^2 + C \| F \|_{L_t^2 L_x^2(S_t)}^2,$$

and, in this case, the first term on the right of this inequality can be bootstrapped. Here, we have used the fact that the Dirichlet boundary condition allows us to control u locally by u' . We have also used that $0 \in \mathcal{K}$, and hence, $1/r$ is bounded on $\mathbb{R}^3 \setminus \mathcal{K}$.

Thus, it immediately follows that if u is a solution to

$$\begin{cases} \square_h u = F + G \\ u|_{\partial \mathcal{K}} = 0, \end{cases} \tag{2.22}$$

and G vanishes unless $|x| \leq 2$, then

$$\begin{aligned}
 & \| \langle x \rangle^{-1/2-} u' \|_{L_t^2 L_x^2(S_t)}^2 + (\log(2+t))^{-1} \| \langle x \rangle^{-1/2} u' \|_{L_t^2 L_x^2(S_t)}^2 + \| u'(t, \cdot) \|_2^2 \\
 & \lesssim \| u'(0, \cdot) \|_2^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\partial_{t,x} u^I| + \frac{|u^I|}{r} \right) |F^I| \, dx \, ds + \| G \|_{L_t^2 L_x^2(S_t)}^2 \\
 & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_\gamma h^{IJ, \gamma\delta}) \partial_\delta u^J \right| \left(|\partial_{t,r} u^I| + \frac{|u^I|}{r} \right) \, dx \, ds \\
 & \quad + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_{t,r} h^{IJ, \gamma\delta}) \partial_\gamma u^J \partial_\delta u^I \right| \, dx \, ds + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |\nabla u| \left(|\nabla u| + \frac{|u|}{r} \right) \, dx \, ds.
 \end{aligned} \tag{2.23}$$

We will use this as a base case for an induction argument to construct higher order energy estimates.

Since ∂_t^j preserves the Dirichlet boundary condition, the estimate (2.23) holds with u replaced by $\partial_t^j u$. Moreover, if we apply elliptic regularity (see, e.g., [21] Lemma 2.3), it follows that

$$\begin{aligned}
 & \sum_{|\mu| \leq N} \| \partial^\mu u' \|_{L_t^2 L_x^2(S_t \cap \{|x| \leq 2\})}^2 \lesssim \sum_{j \leq N} \| \partial_t^j u'(0, \cdot) \|_2^2 \\
 & \quad + \sum_{j,k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\partial_t^k \partial_{t,x} u^I| + \frac{|\partial_t^k u^I|}{r} \right) |\partial_t^j F^I| \, dx \, ds + \sum_{j \leq N} \| \partial_t^j G \|_{L_t^2 L_x^2(S_t)}^2 \\
 & \quad + \sum_{j,k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\partial_t^k \partial_{t,x} u^I| + \frac{|\partial_t^k u^I|}{r} \right) |h^{(\gamma\delta)} \partial_\gamma \partial_\delta \partial_t^j u| \, dx \, ds \\
 & \quad + \sum_{j,k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_\gamma h^{IJ, \gamma\delta}) \partial_t^j \partial_\delta u^J \right| \left(|\partial_t^k \partial_{t,r} u^I| + \frac{|\partial_t^k u^I|}{r} \right) \, dx \, ds \\
 & \quad + \sum_{j,k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_{t,r} h^{IJ, \gamma\delta}) \partial_t^j \partial_\gamma u^J \partial_t^k \partial_\delta u^I \right| \, dx \, ds \\
 & \quad + \sum_{j,k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |\partial_t^j u'| \left(|\partial_t^k u'| + \frac{|\partial_t^k u|}{r} \right) \, dx \, ds + \sum_{|\mu| \leq N-1} \| \partial^\mu \square u \|_{L_t^2 L_x^2(S_t)}.
 \end{aligned} \tag{2.24}$$

It should be noted that we now require additional smoothness of the boundary of \mathcal{K} , rather than C^1 as in Propositions 2.1 and 2.2.

We will need a similar estimate involving the scaling vector field as well as derivatives. In order to obtain this, we employ a technique from [21] which introduces a modified scaling vector field $\tilde{L} = t\partial_t + \eta(x)r\partial_r$, where η is a smooth function with

$\eta(x) \equiv 0$ for $x \in \mathcal{K}$ and $\eta(x) \equiv 1$ for $|x| \geq 1$. Here, of course, we are relying on the assumption that $\mathcal{K} \subset \{|x| \leq 1\}$.

We will look to bound

$$\sum_{\substack{|\mu|+k \leq N \\ k \leq K}} \|L^k \partial^\mu u'\|_{L_t^2 L_x^2(S_t \cap \{|x| < 1\})}. \tag{2.25}$$

By elliptic regularity, this is

$$\begin{aligned} &\lesssim \sum_{\substack{j+k \leq N \\ k \leq K}} \|L^k \partial_t^j u'\|_{L_t^2 L_x^2(S_t \cap \{|x| < 3/2\})} + \sum_{\substack{|\mu|+k \leq N-1 \\ k \leq K}} \|L^k \partial^\mu \square u\|_{L_t^2 L_x^2(S_t)} \\ &\lesssim \sum_{\substack{j+k \leq N \\ k \leq K}} \|(\tilde{L}^k \partial_t^j u)'\|_{L_t^2 L_x^2(S_t \cap \{|x| < 3/2\})} + \sum_{\substack{|\mu|+k \leq N \\ k \leq K-1}} \|L^k \partial^\mu u'\|_{L_t^2 L_x^2(S_t \cap \{|x| < 3/2\})} \\ &\quad + \sum_{\substack{|\mu|+k \leq N-1 \\ k \leq K}} \|L^k \partial^\mu \square u\|_{L_t^2 L_x^2(S_t)}. \end{aligned} \tag{2.26}$$

If $P = P(t, x, D_t, D_x)$ is a differential operator, we fix the notation (as in [21]):

$$[P, h^{\gamma\delta} \partial_\gamma \partial_\delta]u = \sum_{1 \leq I, J \leq D} \sum_{0 \leq \gamma, \delta \leq 3} [P, h^{IJ, \gamma\delta} \partial_\gamma \partial_\delta]u^J.$$

Since

$$\begin{aligned} [\square_h, \tilde{L}^k \partial_t^j]u &= [\square, \tilde{L}^k \partial_t^j]u + [h^{\gamma\delta} \partial_\gamma \partial_\delta, \tilde{L}^k \partial_t^j]u \\ &= [\square, L^k] \partial_t^j u + [\square, (\tilde{L}^k - L^k)] \partial_t^j u + [h^{\gamma\delta} \partial_\gamma \partial_\delta, \tilde{L}^k \partial_t^j]u \end{aligned}$$

and since $\tilde{L}^k \partial_t^j u$ satisfies the Dirichlet boundary condition, in order to bound the first term in the right side of (2.26) we can apply (2.23) with F replaced by

$$\tilde{L}^k \partial_t^j F + [h^{\gamma\delta} \partial_\gamma \partial_\delta, \tilde{L}^k \partial_t^j]u + [\square, L^k] \partial_t^j u$$

and G by

$$\tilde{L}^k \partial_t^j G + [\square, \tilde{L}^k - L^k] \partial_t^j u,$$

which is supported in $|x| < 2$. Thus, it follows that

$$\begin{aligned} &\sum_{\substack{|\mu|+k \leq N \\ k \leq K}} \|L^k \partial^\mu u'\|_{L_t^2 L_x^2(S_t \cap \{|x| < 1\})}^2 \lesssim \sum_{\substack{|\mu|+k \leq N \\ k \leq K}} \|L^k \partial^\mu u'(0, \cdot)\|_2^2 \\ &\quad + \sum_{\substack{|\mu|+j \leq N \\ j \leq K}} \sum_{\substack{|\nu|+k \leq N \\ k \leq K}} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} (|L^j \partial^\mu \partial u^l| + \frac{|L^j \partial^\mu u^l|}{r}) |L^k \partial^\nu F^l| \, dx \, ds \\ &\quad + \sum_{\substack{|\mu|+j \leq N \\ j \leq K}} \sum_{\substack{l+k \leq N \\ k \leq K}} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} (|L^j \partial^\mu \partial u^l| + \frac{|L^j \partial^\mu u^l|}{r}) |[h^{IJ, \gamma\delta} \partial_\gamma \partial_\delta, \tilde{L}^k \partial_t^l]u^J| \, dx \, ds \end{aligned}$$

