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On the standard modules conjecture

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Abstract Let G be a quasi-split p-adic group. Under the assumption that the local coefficients C_ψ defined with respect to ψ-generic tempered representations of standard Levi subgroups of G are regular in the negative Weyl chamber, we show that the standard module conjecture is true, which means that the Langlands quotient of a standard module is generic if and only if the standard module is irreducible.

Let F be a non-archimedean local field of characteristic 0. Let G be the group of points of a quasi-split connected reductive F-group. Fix a F-Borel subgroup $B = TU$ of G and a maximal F-split torus T_0 in T. If M is any semi-standard F-Levi subgroup of G, a standard parabolic subgroup of M will be a F-parabolic subgroup of M which contains $B \cap M$.

Denote by W the Weyl group of G defined with respect to T_0 and by w_0^G the longest element in W. After changing the splitting in U, for any generic representation π of G, one can always find a non-degenerate character ψ of U, which is compatible with w_0^G, such that π is ψ-generic [11, Section 3]. For any semi-standard Levi-subgroup M of G, we will still denote by ψ the restriction of ψ to $M \cap U$. It is compatible with w_0^M. If we write in the sequel that a representation of a F-semi-standard Levi subgroup of G is ψ-generic, then we always mean that ψ is a non-degenerate character of U with the above properties.

Let $P = MU$ be a standard parabolic subgroup of G and T_M the maximal split torus in the center of M. We will write a_M^* for the dual of the real Lie-algebra a_M of $T_M, a_{M,\mathbb{C}}^*$ for its complexification and a_M^{*+} for the positive Weyl chamber in a_M^*

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defined with respect to P. Following [13], we define a map $H_M : M \to a_M$, such that $|\chi(m)|_F = q^{-\langle \chi, H_M(m)\rangle}$ for every F-rational character $\chi \in a_M^*$ of M. If π is a smooth representation of M and $\nu \in a_{M,\mathbb{C}}^*$, we denote by π_ν the smooth representation of M defined by $\pi_\nu(m) = q^{-\langle \nu, H_M(m)\rangle}\pi(m)$. (Remark that, although the sign in the definition of H_M has been changed compared to the one due to Harish–Chandra, the meaning of π_ν is unchanged.) The symbol i_P^G will denote the functor of parabolic induction normalized such that it sends unitary representations to unitary representations, G acting on its space by right translations.

Let τ be a generic irreducible tempered representation of M and $\nu \in a_M^{*+}$. Then the induced representation $i_P^G\tau_\nu$ has a unique irreducible quotient $J(\tau,\nu)$, the so-called Langlands quotient.

The aim of our paper is to prove the *standard module conjecture* [3], which states that

$$J(\tau,\nu) \text{ is generic, if and only if } i_P^G\tau_\nu \text{ is irreducible.}$$

We achieve this aim under the assumption that the local coefficients C_ψ defined with respect to ψ-generic tempered representations of standard Levi subgroups of G are regular in the negative Weyl chamber. This property of the local coefficients C_ψ would be a consequence of Shahidi's tempered L-function conjecture [11, 7.1], which is now known in most cases [6]. Nevertheless, the result that we actually need may be weaker (in particular, we do not need to consider each component r_i of the adjoint representation r separately). So it may be possible to show it independently of the tempered L-function conjecture (see the Remark in **1.6**).

Our conditional proof of the standard module conjecture follows the method developed in [7,8], but using the description of the supercuspidal support of a discrete series representation of G given in [4].

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1. Let $P = MU$ be a standard F-parabolic subgroup of G and (π, V) an irreducible ψ-generic admissible representation of M. The parabolic subgroup of G which is opposite to P will be denoted by $\overline{P} = M\overline{U}$. The set of reduced roots of T_M in Lie(U) will be denoted by $\Sigma(P)$. We will use a superscript G to underline that the corresponding object is defined relative to G.

1.1 For all ν in an open subset of a_M^* we have an intertwining operator $J_{\overline{P}|P}(\pi_\nu) : i_P^G\pi_\nu \to i_{\overline{P}}^G\pi_\nu$. For ν in $(a_M^*)^+$ far away from the walls, it is defined by a convergent integral

$$(J_{\overline{P}|P}(\pi_\nu)v)(g) = \int_{\overline{U}} v(ug)\mathrm{d}u.$$

It is meromorphic in ν and the map $J_{P|\overline{P}}J_{\overline{P}|P}$ is scalar. Its inverse equals Harish–Chandra's μ-function up to a constant and will be denoted $\mu(\pi,\nu)$.