$$\begin{aligned}
 & + \sum_{\substack{|\mu|+j \leq N \\ j \leq K}} \sum_{\substack{l+k \leq N-1 \\ k \leq K-1}} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|L^j \partial^\mu \partial u^l| + \frac{|L^j \partial^\mu u^l|}{r} \right) |L^k \partial_t^l (\square u^l)| \, dx \, ds \\
 & + \sum_{\substack{|\mu|+j \leq N \\ j \leq K}} \sum_{\substack{|\nu|+k \leq N \\ k \leq K}} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_\gamma h^{J, \gamma \delta}) \partial_\delta (\tilde{L}^j \partial^\mu u^l) \right| \left(|L^k \partial^\nu \partial_{t,r} u^l| + \frac{|L^k \partial^\nu u^l|}{r} \right) \, dx \, ds \\
 & + \sum_{\substack{|\mu|+j \leq N \\ j \leq K}} \sum_{\substack{|\nu|+k \leq N \\ k \leq K}} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_{t,r} h^{J, \gamma \delta}) \partial_\gamma (\tilde{L}^j \partial^\mu u^l) \partial_\delta (\tilde{L}^k \partial^\nu u^l) \right| \, dx \, ds \\
 & + \sum_{\substack{|\mu|+j \leq N \\ j \leq K}} \sum_{\substack{|\nu|+k \leq N \\ k \leq K}} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |L^j \partial^\mu u^l| \left(|L^k \partial^\nu u^l| + \frac{|L^k \partial^\nu u^l|}{r} \right) \, dx \, ds \\
 & + \sum_{\substack{|\mu|+k \leq N \\ k \leq K}} \|L^k \partial^\mu G\|_{L_t^2 L_x^2(S_t)}^2 + \sum_{\substack{|\mu|+k \leq N-1 \\ k \leq K}} \|L^k \partial^\mu \square u\|_{L_t^2 L_x^2(S_t)}^2 \\
 & + \sum_{\substack{|\mu|+k \leq N \\ k \leq K-1}} \|L^k \partial^\mu u^l\|_{L_t^2 L_x^2(S_t \cap \{|x| < 3/2\})}^2 \tag{2.27}
 \end{aligned}$$

for solutions u to (2.22). If we argue recursively, the same bound holds with the last term replaced by

$$\sum_{|\mu| \leq N} \|\partial^\mu u^l\|_{L_t^2 L_x^2(S_t \cap \{|x| < 2\})}^2.$$

Thus, in order to control this last term, we may apply (2.24). A similar argument can be used to bound

$$\sum_{|\mu|+k \leq N} \|L^k \partial^\mu u^l(t, \cdot)\|_{L^2(\{|x| < 1\})}.$$

Moreover, since $K \leq N$ is arbitrary, we have shown

Lemma 2.3 *Suppose that \mathcal{K} is a smooth, bounded, star-shaped obstacle as above. Suppose further that the perturbation terms $h^{J, \alpha\beta}$ are as above. Then, if $u \in C^\infty$ solves (2.22) and vanishes for large x for every t and G is supported in $|x| < 2$,*

$$\begin{aligned}
 & \sum_{|\mu|+j \leq N} \|L^j \partial^\mu u^l(t, \cdot)\|_{L^2(\{|x| < 1\})} + \sum_{|\mu|+j \leq N} \|L^j \partial^\mu u^l\|_{L_t^2 L_x^2(S_t \cap \{|x| < 1\})}^2 \\
 & \lesssim \sum_{|\mu|+j \leq N} \|L^j \partial^\mu u^l(0, \cdot)\|_2^2 \\
 & + \sum_{|\mu|+j \leq N} \sum_{|\nu|+k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|L^j \partial^\mu \partial u^l| + \frac{|L^j \partial^\mu u^l|}{r} \right) |L^k \partial^\nu F^l| \, dx \, ds
 \end{aligned}$$

$$\begin{aligned}
 & + \sum_{|\mu|+j \leq N} \sum_{l+k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|L^j \partial^\mu \partial u^l| + \frac{|L^j \partial^\mu u^l|}{r} \right) |[h^{JJ, \gamma \delta} \partial_\gamma \partial_\delta, \tilde{L}^k \partial_t^l] u^j| \, dx \, ds \\
 & + \sum_{|\mu|+j \leq N} \sum_{l+k \leq N-1} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|L^j \partial^\mu \partial u^l| + \frac{|L^j \partial^\mu u^l|}{r} \right) |L^k \partial_t^l (\square u)^l| \, dx \, ds \\
 & + \sum_{|\mu|+j \leq N} \sum_{|v|+k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_\gamma h^{JJ, \gamma \delta}) \partial_\delta (\tilde{L}^j \partial^\mu u^j) \right| \left(|L^k \partial^v \partial_{t,r} u^l| + \frac{|L^k \partial^v u^l|}{r} \right) \, dx \, ds \\
 & + \sum_{|\mu|+j \leq N} \sum_{|v|+k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_{t,r} h^{JJ, \gamma \delta}) \partial_\gamma (\tilde{L}^j \partial^\mu u^j) \partial_\delta (\tilde{L}^k \partial^v u^l) \right| \, dx \, ds \\
 & + \sum_{|\mu|+j \leq N} \sum_{|v|+k \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |L^j \partial^\mu u^j| \left(|L^k \partial^v u^l| + \frac{|L^k \partial^v u^l|}{r} \right) \, dx \, ds \\
 & + \sum_{|\mu|+j \leq N} \|L^j \partial^\mu G\|_{L_t^2 L_x^2(S_t)}^2 + \sum_{|\mu|+j \leq N-1} \|L^j \partial^\mu \square u\|_{L_t^2 L_x^2(S_t)}^2 \\
 & + \sum_{|\mu|+j \leq N-1} \|L^j \partial^\mu \square u(t, \cdot)\|_2^2 \tag{2.28}
 \end{aligned}$$

for any $N \geq 0$ and for every $t > 0$.

We use this to show that a version of (2.23) holds when u is replaced by $\Gamma^\mu u$ where $\Gamma = \{L, \Omega, \partial\}$ is the set of ‘‘admissible’’ vector fields. With η as in the definition of \tilde{L} , we set $\tilde{\Omega}_{ij} = \eta(x)\Omega_{ij}$, $1 \leq i < j \leq 3$ and $\tilde{\partial}_i = \eta(x)\partial_i$, $i = 1, 2, 3$. Similarly, we set $\tilde{\Gamma} = \{\tilde{L}, \tilde{\Omega}, \tilde{\partial}_i, \partial_t\}$ to be the set of boundary-preserving vector fields.

Theorem 2.4 *Suppose \mathcal{K} is a smooth, bounded, star-shaped obstacle as above. Suppose further that the perturbation terms $h^{JJ, \alpha\beta}$ are as above. Then, if $u \in C^\infty$ solves (2.22) and vanishes for large x for every t and G is supported in $|x| < 2$,*

$$\begin{aligned}
 & \sum_{|\mu| \leq N} \|\langle x \rangle^{-1/2} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)}^2 + (\log(2+t))^{-1} \sum_{|\mu| \leq N} \|\langle x \rangle^{-1/2} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)}^2 \\
 & + \sum_{|\mu| \leq N} \|\Gamma^\mu u'(t, \cdot)\|_2 \lesssim \sum_{|\mu| \leq N} \|\Gamma^\mu u'(0, \cdot)\|_2^2 \\
 & + \sum_{|\mu|, |v| \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\Gamma^\mu \partial u^l| + \frac{|\Gamma^\mu u^l|}{r} \right) |\Gamma^v F^l| \, dx \, ds \\
 & + \sum_{|\mu|, |v| \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\Gamma^\mu \partial u^l| + \frac{|\Gamma^\mu u^l|}{r} \right) |[h^{JJ, \gamma \delta} \partial_\gamma \partial_\delta, \Gamma^v] u^l| \, dx \, ds
 \end{aligned}$$

$$\begin{aligned}
 &+ \sum_{|\mu| \leq N, |\nu| \leq N-1} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(|\Gamma^\mu \partial u^I| + \frac{|\Gamma^\mu u^I|}{r} \right) |\Gamma^\nu (\square u)^I| \, dx \, ds \\
 &+ \sum_{|\mu|, |\nu| \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |(\partial_\gamma h^{I, \nu^\delta}) \partial_\delta (\Gamma^\mu u^J)| \left(|\Gamma^\nu \partial u^I| + \frac{|\Gamma^\nu u^I|}{r} \right) \, dx \, ds \\
 &+ \sum_{|\mu|, |\nu| \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left| (\partial_{t,r} h^{I, \nu^\delta}) \partial_\gamma (\Gamma^\mu u^J) \partial_\delta (\Gamma^\nu u^I) \right| \, dx \, ds \\
 &+ \sum_{\substack{|\mu| + |\sigma| \leq N \\ |\nu| \leq N}} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K} \cap \{|x| < 1\}} |\Gamma^\sigma h| |\Gamma^\mu u^I| \left(|\Gamma^\nu u^I| + \frac{|\Gamma^\nu u|}{r} \right) \, dx \, ds \\
 &+ \sum_{|\mu|, |\nu| \leq N} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|h|}{\langle x \rangle} |\Gamma^\mu u^I| \left(|\Gamma^\nu u^I| + \frac{|\Gamma^\nu u|}{r} \right) \, dx \, ds + \sum_{|\mu| \leq N} \|\Gamma^\mu G\|_{L_t^2 L_x^2(S_t)}^2 \\
 &+ \sum_{|\mu| \leq N-1} \|\Gamma^\mu \square u\|_{L_t^2 L_x^2(S_t)}^2 + \sum_{|\mu| \leq N-1} \|\Gamma^\mu \square u(t, \cdot)\|_2^2 \tag{2.29}
 \end{aligned}$$

for any fixed $N \geq 0$ and any $t \geq 0$.

To show this, we argue inductively in N , and the case $N = 0$ clearly follows from (2.23). Let us show the bound for the first term on the left side. Similar arguments will yield the full estimate.

We begin by observing that

$$\begin{aligned}
 \sum_{|\mu| \leq N} \|\langle x \rangle^{-1/2} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)} &\lesssim \sum_{|\mu| \leq N-1} \|\langle x \rangle^{-1/2} \Gamma^\mu (\tilde{\Gamma} u)'\|_{L_t^2 L_x^2(S_t)} \\
 &+ \sum_{|\mu| + j \leq N} \|L^j \partial^\mu u'\|_{L_t^2 L_x^2(S_t \cap \{|x| < 1\})}. \tag{2.30}
 \end{aligned}$$

The bound for the last term clearly follows from (2.28).

To estimate the first term in the right side of (2.30), we begin by noticing that

$$\begin{aligned}
 \square_h \tilde{\Gamma} u &= \tilde{\Gamma} \square_h u + [\square_h, \tilde{\Gamma}] u \\
 &= \tilde{\Gamma} \square_h u + [\square, \tilde{\Gamma}] u + [h^{\nu^\delta} \partial_\gamma \partial_\delta, \tilde{\Gamma}] u \\
 &= \tilde{\Gamma} \square_h u + [\square, \Gamma] u + [\square, \tilde{\Gamma} - \Gamma] u + [h^{\nu^\delta} \partial_\gamma \partial_\delta, \tilde{\Gamma}] u. \tag{2.31}
 \end{aligned}$$

Thus, we will apply the inductive hypothesis to $\tilde{\Gamma} u$ with F replaced by

$$\tilde{\Gamma} F + [\square, \Gamma] u + [h^{\nu^\delta} \partial_\gamma \partial_\delta, \tilde{\Gamma}] u$$

and G by the compactly supported function

$$\tilde{\Gamma} G + [\square, \tilde{\Gamma} - \Gamma] u.$$

It follows that the first term in the right side of (2.30) is dominated by the right side of (2.29) plus

$$\sum_{|\mu|+j \leq N} \|L^j \partial^\mu u'\|^2_{L_t^2 L_x^2(S_t \cap \{|x| < 1\})}$$

since the coefficients of Z are $O(1)$ in $\{|x| < 1\}$. Thus, another application of (2.28) yields the desired estimate.

3 Decay estimates

Classically, the necessary decay to prove long-time existence is afforded to us by the Klainerman-Sobolev inequalities (see [13]; see also [6],[27]). These inequalities, however, require the use of the Lorentz rotations which does not seem permissible in the current setting. In order to get around this, we will rely on decay in $|x|$ (which meshes well with the KSS estimates from the previous section) obtained by a weighted Sobolev inequality and decay in $t - |x|$ that follows from (variants of) estimates of Klainerman and Sideris [15].

3.1 Null form estimates

We begin by providing the well-known decay that is obtained when employing the null condition. The proof that we present is essentially from [26].

Lemma 3.1 *Assume that the null condition, (1.3), is satisfied. Let $c_0 = \min\{c_I/2 : I = 1, \dots, D\}$. Then, for $|x| \geq c_0 t/2$,*

$$\begin{aligned} & |B_{K,\gamma}^{KK,\alpha\beta} \partial_\gamma u^K \partial_\alpha \partial_\beta v^K| \\ & \lesssim \frac{1}{\langle t + |x| \rangle} \left[|\Gamma u^K| |\partial^2 v^K| + |\partial u^K| |\partial \Gamma v^K| + \langle c_K t - r \rangle |\partial u^K| |\partial^2 v^K| \right], \end{aligned} \tag{3.1}$$

$$\begin{aligned} & |B_{K,\gamma}^{KK,\alpha\beta} \partial_\alpha \partial_\gamma u^K \partial_\beta v^K| \\ & \lesssim \frac{1}{\langle t + |x| \rangle} \left[|\Gamma v^K| |\partial^2 u^K| + |\partial v^K| |\partial \Gamma u^K| + \langle c_K t - r \rangle |\partial v^K| |\partial^2 u^K| \right], \end{aligned} \tag{3.2}$$

and

$$\begin{aligned} |B_{K,\gamma}^{KK,\alpha\beta} \partial_\alpha u^K \partial_\beta v^K \partial_\gamma w^K| & \lesssim \frac{1}{\langle t + |x| \rangle} \left[|\Gamma u^K| |\partial v^K| |\partial w^K| + |\partial u^K| |\Gamma v^K| |\partial w^K| \right. \\ & \quad \left. + |\partial u^K| |\partial v^K| |\Gamma w^K| + \langle c_K t - r \rangle |\partial u^K| |\partial v^K| |\partial w^K| \right]. \end{aligned} \tag{3.3}$$

While (3.2) did not appear explicitly in [26], it is used there. It is easily seen to follow from the same argument as the other bounds. We will show (3.1). The other estimates result from similar arguments.