1.2 Put $\widetilde{w} = w_0^G w_0^M$. Then $\widetilde{w}\overline{P}\widetilde{w}^{-1}$ is a standard parabolic subgroup of G. For any $\nu \in a_M^*$ there is a Whittaker functional $\lambda_P(\nu,\pi,\psi)$ on $i_P^G V$. It is a linear functional on $i_P^G V$, which is holomorphic in ν, such that for all $v \in i_P^G V$ and all $u \in U$ one has $\lambda_P(\nu,\pi,\psi)((i_P^G \pi_\nu)(u)v) = \overline{\psi(u)}\lambda_P(\nu,\pi,\psi)(v)$. Remark that by Rodier's theorem [9], $i_P^G \pi_\nu$ has a unique ψ-generic irreducible sub-quotient.

Fix a representative w of $\widetilde{w}$ in K. Let $t(w)$ be the map $i_{\overline{P}}^G V \to i_{w\overline{P}}^G wV$, which sends v to $v(w^{-1}\cdot)$. There is a complex number $C_\psi(\nu,\pi,w)$ [10] such that $\lambda(\nu,\pi,\psi) = C_\psi(\nu,\pi,w)\lambda(w\nu,w\pi,\psi)t(w)J_{\overline{P}|P}(\pi_\nu)$. The function $a_M^* \to \mathbb{C}, \nu \mapsto C_\psi(\nu,\pi,w)$ is meromorphic.

The local coefficient C_ψ satisfies the equality $C_\psi(\cdot,\pi,w)C_\psi(w(\cdot),w\pi,w^{-1}) = \mu(\pi,\nu)$ [10].

1.3 We will use the following criterion which follows easily from the definitions and Rodier's theorem [9]:

Proposition *If (π, V) is an irreducible tempered representation of M and $\nu \in a_M^{*+}$, then the Langlands quotient of the induced representation $i_P^G \pi_\nu$ is ψ-generic if and only if π is ψ-generic and $C_\psi(\cdot,\pi,w)$ is regular in ν.*

1.4 For $\alpha \in \Sigma(P)$, put $w_\alpha = w_0^{M_\alpha} w_0^M$. With this notation, one has the following version of the multiplicative formula for the local coefficient $C_\psi(\cdot,\pi,\psi)$ [10, Proposition 3.2.1]:

Proposition *Let $Q = NV$ be a standard parabolic subgroup of $G, N \subseteq M$, and τ an irreducible generic representation of N, such that π is a sub-representation of $i_{Q\cap M}^M \tau$. Then one has*

$$C_\psi^G(\cdot,\pi,w) = \prod_{\alpha\in\Sigma(Q)-\Sigma(Q\cap M)} C_\psi^{N_\alpha}(\cdot,\tau,w_\alpha).$$

Proof It follows from [10, Proposition 3.2.1] that

$$C_\psi^G(\cdot,\pi,w) = \prod_{\alpha\in\Sigma(P)} C_\psi^{M_\alpha}(\cdot,\tau,w_\alpha).$$

Now fix $\alpha \in \Sigma(P)$. As $J^{M_\alpha}_{\overline{P}\cap M_\alpha|P\cap M_\alpha}(\pi_\nu)$ [resp. $\lambda^{M_\alpha}(\nu,\pi,\psi)$] equals the restriction of $J^{M_\alpha}_{\overline{Q}\cap M_\alpha|Q\cap M_\alpha}(\tau_\nu)$ [resp. $\lambda^{M_\alpha}(\nu,\tau,\psi)$] to the space of $i^{M_\alpha}_{P\cap M_\alpha}\pi_\nu \subseteq i^{M_\alpha}_{Q\cap M_\alpha}\tau_\nu$, it follows that

$$C_\psi^{M_\alpha}(\nu,\pi,w_\alpha) = C_\psi^{N_\alpha}(\nu,\tau,w_\alpha).$$

Applying the above product formula to the expression on the right, one gets the required identity. □

1.5 Recall the tempered L-function conjecture [10,11, 7.1]: if σ is an irreducible tempered representation of M and M is a maximal Levi-subgroup of G_1, then for every component r_i of the adjoint representation r the L-function $L(s,\sigma,r_{G_1,i})$ is holomorphic for $\Re(s) > 0$.

Proposition *Let π be an irreducible generic tempered representation of M. Assume that the tempered L-function holds for π relative to any $M_\alpha, \alpha \in \Sigma(P)$.*

Then $C_\psi(w(\cdot),w\pi,w^{-1})$ is regular in $(a_M^*)^+$.