It suffices to consider the case $|t, x| \geq 1$ as the bounds are otherwise trivial. We first decompose the spatial gradient into radial and angular parts:

$$\nabla_x = \frac{x}{r} \partial_r - \frac{x}{r^2} \wedge \Omega,$$

where $\Omega = x \wedge \partial_x$ and \wedge denotes the usual vector cross product. Introducing the operators $D^\pm = \frac{1}{2}(\partial_t \pm c_K \partial_r)$ and the null vectors $Y^\pm = (1, \pm x/c_K r)$, we can further decompose

$$(\partial_t, \nabla_x) = (Y^- D^- + Y^+ D^+) - \left(0, \frac{x}{r^2} \wedge \Omega\right),$$

or alternately

$$\begin{aligned} \partial_{t,x} &= Y^- D^- - \frac{c_K t - r}{c_K t + r} Y^+ D^- + \frac{c_K}{c_K t + r} Y^+ L - \left(0, \frac{x}{r^2} \wedge \Omega\right) \\ &= Y^- D^- + R, \end{aligned}$$

where

$$|Ru| \lesssim \langle r \rangle^{-1} |\Gamma u| + \frac{\langle c_K t - r \rangle}{t + r} |\partial u|. \tag{3.4}$$

Thus, we have

$$\begin{aligned} B_{K,\gamma}^{KK,\alpha\beta} \partial_\gamma u^K \partial_\alpha \partial_\beta v^K &= B_{K,\gamma}^{KK,\alpha\beta} [Y_\alpha^- Y_\beta^- Y_\gamma^- D^- u^K (D^-)^2 v^K + R_\gamma u^K \partial_\alpha \partial_\beta v^K \\ &\quad + Y_\gamma^- D^- u^K R_\alpha \partial_\beta v^K + Y_\gamma^- D^- u^K Y_\alpha^- D^- R_\beta v^K]. \end{aligned}$$

Since $(Y_0^-)^2/c_K^2 - (Y_1^-)^2 - (Y_2^-)^2 - (Y_3^-)^2 = 0$, by (1.3), the first term must vanish. The remaining bounds follow from (3.4).

3.2 Weighted Sobolev estimates

The first estimate is a now standard weighted Sobolev inequality. See [13]. The reader is also encouraged to see [11] for the first example of how this decay can be paired with KSS estimates to yield long time existence for nonlinear equations.

Lemma 3.2 *Suppose $h \in C^\infty(\mathbb{R}^3)$. Then, for $R > 1$*

$$\|h\|_{L^\infty(R/2 < |x| < R)} \lesssim R^{-1} \sum_{|\alpha| \leq 2} \|Z^\alpha h\|_{L^2(R/4 < |x| < 2R)}. \tag{3.5}$$

For $|x| \in (R/2, R)$, we apply Sobolev’s estimate for $\mathbb{R} \times S^2$ to see that

$$|h(x)| \lesssim \sum_{|\alpha|+j \leq 2} \left(\int_{|x|-1/4}^{|x|+1/4} \int_{S^2} |\partial_r^j \Omega^\alpha h(r\omega)|^2 dr d\omega \right)^{1/2}.$$

Since the volume element in \mathbb{R}^3 is a constant times $r^2 dr d\omega$, this is dominated by the right side of (3.5) as desired.

The second of the necessary Sobolev type estimates follows essentially from that in [25] (Lemma 3.3).

Lemma 3.3 *Let $u \in C^\infty(\mathbb{R}^3 \setminus \mathcal{K})$ and suppose that u vanishes on $\partial\mathcal{K}$ and for large x for every t . Then,*

$$r^{1/2}|u(t,x)| \lesssim \sum_{|\mu| \leq 1} \|Z^\mu u'(t, \cdot)\|_2. \tag{3.6}$$

Moreover,

$$r^{1/2} \sum_{|\nu| \leq N} |\Gamma^\nu u(t,x)| \lesssim \sum_{|\nu| \leq N+1} \|\Gamma^\nu u'(t, \cdot)\|_2 \tag{3.7}$$

for any $N \geq 0$.

We first note that the Dirichlet boundary condition allows us to control u locally by u' . Thus, over $|x| \leq 1$, the result follows trivially from the standard Sobolev estimates.

In the remaining region, $|x| \geq 1$, (3.6) is a consequence of the arguments in [25]. We write $x = r\omega$ where $\omega \in S^2$ (and $d\omega$ denotes the surface measure of this unit sphere). We begin by noting that

$$r^2|u(t,x)|^4 \lesssim \sum_{|\mu| \leq 1} r^2 \|\Omega^\mu u(t, r \cdot)\|_{L^4(S^2)}^4$$

follows from a basic Sobolev estimate. By the fundamental theorem of calculus, Hölder’s inequality, and the standard Sobolev estimate $\|h\|_6 \lesssim \|\nabla h\|_2$, it follows that

$$\begin{aligned} r^2 \int_{S^2} |v(t, r\omega)|^4 d\omega &\lesssim r^2 \int_r^\infty \int_{S^2} |\partial_r v(\rho\omega)| |v(\rho\omega)|^3 d\rho d\omega \\ &\lesssim \|\partial_r v(t, \cdot)\|_2 \|v(t, \cdot)\|_6^3 \lesssim \|\nabla_x v(t, \cdot)\|_2^4 \end{aligned}$$

which yields (3.6) when v is replaced by $\Omega^\mu u$.

When $|x| \geq 1$, (3.7) follows from the same argument as that for (3.6). Since the coefficients of Z are $O(1)$ for $|x| \leq 1$, it only remains to show that

$$\sum_{|\nu| \leq N} |L^\nu u(t,x)| \lesssim \sum_{|\nu| \leq N+1} \|\Gamma^\nu u'(t, \cdot)\|_2, \quad |x| \leq 1. \tag{3.8}$$

By Sobolev’s lemma, we have that for $|x| \leq 1$,

$$\sum_{|\nu| \leq N} |L^\nu u(t,x)| \lesssim \sum_{|\nu| \leq N+1} \|\Gamma^\nu u'(t, \cdot)\|_2 + \sum_{|\nu| \leq N} \|L^\nu u(t, \cdot)\|_{L^2(\{|x| < 2\})}.$$

The last term in the right side is

$$\lesssim \sum_{|\nu| \leq N} \|(t\partial_t)^\nu u(t, \cdot)\|_{L^2(\{|x| < 2\})} + \sum_{|\nu| \leq N-1} \|L^\nu u'(t, \cdot)\|_{L^2(\{|x| < 2\})}.$$

Since ∂_t preserves the Dirichlet boundary conditions, it follows from the Fundamental Theorem of Calculus that the former term is

$$\lesssim \sum_{|\nu| \leq N} \|(t\partial_t)^\nu u'(t, \cdot)\|_{L^2(\{|x| < 2\})} \lesssim \sum_{|\nu| + |\mu| \leq N} \|L^\nu \partial^\mu u'(t, \cdot)\|_{L^2(\{|x| < 2\})}$$

which completes the proof of (3.7).

3.3 Klainerman-Sideris estimates

We finally present some estimates from [15] and some consequences of these estimates. These estimates are the ones that provide any required decay in the time variable t .

We begin with the following basic estimate from [15],

$$\langle c_K t - r \rangle \left(|\partial_t \partial u^K| + |\Delta u^K| \right) \lesssim \sum_{|\mu| \leq 1} |\Gamma^\mu u'| + \langle t + r \rangle |\square u|. \tag{3.9}$$

Moreover, using integration by parts, it was shown that

$$\| \langle c_K t - r \rangle \partial^2 v^K(t, \cdot) \|_2 \lesssim \sum_{|\mu| \leq 1} \| \Gamma^\mu v'(t, \cdot) \|_2 + \| \langle t + r \rangle \square v(t, \cdot) \|_2 \tag{3.10}$$

when there is no boundary. Moreover, if one applies the boundaryless analog of (3.6) to $\langle c_K t - r \rangle \partial v^K$ and uses (3.10), the following is obtained,

$$r^{1/2} \langle c_K t - r \rangle |\partial v^K(t, x)| \lesssim \sum_{|\mu| \leq 2} \| \Gamma^\mu v'(t, \cdot) \|_2 + \sum_{|\mu| \leq 1} \| \langle t + r \rangle \Gamma^\mu \square v(t, \cdot) \|_2, \tag{3.11}$$

which first appeared in Hidano and Yokoyama [5].