Proof Denote by $(a_M^{M_\alpha})^*$ the annihilator of a_{M_α} in a_M^* and by $\widetilde{\alpha}$ the element in $(a_M^{M_\alpha})^*$ deduced from α as in [11, (3.11)]. Let $\lambda \in (a_M^*)^+$. Denote by $s_\alpha \widetilde{\alpha}, s_\alpha \in \mathbb{R}$, the orthogonal projection of λ on $(a_M^{M_\alpha})^*$. Then $s_\alpha > 0$. By Proposition 1.4 applied to $\tau = \pi$,

$$C_\psi(w(\lambda), w\pi, w^{-1}) = \prod_{\alpha \in \Sigma(P)} C_\psi^{M_{w\alpha}}(-s_\alpha \widetilde{w\alpha}, w\pi, w_{w\alpha}).$$

By [11, 3.11, 7.8.1 and 7.3], as meromorphic functions in s,

$$C_\psi^{M_{w\alpha}}(s\, \widetilde{w\alpha}, w\pi, w_{w\alpha}) = * \prod_i \frac{L(1 - is, w\pi, r_{M_{w\alpha},i})}{L(is, w\pi, r_{M_{w\alpha},i})},$$

where $*$ denotes a monomial in $q^{\pm s}$.

Now $L(\cdot, w\pi, r_{M_{w\alpha},i})$ is regular in $1 + is_\alpha$ by assumption. As $1/L(\cdot, w\pi, r_{M_{w\alpha},i})$ is polynomial, this proves the proposition. □

Remark 1.6 In fact, what is really needed to prove the above proposition is a result that may be weaker than the tempered L-function conjecture: suppose for simplicity that π is square integrable and choose a standard parabolic subgroup $Q = NV$ of G and a unitary supercuspidal representation σ of $N, N \subseteq M$, and $\nu \in a_N^*$, such that π is a sub-representation of $i_{Q\cap M}^M \sigma_\nu$. Then, by **1.4**,

$$C_\psi^G(s\widetilde{\alpha}, \pi, w) = \prod_{\alpha \in \Sigma(Q) - \Sigma(Q\cap M)} C_\psi^{M_\alpha}(s\widetilde{\alpha} + \nu, \sigma, w_\alpha).$$

This is, up to a meromorphic function on the real axes, equal to

$$\prod_{\alpha \in \Sigma(Q) - \Sigma(Q\cap M)} \frac{L(1 - i_\alpha s, \sigma_\nu, r_{M_{w_\alpha}, i_\alpha})}{L(i_\alpha s, \sigma_\nu, r_{M_{w_\alpha}, i})}$$

where $i_\alpha \in \{1, 2\}$ and $i_\alpha = i_\beta$ if α and β are conjugated.

Now let $\Sigma_{\mathcal{O}}(N)$ be the set of reduced roots α of T_N in Lie(V), such that Harish–Chandra's μ-function μ^{N_α} defined with respect to N_α and σ has a pole. The set of these roots forms a root system [12, 3.5]. Denote by $\Sigma_{\mathcal{O}}(Q)$ the subset of those roots, which are positive for Q and by $\Sigma_{\mathcal{O}}(Q \cap M)$ the subset of those roots in $\Sigma_{\mathcal{O}}(Q)$, which belong to M. Then the above product equals, up to a holomorphic function, to

$$\prod_{\alpha \in \Sigma_{\mathcal{O}}(Q) - \Sigma_{\mathcal{O}}(Q\cap M)} \frac{1 - q^{-i_\alpha(s + \langle\alpha^\vee, \nu\rangle)}}{1 - q^{-1 + i_\alpha(s + \langle\alpha^\vee, \nu\rangle)}}. \qquad (*)$$

So, what we have to know, is that this meromorphic function is holomorphic for $s < 0$. The fact that σ_ν lies in the supercuspidal support of a discrete series means by the main result of [4], that σ_ν is a pole of order $rk_{ss}(M) - rk_{ss}(N)$ of Harish–Chandra's μ-function μ^M. This is equivalent to saying that ν is a pole of order $rk_{ss}(M) - rk_{ss}(N)$ of the meromorphic function

$$\prod_{\alpha \in \Sigma_{\mathcal{O}}(Q\cap M)} \frac{1 - q^{\pm i_\alpha(\langle\alpha^\vee, \cdot\rangle)}}{1 - q^{-1 \pm i_\alpha(\langle\alpha^\vee, \cdot\rangle)}},$$

where the labels i_α satisfy the same condition as above. So one could state the independent combinatorial question, whether the product (*) is holomorphic for $s < 0$ if ν has the above properties.