When there is a boundary, the integration by parts argument in [15] does not yield (3.10). We will, however, require analogous estimates. The first three are from [19]. The first follows from applying (3.10) to $\eta(x)u(t, x)$ where η is a smooth cutoff that vanishes for $|x| \leq 1$ and is identically one when $|x| \geq 3/2$.

$$\begin{aligned} \sum_{|\mu| \leq N} \| \langle c_K t - r \rangle \Gamma^\mu \partial^2 u^K(t, \cdot) \|_2 &\lesssim \sum_{|\mu| \leq N+1} \| \Gamma^\mu u'(t, \cdot) \|_2 \\ &+ \sum_{|\mu| \leq N} \| \langle t + r \rangle \Gamma^\mu \square u(t, \cdot) \|_2 + t \sum_{|\mu| \leq N} \| \Gamma^\mu u'(t, \cdot) \|_{L^2(\{|x| < 1\})}. \end{aligned} \tag{3.12}$$

Moreover, by combining (3.5), (3.12), and elliptic regularity (cf. [19]), one obtains

$$\begin{aligned} r \langle c_K t - r \rangle \sum_{|\mu| \leq N} |\Gamma^\mu \partial^2 u^K| &\lesssim \sum_{|\mu| \leq N+3} \| \Gamma^\mu u'(t, \cdot) \|_2 \\ &+ \sum_{|\mu| \leq N+2} \| \langle t + r \rangle \Gamma^\mu \square u(t, \cdot) \|_2 + t \sum_{|\mu| \leq N} \| \Gamma^\mu u'(t, \cdot) \|_{L^2(\{|x| < 1\})}. \end{aligned} \tag{3.13}$$

Finally, by applying (3.6) to the cutoff solution and using (3.12), we can obtain the following analog of the estimate from [5].

$$\begin{aligned} r^{1/2} \langle c_K t - r \rangle \sum_{|\mu| \leq N} |\Gamma^\mu \partial u^K| &\lesssim \sum_{|\mu| \leq N+2} \| \Gamma^\mu u'(t, \cdot) \|_2 + \sum_{|\mu| \leq N+1} \| \langle t + r \rangle \Gamma^\mu \square u(t, \cdot) \|_2 \\ &+ t \sum_{|\mu| \leq N} \| \Gamma^\mu u'(t, \cdot) \|_{L^2(\{|x| < 1\})}. \end{aligned} \tag{3.14}$$

As in [21], in a region $|x| \geq (c_0/2)t$, the boundary terms are no longer required. In particular, we have

$$\sum_{|\mu| \leq N} \|(c_K t - r)\partial^2 \Gamma^\mu u^K(t, \cdot)\|_{L^2(|x| \geq c_0 t/2)} \lesssim \sum_{|\mu| \leq N+1} \|\Gamma^\mu u'(t, \cdot)\|_2 + \sum_{|\mu| \leq N} \|(t+r)\Gamma^\mu \square u(t, \cdot)\|_{L^2(|x| \geq c_0 t/4)}, \tag{3.15}$$

$$r\langle c_K t - r \rangle \sum_{|\mu| \leq N} |\partial^2 \Gamma^\mu u^K(t, x)| \lesssim \sum_{|\mu| \leq N+3} \|\Gamma^\mu u'(t, \cdot)\|_2 + \sum_{|\mu| \leq N+2} \|(t+r)\Gamma^\mu \square u(t, \cdot)\|_{L^2(|x| \geq c_0 t/4)}, \tag{3.16}$$

and

$$r^{1/2}\langle c_K t - r \rangle \sum_{|\mu| \leq N} |\partial \Gamma^\mu u^K(t, x)| \lesssim \sum_{|\mu| \leq N+2} \|\Gamma^\mu u'(t, \cdot)\|_2 + \sum_{|\mu| \leq N+1} \|(t+r)\Gamma^\mu \square u(t, \cdot)\|_{L^2(|x| \geq c_0 t/4)}. \tag{3.17}$$

Indeed, we now fix $\eta \in C^\infty(\mathbb{R}^3)$ satisfying $\eta(x) \equiv 1, |x| > 1/2$ and $\eta(x) \equiv 0$ for $|x| < 1/4$. We then set $v(t, x) = \eta(x/(c_0(t)))u(t, x)$ and apply (3.10) and (3.11).

4 Global existence

In this section, we prove our main result, Theorem 1.1. Here, we shall choose $N = 30$, but this is far from optimal. As in [26], the proof proceeds by examining a coupling between a low-order energy and a higher-order energy.

$$\sum_{|\mu| \leq 20} \left(\|\Gamma^\mu u'(t, \cdot)\|_2 + \|\langle x \rangle^{-5/8} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)} \right) \leq A\varepsilon \tag{4.1}$$

$$\sum_{|\mu| \leq 30} \left(\|\Gamma^\mu u'(t, \cdot)\|_2 + \|\langle x \rangle^{-5/8} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)} \right) \leq B\varepsilon(1+t)^{c\varepsilon}. \tag{4.2}$$

Here, A is chosen to be 10 times greater than the square root of the implicit constant in (2.29). The exponent $5/8$ was chosen to make the argument explicit. The same argument would hold for sufficiently small ε for any exponent p with $1/2 < p < 3/4$.

There are two steps required in order to complete the continuity argument:

- (i.) Show (4.1) holds with A replaced by $A/2$,
- (ii.) Show that (4.2) follows from (4.1).

Throughout the remainder of the argument, we will be applying (2.29) with

$$h^{IJ, \alpha\beta} = -B_{K, \gamma}^{IJ, \alpha\beta} \partial_\gamma u^K$$

and $F = G = 0$.

4.1 Preliminaries

Before beginning the proofs of (i.) and (ii.), we establish some preliminary estimates. These are shown assuming (4.1), and both are used to control terms that appear after applications of the decay estimates.

The first is a lower order version. We will establish:

$$\sum_{|\mu| \leq 19} \|\langle t+r \rangle \Gamma^\mu \square u(t, \cdot)\|_2 \lesssim \varepsilon^2 + t\varepsilon \sum_{|\mu| \leq 11} \|\Gamma^\mu u'(t, \cdot)\|_{L^2(|x| < 1)}. \tag{4.3}$$

The left side of (4.3) is clearly controlled by

$$\sum_{|\mu| \leq 11, |\nu| \leq 20} \|\langle t+r \rangle \Gamma^\mu u'(t, \cdot) \Gamma^\nu u'(t, \cdot)\|_2.$$

When the norm is taken over $|x| \geq c_0 t/2$, we can apply (3.5) and (4.1) to see that this is $O(\varepsilon^2)$. When the norm is over $|x| \leq c_0 t/2$, we apply (3.14) to see that this is

$$\begin{aligned} & \sum_{|\nu| \leq 20} \|\Gamma^\nu u'(t, \cdot)\|_2 \left(\sum_{|\mu| \leq 14} \|\Gamma^\mu u'(t, \cdot)\|_2 + \sum_{|\mu| \leq 13} \|\langle t+r \rangle \Gamma^\mu \square u(t, \cdot)\|_2 \right) \\ & + t \sum_{|\mu| \leq 11} \|\Gamma^\mu u'(t, \cdot)\|_{L^2(|x| < 1)} \lesssim \varepsilon^2 + \varepsilon \sum_{|\mu| \leq 13} \|\langle t+r \rangle \Gamma^\mu \square u(t, \cdot)\|_2 \\ & + \varepsilon t \sum_{|\mu| \leq 11} \|\Gamma^\mu u'(t, \cdot)\|_{L^2(|x| < 1)}. \end{aligned}$$

The last inequality follows from (4.1). Since the second term on the right can be bootstrapped if ε is sufficiently small, we see that this yields (4.3).

From this proof, it is easy to see that we also have

$$\sum_{|\mu| \leq 19} \|\langle t+r \rangle \Gamma^\mu \square u(t, \cdot)\|_{L^2(|x| \geq c_0 t/4)} \lesssim \varepsilon^2. \tag{4.4}$$

We will additionally require the related higher order estimate

$$\sum_{|\mu| \leq 29} \|\langle t+r \rangle \Gamma^\mu \square u(t, \cdot)\|_2 \lesssim \left(\varepsilon + t \sum_{|\mu| \leq 15} \|\Gamma^\mu u'(t, \cdot)\|_{L^2(|x| < 1)} \right) \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(t, \cdot)\|_2. \tag{4.5}$$

Plugging in our nonlinearity in the left, this is

$$\lesssim \sum_{|\mu| \leq 30, |\nu| \leq 15} \|\langle t+r \rangle \Gamma^\mu u' \Gamma^\nu u'\|_2.$$

If $|x| \geq c_0t/2$, applying (3.5) and (4.1) results in the bound

$$\lesssim \varepsilon \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(t, \cdot)\|_2.$$

If $|x| < c_0t/2$, we apply (3.14) to see that this is

$$\begin{aligned} &\lesssim \sum_{|\mu| \leq 17} \|\Gamma^\mu u'(t, \cdot)\| \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(t, \cdot)\|_2 + \sum_{|\mu| \leq 16} \|(t+r)\Gamma^\mu \square u(t, \cdot)\|_2 \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(t, \cdot)\|_2 \\ &+ \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(t, \cdot)\|_2 \sum_{|\nu| \leq 15} t \|\Gamma^\nu u'(t, \cdot)\|_{L^2(|x| < 1)}. \end{aligned}$$

By applying (4.1) and (4.3), we indeed see that (4.5) follows.

Again, the same proof also yields

$$\sum_{|\mu| \leq 29} \|(t+r)\Gamma^\mu \square u(t, \cdot)\|_{L^2(|x| \geq c_0t/2)} \lesssim \varepsilon \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(t, \cdot)\|_2. \tag{4.6}$$

4.2 Low order energy

Here, while assuming (4.1) and (4.2), we must show that (4.1) holds with A replaced by $A/2$. Using (1.4) and (2.29), the square of the left side of (4.1) is easily seen to be

$$\begin{aligned} &\leq (A\varepsilon/10)^2 + C \sum_{|\mu| \leq 20} \sum_{|\nu| \leq 19, |\sigma| \leq 10} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |\Gamma^\mu \partial u^I| |\tilde{B}_{K,\gamma}^{I,\alpha\beta} \partial_\gamma (\Gamma^\sigma u^K) \partial_\alpha \partial_\beta (\Gamma^\nu u^J)| \, dx \, ds \\ &+ C \sum_{|\mu|, |\sigma| \leq 20} \sum_{|\nu| \leq 10} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |\Gamma^\nu \partial u^I| |\tilde{B}_{K,\gamma}^{I,\alpha\beta} \partial_\gamma (\Gamma^\sigma u^K) \partial_\alpha \partial_\beta (\Gamma^\nu u^J)| \, dx \, ds \\ &+ C \sum_{|\mu|, |\nu| \leq 20} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |\Gamma^\mu \partial u^I| |\tilde{B}_{K,\gamma}^{I,\alpha\beta} \partial_\alpha \partial_\gamma u^K \partial_\beta (\Gamma^\nu u^J)| \, dx \, ds \\ &+ C \sum_{|\mu|, |\nu| \leq 20} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} |\tilde{B}_{K,\gamma}^{I,\alpha\beta} \partial_\gamma \partial_{t,r} u^K \partial_\alpha (\Gamma^\mu u^I) \partial_\beta (\Gamma^\nu u^J)| \, dx \, ds \\ &+ C \sum_{|\mu| \leq 20} \sum_{|\nu| \leq 19} \sum_{|\sigma| \leq 11} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|\Gamma^\mu u|}{r} |\Gamma^\nu u''| |\Gamma^\sigma u'| \, dx \, ds \\ &+ C \sum_{|\mu|, |\nu| \leq 20} \sum_{|\sigma| \leq 11} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{|\Gamma^\mu u|}{r} |\Gamma^\nu u'| |\Gamma^\sigma u''| \, dx \, ds \\ &+ C \sum_{|\mu|, |\nu| \leq 20} \sum_{|\sigma| \leq 11} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K} \cap \{|x| < 1\}} |\Gamma^\sigma u'| |\Gamma^\mu u'| \left(|\Gamma^\nu u'| + \frac{|\Gamma^\nu u|}{r} \right) \, dx \, ds \end{aligned}$$

$$\begin{aligned}
 &+ C \sum_{|\mu|,|\nu|\leq 20} \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \frac{1}{\langle x \rangle} |u'| |\Gamma^\mu u'| \left(|\Gamma^\nu u'| + \frac{|\Gamma^\nu u|}{r} \right) dx ds \\
 &+ C \sum_{|\mu|\leq 20} \sum_{|\nu|\leq 11} \|\Gamma^\nu u'\| \|\Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)}^2 + C \sum_{|\mu|\leq 20} \sum_{|\nu|\leq 11} \|\Gamma^\nu u'(t, \cdot)\| \|\Gamma^\mu u'(t, \cdot)\|_2^2.
 \end{aligned} \tag{4.7}$$

Due to constants that are introduced when Γ^μ commutes with ∂_α , the coefficients $B_{K,\gamma}^{II,\alpha\beta}$ become new constants $\tilde{B}_{K,\gamma}^{II,\alpha\beta}$. By, e.g., Lemma 4.1 of [26], it is known that Γ preserves the null forms. Thus, if $B_{K,\gamma}^{KK,\alpha\beta}$ satisfies (1.3), then so do the $\tilde{B}_{K,\gamma}^{KK,\alpha\beta}$.

In order to complete the proof, we will show that every term in (4.7) except for the first is $O(\varepsilon^3)$ if ε is sufficiently small. By (3.5), a Hardy inequality, and the Schwarz inequality, the eighth and ninth terms above are dominated by

$$C \sum_{|\mu|\leq 20} \left(\sup_{0\leq s\leq t} \|\Gamma^\mu u'(s, \cdot)\|_2 \right) \sum_{|\nu|\leq 20} \|\langle x \rangle^{-1} \Gamma^\nu u'\|_{L_t^2 L_x^2(S_t)}^2$$

which is $O(\varepsilon^3)$ by (4.1). We may similarly apply (3.5) to see that the last two terms are

$$\lesssim \sum_{|\mu|\leq 20} \left(\sup_{0\leq s\leq t} \|\Gamma^\mu u'(s, \cdot)\|_2^2 \right) \left(\sum_{|\nu|\leq 20} \|\langle x \rangle^{-1} \Gamma^\nu u'\|_{L_t^2 L_x^2(S_t)}^2 + \sum_{|\nu|\leq 20} \|\Gamma^\nu u'(t, \cdot)\|_2^2 \right)$$

which is $O(\varepsilon^4)$ by (4.1).

It thus suffices to show that the second through the seventh terms in (4.7) are $O(\varepsilon^3)$. We shall examine the regions $|x| \leq c_0s/2$ and $|x| \geq c_0s/2$ separately. Here, c_0 is as in Lemma 3.1.