This would be enough to prove **1.5** unconditionally, but by this way one would perhaps not get the tempered L-function conjecture itself, because in that case one has to consider each component r_i of the adjoint representation r individually.

2. In this section we make the following assumption on G (see the Remark in **1.5** for what we actually need):

(TL) If M is a semi-standard Levi subgroup of G and if π is an irreducible generic tempered representation of M then $L(s,\pi,r_i)$ is regular for $\Re(s) > 0$ for every i.

We give a proof of the following lemma only for completeness:

Lemma 2.1 *Let $P = MU$ be a F-standard parabolic subgroup of G and σ an irreducible supercuspidal representation of M. If the induced representation $i_P^G\sigma$ has a sub-quotient, which lies in the discrete series of G, then any tempered sub-quotient of $i_P^G\sigma$ lies in the discrete series of G.*

Proof If $i_P^G\sigma$ has an irreducible sub-quotient, which is square-integrable, then by the main result of [4,5], σ is a pole of order $rk_{ss}(G) - rk_{ss}(M)$ of μ and $\sigma_{|A_M}$ is unitary. It follows that the central character of $i_P^G\sigma$ is unitary, too, which implies that the central character of any irreducible sub-quotient of $i_P^G\sigma$ is unitary. In particular, any essentially tempered irreducible sub-quotient of $i_P^G\sigma$ is tempered.

So, if τ is an irreducible tempered sub-quotient of $i_P^G\sigma$, then there is a F-parabolic subgroup $P' = M'U'$ of G and a square-integrable representation σ' of M', such that τ is a sub-representation of $i_{P'}^G\sigma'$. The supercuspidal support of σ' and the W-orbit of σ share a common element σ_0.

By the invariance of Harish–Chandra's μ-function, σ_0 is still a pole of μ of the order given above. As this order is maximal and the central character of σ' must be unitary, this implies that M' must be equal to G, and consequently $\tau = \sigma'$ is square-integrable. □

Theorem 2.2 *Let G be a group that satisfies property (TL). Let $P = MU$ be a F-standard parabolic subgroup of G and σ be an irreducible ψ-generic supercuspidal representation of M.*

If the induced representation $i_P^G\sigma$ has a sub-quotient, which lies in the discrete series of G (resp. is tempered), then any irreducible ψ-generic sub-quotient of $i_P^G\sigma$ lies in the discrete series of G (resp. is tempered).

Proof First assume that $i_P^G\sigma$ has a sub-quotient, which lies in the discrete series of G. Let (π, V) be an irreducible, admissible ψ-generic representation of G, which is a sub-quotient of $i_P^G\sigma$. By the Langlands quotient theorem, there is a standard parabolic subgroup $P_1 = M_1U_1$ of G, an irreducible tempered representation τ of M_1 and $\nu \in (a_{M_1}^*)^+$, such that π is the unique irreducible quotient of $i_{P_1}^G\tau_\nu$.

As any representation in the supercuspidal support of τ_ν must lie in the supercuspidal support of π, any such representation must be conjugated to σ. So, after conjugation by an element of G, we can assume that $M \subseteq M_1$ and that τ_ν is a sub-representation of $i_{P\cap M_1}^{M_1}\sigma$. (One can find a standard parabolic subgroup $P' = M'U'$ of G and an irreducible supercuspidal representation σ' of M' such that τ_ν is a sub-representation of $i_{P'\cap M_1}^{M_1}\sigma'$ [2, 6.3.7]. By the above remarks, P and σ can be replaced by P' and σ' to get a situation as claimed.)

We will actually show that $P_1 = G$, which means that π is tempered and by **2.1** in fact square-integrable.

Following **1.3**, τ must be ψ-generic and it is enough to show that $C_\psi(\cdot,\tau,w)$ has a pole in ν, if $P_1 \neq G$.

For this we will use the assumption that $i_P^G\sigma$ has an irreducible sub-quotient which is square-integrable. By the main result of [4,5] this implies that μ has a pole of order equal to $rk_{ss}G - rk_{ss}M$ in σ. Remark that μ^{M_1} can have at most a pole of order $rk_{ss}M_1 - rk_{ss}M$ in σ [4,5, Corollary 8.7]. The order of the pole of μ in τ_ν is equal to the one of μ/μ^{M_1} in σ, as τ_ν is a sub-representation of $i_{P\cap M_1}^{M_1}\sigma$. It follows that the order of this pole must be > 0, if $P_1 \neq G$. As

$$\mu(\tau,\cdot) = C_\psi(\cdot,\tau,w)C_\psi(w(\cdot),w\tau,w^{-1}),$$

it follows that either $C_\psi(\cdot,\tau,w)$ or $C_\psi(w(\cdot),w\tau,w^{-1})$ must have a pole in ν. As $\nu \in (a_M^*)^+$, it follows from **1.5** that $C_\psi(w(\cdot),w\tau,w^{-1})$ cannot have a pole in ν. So $C_\psi(\cdot,\tau,w)$ does. This gives us the desired contradiction.

Now assume that $i_P^G\sigma$ only has a tempered sub-quotient τ. Then there is a standard parabolic subgroup $P_1 = M_1U_1$ of G and a discrete series representation π_1 of M_1, such that τ is a sub-representation of $i_{P_1}^G\pi_1$. As the supercuspidal support of π_1 is contained in the G-conjugacy class of σ, it follows that there is a standard Levi subgroup $M' \supseteq M$, such that $i_{P\cap M'}^{M'}\sigma$ has a discrete series sub-quotient.

By what we have just shown, there exists a unique ψ-generic subquotient π' of $i_{P\cap M'}^{M'}\sigma$, which lies in the discrete series.

As $i_P^G\sigma$ and $i_{P'}^G\pi'$ have each one a unique irreducible ψ-generic sub-quotient and any sub-quotient of $i_{P'}^G\pi'$ is a sub-quotient of $i_P^G\sigma$, these irreducible ψ-generic sub-quotients must be equal and therefore tempered. □

Theorem 2.3 *Let G be a group that satisfies property (TL). Let $P = MU$ be a F-standard Levi subgroup of G, τ an irreducible tempered generic representation of M and $\nu \in a_M^{*+}$.*

Then the Langlands quotient $J(\tau,\nu)$ is generic, if and only if $i_P^G\tau_\nu$ is irreducible.

Proof As $i_P^G\tau_\nu$ always has a generic sub-quotient, one direction is trivial. So, assume $i_P^G\tau_\nu$ is reducible. We will show that $\pi = J(\nu,\tau)$ is not ψ-generic for any ψ.

We can consider (and will) $\nu_\pi := \nu$ as an element of a_T^*. We denote by $<$ the partial order on a_T^* explained in (1, Chapter XI, 2.1) (for our purpose it is not important to write it explicitly).

Let π' be an irreducible sub-quotient of $i_P^G\tau_\nu$, which is not isomorphic to π. Let $P' = M'U'$ be a F-standard parabolic subgroup, τ' an irreducible tempered representation of M' and $\nu' \in a_{M'}^{*+}$, such that $\pi' = J(\nu',\tau')$. Let $\nu_{\pi'} := \nu'$. Then [1, XI, Lemma 2.13]

$$\nu_{\pi'} < \nu_\pi. \tag{2.1}$$

Choose an F-standard parabolic subgroup $P_1 = M_1U_1, M_1 \subseteq M$, with an irreducible ψ–generic supercuspidal representation σ of M_1, such that τ is a sub-quotient of $i_{P_1\cap M}^M\sigma$.

Then σ_ν lies as well in the supercuspidal support of π as in the supercuspidal support of π'. It lies also in the G-conjugacy class of the supercuspidal support of $\tau'_{\nu'}$ and τ_ν. Let π_ψ be the unique ψ-generic irreducible sub-quotient of $i_{P_1}^G\sigma_\nu$. By **2.2**, the unique

ψ-generic irreducible sub-quotient τ'' of $i_{P_1 \cap M'}^{M'} \sigma$ is tempered. The induced representation $i_{P'}^G \tau''_{\nu'}$ admits a unique ψ-generic irreducible sub-quotient, which is equal to the unique ψ-generic sub-quotient of $i_{P_1}^G \sigma_\nu$. Let $\pi'' = J(\nu', \tau'')$ be the Langlands quotient of $i_{P'}^G \tau''_{\nu'}$. Since (2.1) implies $\nu_{\pi''} = \nu_{\pi'} < \nu_\pi, \pi$ cannot be a sub-quotient of $i_{P'}^G \tau''_{\nu'}$ by [1, XI, Lemma 2.13]. Therefore, π is not ψ-generic. □

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