4.2.1 In the region $|x| \leq c_0s/2$

This is the easier case. We look at the remaining terms in (4.7) when the spatial integrals are restricted to $|x| \leq c_0s/2$. By the Schwarz inequality and a Hardy inequality, these terms are

$$\begin{aligned}
 &\lesssim \left(\sum_{|\mu|\leq 20} \sup_{0\leq s\leq t} \|\Gamma^\mu u'(s, \cdot)\|_2 \right) \left[\int_0^t \sum_{|\nu|\leq 20, |\sigma|\leq 10} \|\Gamma^\sigma u'' \Gamma^\nu u'(s, \cdot)\|_{L^2(|x|\leq c_0s/2)} ds \right. \\
 &\qquad \qquad \qquad \left. + \int_0^t \sum_{|\nu|\leq 19, |\sigma|\leq 10} \|\Gamma^\sigma u' \Gamma^\nu u''(s, \cdot)\|_{L^2(|x|\leq c_0s/2)} ds \right] \\
 &\lesssim \varepsilon \left[\int_0^t \sum_{|\nu|\leq 20, |\sigma|\leq 10} \|\Gamma^\sigma u'' \Gamma^\nu u'(s, \cdot)\|_{L^2(|x|\leq c_0s/2)} ds \right. \\
 &\qquad \qquad \qquad \left. + \int_0^t \sum_{|\nu|\leq 19, |\sigma|\leq 10} \|\Gamma^\sigma u' \Gamma^\nu u''(s, \cdot)\|_{L^2(|x|\leq c_0s/2)} ds \right].
 \end{aligned} \tag{4.8}$$

The second inequality follows from (4.1).

For the first term on the right of (4.8), we apply (3.13) and (4.3) to see that this term is

$$\begin{aligned} &\lesssim \varepsilon^2 \sum_{|\mu| \leq 20} \int_0^t \frac{1}{\langle s \rangle} \|\langle x \rangle^{-1} \Gamma^\mu u'(s, \cdot)\|_2 \, ds \\ &\quad + \varepsilon \sum_{|\mu| \leq 20, |v| \leq 11} \int_0^t \|\Gamma^v u'(s, \cdot)\|_{L^2(|x| < 1)} \|\langle x \rangle^{-1} \Gamma^\mu u'(s, \cdot)\|_2 \, ds. \end{aligned}$$

Here, we have also applied the bound (4.1). By the Schwarz inequality and (4.1), this is indeed $O(\varepsilon^3)$. For the second term on the right of (4.8), we apply (3.5) and (3.12) to the two factors respectively, yielding the same bound as above for the first term in the right of (4.8), but with the weights $\langle x \rangle^{-1}$ replaced by $\langle x \rangle^{-1+}$. Thus, this term is also $O(\varepsilon^3)$ as desired.

4.2.2 In the region $|x| > c_0s/2$

We first show that the sixth and seventh terms in (4.7) are $O(\varepsilon^3)$ when the spatial integral is taken over $|x| > c_0s/2$. Indeed, we can apply (3.7) to see that these terms are

$$\lesssim \sum_{|\mu| \leq 21} \sum_{|v| \leq 20} \int_0^t \frac{1}{(1+s)^{1/4}} \|\Gamma^\mu u'(s, \cdot)\|_2 \|\langle x \rangle^{-5/8} \Gamma^v u'(s, \cdot)\|_2^2 \, ds.$$

Thus, by (4.1) and (4.2), this is indeed $O(\varepsilon^3)$ for ε sufficiently small.

For the remaining terms (the second, third, fourth, and fifth terms in (4.7)), there are two cases: (1) when all three wave speeds are the same, $(I, J, K) = (I, I, I)$, and (2) when there is a wave speed that is distinct from the other two.

In case (1), the null form bounds (3.1), (3.2), and (3.3) apply. In the region $|x| \geq c_0s/2$, these terms are

$$\begin{aligned} &\lesssim \int_0^t \int_{|x| \geq c_0s/2} \frac{1}{\langle s+r \rangle} \left(\sum_{|\mu| \leq 11} |\Gamma^\mu u'| \sum_{|v| \leq 20} |\Gamma^v u| \sum_{|\sigma| \leq 20} |\Gamma^\sigma u'| \right) \, dx \, ds \\ &\quad + \int_0^t \int_{|x| \geq c_0s/2} \frac{\langle c_K s - r \rangle}{\langle s+r \rangle} \left(\sum_{|\mu| \leq 11} |\Gamma^\mu \partial u^K| \right)^2 \sum_{|v| \leq 20} |\Gamma^v \partial u^K| \, dx \, ds. \end{aligned} \tag{4.9}$$

Applying (3.7), it follows as above that the first term is

$$\lesssim \int_0^t \int_{|x| \geq c_0s/2} \frac{1}{(1+s)^{1/4}} \sum_{|\mu| \leq 21} \|\Gamma^\mu u'(s, \cdot)\|_2 \sum_{|\mu| \leq 20} \|\langle x \rangle^{-5/8} \Gamma^\mu u'(s, \cdot)\|_2^2 \, ds$$

which is easily seen to be $O(\varepsilon^3)$ using (4.1) and (4.2). For the second term in (4.9), we apply (3.17), (4.1), and (4.4) to see that it is

$$\lesssim \varepsilon \int_0^t \int_{|x| \geq c_0s/2} \frac{1}{\langle x \rangle^{-3/2}} \sum_{|\mu| \leq 20} |\Gamma^\mu u'(s, x)|^2 \, dx \, ds,$$

which is $O(\varepsilon^3)$ by (4.1). This concludes the proof of the same speed case (1).

We next examine case (2), the multiple speed case. Here, we must bound

$$\int_0^t \int_{|x| \geq c_0 s/2} \sum_{|\mu| \leq 20} |\Gamma^\mu \partial u^I| \sum_{|\nu| \leq 20} |\Gamma^\nu \partial u^J| \sum_{|\sigma| \leq 20} |\Gamma^\sigma \partial u^K| \, dx \, ds \tag{4.10}$$

with $(I, K) \neq (K, J)$. For simplicity of exposition, we assume $I \neq K, I = J$. The other cases follow from symmetric arguments. We fix $\delta < |c_I - c_K|/3$. Thus, $\{|x| \in [(c_I - \delta)s, (c_I + \delta)s]\} \cap \{|x| \in [(c_K - \delta)s, (c_K + \delta)s]\} = \emptyset$, and it suffices to show that (4.10) is $O(\varepsilon^3)$ when the spatial integral is taken over the complements of these sets separately. We will show the bound over $\{|x| \notin [(c_K - \delta)s, (c_K + \delta)s]\} \cap \{|x| \geq c_0 s/2\}$. Again, the remainder of the necessary argument follows symmetrically.

If we apply (3.17) and (4.6), we have

$$\begin{aligned} & \int_0^t \int_{\{|x| \notin [(c_K - \delta)s, (c_K + \delta)s]\} \cap \{|x| \geq c_0 s/2\}} \left(\sum_{|\mu| \leq 20} |\Gamma^\mu \partial u^I| \right)^2 \sum_{|\nu| \leq 20} |\Gamma^\nu \partial u^K| \, dx \, ds \\ & \lesssim \int_0^t \frac{1}{(1+s)^{1/4}} \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(s, \cdot)\|_2 \int_{\mathbb{R}^3 \setminus \mathcal{K}} \left(\langle x \rangle^{-5/8} \sum_{|\mu| \leq 20} |\Gamma^\mu u'(s, x)| \right)^2 \, dx \, ds \end{aligned}$$

since $\langle c_K s - r \rangle \gtrsim (1+s)$ on the domain of integration in the left. Thus, by (4.1) and (4.2), this is also $O(\varepsilon^3)$ for sufficiently small ε , which completes the proof of (i).

4.3 High order energy

Here, we shall show that (4.2) follows from (4.1). By (1.4) and (2.29), the square of the left side of (4.2) is

$$\begin{aligned} & \lesssim \varepsilon^2 + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \sum_{|\mu| \leq 15} |\Gamma^\mu u'| \sum_{|\nu| \leq 29} |\Gamma^\nu u''| \sum_{|\sigma| \leq 30} |\Gamma^\sigma u'| \, dx \, ds \\ & + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \sum_{|\mu| \leq 15} |\Gamma^\mu u'| \sum_{|\nu| \leq 29} |\Gamma^\nu u''| \frac{\sum_{|\sigma| \leq 30} |\Gamma^\sigma u|}{r} \, dx \, ds \\ & + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \sum_{|\mu| \leq 15} |\Gamma^\mu u''| \left(\sum_{|\nu| \leq 30} |\Gamma^\nu u'| \right)^2 \, dx \, ds \\ & + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \sum_{|\mu| \leq 15} |\Gamma^\mu u''| \sum_{|\nu| \leq 30} |\Gamma^\nu u'| \frac{\sum_{|\sigma| \leq 30} |\Gamma^\sigma u|}{r} \, dx \, ds \\ & + \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \langle x \rangle^{-1} \sum_{|\mu| \leq 15} |\Gamma^\mu u'| \left(\sum_{|\nu| \leq 30} |\Gamma^\nu u'| \right)^2 \, dx \, ds \end{aligned}$$

$$\begin{aligned}
 &+ \int_0^t \int_{\mathbb{R}^3 \setminus \mathcal{K}} \langle x \rangle^{-1} \sum_{|\mu| \leq 15} |\Gamma^\mu u'| \sum_{|\nu| \leq 30} |\Gamma^\nu u'| \frac{\sum_{|\sigma| \leq 30} |\Gamma^\sigma u|}{r} \, dx \, ds \\
 &+ \sum_{|\mu| \leq 15} \sum_{|\nu| \leq 30} \|\Gamma^\mu u'\| \|\Gamma^\nu u'\|_{L_t^2 L_x^2(S_t)}^2 + \sum_{|\mu| \leq 15} \sum_{|\nu| \leq 30} \|\Gamma^\mu u'\| \|\Gamma^\nu u'\|(t, \cdot)\|_2^2. \tag{4.11}
 \end{aligned}$$

The last eight terms in (4.11) will be referred to as *I, II, . . . , VIII* respectively. Terms *I* and *III* are the key terms; the others are technical terms that result from our analysis of the perturbed KSS estimates.

We start by bounding the terms *I* and *II*. When the spatial integrals are over $|x| \geq c_0s/2$, it follows from (3.5), the Schwarz inequality, and a Hardy inequality that these terms are

$$\lesssim \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 17} \|\Gamma^\mu u'(s, \cdot)\|_2 \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(s, \cdot)\|_2^2 \, ds.$$

When the spatial integrals are instead over $|x| < c_0s/2$, we have that $\langle c_K s - r \rangle \gtrsim (1+s)$ for any $K = 1, \dots, D$. Thus, by (3.5), (3.12), and (4.5), these terms are

$$\begin{aligned}
 &\lesssim \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 17} \|\langle x \rangle^{-1} \Gamma^\mu u'(s, \cdot)\|_2 \left(\sum_{|\nu| \leq 30} \|\Gamma^\nu u'(s, \cdot)\|_2 + s \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(t, \cdot)\|_{L^2(|x| < 1)} \right) \\
 &+ s \sum_{|\nu| \leq 16} \|\Gamma^\nu u'(s, \cdot)\|_{L^2(|x| < 1)} \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(s, \cdot)\|_2 \Big) \sum_{|\sigma| \leq 30} \|\Gamma^\sigma u'(s, \cdot)\|_2 \, ds \\
 &\lesssim \varepsilon \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(s, \cdot)\|_2^2 \, ds \\
 &+ \varepsilon \sum_{|\nu| \leq 30} \|\Gamma^\nu u'\|_{L_t^2 L_x^2([0,t] \times \{|x| < 1\})} \sup_{0 \leq s \leq t} \sum_{|\sigma| \leq 30} \|\Gamma^\sigma u'(s, \cdot)\|_2 \\
 &+ \varepsilon^2 \sup_{0 \leq s \leq t} \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(s, \cdot)\|_2^2.
 \end{aligned}$$

Here, we have applied the Schwarz inequality and (4.1). Thus, it follows that

$$\begin{aligned}
 I + II &\lesssim \varepsilon \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(s, \cdot)\|_2^2 \, ds + \varepsilon \sup_{0 \leq s \leq t} \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(s, \cdot)\|_2^2 \\
 &+ \varepsilon \sum_{|\mu| \leq 30} \|\langle x \rangle^{-5/8} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)}^2. \tag{4.12}
 \end{aligned}$$

The last two terms can be bootstrapped and absorbed into the left side of (4.11)

The bound for the next two terms in (4.11) is similar. Again, in $|x| \geq c_0s/2$, by an application of (3.5) and (4.1), we have that these terms are

$$\lesssim \varepsilon \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(s, \cdot)\|_2^2 \, ds.$$

When the inner integrals are over $|x| < c_0s/2$, we may apply (3.13) and (4.3) to see that these terms are

$$\begin{aligned} &\lesssim \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 18} \|\Gamma^\mu u'(s, \cdot)\|_2 \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(s, \cdot)\|_2^2 ds \\ &+ \sum_{|\mu| \leq 15} \|\Gamma^\mu u'\|_{L_t^2 L_x^2(S_t \cap \{|x| < 1\})} \sum_{|\nu| \leq 30} \|\langle x \rangle^{-1} \Gamma^\nu u'\|_{L_t^2 L_x^2(S_t)} \sup_{0 \leq s \leq t} \sum_{|\sigma| \leq 30} \|\Gamma^\sigma u'(s, \cdot)\|_2. \end{aligned}$$

Here, we have also applied the Schwarz inequality and a Hardy inequality. Thus, it follows from (4.1) that

$$\begin{aligned} III + IV &\lesssim \varepsilon \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(s, \cdot)\|_2^2 ds + \varepsilon \sup_{0 \leq s \leq t} \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(s, \cdot)\|_2^2 \\ &+ \varepsilon \sum_{|\mu| \leq 30} \|\langle x \rangle^{-5/8} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)}^2, \end{aligned} \tag{4.13}$$

and again, the last two terms will be bootstrapped.

By the Schwarz inequality, a Hardy inequality, (3.5), and (4.1), we easily obtain

$$V + VI \lesssim \varepsilon \sum_{|\mu| \leq 30} \|\langle x \rangle^{-1} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)} \sup_{0 \leq s \leq t} \sum_{|\nu| \leq 30} \|\Gamma^\nu u'(s, \cdot)\|_2, \tag{4.14}$$

which will also be bootstrapped. Using (3.5), the last two terms are also easily handled, resulting in

$$VII + VIII \lesssim \varepsilon^2 \left(\sum_{|\mu| \leq 30} \|\langle x \rangle^{-1} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)}^2 + \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(t, \cdot)\|_2^2 \right). \tag{4.15}$$

If we use the estimates (4.12)–(4.15) in (4.11) and bootstrap the appropriate terms, it follows that

$$\begin{aligned} &\sum_{|\mu| \leq 30} \|\Gamma^\mu u'(t, \cdot)\|_2^2 + \sum_{|\mu| \leq 30} \|\langle x \rangle^{-5/8} \Gamma^\mu u'\|_{L_t^2 L_x^2(S_t)}^2 \\ &\lesssim \varepsilon^2 + \varepsilon \int_0^t \frac{1}{1+s} \sum_{|\mu| \leq 30} \|\Gamma^\mu u'(s, \cdot)\|_2^2 ds. \end{aligned} \tag{4.16}$$

The desired bound, (4.2), then follows from Gronwall’s inequality, which completes the proof.

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